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1 **A novel method for determining the feasible integral self-stress states for tensegrity**  
2 **structures**

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8

9 **Abstract**

10 The form-finding analysis is a crucial step for determining the stable self-equilibrated states for  
11 tensegrity structures, in the absence of external loads. This form-finding problem leads to the  
12 evaluation of both the self-stress in the elements and the shape of the tensegrity structure. This paper  
13 presents a novel method for determining feasible integral self-stress states for tensegrity structures, that  
14 is self-equilibrated states consistent with the unilateral behaviour of the elements, struts in compression  
15 and cables in tension, and with the symmetry properties of the structure. In particular, once defined the  
16 connectivity between the elements and the nodal coordinates, the feasible self-stress states are  
17 determined by suitably investigating the Distributed Static Indeterminacy (DSI). The proposed method  
18 allows for obtaining feasible integral self-stress solutions by a unique Singular Value Decomposition  
19 (SVD) of the equilibrium matrix, whereas other approaches in the literature require two SVD.  
20 Moreover, the proposed approach allows for effectively determining the Force Density matrix, whose  
21 properties are strictly related to the super-stability of the tensegrity structures. Three tensegrity

22 structures were studied in order to assess and discuss the efficiency and accuracy of the proposed  
23 innovative method.

24 [Paper included in the Special Issue entitled: "Shell and Spatial Structures: Between New Developments  
25 and Historical Aspects"](#).

## 26 **Keywords**

27 Tensegrity structures, self-equilibrium, feasible self-stress states.

## 28 **1. Introduction**

29 Tensegrity structures are an intriguing class of reticulated systems and hold promising possibilities in  
30 different applications: from architecture [1,2] to civil engineering [3–6], from biology [7,8] to  
31 aerospace [9–11], as well as from robotics [12–15] to the design of metamaterials [16–20].

32 Originally proposed by Buckminster Fuller [21], tensegrity structures can be defined as a, usually free-  
33 standing, pre-stressed, pin-jointed system, composed by a network of tensile elements (cables) within a  
34 discontinuous set of compressed elements (struts). The initial pre-stressed condition allows for the  
35 rigidity and the stability of the tensegrity structures [22].

36 It is evident that the mechanical behaviour of these structures is highly dependent on the self-stress  
37 states [23]. Thus a complete analysis of tensegrity structures is made of two key points: first, the form-  
38 finding problem, and then the study of the response to the external loads [24].

39 The process of form-finding depends on the initial input parameters, that is, the geometry of the  
40 structure and the level of the self-stress in the elements [25,26]. Commonly, both the geometry and the  
41 self-stress are unknown variables of the problem. If only the latter is known, i.e. the internal forces in  
42 the elements in the self-equilibrium state are defined, the problem reduces to the seeking of the nodal  
43 coordinates of the structure, which can be determined from the analysis of the equilibrium states. On

44 the other hand, if the geometry of the tensegrity structure is known, that is, the nodal coordinates and  
45 the connectivity between elements are prescribed, the problem turns out to be the initial self-stress  
46 identification (*force-finding* problem) [27].

47 In the latter case, however, difficulties arise with the evaluation of the level of the self-stress and then  
48 of suitable self-stress vectors which taking into account both the unilateral behaviour of the elements  
49 and the self-equilibrium of the tensegrity structure [28]. This happens, especially, for tensegrity  
50 structures with multiple independent self-stress states [29]. Indeed, in general, the independent self-  
51 stress modes obtained from the null-space of the equilibrium matrix do not meet the predefined  
52 unilateral behaviour of the elements [30]. Thus, it is necessary to determine a special combination of  
53 such independent self-stress modes in order to define possible feasible self-stress states [31].

54 It is worth to recall that, a *feasible self-stress state* is a self-stress state consistent both with the self-  
55 equilibrium of the tensegrity structure and the unilateral behaviour of the elements, that is, cables in  
56 tension and struts in compression [27]. If a feasible self-stress state also satisfies the symmetry  
57 properties of the structure, it takes the name of *feasible integral self-stress state* [32].

58 In the recent past, various efficient analytical or numerical form-finding methods [33] have been  
59 proposed: among the others, Force Density Method (FDM) [34–36], programming method [37–40],  
60 dynamics relaxation method [41,42], finite-element method [43,44], optimization-based method [45–  
61 47].

62 In the present work, the FDM has been used in order to tackle the self-equilibrium problem for  
63 tensegrity structures.

64 The concept of the force density, originally proposed in [48], corresponds to the ratio between the  
65 internal forces in the elements and their lengths. Such quantities are clearly affected by the sign, i.e.

66 positive for cables and negative for struts. By considering the force densities of the elements, the non-  
67 linear problem of the equilibrium can be neatly linearized [49].

68 Many researchers have made considerable efforts for improving the application of the FDM to the  
69 form-finding of tensegrity structures. Among them, Xian et al. [50] proposed an optimization approach  
70 based on the FDM and the mixed-integer nonlinear programming for the design of tensegrity  
71 structures. The member connectivity, as well as the nodal coordinates and force densities, are  
72 simultaneously used as design variables.

73 Cai et al. [51] studied the form-finding problem of tensegrity structures with multiple equilibrium  
74 modes by means of an equivalent optimization problem of an energy-based objective function with  
75 Lagrange multipliers. Different structural modes corresponding to different symmetry grouping  
76 conditions were achieved.

77 Also, Cai and Feng [52] proposed an efficient form-finding method based on the optimization method;  
78 here, the force densities of the elements of a tensegrity structure are obtained by minimizing a special  
79 objective function, which satisfies the non-degeneracy necessary condition for the force density matrix.

80 Zhang and Ohsaki [34] presented a numerical method for the form-finding of tensegrity structures. In  
81 particular, eigenvalue analysis and spectral decomposition were carried out iteratively to find the  
82 feasible set of force densities that satisfies the requirement on the rank deficiency of the equilibrium  
83 matrix with respect to the nodal coordinates.

84 In addition, Zhang et al. [25] presented a highly efficient form-finding method for tensegrity systems  
85 based on the structural stiffness matrix defined as the derivative of the out-of-balance force vector with  
86 respect to the nodal coordinate vector.

87 Lee et al [53] have studied the truncated polyhedral tensegrity structures by means of a generalized  
88 form-finding procedure by using the FDM combined with a genetic algorithm. Additionally, Gan et al.  
89 [54] suggested a novel and versatile numerical technique for determining a self-stress state in a  
90 combination with a genetic algorithm as a form-finding procedure for an irregular tensegrity structure.

91 Yuan et al. [55] presented a novel and versatile form-finding method for tensegrity structures based on  
92 nonlinear equilibrium equations where the nodal coordinates vectors are variables. The input parameters  
93 for the form-finding method are the topology, the initial configuration of the structure, the rest lengths,  
94 and the axial stiffness of elements.

95 Koohestani [56] utilized the Faddeev-LeVerrier algorithm to generate relationships between force  
96 densities of elements, providing explicit analytical conditions for self-stressed states. This method only  
97 requires sum and multiplications as major computational operations and overcomes complicated  
98 triangular factorizations and eigenvalue decompositions of the symbolic force density matrix.

99 Moreover, Gomez Estrada et al. [57] proposed a numerical form-finding procedure which only requires  
100 the specification of the type of each member, i.e. cable or strut, and the connectivity of the nodes.  
101 Iterative adjustment of the member forces are made until the state of self-stress is found.

102 Moreover, for describing the mechanical behaviour of tensegrity structures [58–60], the static and  
103 kinematic indeterminacy evaluation can be effectively used as a method for structural identification.  
104 For defining the contribution of each element to the total degree of indeterminacy of the structure, also  
105 taking into account the influence of the material properties, it can be used the *distributed static*  
106 *indeterminacy* (DSI) value [61]. Thus, DSI can represent the mechanical behaviour of flexible  
107 structures in the primary design. Moreover, in [61] a unified method for the DSI evaluation is  
108 proposed, both for kinematically determinate and indeterminate structures. It has been highlighted that

109 since DSI takes into account symmetry properties, a simple but efficient grouping criterion of the  
110 elements of the structure can be established for improving the efficiency of the force-finding method.

111 Notice that DSI values are related to both geometric and stiffness symmetry properties of the structure;  
112 moreover, stiffness symmetry (depending on the axial stiffness of the elements) can be inconsistent  
113 with the geometric symmetry (depending only on the position of the elements).

114 The application of DSI suggested in [61] concerns the use of DSI as simple and efficient grouping  
115 criterion into a specific Force Density Method called *Double Singular Value Decomposition* (DSVD)  
116 [62]. Moreover, DSI values were used as symmetry indicators for generating an initial group  
117 classification of the elements of a cable-strut structure for performing a DSVD [63]. However such  
118 initial group clustering of elements only reduces the iteration time of seeking a proper grouping scheme  
119 for the DSVD.

120 Moreover, once obtained the self-stress states in the elements **by using the proposed approach**, it is  
121 possible to determine the Force Density matrix [66], whose characteristics are crucial for studying the  
122 self-equilibrium problem and the stability conditions for tensegrity structures [67,68].

123 Many authors studied different kinds of problems related to the form-finding of the tensegrity  
124 structures based on the properties of the Force Density matrix.

125 Chen et al. [67] pointed out an improved symmetry method for the analytical form-finding of tensegrity  
126 structures based on the group representation theory and the FDM. This approach requires only to  
127 specify the symmetry properties and the connectivity of the structure. However, with the increase of the  
128 element type, the computational complexity of the determination of the Force Density matrix increases.

129 Based on the characteristic polynomial of the symbolic Force Density matrix, a general analytical  
130 scheme for tensegrity form-finding analysis was proposed by Zhang et al. [68]. Also for this case, the

131 proposed method requires high computational efforts as the geometrical complexity of the structure  
132 increases

133 Tran and Lee [69] presented a numerical method for form-finding of tensegrity structures in which the  
134 topology and the types of members are the only required information; the eigenvalue decomposition of  
135 the Force Density matrix and the single value decomposition of the equilibrium matrix are performed  
136 iteratively.

137 Another relevant issue concerns stability conditions. In this case, the Force Density matrix plays a  
138 fundamental role in the analysis of the necessary and sufficient conditions for the super-stability, i.e.,  
139 the property for a tensegrity structure to be stable irrespectively of the selection of materials and of the  
140 level of self-stress in the elements [70]. Indeed, a  $d$ -dimensional tensegrity structure is said to be super-  
141 stable if the Force Density matrix is positive semi-definite and its rank deficiency is equal to  $d + 1$ , and  
142 it has a *non-degenerate* geometry in the  $d$ -dimensional space [70].

143 In the literature, to the best of the Author's knowledge, the force-finding problem for tensegrity with  
144 multiple independent self-stress modes has been carried out by using cumbersome approaches:  
145 optimization techniques, mixed-integer nonlinear programming strategies, spectral decompositions,  
146 stiffness matrix evaluations and numerical iterative procedures.

147 Thus, as mentioned above, a more efficient algorithm for determining the feasible integral self-stress  
148 states for tensegrity structures by using the DSI values needs to be investigated. In particular, it should  
149 be avoided the second *Singular Value Decomposition* (SVD) for reducing time-consuming inherent the  
150 grouping operation.

