On non-singular Hermitian varieties of $\mathrm{PG}(4, q^2)$

Angela Aguglia* Francesco Pavese†

Abstract

We provide a characterization of the non-singular Hermitian variety of $\mathrm{PG}(4, q^2)$ as a hypersurface of degree $q+1$ over $\mathrm{GF}(q^2)$ with $q^7+q^5+q^2+1$ rational points, which does not contain linear subspaces of dimension greater than 1 and having exactly a line in common with at least a plane of $\mathrm{PG}(4, q^2)$.

Keywords: unital, hermitian variety, algebraic hypersurface.

1 Introduction

Let $\mathrm{PG}(r, q)$ be the Desarguesian projective space of dimension r over the finite field $\mathrm{GF}(q)$ of order q and denote by $X = (x_0, x_1, \ldots, x_r)$ homogeneous coordinates for its points. A *projective hypersurface* $\mathcal{H}$ of $\mathrm{PG}(r, q)$ is

$$\mathcal{H} = \{(x_0, x_1, \ldots, x_r) \in \mathrm{PG}(r, q) \mid F(x_0, x_1, \ldots, x_r) = 0\},$$

where F is a homogeneous polynomial of $\mathrm{GF}(q)[X_0, X_1, \ldots, X_r]$. The degree of F is the *degree* of the hypersurface $\mathcal{H}$ with equation $F(X_0, \ldots, X_r) = 0$. When $r = 2$, a projective hypersurface $\mathcal{H}$ is called a *projective plane curve*, whereas when $r = 3$, $\mathcal{H}$ is called a *projective surface*. Suppose that $\mathcal{H}$ is a hypersurface of $\mathrm{PG}(r, q)$ with equation $F = 0$. In order to understand the geometry of $\mathcal{H}$, the zeros of F over $\mathrm{GF}(q)$ and any extension of $\mathrm{GF}(q)$ are required. Thus, $\mathcal{H}$ is viewed as a hypersurface over the algebraic closure of $\mathrm{GF}(q)$ and a point of $\mathrm{PG}(r, q^i)$ in $\mathcal{H}$ is called a $\mathrm{GF}(q^i)$*-point*. A $\mathrm{GF}(q)$-point of $\mathcal{H}$ is also said a *rational point* of $\mathcal{H}$. Throughout this paper, the number of $\mathrm{GF}(q^i)$-points of $\mathcal{H}$ will be denoted by $N_{q^i}(\mathcal{H})$.

A non-singular Hermitian variety $\mathcal{H}(r, q^2)$ of $\mathrm{PG}(r, q^2)$ is the set of absolute points of a Hermitian polarity of $\mathrm{PG}(r, q^2)$. On the other hand, from [4], a non-singular Hermitian variety of $\mathrm{PG}(r, q^2)$ is projectively equivalent to the hypersurface of $\mathrm{PG}(r, q^2)$ of degree $q+1$ having equation

$$X_0^{q+1} + \cdots + X_r^{q+1} = 0.$$

*Dipartimento di Meccanica, Matematica e Management, Politecnico di Bari, Via Orabona 4, 70125 Bari, Italy; *e-mail*: angela.aguglia@poliba.it

†Dipartimento di Meccanica, Matematica e Management, Politecnico di Bari, Via Orabona 4, 70125 Bari, Italy; *e-mail*: francesco.pavese@poliba.it

Mathematics Subject Classification (2010): Primary 51E21; Secondary 51E15 51E20

In [10, 11] it has been proved that if $\mathcal{X}$ is a hypersurface of degree $q+1$ in $\mathrm{PG}(r,q^2)$, $r \geq 3$ odd, such that it has $(q^{r+1}+(-1)^r)(q^r-(-1)^r)/(q^2-1)$ rational points and does not contain linear subspaces of dimension greater than $\frac{r-1}{2}$, then $\mathcal{X}$ is a non-singular Hermitian variety of $\mathrm{PG}(r,q^2)$. This result generalizes the characterization obtained in [6] for the Hermitian curve of $\mathrm{PG}(2,q^2)$, $q \neq 2$.