151 In this paper, an innovative and efficient method for determining feasible integral self-stress states for  
152 tensegrity structures is proposed by considering the Distributed Static Indeterminacy (DSI) evaluation.

153 The only required initial data are the topology of the structure, i.e. the connectivity relations between  
154 the elements and their types (cables or struts), and the nodal coordinates.

155 Two advantages of the proposed approach can be remarked. First, a unique (SVD) of the equilibrium  
156 matrix has to be carried out for determining the independent self-stress modes, which span the null-  
157 space of this matrix, then through the DSI evaluation it is possible to determine the feasible self-stress  
158 states. To this aim, a linear combination of the independent self-stress modes consistent with the  
159 flexibility properties of the elements of the tensegrity structure can be evaluated. From this stems the  
160 second advantage consisting in the possibility of obtaining different feasible self-stress states according  
161 to the design needs by choosing the material parameters of the elements, that is the Young's modulus  
162 and the cross-sectional area.

163 Such innovative method can be especially useful for the analysis of tensegrity structures with multiple  
164 independent self-stress states. Unlike the existing methods in literature [27,64,65], in the proposed  
165 approach the combined conditions coming from the stiffness symmetry and the geometry symmetry of  
166 the tensegrity structure can be satisfied without using further grouping operations, which usually are  
167 inferred from a visual inspection of the structure.

168 Furthermore, it can be noted that the Force Density matrix is strictly related to the connectivity  
169 properties of the system, i.e. the relations between the elements of the structure and the nodes, and to  
170 the level of the self-stress in the elements.

171 The approach here proposed effectively allows for determining the Force Density matrix and its  
172 properties with a low computational cost. It reveals to be useful for all the analysis for the tensegrity  
173 structures above recalled: the form-finding analysis, the investigation of the super-stability conditions,  
174 and the study of the relations between elements and of the self-stress level according to the actual axial  
175 stiffness of the elements.

176 The paper is organized as follows: Section 2 briefly introduces the basic idea of the FDM. In Section 3,  
 177 the concept of the DSI is recalled and its application to the tensegrity structures is explained. Section 4  
 178 is devoted to the description of the novel method here proposed. Section 5 recalls the definition of the  
 179 Force Density matrix and illustrates its formulation according to the proposed approach. Finally, for  
 180 validating the method several well-known tensegrity structures are studied in Section 6.

## 181 2. Force Density Method

182 In this Section, we briefly recall the self-equilibrium problem for tensegrity structures. The following  
 183 assumptions are made:

- 184 • elements (struts and cables) are rectilinear and connected only at their ends by pin-joints;
- 185 • nodal coordinates and nodal connectivity are given;
- 186 • no external loads are applied;
- 187 • the cross-sectional area  $A$  of each element remains unchanged under the pre-stress.

188 We consider a tensegrity structure with  $e$  elements ( $s_t$  struts and  $c$  cables, that is,  $s_t + c = e$ ) connected  
 189 to  $n$  nodes ( $e < 3n$ ). Nodal coordinates are expressed in a Cartesian orthogonal reference  
 190 system  $O\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$  and are collected in three vectors  $\mathbf{x}$ ,  $\mathbf{y}$  and  $\mathbf{z} \in \mathbb{R}^n$ , respectively.

191 By the Graph Theory [71], member connectivity relations can be expressed by means of the so-called  
 192 *Connectivity matrix*  $\mathbf{C} \in \mathbb{R}^{e \times n}$  [36]. In particular, if the member  $k$  connects the node  $i$  to the node  $j$ , then  
 193 the  $k$ -th row of  $\mathbf{C}$  has only two non-zero entries in the  $i$ -th and  $j$ -th position ( $i < j$ ), which are equal to 1  
 194 and -1 respectively. Hence:

$$[\mathbf{C}]_{k,p} = \begin{cases} +1 & \text{if } p = i \\ -1 & \text{if } p = j \\ 0 & \text{otherwise} \end{cases} \quad k = 1, \dots, e, p = 1, \dots, n. \quad (1)$$

195 Furthermore, the length  $l_k$  of the  $k$ -th member of the structure can be expressed as:

$$l_k = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2}. \quad (2)$$

196 For our purposes, the matrix  $\mathbf{L} \in \mathbb{R}^{\text{exc}}$  is defined as the diagonal matrix by collecting the lengths of the  
 197 elements.

198 The self-equilibrium problem can be solved by using FDM. To this aim, for the  $k$ -th element of the  
 199 structure it is possible to define the force density  $q_k$ :

$$q_k = \frac{t_k}{l_k}, \quad (3)$$

200 where  $t_k$  denotes the internal force in the element  $k$  ( $t_k$  is positive for cables and negative for struts) in  
 201 the self-stress state. Force densities of the elements can be grouped in the vector  $\mathbf{q} \in \mathbb{R}^c = \{q_1, q_2, \dots,$   
 202  $q_k\}$ , whose matrix diagonalization is  $\mathbf{Q} \in \mathbb{R}^{\text{exc}}$ , i.e.,  $\mathbf{Q} = \text{diag}(\mathbf{q})$ .

203 Considering both Eq. (1) and Eq. (3), the equilibrium equations for the tensegrity structure in the three  
 204 directions  $\mathbf{e}_x$ ,  $\mathbf{e}_y$ , and  $\mathbf{e}_z$  can be then expressed in the following matrix linear form [33]:

$$\begin{cases} \mathbf{C}^T \mathbf{Q} \mathbf{C} \mathbf{x} = \mathbf{0} \\ \mathbf{C}^T \mathbf{Q} \mathbf{C} \mathbf{y} = \mathbf{0}, \\ \mathbf{C}^T \mathbf{Q} \mathbf{C} \mathbf{z} = \mathbf{0} \end{cases} \quad (4)$$

205 where the superscript ‘‘T’’ indicates the usual matrix transposition operation.

206 Alternatively, by considering the element internal forces vector  $\mathbf{t} \in \mathbb{R}^c = \{t_1, t_2, \dots, t_k\}$ , the equilibrium  
 207 equations in Eq. (4) can be written as:

$$\begin{cases} \mathbf{C}^T \text{diag}(\mathbf{C} \mathbf{x}) \mathbf{L}^{-1} \mathbf{t} = \mathbf{0} \\ \mathbf{C}^T \text{diag}(\mathbf{C} \mathbf{y}) \mathbf{L}^{-1} \mathbf{t} = \mathbf{0}. \\ \mathbf{C}^T \text{diag}(\mathbf{C} \mathbf{z}) \mathbf{L}^{-1} \mathbf{t} = \mathbf{0} \end{cases} \quad (5)$$

208 By introducing the *equilibrium matrix*  $\mathbf{A} \in \mathbb{R}^{3n \times e}$  [66], Eq. (5) can be rewritten in a compact form as:

$$\mathbf{A}\mathbf{t} = \mathbf{0}, \quad (6)$$

209 where the equilibrium matrix  $\mathbf{A}$  can be expressed as:

$$\mathbf{A} = \begin{bmatrix} \mathbf{C}^T \text{diag}(\mathbf{C}\mathbf{x})\mathbf{L}^{-1} \\ \mathbf{C}^T \text{diag}(\mathbf{C}\mathbf{y})\mathbf{L}^{-1} \\ \mathbf{C}^T \text{diag}(\mathbf{C}\mathbf{z})\mathbf{L}^{-1} \end{bmatrix}. \quad (7)$$

210 Let  $r_A$  be the rank of  $\mathbf{A}$ ; if  $r_A < e$ , non-trivial solutions exist. These non-trivial solutions correspond to  $s$   
 211 *independent self-stress modes*, which can be viewed as the bases of the vector space of the internal  
 212 forces in the elements, with:

$$s = e - r_A \geq 1. \quad (8)$$

213 Hence, it is possible to define a matrix  $\mathbf{S} \in \mathbb{R}^{e \times s}$  whose  $i$ -th column is the  $\mathbf{s}_i$  independent self-stress  
 214 mode, i.e.:

$$\mathbf{S} = [\mathbf{s}_1, \mathbf{s}_2, \dots, \mathbf{s}_s] = \begin{bmatrix} s_{11} & s_{21} & \cdots & s_{s1} \\ s_{12} & s_{22} & \cdots & s_{s2} \\ \vdots & \vdots & \cdots & \vdots \\ s_{1e} & s_{2e} & \cdots & s_{se} \end{bmatrix}. \quad (9)$$

215 A general solution of Eq. (6) can be determined as a linear combination of  $s$  independent self-stress  
 216 modes [29], that is:

$$\mathbf{t} = \mathbf{S}\boldsymbol{\alpha}, \quad (10)$$

217 where  $\alpha_i, i = 1, 2, \dots, s$ , are arbitrary real coefficients of the linear combination collected in the vector  $\boldsymbol{\alpha}$   
 218  $\in \mathbb{R}^s$ .

219 If the null-space of the equilibrium matrix  $\mathbf{A}$  is spanned by a unique independent self-stress mode, i.e.  
220 if  $s = 1$ , then such vector represents the only feasible self-stress state of the structure. In this case, the  
221 matrix  $\mathbf{S}$  becomes a column vector. It can be noted that in this case, the independent self-stress mode  
222 should be consistent with the unilateral behaviour of the elements for determining the feasible self-  
223 stress states.

224 If there are multiple independent self-stress modes, i.e. if  $s > 1$ , then it is necessary to calculate suitable  
225 linear combinations of these bases by means of Eq. (10) since such modes, usually, do not satisfy the  
226 unilateral behaviour of the elements as well as the symmetry of the structure.

227 Indeed, independent self-stress modes resulting from the null-space of the equilibrium matrix usually  
228 only satisfy the nodal equilibrium conditions, thus cannot be utilized directly. On the other hand,  
229 unilateral conditions related to the mechanical behaviour of struts and cables are not considered in the  
230 formulation of the matrix  $\mathbf{A}$ .

231 However, for statically indeterminate structures ( $s > 1$ ) exhibiting symmetry properties, as is often the  
232 case for tensegrity structures, many elements can be collected into suitable groups according to the  
233 symmetry [28]. In this vein, the evaluation of Eq. (10) can be simplified taking into account the  
234 symmetry constraints of the geometry of the structure, that is, the same self-stress can be assigned to  
235 elements in the same symmetric position. Thus, it can be viewed as a constraint on the self-stress  
236 distribution in the elements of the structure.

237 Definitively, the aim is the evaluation of the self-stress distribution in the elements consistent with the  
238 symmetry properties of the structure and their unilateral behaviour, that is the feasible integral self-  
239 stress states.

240 **3. Distributed static indeterminacy**

241 Let  $\mathbf{d} \in \mathbb{R}^{3n}$ , and  $\mathbf{e} \in \mathbb{R}^e$  denote the vector of infinitesimal nodal displacements and the vector of member  
242 elongations, respectively. It is possible to define the relations among such kinematic variables in terms  
243 of the *compatibility matrix*  $\mathbf{B} \in \mathbb{R}^{e \times 3n}$  [60] such that:

$$\mathbf{B}\mathbf{d} = \mathbf{e}. \quad (11)$$

244 From the principle of virtual work, it follows that  $\mathbf{B} = \mathbf{A}^T$  [58]. Let  $r_B$  be the rank of  $\mathbf{B}$  ( $r_B = r_A$ ); then  
245 the number  $m$  of the possible *mechanisms* which span the null-space of  $\mathbf{B}$  is  $m = 3n - r_B$ . Moreover, the  
246 number  $m_i$  of the *infinitesimal mechanisms* can be obtained by excluding the rigid-body motions in the  
247 three-dimensional space, i.e.,  $m_i = m - 6$ .

248 Taking into account the effects of initial elongations  $e_k$ ,  $k = 1, 2, \dots, e$ , under the pre-stress, and by  
249 assembling the initial elongations vector  $\mathbf{e}_0 \in \mathbb{R}^e$ , constitutive equations for the tensegrity structures can  
250 be then expressed as [61]:

$$\mathbf{e} = \mathbf{e}_0 + \mathbf{F}\mathbf{t}, \quad (12)$$

251 where  $\mathbf{F} \in \mathbb{R}^{e \times e}$  is the diagonal *flexibility matrix*, whose  $k$ -th diagonal entry is  $l_k/E_k A_k$ , with  $E_k$  and  $A_k$  the  
252 Young's modulus and the cross-sectional area of the element, respectively.

253 Moreover, in the standard linear algebraic theory of vector spaces, it results that all the information  
254 required for the analysis of a framework are contained in the four fundamental vector spaces associated  
255 with the equilibrium matrix  $\mathbf{A}$  (for further details about their kinematic and static interpretation [59]).

256 In particular, the row-space, the null-space, the column-space and the left null-space of  $\mathbf{A}$ , can be  
257 associated with the equilibrium matrix. In particular, the left null-space and the null-space of  $\mathbf{A}$  are  
258 spanned by the  $m_i$  infinitesimal mechanisms and the  $s$  independent self-stress modes, respectively. For

259 what considered below, it is possible to recall the well-known properties of orthogonality among such  
 260 vector subspaces [59], thus it is possible to write:

$$\mathbf{S}^T(\mathbf{e}_0 + \mathbf{Ft}) = \mathbf{0}, \quad (13)$$

261 and substituting Eq. (9) in (13):

$$\mathbf{S}^T(\mathbf{e}_0 + \mathbf{FS}\mathbf{a}) = \mathbf{0}. \quad (14)$$

262 It is possible to recall that for a full rank matrix  $\mathbf{P} \in \mathbb{R}^{i \times j}$ , with  $j \leq i$ , the square matrix  $\mathbf{P}^T\mathbf{P}$  is always  
 263 positive definite. Moreover, let  $\mathbf{Q} \in \mathbb{R}^{i \times i}$  symmetric and positive definite, then  $\mathbf{P}^T\mathbf{Q}\mathbf{P}$  is a symmetric,  
 264 non-singular, positive definite matrix. Thus, the matrix  $\mathbf{S}^T\mathbf{FS}$  is a symmetric, non-singular, positive  
 265 definite matrix.

266 Therefore, from Eq. (14) it is possible to determine the vector  $\mathbf{a}$  as:

$$\mathbf{a} = -(\mathbf{S}^T\mathbf{FS})^{-1} \mathbf{S}^T \mathbf{e}_0. \quad (15)$$

267 Hence, the element internal forces vector  $\mathbf{t}$  can be obtained by substituting Eq. (15) into Eq. (10):

$$\mathbf{t} = -\mathbf{S}(\mathbf{S}^T\mathbf{FS})^{-1} \mathbf{S}^T \mathbf{e}_0. \quad (16)$$

268 By introducing the *diagonal stiffness matrix*  $\mathbf{K} \in \mathbb{R}^{exc}$ , such that  $\mathbf{K} = \mathbf{F}^{-1}$ , (the  $k$ -th diagonal entry of  $\mathbf{K}$  is  
 269  $E_k A_k / l_k$ ) Eq. (15) can be rewritten as:

$$\mathbf{t} = -\mathbf{K} \left[ \mathbf{FS}(\mathbf{S}^T\mathbf{FS})^{-1} \mathbf{S}^T \right] \mathbf{e}_0 = -\mathbf{K}\mathbf{\Omega} \mathbf{e}_0, \quad (17)$$

270 where the square matrix  $\mathbf{\Omega} \in \mathbb{R}^{exc}$  ( $=\mathbf{FS}(\mathbf{S}^T\mathbf{FS})^{-1}\mathbf{S}^T$ ) correlates different aspects of the structure: the  
 271 geometrical configuration, the topology and the stiffness properties of the elements, defined by the

272 designers. Since the matrix  $\mathbf{S}^T\mathbf{F}\mathbf{S}$  in Eq. (15) is always positive definite; Eq. (17) is applicable for both  
 273 kinematically determinate and indeterminate structures. Equation (17) is a constitutive equation  
 274 describing the relation between the internal forces in the elements and their initial elongation. From the  
 275 definition of the square matrix  $\mathbf{\Omega}$ , it results that  $\mathbf{\Omega}$  is an idempotent singular matrix, that is,  $\mathbf{\Omega}^2$  is equal  
 276 to  $\mathbf{\Omega}$ , hence its eigenvalues are either 0 or 1. Furthermore, the rank of  $\mathbf{\Omega}$  is equal to the sum of its  
 277 eigenvalues, or equivalently, is equal to its trace. The sum of all the main diagonal elements  $\gamma_i$   
 278 ( $i=1,2,\dots,e$ ) is equal, thus, to the total degree  $s$  of static indeterminacy of the structure. Such diagonal  
 279 entries  $\gamma_i$ , collected into the vector  $\boldsymbol{\omega} \in \mathbb{R}^e$ , are defined in the literature as *Distributed Static*  
 280 *Indeterminacies* (DSI) [63]: indeed  $\gamma_i$  represents the contribution of the  $i$ -th element of the structure to  
 281 its total degree of static indeterminacy.

282 Moreover, it is possible to show that elements having the same symmetry properties have the same DSI  
 283 values; indeed, DSI can be viewed as an indicator of the symmetry properties of the structure [61].

284 Finally, if the flexibility matrix  $\mathbf{F}$  is equal to the identity matrix  $\mathbf{I}$ , then the matrix  $\mathbf{\Omega}$  becomes the  
 285 matrix  $\mathbf{\Omega}_m = \mathbf{S}(\mathbf{S}^T\mathbf{S})^{-1}\mathbf{S}^T \in \mathbb{R}^{e \times e}$ , whose diagonal terms can be collected in the vector  $\boldsymbol{\omega}_m \in \mathbb{R}^e$ . It can be  
 286 noted that the matrix  $\mathbf{\Omega}_m$ , in addition to the above-recalled algebraic properties of the matrix  $\mathbf{\Omega}$ , is  
 287 characterized by the further property of being always symmetric. In this particular case, the matrix  $\mathbf{\Omega}_m$   
 288 is not affected by the axial stiffness properties of the elements; hence, it is strictly related to the self-  
 289 equilibrium conditions of the structure.

#### 290 4. The new approach for the determination of the feasible integral self-stress states

291 We consider a test vector  $\mathbf{t}_p \in \mathbb{R}^e$ , consistent with the sign of the internal forces in the elements, i.e.  
 292 positive in the cables and negative in the struts, built as follows:

$$\mathbf{t}_{p,i} = \begin{cases} +1 & \text{if element } i \text{ is a cable} \\ -1 & \text{if element } i \text{ is a strut} \end{cases}, \quad i = 1, \dots, e. \quad (18)$$

293 By considering a single element of the structure subjected to an initial elongation, it results that  
 294 shortening generates tension, while extension creates compression. Thus, an initial elongations vector  
 295  $\mathbf{e}_0$  can be associated to the test vector  $\mathbf{t}_p$ :

$$\mathbf{e}_0 = -\mathbf{F}\mathbf{t}_p. \quad (19)$$

296 Substituting Eq. (19) into Eq. (17), we have:

$$\mathbf{t} = \mathbf{K}\mathbf{\Omega}\mathbf{F}\mathbf{t}_p. \quad (20)$$

297 It is easy to prove that  $\mathbf{K}\mathbf{\Omega}\mathbf{F}$  is equal to  $\mathbf{\Omega}^T$  (see Appendix A); therefore, Eq. (20) can be rearranged as:

$$\mathbf{t}_n = \mathbf{\Omega}^T \mathbf{t}_p. \quad (21)$$

298 From the definition of the matrix  $\mathbf{\Omega}$  and from Eq. (18), the internal forces vector  $\mathbf{t}_n$  obtained from the  
 299 Eq. (21) takes into account both the unilateral behaviour of the elements and the self-equilibrium  
 300 conditions of the structure. Moreover, as it results from the numerical experiments performed in  
 301 Section 6, symmetric assignments of the axial stiffness of the elements lead to a symmetric distribution  
 302 of the internal forces in the elements. Thus, such a vector represents a feasible integral self-stress vector  
 303 for the tensegrity structure. Moreover, it is worth to observe that, since the definition of the matrix  $\mathbf{\Omega}$ ,  
 304 the Eq. (21) is strictly related to the material properties of the elements, represented by the matrix  $\mathbf{F}$ .

305 In order to verify the accuracy of the numerical analyses performed in Section 6, the vector  $\boldsymbol{\varepsilon}_u \in \mathbb{R}^e$   
 306 represents the unbalanced residual normalized internal forces defined as:

$$\boldsymbol{\varepsilon}_u = \mathbf{A}\mathbf{t}, \quad (22)$$

307 and its Euclidean norm can be used to evaluate the numerical errors.

308 The proposed method, coded using Wolfram Mathematica 11.0, can be outlined as follows. Assigned  
309 the element connectivity, by means of the matrix  $\mathbf{C}$ , and the geometry of the structure in terms of the  
310 nodal coordinate vectors  $\mathbf{x}$ ,  $\mathbf{y}$  and  $\mathbf{z}$ , then:

311 **Step 1:** Assemble the equilibrium matrix  $\mathbf{A}$  by using Eq. (7).

312 **Step 2:** Define the material parameters of the elements, that is, the Young's modulus  $E_k$  and the cross-  
313 sectional area  $A_k$ , and then construct the flexibility matrix  $\mathbf{F}$ .

314 **Step 3:** Collect the prototype vector  $\mathbf{t}_p$ , according to the unilateral behaviour of the elements, by means  
315 of Eq. (18).

316 **Step 4:** Determine the null-space of the equilibrium matrix  $\mathbf{A}$  and then assemble the self-stress matrix  
317  $\mathbf{S}$ .

318 **Step 5:** Calculate the matrix  $\mathbf{\Omega}$  and thereafter evaluate the feasible self-stress states  $\mathbf{t}_n$  by using the Eq.  
319 (21).

320 **Step 6:** Compute the norm of the unbalanced residual normalized internal forces vectors in order to  
321 verify the accuracy of the analyses.

322 It is worth to note that for tensegrity structures with a unique independent self-stress mode, that is  $s = 1$ ,  
323 the feasible self-stress states calculated by using the Eq. (21) is obviously not affected by the  
324 assignments of the axial stiffness of the elements. Conversely, for tensegrity structures with multiple  
325 independent self-stress modes, that is, for  $s > 1$ , different choices of the axial stiffness of the elements  
326 lead to different linear combinations of the above-mentioned independent self-stress modes, hence, to  
327 different self-stress states.