In this article we deal with the 4-dimensional projective case. Our main result is achieved by combining geometric and combinatorial arguments with algebraic geometry.

Theorem 1.1. *Let $\mathcal{H}$ be a hypersurface of $\mathrm{PG}(4,q^2)$, $q > 3$, defined over $\mathrm{GF}(q^2)$, without $\mathrm{GF}(q^2)$-hyperplane components and not containing planes. If the degree of $\mathcal{H}$ is $q+1$ and the number of its rational points is $q^7+q^5+q^2+1$, then every plane of $\mathrm{PG}(4,q^2)$ meets $\mathcal{H}$ in at least q^2+1 rational points. If there is at least a plane π such that $N_{q^2}(\pi \cap \mathcal{H}) = q^2+1$, then $\mathcal{H}$ is a non-singular Hermitian variety of $\mathrm{PG}(4,q^2)$.*

Our result improves the characterization provided in [1] where it is also required that the hypersurface $\mathcal{H}$ has two intersection numbers with hyperplanes of $\mathrm{PG}(4,q)$.

2 Background

In this section we collect some useful information and results that will be crucial to obtain our result.

Let $\mathcal{H}(r,q^2)$ be a non-singular Hermitian variety of $\mathrm{PG}(r,q^2)$. The number of rational points of $\mathcal{H}(r,q^2)$ equals

$$(q^{r+1}+(-1)^r)(q^r-(-1)^r)/(q^2-1).$$

Any line of $\mathrm{PG}(r,q^2)$ meets $\mathcal{H}(r,q^2)$ in $1, q+1$ or q^2+1 points. The maximal dimension of a projective subspace contained in $\mathcal{H}(r,q^2)$ is $(r-2)/2$, if r is even, or $(r-1)/2$, if r is odd. The latter subspaces are called *generators* of $\mathcal{H}(r,q^2)$. The generators of $\mathcal{H}(r,q^2)$ through a point P of $\mathcal{H}(r,q^2)$ span a hyperplane of $\mathrm{PG}(r,q^2)$. The hyperplane containing these generators is the *tangent hyperplane* of P with respect to the unitary polarity of $\mathrm{PG}(r,q^2)$ defining $\mathcal{H}(r,q^2)$ and it meets $\mathcal{H}(r,q^2)$ in a cone having as vertex the point P and as base a non-singular Hermitian variety of $\mathrm{PG}(r-2,q^2)$. Every hyperplane of $\mathrm{PG}(r,q^2)$, which is not tangent, meets $\mathcal{H}(r,q^2)$ in a non-singular Hermitian variety $\mathcal{H}(r-1,q^2)$, and is called a *secant hyperplane* of $\mathcal{H}(r,q^2)$. It follows that a tangent hyperplane contains

$$1+q^2(q^{r-1}+(-1)^r)(q^{r-2}-(-1)^r)/(q^2-1)$$

rational points of $\mathcal{H}(r,q^2)$, whereas a secant hyperplane contains

$$(q^r+(-1)^{r-1})(q^{r-1}-(-1)^{r-1})/(q^2-1)$$

rational points of $\mathcal{H}(r,q^2)$.

We will need the following results.

Lemma 2.1 ([13])**.** *Let $\mathcal{C}$ be an absolutely irreducible non-singular algebraic curve of genus g in* $\mathrm{PG}(2,q)$*. If there is on $\mathcal{C}$ a base-point-free linear system defined over the algebraic closure of* $\mathrm{GF}(q)$ *of degree n and dimension 2 with Frobenius order sequence $\nu_0,\dots,\nu_n$, then*

$$N_q(\mathcal{C}) \leq 2^{-1}\{\nu_1(2g-2)+(q+2)n\} \tag{2.1}$$

Lemma 2.2 ([12])**.** *Let d be an integer with $1 \leq d \leq q+1$ and $\mathcal{C}$ be a curve of degree d in* $\mathrm{PG}(2,q)$ *defined over* $\mathrm{GF}(q)$*, which may have* $\mathrm{GF}(q)$*-linear components. Then the number of its rational points is at most $dq+1$ and $N_q(\mathcal{C}) = dq+1$ if and only if $\mathcal{C}$ is a pencil of d lines of* $\mathrm{PG}(2,q)$*.*