328 Conclusively, it can be noted that symmetric distributions of the axial stiffness of the elements  
 329 correspond to symmetric internal forces in the elements, thus lead to feasible integral self-stress states  
 330 consistent both with the stiffness symmetry and the geometrical symmetry properties of the structure.

### 331 **5. An efficient approach for determining the Force Density Matrix**

332 In this section, we briefly recall the definition of the Force Density matrix [66], always a square  
 333 symmetric matrix. In particular, by using the Eq. (4), the equilibrium equations for the tensegrity  
 334 structure, projected in the three directions  $\mathbf{e}_x$ ,  $\mathbf{e}_y$ , and  $\mathbf{e}_z$ , can be then expressed in the following matrix  
 335 linear form:

$$\begin{cases} \mathbf{D}_s \mathbf{x} = \mathbf{0} \\ \mathbf{D}_s \mathbf{y} = \mathbf{0}, \\ \mathbf{D}_s \mathbf{z} = \mathbf{0} \end{cases} \quad (23)$$

336 with  $\mathbf{D}_s \in \mathbb{R}^{\text{exc}}$  the Force Density matrix, defined as follows:

$$\mathbf{D}_s = \mathbf{C}^T \mathbf{Q} \mathbf{C}. \quad (24)$$

337 *A non-degenerate tensegrity structure is super-stable, if the rank deficiency  $n_D$  of the Force Density*  
 338 *matrix, that is the number of its null eigenvalues, is equal to  $d + 1$  ( $\lambda_1 = \dots = \lambda_{d+1} = 0$ ), and the remaining*  
 339 *eigenvalues are strictly positive ( $0 < \lambda_{d+2} \leq \dots \leq \lambda_e$ ).*

340 Eq. (24) represents the standard formulation of the Force Density matrix  $\mathbf{D}_s$ . The major difficulties in  
 341 evaluating  $\mathbf{D}_s$  through Eq. (24) comes from the determination of the components of the diagonal matrix  
 342  $\mathbf{Q}$ , especially for tensegrity structures with multiple independent self-stress states, as explained in the  
 343 previous section.

344 We recall that independent self-stress modes, usually, are not consistent with the signs of the internal  
 345 forces in the elements, i.e. positive for cables and negative for struts, since the self-equilibrium

346 conditions do not take into account the unilateral behaviour of the elements. Moreover, it can be  
 347 observed that for a tensegrity structure with multiple independent self-stress states the Force Density  
 348 matrix obtained by considering a single independent self-stress mode should be indefinite, that is, it is  
 349 neither positive semi-definite nor negative semi-definite.

350 For these reasons, here an alternative formulation of the Force Density matrix  $\mathbf{D}_s$  is proposed. Indeed,  
 351 by recalling the algorithm for the determination of the feasible integral self-stress states proposed in the  
 352 previous section, and by using the Eq. (21) and Eq.(3), it is possible to rewrite  $\mathbf{D}_s$  as follows:

$$\begin{aligned} \mathbf{D}_s &= \mathbf{C}^T \mathbf{Q} \mathbf{C} = \mathbf{C}^T \text{diag}(\mathbf{L}^{-1} \mathbf{t}_n) \mathbf{C} = \\ &= \mathbf{C}^T \text{diag}(\mathbf{L}^{-1} \mathbf{\Omega}^T \mathbf{t}_p) \mathbf{C}, \end{aligned} \quad (25)$$

353 where it is recalled that the diagonal matrix  $\mathbf{Q}$  is equal to  $\text{diag}(\mathbf{L}^{-1} \mathbf{t}_n)$ .

354 Since our approach allows for efficiently evaluating the feasible integral self-stress states of a  
 355 tensegrity structure (see Sect. 4), it can now effectively employed also for calculating the Force Density  
 356 matrix that can be determined by performing a unique SVD of the equilibrium matrix of the structure  
 357 and by evaluating the DSI values of the elements.

358 Notice that, different choices of the feasible self-stress states lead to different Force Density matrices  
 359 and, thus, to different eigenvalues.

360 Since the geometry and the connectivity properties of the structure are given, the equilibrium matrix  $\mathbf{A}$ ,  
 361 see Eq. (7), remains unchanged, that is, such matrix is constant in the force-finding problem in the Eq.  
 362 (6).

363 Thus, the *force-density* problem in the Eq. (23) can be seen as an alternative representation of the self-  
 364 equilibrium problem represented by Eq. (7).

365 Hence, rank deficiency, as well as, the sign of non-zero eigenvalues of the Force Density matrix,  
366 evaluated by using Eq. (25), remain unvaried irrespective of any feasible self-stress state considered.

367 Hence, by using the Eq. (25) it is possible to effectively evaluate the matrix  $\mathbf{D}_s$  also for verifying the  
368 super-stability conditions for tensegrity structures.

## 369 **6. Numerical examples**

370 In this section, three well-known tensegrity structures have been studied in order to compare the results  
371 available in the literature with the results obtained by means of the new method here proposed. In  
372 particular, we analyze the following tensegrity structures: the Quadruplex; the Snelson's X beam with  
373 three modules; the Octahedral cell.

374 Different assignments of the Young's modulus and the cross-sectional area of the elements were made  
375 in order to calculate the corresponding different feasible self-stress states by evaluating the DSI vector  
376  $\boldsymbol{\omega}$  and the corresponding internal force vector  $\mathbf{t}$ .

377 Specifically, for each of the three tensegrity structures, the analysis was conducted by considering five  
378 different conditions:

379 1) the case in which the flexibility matrix is equal to the identity matrix, that is,  $\mathbf{F} = \mathbf{I}$ ; in this case, the  
380 DSI vector coincides with  $\boldsymbol{\omega}_m$  and the corresponding internal force vector is denoted by  $\mathbf{t}_{nm}$ ;

381 2) a possible assignment of the axial stiffness of the elements which lead to the results reported by the  
382 literature; in such case, the vector named  $\boldsymbol{\omega}$  (*literature*) and the vector termed  $\mathbf{t}$  (*literature*) were  
383 calculated

384 3), 4) and 5) two symmetric distribution (called n1 and n2) of the axial stiffness of the elements and a  
385 not-symmetric distribution (called n3) of the axial stiffness of the elements, which yield to the

386 determination of the DSI vectors  $\mathbf{w}_{n1}$ ,  $\mathbf{w}_{n2}$ ,  $\mathbf{w}_{n3}$  and of the related internal force vectors  $\mathbf{t}_{n1}$ ,  $\mathbf{t}_{n2}$ ,  $\mathbf{t}_{n3}$ ,  
387 respectively.

388 Furthermore, for both the Snelson's X beam with three modules and the Octahedral cell (tensegrity  
389 structures with multiple independent self-stress modes) three further distributions of the axial stiffness  
390 of the elements, which also allow for determining results equal to those reported by the literature, have  
391 been considered.

392 Moreover, for the tensegrity structures analysed, the Force Density matrices have been calculated.  
393 Their rank deficiencies, as well as, their eigenvalues have been determined in order to evaluate the  
394 super-stability conditions of the structures.

395 Finally, in order to compare the results corresponding to different stiffness properties, the internal force  
396 vectors have been normalized, and the force densities of the elements have been normalized respect to  
397 the force density of the elements belonging to the first group.

#### 398 *6.1. Quadruplex*

399 The tensegrity Quadruplex analyzed, see Fig. 1, consists of  $n = 8$  nodes and  $e = 16$  elements, i.e. 4  
400 struts and 12 cables, and its geometrical configuration can be found in [61]. In particular, the top square  
401 base and the bottom square base are rotated with respect to each other by a twist angle equal to  $\pi/4$ ;  
402 such bases are inscribed in a circle of radius equal to 707 mm, the height of the prism is equal to 1000  
403 mm. In Table 1 are shown the assignments of the axial stiffness of the elements of the Quadruplex,  
404 whereas the corresponding DSI values of the elements are shown in Fig. 2.

405

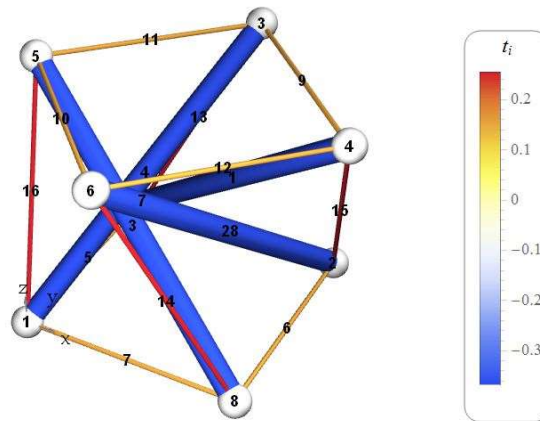


Fig. 1. Quadruplex, perspective view. Thick cylinders represent the struts. Different colours have been assigned according to the value of the internal forces in the elements, which are labelled according to the connectivity matrix

406

**Table 1**

Axial stiffness of the elements of the Quadruplex

	$\mathbf{F} = \mathbf{I}$	<i>literature</i>	n1	n2	n3	
Element	$E_k A_k$ (N)	$E_k A_k$ (N)	$E_k A_k$ (N)	$E_k A_k$ (N)	$E_k A_k$ (N)	Element
struts (1-4)	1645.33	$10^6$	$10^6$	$10^6$	$10^6$	struts (1-3)
cables (5-12)	1000	$49 \cdot 10^3$	$49 \cdot 10^3$	$24.5 \cdot 10^3$	$1.5 \cdot 10^6$	strut (4)
cables (13-16)	1137.05	$49 \cdot 10^3$	$24.5 \cdot 10^3$	$49 \cdot 10^3$	$24.5 \cdot 10^3$	cables (5-10)
					$49 \cdot 10^3$	cables (11-12)
					$49 \cdot 10^3$	cables (13-15)
					$39.2 \cdot 10^3$	cable (16)

407

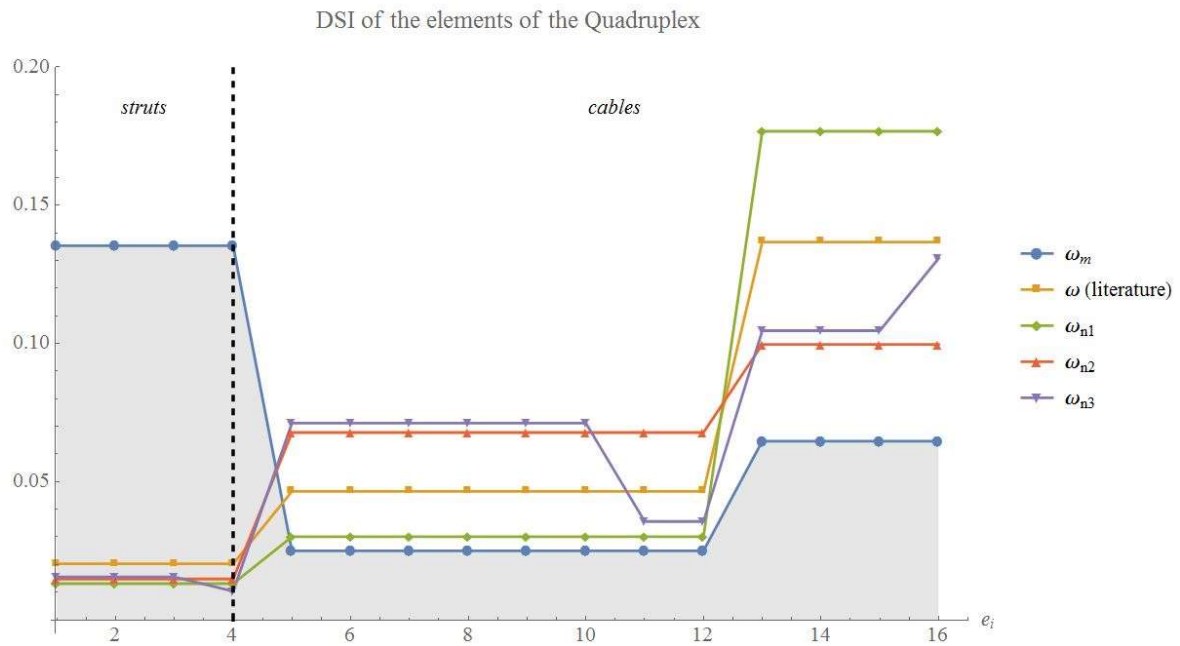


Fig. 2. DSI of the elements of the Quadruplex for different assignments of the axial stiffness.