Lemma 2.3 ([8])**.** *Let d be an integer with $2 \leq d \leq q+2$, and $\mathcal{C}$ a curve of degree d in* $\mathrm{PG}(2,q)$ *defined over* $\mathrm{GF}(q)$ *without* $\mathrm{GF}(q)$*-linear components. Then $N_{q^2}(\mathcal{C}) \leq (d-1)q+1$, except for a class of plane curves of degree 4 over* $\mathrm{GF}(4)$ *having 14 rational points.*

For generalities on curves in projective planes the reader is also referred to [5].

Lemma 2.4 ([9])**.** *Let $\mathcal{S}$ be a surface of degree d in* $\mathrm{PG}(3,q)$ *over* $\mathrm{GF}(q)$ *without* $\mathrm{GF}(q)$*-plane components. Then*

$$N_{q^2}(\mathcal{S}) \leq (d-1)q^2+dq+1$$

Lemma 2.5 ([10])**.** *Let $\mathcal{S}$ be a surface in* $\mathrm{PG}(3,q^2)$ *defined over* $\mathrm{GF}(q^2)$ *without* $\mathrm{GF}(q^2)$*-plane components. If the degree of $\mathcal{S}$ is $q+1$ and the number of its rational points is $(q^3+1)(q^2+1)$ then $\mathcal{S}$ is a non-singular Hermitian surface over* $\mathrm{GF}(q^2)$*.*

Lemma 2.6 ([6])**.** *Suppose $q \neq 2$. Let $\mathcal{C}$ be a plane curve over* $\mathrm{GF}(q^2)$ *of degree $q+1$ without* $\mathrm{GF}(q^2)$*-line components. If $\mathcal{C}$ has q^3+1 rational points, then $\mathcal{C}$ is a Hermitian curve.*

Lemma 2.7 ([3])**.** *A subset of points of* $\mathrm{PG}(r,q^2)$ *having the same intersection numbers with respect to hyperplanes and spaces of codimension 2 as non-singular Hermitian varieties, is a non-singular Hermitian variety of* $\mathrm{PG}(r,q^2)$*.*

From [7, Theorem 2.25] we have the following result.

Lemma 2.8. *If W is a set of q^5+q^2+1 points of* $\mathrm{PG}(3,q^2)$*, $q>2$, such that every line of* $\mathrm{PG}(3,q^2)$ *meets W in $1, q+1$ or q^2+1 points, then W is a cone with vertex a point and base a unital.*

Finally, we recall that a *blocking set with respect to lines* of $\mathrm{PG}(2,q)$ is a point set which blocks the lines, i.e., intersects each line of $\mathrm{PG}(2,q)$ in at least one point.

3 A characterization of $\mathcal{H}(4,q^2)$

In this section we provide a proof of Theorem 1.1. Thus, we denote by $\mathcal{H}$ a hypersurface of $\mathrm{PG}(4,q^2)$, $q>3$, defined over $\mathrm{GF}(q^2)$, without $\mathrm{GF}(q^2)$-hyperplane components and not containing planes.

Lemma 3.1. *$\mathcal{H}$ is a blocking set with respect to lines of* $\mathrm{PG}(4,q^2)$.

Proof. Suppose on the contrary that there is a line r of $\mathrm{PG}(4,q^2)$ which does not meet $\mathcal{H}$. Let α be a plane containing r. The algebraic plane curve $C=\alpha\cap\mathcal{H}$ of degree $q+1$ cannot have $\mathrm{GF}(q^2)$-linear components and hence it has at most q^3+1 points because of Lemma 2.3. If C had q^3+1 rational points, then from Lemma 2.6, C would be a Hermitian curve with an external line, a contradiction. Thus $N_{q^2}(C)\leq q^3$. We are going to show that C contains at most q^3-1 rational points. Assume that $N_{q^2}(C)=q^3$. We prove the following properties.

(P1) *C is an absolutely irreducible non-singular curve of α.*
Let C have equation $F(x,y,z)=0$, where F is a homogeneous polynomial of degree $q+1$ and let $F_1,\dots,F_n$ be the different irreducible factors of F over the algebraic closure of $\mathrm{GF}(q^2)$. Let us denote by C_i the component of C of equation $F_i=0$ and degree $d_i>1$. We have $|C_i|\leq(d_i-1)q^2+1$ and hence

$$q^3=N_{q^2}(C)\leq\sum_{i=1\dots n}|C_i|\leq q^2\sum_{i=1\dots n}d_i-q^2n+n=q^2(q+1)-q^2n+n.$$

Therefore $n=1$ that is, C is absolutely irreducible which means irreducible over the algebraic closure of $\mathrm{GF}(q^2)$.

Let s' be the number of singular rational points and let N be the number of non-singular rational points of C. Now, if C had a singular point P in α then by considering all lines of α through P we would obtain $N_{q^2}(C)\leq(q^2+1)(q-1)+1$ which is not possible. Hence $s'=0$.

Let M denote the number of rational points on a non-singular model of C. From [4, p.58] it follows that

$$N_{q^2}(C)=N+s'=N\leq M.$$

By the Hasse-Weil bound [4, Corollary 2.27(ii)] we get

$$q^3=N_{q^2}(C)\leq M\leq q^2+1+2gq,$$

where g is the genus of C. Hence $g\geq q(q-1)/2$, that is $g=q(q-1)/2$ and thus C is non-singular and (P1) is proved.

(P2) *The tangent line of C at a generic point $P\in C$ intersects C in P with multiplicity q.*
From (P1) we know that C is absolutely irreducible, non singular and of genus $g=q(q-1)/2$. We apply Lemma 2.1 by considering as a linear system the one defined on C by all lines of α. In this case $N_{q^2}(C)=q^3$, $2g-2=q^2-q-2$, $r=2$ and $n=q+1$. Thus

$$2q^3\leq\nu_1(q^2-q-2)+(q^2+2)(q+1),$$

which gives $\nu_1\geq q-2/(q^2-q-2)$ and hence $\nu_1\geq q$. On the other hand $\nu_1=1$ or $\nu_1=\epsilon_2$, where $\epsilon_2=i(\ell,C;P)$ is the order of contact of the tangent line ℓ at a generic point P of C. So $\nu_1=\epsilon_2$. Furthermore $\epsilon_2=2$ or ϵ_2 is a power of the characteristic p, see [13]. It also holds $\epsilon_2\leq\deg C=q+1$. Thus $\nu_1=\epsilon_2=q$ and property (P2) follows.

Now we recall that $\epsilon_2\leq\epsilon_2(Y)$, with $\epsilon_2(Y)=i(\ell,C;Y)$, where ℓ is the tangent line to C at Y and $Y\in\alpha\cap C$. Since $\epsilon_2(Y)\leq\deg C$, we have either $i(\ell,C;Y)=q$ or $i(\ell,C;Y)=q+1$.

Suppose first that there is a rational point Y of C with tangent line ℓ for which $i(\ell, C; Y) = q$. This implies that ℓ intersects C in exactly one other point Y_1 of α. On the other hand, all lines intersect C in at most $q+1$ distinct rational points as $\deg C = q+1$. Thus, considering all lines through Y_1 and taking into account that the tangent line to C at Y_1 has at most two rational points in common with C, we get $q^3 = N_{q^2}(C) \leq (q^2-1)q + 3$, that is $q \leq 3$, which is excluded.