408

409 As it can be observed, by increasing the axial stiffness of an element of the structure, its DSI value  
 410 decreases whereas the DSI values of the other elements increase. Moreover, symmetric assignments of  
 411 the axial stiffness, namely the first four cases analyzed, lead to a symmetric distribution of DSI values.

412 The rank of the equilibrium matrix  $\mathbf{A}$  is 15, thus the structure has one self-stress mode, i.e.  $s = 1$ , and it  
 413 possesses 3 infinitesimal mechanisms. Furthermore, by using Eq. (21), it is possible to evaluate the  
 414 feasible self-stress states, see Fig. 3. The figure clearly shows that, as expected, the normalized internal  
 415 force vectors do not change as the axial stiffness's of the elements vary, also for not-symmetric  
 416 distribution of the axial stiffness.

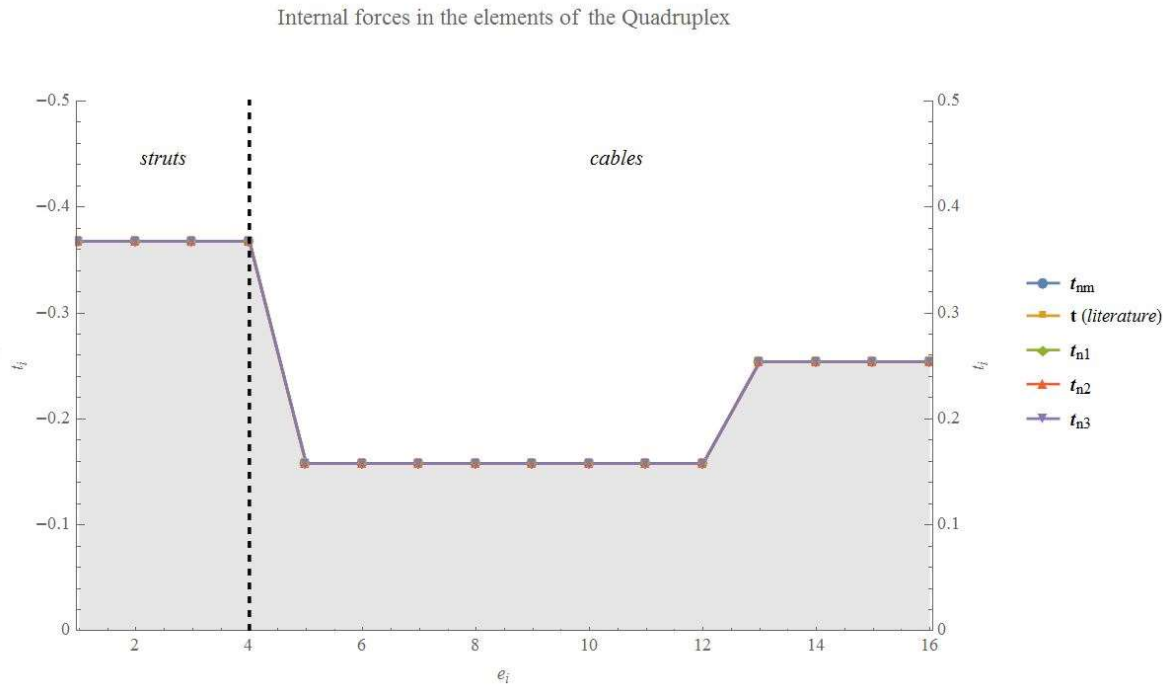


Fig. 3. Internal forces in the elements of the Quadruplex as the axial stiffness of the elements vary

417

418 As it can be noted in Fig. 3, the elements of the Quadruplex can be collected in three groups according  
 419 to their internal forces, as well as to their force densities: struts (1-4), cables (5-12) and cables (13-16).

420 In particular, the normalized force density of the elements of the first group is equal to -1, the  
 421 normalized force density of the elements of the second group is equal to about 0.7071, and the  
 422 normalized force density of the elements of the third group is equal to 1.

423 Moreover, from the analysis of the Force Density matrix, it results that the Quadruplex is a super-stable  
 424 tensegrity structure. Indeed, its Force Density matrix is a positive semi-definite matrix with four zero  
 425 eigenvalues, as shown in Fig. 4. Obviously, the corresponding eigenvalues evaluated in the five  
 426 different assignments of the axial stiffness of the elements are identical.

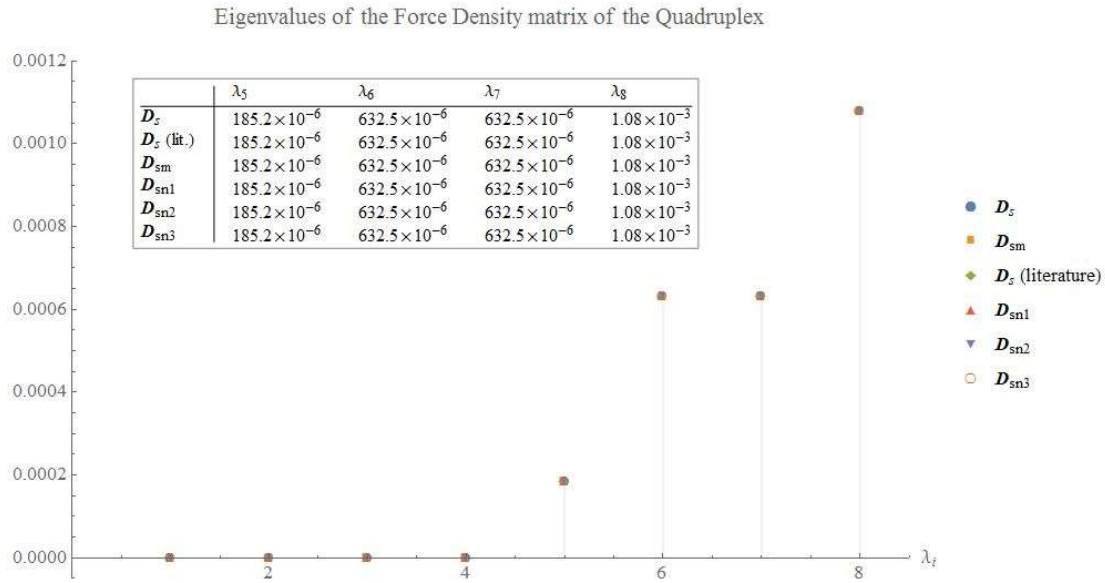


Fig. 4. Eigenvalues of the Force Density matrices of the Quadruplex

427

428 Finally, the norms of the unbalanced residual normalized internal forces vectors are calculated in order  
 429 to verify the accuracy of the method. As it is shown in Table 2, such norms are close to zero.

**Table 2**

Norm of the unbalanced residual normalized internal forces vectors of the Quadruplex

	$\mathbf{t}_{nm}$	$\mathbf{t}$ ( <i>literature</i> )	$\mathbf{t}_{n1}$	$\mathbf{t}_{n2}$	$\mathbf{t}_{n3}$
$[[\boldsymbol{\epsilon}_u]]$	$5.54 \cdot 10^{-16}$	$5.73 \cdot 10^{-16}$	$5.76 \cdot 10^{-16}$	$6.03 \cdot 10^{-16}$	$5.82 \cdot 10^{-16}$

430

### 431 6.2. Snelson's X beam with three modules

432 The Snelson's X beam shown in Fig. 5 is made of three modules; its topology and geometry are  
 433 described in [27]. The Snelson's elementary module has dimensions in  $x$  and  $y$  directions equal to 3000  
 434 mm and 2000 mm, respectively. The tensegrity structure has 8 nodes and it is composed of 16  
 435 elements, 10 cables and 6 struts. From the analysis of the null-space of the equilibrium matrix  $\mathbf{A}$ , it

436 results that its rank is equal to 13, thus the tensegrity structure has 3 independent self-stress states.  
 437 Moreover, the number of the infinitesimal mechanisms is equal to 0, that is the Snelson's X tensegrity  
 438 beam analyzed is kinematically determinate. The normalized feasible self-stress states evaluated by  
 439 means of the algorithm presented in [32] are displayed in Fig. 5.

440

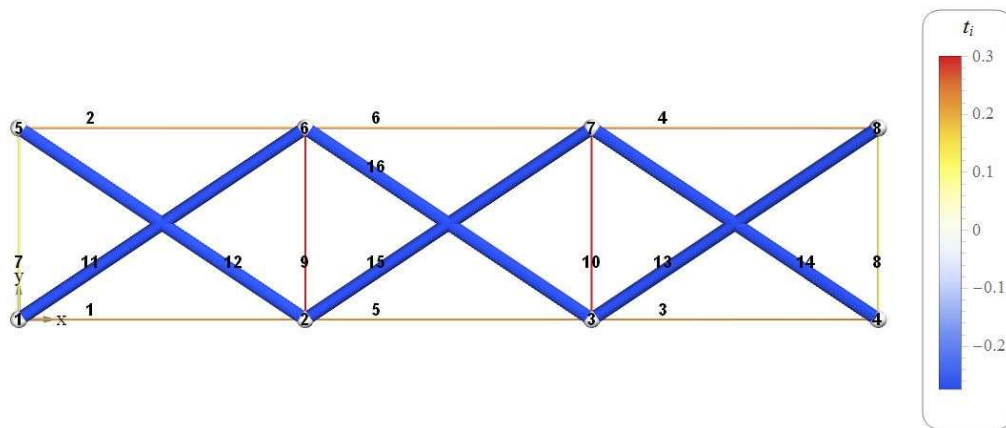


Fig. 5. Snelson's X beam with three modules, perspective view. Thick cylinders represent the struts. Different colours have been assigned according to the value of the internal forces in the elements, which are labelled according to the connectivity matrix

441

442 In Table 3 are listed the assignments of the axial stiffness of the elements of the Snelson's X beam,  
 443 whereas the related DSI values of the elements are shown in Fig. 6.

444 In order to evaluate how the internal forces vary as the stiffness properties of a single group of the  
 445 elements change, the case n1 and the case n2 differs only for the fact that in the case n2 the stiffness  
 446 assigned to the cables 9-10 is greater than the stiffness assigned to the same elements in the case n1.