Finally, assume that $i(\ell, C; Y) = q+1$, where ℓ is the tangent line of C at $Y \in C \cap \alpha$. Suppose that no line through Y meets C in q distinct rational points. Denote by s the number of lines through Y intersecting C in $q+1$ distinct rational points and by t the remaining lines through Y and different from ℓ. Hence $s + t = q^2$ and

$$q^3 = N_{q^2}(C) \leq sq + t(q-2) + 1 = q^3 - 2t + 1,$$

which gives $t = 0$. Therefore $q^3 = N_{q^2}(C) = q^2(q) + 1$, a contradiction. This means that at least one line through Y, say ℓ_1, intersects C in q distinct rational points $Y, Y_2, \ldots, Y_q$ and at one of these points the intersection multiplicity has to be 2. Suppose $i(\ell_1, C, Y_2) = 2$. Thus ℓ_1 is the tangent line of C at Y_2. On the other hand $i(\ell_1, C, Y_2) \geq q$ and hence we obtain $q \leq 2$, a contradiction.

Thus every plane through r meets $\mathcal{H}$ in at most q^3-1 rational points and hence, by considering all planes through r, we get the following bound for the number of rational points of $\mathcal{H}$: $N_{q^2}(\mathcal{H}) = q^7 + q^5 + q^2 + 1 \leq (q^4 + q^2 + 1)(q^3 - 1)$, which is impossible. Therefore there are no external lines to $\mathcal{H}$ and so $\mathcal{H}$ is a blocking set w.r.t. lines of $\mathrm{PG}(4, q^2)$. $\square$

Lemma 3.2. *Let π be a plane of $\mathrm{PG}(4, q^2)$, $q > 2$, meeting $\mathcal{H}$ in exactly $q^2 + 1$ rational points. Each hyperplane of $\mathrm{PG}(4, q^2)$ containing π intersects $\mathcal{H}$ in a cone over a non-degenerate Hermitian curve.*

Proof. By Lemma 3.1, $\pi \cap \mathcal{H}$ is a blocking set with respect to lines of π. Since $N_{q^2}(\pi \cap \mathcal{H}) = q^2+1$, from [2], it follows that π meets $\mathcal{H}$ in a line, say ℓ. Bounding the number of rational points of $\mathcal{H}$ by using all planes through ℓ and taking into account Lemma 2.2 yield

$$N_{q^2}(\mathcal{H}) = q^7 + q^5 + q^2 + 1 \leq (q^4 + q^2)q^3 + q^2 + 1 = q^7 + q^5 + q^2 + 1.$$

This means that there is exactly one plane through ℓ intersecting $\mathcal{H}$ in $q^2 + 1$ rational points, namely π, and all the remaining planes through ℓ meet $\mathcal{H}$ in $q^3 + q^2 + 1$ rational points. By Lemma 2.2, if α is a plane through ℓ such that $N_{q^2}(\alpha \cap \mathcal{H}) = q^3 + q^2 + 1$ then $\alpha \cap \mathcal{H}$ consists of $q + 1$ $\mathrm{GF}(q^2)$-lines of a pencil.

Let S be a solid containing the plane π and set $S \cap \mathcal{H} = W$. In this case, counting the number of rational points of W by using all planes in S through ℓ, we get

$$N_{q^2}(W) = q^5 + q^2 + 1.$$

We also see that there is a set $\mathcal{L}$ of $q^3 + 1$ lines divided into q^2 planar pencils, where each of these pencils consists of the line ℓ and other q lines that are concurrent at a point of ℓ. Moreover, every point of W not on ℓ lies on exactly one line of $\mathcal{L}$ and no other line of S is contained in W.

Indeed, assume on the contrary that a further line r is contained in W, then r has to be disjoint from ℓ and therefore $r \cap \pi \in W \setminus \{\ell\}$, a contradiction.

We claim that a line r of S meets W in $1, q+1$ or q^2+1 rational points. Indeed, if $|\ell \cap r| \neq 0$ and $\ell \neq r$, then the plane α containing ℓ and r either coincides with π and hence $r \cap W$ consists of one rational point or $\alpha \neq \pi$ and r meets each of the $q+1$ lines of α. Hence $N_{q^2}(r \cap W) \in \{1, q+1, q^2+1\}$. If r is disjoint from ℓ, then let P be a point of ℓ such that no line of $\mathcal{L} \setminus \{\ell\}$ meets P and let σ be the plane containing the point P and the line r. Since $N_{q^2}(\sigma \cap W) = q^3+1$, from Lemma 2.6, we have that $\sigma \cap W$ is a Hermitian curve and hence $N_{q^2}(r \cap W) \in \{1, q+1\}$. By Lemma 2.8, it turns out that W is a cone projecting a non-degenerate Hermitian curve. $\square$

Lemma 3.3. *If there exists a plane π of* $\mathrm{PG}(4, q^2)$ *such that $N_{q^2}(\pi \cap \mathcal{H}) = q^2+1$, then each hyperplane of* $\mathrm{PG}(4, q^2)$ *meets $\mathcal{H}$ either in a non-singular Hermitian surface or in a cone over a non-singular Hermitian curve.*

Proof. Let S be a hyperplane of $\mathrm{PG}(4, q^2)$ through the plane π. By Lemma 3.2, $S \cap \mathcal{H} = W$ is a Hermitian cone, i.e., a cone having as vertex a point and as base a non-degenerate Hermitian curve. In particular each plane in S meets $\mathcal{H}$ in either q^2+1 or q^3+1 or q^3+q^2+1 rational points. We consider all hyperplanes passing through a given plane α in S. If α is a plane of S meeting $\mathcal{H}$ in q^2+1 rational points, then, by Lemma 3.2, each hyperplane through α meets $\mathcal{H}$ in a Hermitian cone.

Now assume that α intersects $\mathcal{H}$ in q^3+q^2+1 rational points which form a pencil of $q+1$ lines, say $\mathcal{F}$. Let $S' \neq S$ be a hyperplane through α and let ℓ' denote a line of $\mathcal{F}$. From the proof of Lemma 3.2, we have that the line ℓ' is contained in exactly one plane, say π', meeting $\mathcal{H}$ in q^2+1 rational points, whereas all other planes through ℓ' intersect $\mathcal{H}$ in q^3+q^2+1 rational points. Note that π' is a plane contained in S and hence

$$N_{q^2}(S' \cap \mathcal{H}) = (q^2+1)q^3 + q^2 + 1 = q^5 + q^3 + q^2 + 1.$$

Furthermore, $S' \cap \mathcal{H}$ is a surface of degree $q+1$ of S' without $\mathrm{GF}(q^2)$-plane components and hence by Lemma 2.5, $S' \cap \mathcal{H}$ turns out to be a non-degenerate Hermitian surface of S'.

Finally assume α to be a plane intersecting $\mathcal{H}$ in q^3+1 rational points. As before, let S' denote a hyperplane containing α. Since $S' \cap \mathcal{H}$ is a surface of S' of degree $q+1$ without $\mathrm{GF}(q^2)$-plane components, by Lemma 2.4, it has at most $q^5+q^3+q^2+1$ rational points. Let us consider a second plane γ in S meeting $\mathcal{H}$ in q^2+1 rational points and thus satisfying $\gamma \cap \mathcal{H} \neq \gamma \cap S'$. Note that a hyperplane passing through γ meets $\mathcal{H}$ in a Hermitian cone. Therefore, if there is at least one hyperplane containing γ which intersects $S' \cap W$ in q^2+1 rational points, then $S' \cap \mathcal{H}$ has to be a Hermitian cone.

Suppose, by contradiction, that each hyperplane through γ meets S' in a plane which has either q^3+1 or q^3+q^2+1 rational points in common with $\mathcal{H}$. We denote by x the number of hyperplanes through γ meeting $\mathcal{H}$ in q^3+1 rational points and by y the number of the remaining hyperplanes through γ. We have $x+y = q^2+1$ and

$$N_{q^2}(S' \cap \mathcal{H}) = xq^3 + y(q^3+q^2) + 1 \leq q^5 + q^3 + q^2 + 1,$$

yielding $y \leq 1$. Hence, $y = 0$ or $y = 1$. Assume $y = 0$. In this case $x = q^2 + 1$ and $N_{q^2}(S' \cap \mathcal{H}) = q^5 + q^3 + 1$. In particular, $\mathcal{H}$ turns out to have three possible intersection numbers with respect to hyperplanes of $\mathrm{PG}(4, q^2)$. If we denote by x_i the number of hyperplanes meeting $\mathcal{H}$ in i rational points with $i \in \{q^5 + q^2 + 1, q^5 + q^3 + 1, q^5 + q^3 + q^2 + 1\}$, double counting arguments give the following equations for the integers x_i.