447 As well as for Quadruplex, by increasing the axial stiffness of cables 9-10, their DSI values decrease,  
 448 whereas DSI values of the other elements increase.

**Table 3**

Axial stiffness of the elements of the Snelson's X beam with three modules

	<b>F = I</b>	<i>literature</i>	n1	n2	n3	
Element	$E_k A_k$ (N)	$E_k A_k$ (N)	$E_k A_k$ (N)	$E_k A_k$ (N)	$E_k A_k$ (N)	Element
cables (1-4)	3000	$9.714 \cdot 10^6$	$3.238 \cdot 10^6$	$3.238 \cdot 10^6$	$9.714 \cdot 10^6$	cables (1-2)
cables (5-6)	3000	$6.48 \cdot 10^5$	$3.238 \cdot 10^6$	$3.238 \cdot 10^6$	$3.238 \cdot 10^6$	cables (3-6)
cables (7-8)	2000	$18.78 \cdot 10^6$	$3.238 \cdot 10^6$	$3.238 \cdot 10^6$	$6.476 \cdot 10^6$	cables (7-9)
cables (9-10)	2000	$24 \cdot 10^6$	$3.238 \cdot 10^6$	$16.19 \cdot 10^6$	$12.952 \cdot 10^6$	cable (10)
struts (11-14)	3605.55	$19.428 \cdot 10^7$	$19.428 \cdot 10^7$	$19.428 \cdot 10^7$	$16.19 \cdot 10^7$	struts (11-12)
struts (15-16)	3605.55	$19.428 \cdot 10^7$	$19.428 \cdot 10^7$	$19.428 \cdot 10^7$	$19.428 \cdot 10^7$	struts (13-16)

449

450

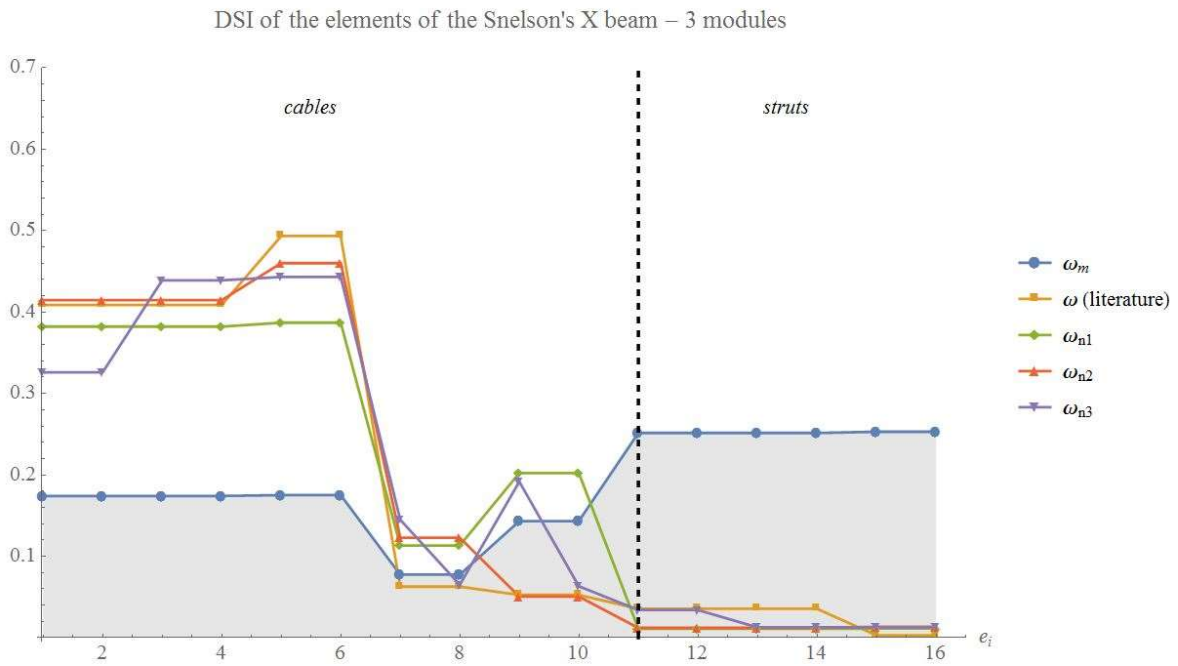


Fig. 6. DSI of the elements of the Snelson's X beam with three modules for different assignments of the axial stiffness.

451

452 It is worth to note that the feasible self-stress states obtained in literature, see [27], can be obtained for  
 453 several distributions of the stiffness of the elements; three examples are listed in Table 4 (named exS1,  
 454 exS2, exS3).

**Table 4**

Axial stiffness of the elements of the Snelson's X beam with three modules which lead to the results obtained in the literature

	exS1	exS2	exS3
Element	$E_k A_k$ (N)	$E_k A_k$ (N)	$E_k A_k$ (N)
cables (1-4)	$1.44 \cdot 10^6$	$3.238 \cdot 10^4$	$19.428 \cdot 10^6$
cables (5-6)	$4.69 \cdot 10^5$	$8.42 \cdot 10^3$	$5.11 \cdot 10^6$
cables (7-8)	$64.76 \cdot 10^6$	$16.19 \cdot 10^6$	$16.19 \cdot 10^6$
cables (9-10)	$64.76 \cdot 10^6$	$16.19 \cdot 10^6$	$3.238 \cdot 10^6$
struts (11-14)	$25.9 \cdot 10^7$	$97.14 \cdot 10^6$	$19.428 \cdot 10^7$
struts (15-16)	$25.9 \cdot 10^7$	$12.95 \cdot 10^7$	$97.14 \cdot 10^6$

455

456 The analyses of the feasible self-stress states obtained by using the proposed method lead to the  
 457 normalized internal forces in the elements shown in Fig. 7.

458

Internal forces in the elements of the Snelson's X beam – 3 modules

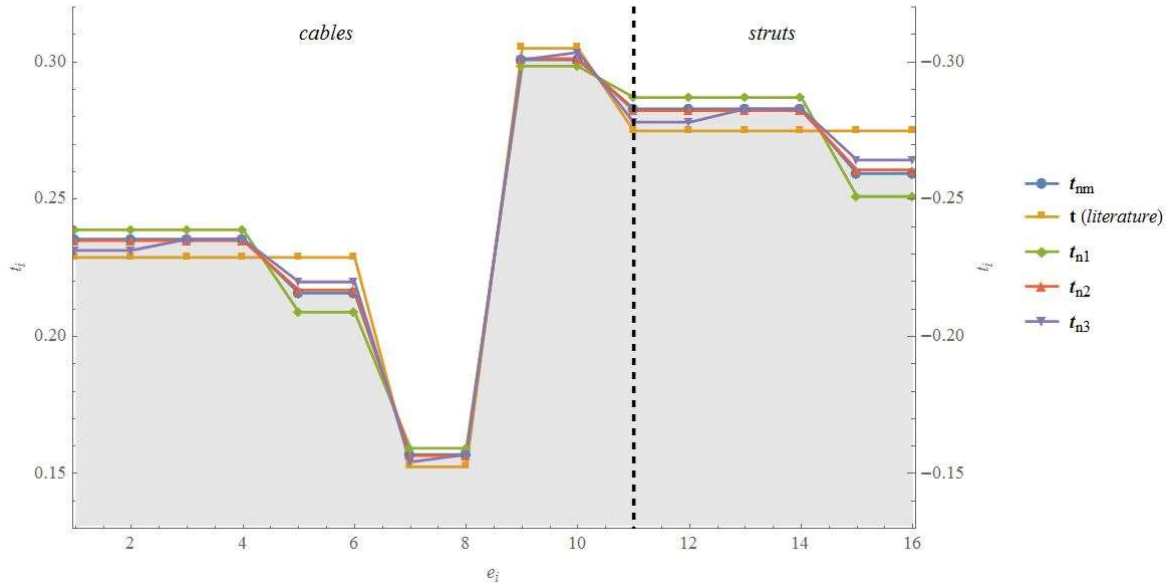


Fig. 7. Internal forces in the elements of the tensegrity Snelson's X beam with three modules as the axial stiffness of the elements vary (note that the origin of the vertical axes is 0.13)

459

460 It can be observed that for  $\mathbf{F} = \mathbf{I}$ , for the case named n1, as well as for the case n2, although the  
 461 distribution of the axial stiffness of the elements is different, the elements of the tensegrity structure  
 462 can be collected in the same groups according to the normalized internal forces. This happens because  
 463 the matrix  $\mathbf{\Omega}$  takes into account not only the stiffness symmetry but also the geometric symmetry  
 464 properties of the structure.

465 By considering both the case n1 and n2, it can be seen that by increasing the axial stiffness of the  
 466 cables 9-10, their internal forces increase. Simultaneously, the tensile internal forces in the cables 1-4  
 467 and in the cables 7-8, as well as the compressive internal forces in the struts 11-14 decrease. At the  
 468 same time, the tensile internal forces in the cables 5-6, as well as, the compressive internal forces in the  
 469 struts 15-16 increase.

470 Such sensitivity analyses can be easily conducted by varying the Young's modulus and the cross-  
 471 sectional area either of a unique element of the structure or of a single group of the elements.

472 Moreover, the same behaviour can be noted by examining the force densities of the elements, listed in  
 473 Table 5, normalized respect to the force density of the first group. Such feasible force densities are in  
 474 perfect agreement with the results obtained in [27] (refer to Table III in the reference).

475

**Table 5**

Force densities of the elements of the Snelson's X beam with three modules, normalized with respect to the force density of the first group

	$\mathbf{F} = \mathbf{I}$	<i>literature</i>	n1	n2	n3	
Element	$q_k$	$q_k$	$q_k$	$q_k$	$q_k$	Element
cables (1-4)	1	1	1	1	1	cables (1-2)
cables (5-6)	0.92	1	0.87	0.92	1.02	cables (3-4)
cables (7-8)	1	1	1	1	0.95	cables (5-6)
cables (9-10)	1.92	2	1.87	1.92	1	cable (7)
struts (11-14)	-1	-1	-1	-1	1.02	cable (8)
struts (15-16)	-0.92	-1	-0.87	-0.92	1.95	cable (9)
					1.97	cable (10)
					-1	struts (11-12)
					-1.02	struts (13-14)
					-0.95	struts (15-16)

476

477 The eigenvalues of the Force Density matrices  $\mathbf{D}_s$ , calculated by using the Eq. (25), are shown in Fig. 8.  
 478 It can be observed that these matrices are semi-positive definite and their rank deficiencies are equal to  
 479 5. However, such Snelson's X beam has a degenerate geometry in a three-dimensional space, thus, it is  
 480 not super-stable.