$$\begin{cases} \sum_i x_i = q^8 + q^6 + q^4 + q^2 + 1 \\ \sum_i ix_i = N_{q^2}(\mathcal{H})(q^6 + q^4 + q^2 + 1) \\ \sum_{i=1} i(i-1)x_i = N_{q^2}(\mathcal{H})(N_{q^2}(\mathcal{H}) - 1)(q^4 + q^2 + 1). \end{cases} \tag{3.1}$$

Solving (3.1) we obtain $x_{q^5+q^3+1} = 0$. Thus we have that necessarily $y = 1$, $N_{q^2}(S' \cap \mathcal{H}) = q^5 + q^3 + q^2 + 1$ and the result follows from Lemma 2.5. $\square$

Proof of Theorem 1.1 From Lemma 3.1, the hypersurface $\mathcal{H}$ meets every plane in at least $q^2 + 1$ rational points. Assume that there is at least a plane meeting $\mathcal{H}$ in precisely $q^2 + 1$ rational points. Then, from Lemma 3.3, $\mathcal{H}$ has the same intersection numbers with respect to hyperplanes and planes as a non-singular Hermitian variety of $\mathrm{PG}(4, q^2)$, hence by Lemma 2.7, $\mathcal{H}$ is a $\mathcal{H}(4, q^2)$ and our theorem follows.

References

[1] A. Aguglia *Characterizing Hermitian varieties in 3- and 4-dimensional projective spaces* Australas. J. Combin. **107** (2019), 1–8.

[2] R. C. Bose, R. C. Burton, A characterization of flat spaces in a finite geometry and the uniqueness of the Hamming and the MacDonald codes, *J. Combin. Theory* **1** (1966), 96–104.

[3] S. De Winter, J. Schillewaert, Characterizations of finite classical polar spaces by intersection numbers with hyperplanes and spaces of codimension 2, *Combinatorica* **30** (2010), n. 1, 25–45.

[4] J. W. P. Hirschfeld, *Projective geometries over finite fields*, Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 1998.

[5] J. W. P. Hirschfeld, G. Korchmáros, F. Torres, *Algebraic Curves over a Finite Field*, Princeton Series in Applied Mathematics, Princeton University Press, Princeton, NJ, 2008.

[6] J. W. P. Hirschfeld, L. Storme, J. A. Thas, J. F. Voloch, A characterization of Hermitian curves, *J. Geom.* **41** (1991), n. 1-2, 72–78.

[7] J. W. P. Hirschfeld, J. A. Thas, *General Galois geometries*, Springer Monographs in Mathematics, Springer, London, 2016.

[8] M. Homma, S. J. Kim, Around Sziklai's conjecture on the number of points of a plane curve over a finite field, *Finite Fields Appl.* **15** (2009), no. 4, 468–474.

[9] M. Homma, S. J. Kim, An elementary bound for the number of points of a Hypersurface over a finite field, *Finite Fields Appl.* **20** (2013), 76–83.

[10] M. Homma, S. J. Kim, The characterization of Hermitian surfaces by the number of points, *J. Geom.* **107** (2016), 509–521.

[11] M. Homma, S. J. Kim, Number of points of a nonsingular hypersurface in an odd-dimensional projective space, *Finite Fields Appl.* **48** (2017), 395–419.

[12] B. Segre, Le geometrie di Galois, *Ann. Mat. Pura Appl.* **48** (1959), n. 4, 1–96.

[13] K. O. Stöhr, J. F. Voloch, Weierstrass points and curves over finite fields, *Proc. London Math. Soc.* **52** (1986), n. 3, 1–19.