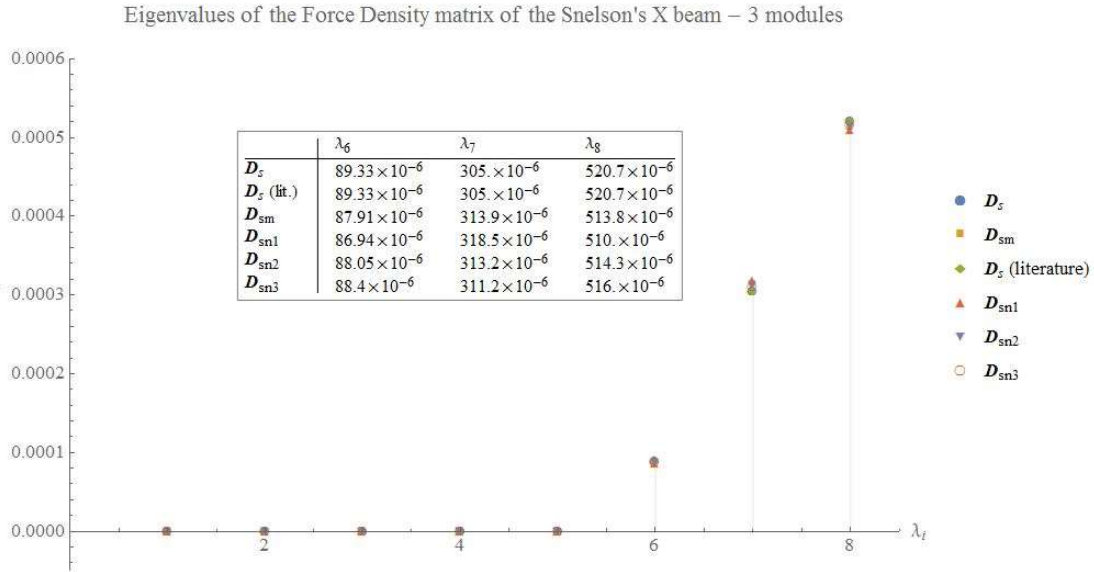


Fig. 8. Eigenvalues of the Force Density matrices of the tensegrity Snelson's X beam with three modules

481

482 The norms of the unbalanced residual normalized internal forces vectors  $[[\epsilon_u]]$  are shown in Table 6,  
 483 and it can be observed that such values are extremely close to 0, which demonstrates the accuracy of  
 484 the proposed method.

**Table 6**

Norm of the unbalanced residual normalized internal forces vectors of the Snelson's X beam with three modules

$t_{nm}$	$t$ (literature)	$t_{n1}$	$t_{n2}$	$t_{n3}$
----------	------------------	----------	----------	----------

$[[\epsilon_u]]$	$2.54 \cdot 10^{-16}$	$4.74 \cdot 10^{-16}$	$2.35 \cdot 10^{-16}$	$2.15 \cdot 10^{-16}$	$3.04 \cdot 10^{-16}$
------------------	-----------------------	-----------------------	-----------------------	-----------------------	-----------------------

485

486 6.3. *Octahedral cell*

487 The Octahedral cell, shown in Fig. 9, is made of 6 nodes and 15 elements, 12 cables and 3 struts. Its  
 488 topology and geometry are illustrated in [29,72]. In particular, the length of the vertical strut (strut 15)  
 489 is equal to 1000 mm, whereas the lengths of the horizontal struts (struts 13-14) are equal to about  
 490 666.667 mm. The feasible self-stress states presented in the literature [29,72] are also shown in Fig. 9.

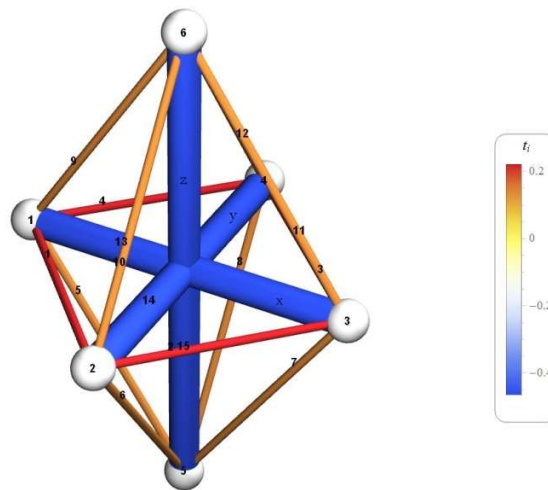


Fig. 9. Octahedral cell, perspective view. Thick cylinders represent the struts. Different colours have been assigned according to the value of the internal forces in the elements, which are labelled according to the connectivity matrix

491

492 The analysis of the equilibrium matrix  $\mathbf{A}$  conducts to 3 independent self-stress modes, that is  $s = 3$ , and  
 493 0 infinitesimal mechanisms, thus the Octahedral cell is a statically indeterminate and kinematically  
 494 determinate tensegrity structure.

495 In Table 7 are listed the distribution of the axial stiffness of the elements of the Octahedral cell,  
 496 whereas the related DSI values of the elements are illustrated in Fig. 10Fig. 9. In particular, the case n2  
 497 differs from the case n1 only for the axial stiffness of the cables 1-4.

498

**Table 7**

Axial stiffness of the elements of the Octahedral cell

	<b>F = I</b>	<i>literature</i>	n1	n2	n3	
Element	$E_k A_k$ (N)	$E_k A_k$ (N)	$E_k A_k$ (N)	$E_k A_k$ (N)	$E_k A_k$ (N)	Element
cables (1-4)	471.405	$3.238 \cdot 10^6$	$3.238 \cdot 10^6$	$16.19 \cdot 10^6$	$9.714 \cdot 10^6$	cables (1-2)
cables (5-12)	600.925	$19.99 \cdot 10^6$	$3.238 \cdot 10^6$	$3.238 \cdot 10^6$	$3.238 \cdot 10^6$	cables (3-12)
struts (13-14)	666.667	$48.57 \cdot 10^6$	$65.94 \cdot 10^6$	$65.94 \cdot 10^6$	$65.94 \cdot 10^6$	strut (13)
strut (15)	1000	$48.57 \cdot 10^6$	$65.94 \cdot 10^6$	$65.94 \cdot 10^6$	$32.97 \cdot 10^6$	struts (14-15)

499

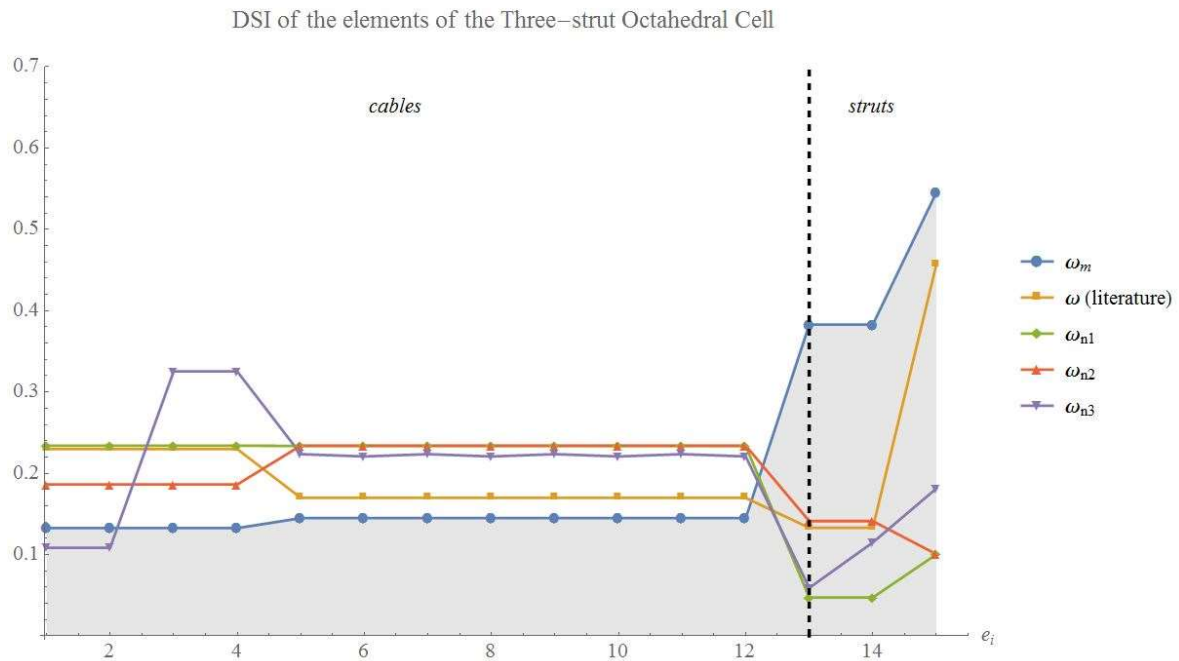


Fig. 10. DSI of the elements of the Octahedral cell for different assignments of the axial stiffness.

500

501 Also for the Octahedral cell it can be observed that by increasing the axial stiffness of the cables 1-4  
 502 their DSI values decrease, whereas DSI values of the remaining elements increase. Moreover, also in  
 503 this case the feasible self-stress states reported in the literature can be obtained for several assignments  
 504 of the axial stiffness of the elements of the structure. In particular, in Table 8 are listed three possible  
 505 assignments (termed exO1, exO2, exO3).

506

**Table 8**

Axial stiffness of the elements of the Octahedral cell which also lead to the results obtained in the literature

	exO1	exO2	exO3
--	------	------	------

Element	$E_k A_k$ (N)	$E_k A_k$ (N)	$E_k A_k$ (N)
cables (1-4)	$3.238 \cdot 10^5$	$5.05 \cdot 10^6$	$1.619 \cdot 10^6$
cables (5-12)	$16.19 \cdot 10^6$	$48.57 \cdot 10^6$	$43.472 \cdot 10^6$
struts (13-14)	$97.14 \cdot 10^6$	$12.952 \cdot 10^7$	$19.428 \cdot 10^7$
strut (15)	$40.534 \cdot 10^6$	$12.952 \cdot 10^7$	$11.333 \cdot 10^7$

507

508 As it can be seen in Fig. 11, for  $\mathbf{F}$  equal to the identity matrix, as well as for the cases named n1 and n2,  
509 the elements of the Octahedral cell can be collected in the same groups according to the normalized  
510 internal forces. In particular, four groups can be identified: cables 1-4, cables 5-12, struts 13-14 and  
511 strut 15. Such a grouping scheme is consistent with both the geometrical symmetry and the stiffness  
512 symmetry of the structure.

513 Moreover, by comparing the case n1 with the case n2, it emerges that by increasing the axial stiffness  
514 of the cables 1-4 their tensile internal forces decrease, as well as the compressive internal forces in the  
515 horizontal struts 13-14 decrease. At the same time, the tensile internal forces in the cables 5-12 and the  
516 compressive internal force in the vertical strut 15 increases.

517

Internal forces in the elements of the Octahedral Cell

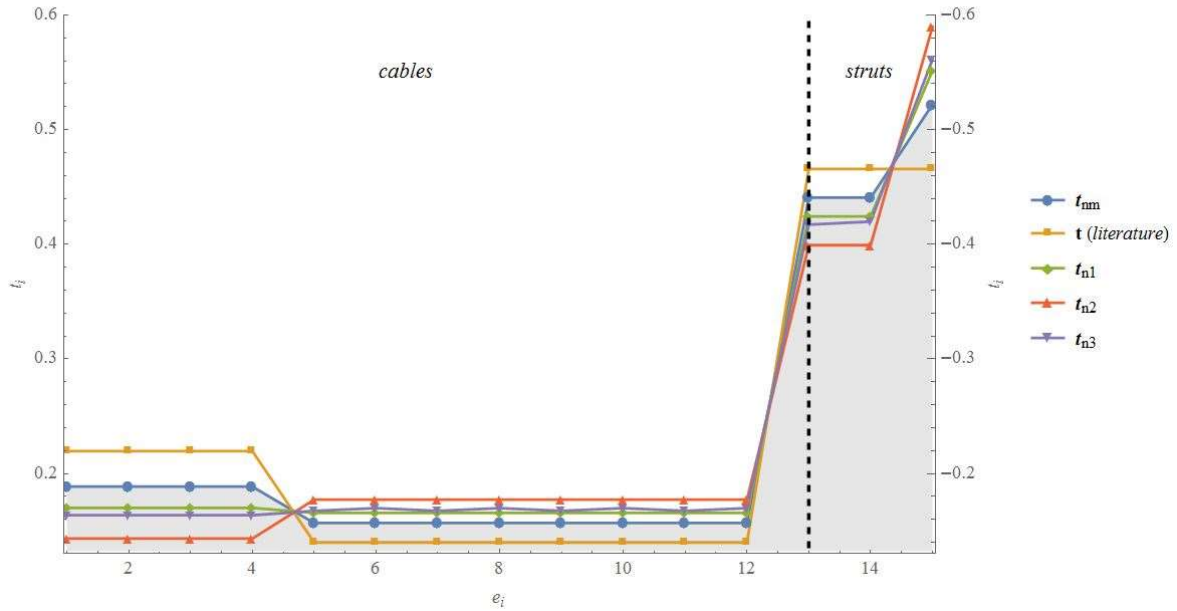


Fig. 11. Internal forces in the elements of the Octahedral cell as the axial stiffness of the elements vary (note that the origin of the vertical axes is 0.13)

518

519 The normalized force densities of the elements of the Octahedral cell consistent with the different  
 520 assignments of the axial stiffness of the members of the structure are listed in Table 9.

521

**Table 9**

Force densities of the elements of the Octahedral cell, normalized with respect to the force density of the first group

	$\mathbf{F} = \mathbf{I}$	<i>literature</i>	n1	n2	n3	
Element	$q_k$	$q_k$	$q_k$	$q_k$	$q_k$	Element
cables (1-4)	1	1	1	1	1	cables (1-4)

cables (5-12)	0.65	0.5	0.76	0.92	0.8	cables (5,7,9,11)
struts (13-14)	-1.65	-1.5	-1	-1.76	0.81	cables (6,8,10,12)
strut (15)	-1.3	-1	-0.87	-1.53	-1.8	strut (13)
					-1.81	strut (14)
					-1.62	strut (15)

522

523 By using the Eq. (25) it is possible to determine the Force Density matrices for each of the feasible self-  
 524 stress states; their eigenvalues are shown in Fig. 12.

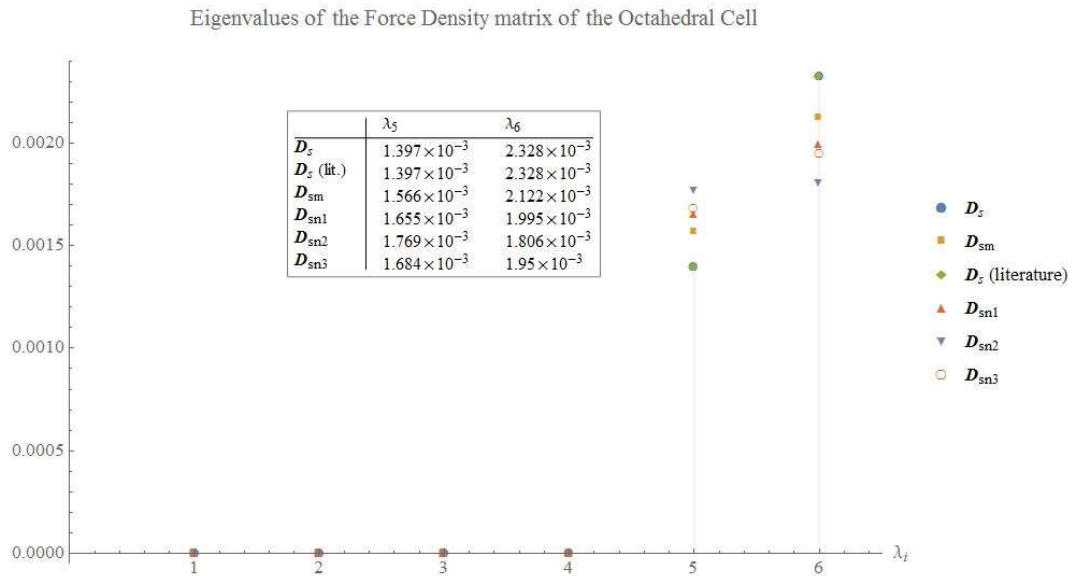


Fig. 12. Eigenvalues of the Force Density matrices of the tensegrity Octahedral cell

525

526 It can be noted that the Octahedral cell is a super-stable tensegrity structure; indeed, it has a non-  
 527 degenerate geometry in the three-dimensional space, and its Force Density matrix is semi-positive

528 definite with rank deficiency equal to 4. Such conditions occur for each of the axial stiffness  
 529 assignments of the elements, thus for each feasible self-stress states obtained by using the proposed  
 530 approach.

531 Finally, it is possible to calculate the norm of the unbalanced residual normalized internal forces  
 532 vectors  $[[\boldsymbol{\varepsilon}_u]]$ , see Table 10, showing the accuracy of the proposed method.

533

**Table 10**

Norm of the unbalanced residual normalized internal forces vectors of the Octahedral cell

	$\mathbf{t}_{nm}$	$\mathbf{t}$ ( <i>literature</i> )	$\mathbf{t}_{n1}$	$\mathbf{t}_{n2}$	$\mathbf{t}_{n3}$
$[[\boldsymbol{\varepsilon}_u]]$	$6.38 \cdot 10^{-16}$	$8.05 \cdot 10^{-16}$	$6.23 \cdot 10^{-16}$	$6.76 \cdot 10^{-16}$	$6.27 \cdot 10^{-16}$

534

## 535 7. Discussion and conclusions

536 A novel efficient method has been proposed for determining feasible self-stress states for tensegrity  
 537 structures by investigating the Distributed Static Indeterminacy of the tensegrity structure.

538 The proposed methods have some advantages over the existing form-finding methods; in particular: (i)  
 539 it allows for evaluating feasible self-stress states by performing only a unique Singular Value  
 540 Decomposition of the equilibrium matrix  $\mathbf{A}$ ; (ii) it only requires, as preliminary information, the  
 541 connectivity of the elements and their type, i.e. cable or struts, and the nodal coordinates; (iii) it is  
 542 possible to obtain several feasible self-stress states as linear combinations of the independent self-stress  
 543 modes according to the assignments of the axial stiffness of the elements.

544 This approach becomes particularly efficient for tensegrity structures with multiple self-stress states, as  
545 shown in the examined examples since it is not necessary to perform further SVD decompositions or to  
546 initialize grouping operations of the elements.

547 Indeed, such approach consists of determining suitable linear combinations of the independent self-  
548 stress modes according to the axial stiffness of the elements. Thus, it overcomes difficulties arising  
549 with complicated optimization techniques, mixed-integer nonlinear programming strategies, spectral  
550 decompositions, stiffness matrix evaluations and numerical iterative procedures presented in the  
551 literature. This main feature reduces the time consuming of the computational operations.

552 Moreover, it emerges that different feasible integral self-stress states can be easily obtained. In fact,  
553 feasible self-stress states consistent with the symmetry properties of the structures can be simply  
554 calculated by considering symmetric axial stiffness assignments to the elements of the tensegrity  
555 structure.

556 From the knowledge of the independent self-stress modes, and the evaluation of the Distributed Static  
557 Indeterminacy values related to the axial stiffness of each element, it is possible to address the self-  
558 stress identification of the tensegrity structures. Indeed, the proposed procedure bypass the element  
559 grouping operations needed in most of the state-of-the-art methods.

560 Furthermore, since DSI values are indicators that reflect the combined influence of the geometry,  
561 topology and axial stiffness of each element, different choices of element's stiffness lead to different  
562 feasible self-stress states. In particular, the load-bearing capacity of an element becomes lower as its  
563 DSI value increases. Thus, once evaluated the DSI of the elements of the structure, it is possible to  
564 calibrate each axial stiffness for achieving the desired mechanical behaviour of the entire structure.

565 The numerical analyses have shown that the norms of the unbalanced residual internal forces, evaluated  
566 for each case, are extremely close to zero, thus the accuracy of the proposed method has been proved.

567 Furthermore, the proposed approach allows to effectively determine the Force Density matrix of the  
568 structure, in order to evaluate the conditions of the super-stability for the tensegrity structure.

569 Moreover, the innovative method can be generalized by using a parametric description of the geometry  
570 of the structures in order to study how the internal forces in the elements vary as the geometrical  
571 parameters change, [also for large-scale tensegrity structures](#).

572 **Appendix A**

573 We can explicitly calculate  $\mathbf{K}\boldsymbol{\Omega}\mathbf{F}$  in terms of the matrix  $\mathbf{S}$ . In particular, we have:

$$\begin{aligned}\mathbf{K}\boldsymbol{\Omega}\mathbf{F} &= \mathbf{K}\mathbf{F}\mathbf{S}\left(\mathbf{S}^T\mathbf{F}\mathbf{S}\right)^{-1}\mathbf{S}^T\mathbf{F} = \\ &= \mathbf{S}\left(\mathbf{S}^T\mathbf{F}\mathbf{S}\right)^{-1}\mathbf{S}^T\mathbf{F}.\end{aligned}\tag{A1}$$

574 Then we evaluate the transpose of the matrix  $\boldsymbol{\Omega}$ :

$$\begin{aligned}\boldsymbol{\Omega}^T &= \left(\mathbf{F}\mathbf{S}\left(\mathbf{S}^T\mathbf{F}\mathbf{S}\right)^{-1}\mathbf{S}^T\right)^T = \\ &= \left(\left(\mathbf{S}^T\mathbf{F}\mathbf{S}\right)^{-1}\mathbf{S}^T\right)^T\left(\mathbf{F}\mathbf{S}\right)^T = \mathbf{S}\left(\left(\mathbf{S}^T\mathbf{F}\mathbf{S}\right)^{-1}\right)^T\mathbf{S}^T\mathbf{F} = \\ &= \mathbf{S}\left(\left(\mathbf{S}^T\mathbf{F}\mathbf{S}\right)^T\right)^{-1}\mathbf{S}^T\mathbf{F} = \mathbf{S}\left(\left(\mathbf{F}\mathbf{S}\right)^T\mathbf{S}\right)^{-1}\mathbf{S}^T\mathbf{F} = \\ &= \mathbf{S}\left(\mathbf{S}^T\mathbf{F}\mathbf{S}\right)^{-1}\mathbf{S}^T\mathbf{F}.\end{aligned}\tag{A2}$$

575 Hence, from Eq. (A1) and Eq. (A2) it is clear that  $\mathbf{K}\boldsymbol{\Omega}\mathbf{F}$  is equal to  $\boldsymbol{\Omega}^T$ .

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