Standing waves for a Gauged Nonlinear Schrödinger Equation with a vortex point

STANDING WAVES FOR A GAUGED NONLINEAR SCHRÖDINGER EQUATION WITH A VORTEX POINT

YONGSHENG JIANG[1], ALESSIO POMPONIO[2], AND DAVID RUIZ[3]

ABSTRACT. This paper is motivated by a gauged Schrödinger equation in dimension 2. We are concerned with radial stationary states under the presence of a vortex at the origin. Those states solve a nonlinear nonlocal PDE with a variational estructure. We will study the global behavior of that functional, extending known results for the regular case.

1. INTRODUCTION

In this paper we are concerned with a planar gauged Nonlinear Schrödinger Equation:

$$iD_0\phi + (D_1D_1 + D_2D_2)\phi + |\phi|^{p-1}\phi = 0. \tag{1}$$

Here $t \in \mathbb{R}$, $x = (x_1, x_2) \in \mathbb{R}^2$, $\phi : \mathbb{R}\times\mathbb{R}^2 \to \mathbb{C}$ is the scalar field, $A_\mu : \mathbb{R}\times\mathbb{R}^2 \to \mathbb{R}$ are the components of the gauge potential and $D_\mu = \partial_\mu + iA_\mu$ is the covariant derivative ($\mu = 0,\ 1,\ 2$).

The modified gauge field equation proposes the following equation for the gauge potential, including the so-called Chern-Simons term (see [7,26]):

$$\partial_\mu F^{\mu\nu} + \frac{1}{2}\kappa\epsilon^{\nu\alpha\beta}F_{\alpha\beta} = j^\nu, \ \text{ with } \ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{2}$$

In the above equation, κ is a parameter that measures the strength of the Chern-Simons term. As usual, $\epsilon^{\nu\alpha\beta}$ is the Levi-Civita tensor, and super-indices are related to the Minkowski metric with signature $(1, -1, -1)$. Finally, j^μ is the conserved matter current,

$$j^0 = |\phi|^2,\ j^i = 2\mathrm{Im}\left(\bar{\phi}D_i\phi\right).$$

At low energies, the Maxwell term in (2) becomes negligible and can be dropped, giving rise to:

$$\frac{1}{2}\kappa\epsilon^{\nu\alpha\beta}F_{\alpha\beta} = j^\nu. \tag{3}$$

See [9,10,14,15,16] for the discussion above. If we fix $\kappa = 2$, equations (1) and (3) lead us to the problem:

$$\begin{aligned} &iD_0\phi + (D_1D_1 + D_2D_2)\phi + |\phi|^{p-1}\phi = 0,\\ &\partial_0A_1 - \partial_1A_0 = \mathrm{Im}(\bar{\phi}D_2\phi),\\ &\partial_0A_2 - \partial_2A_0 = -\mathrm{Im}(\bar{\phi}D_1\phi),\\ &\partial_1A_2 - \partial_2A_1 = \tfrac{1}{2}|\phi|^2. \end{aligned} \tag{4}$$

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As usual in Chern-Simons theory, problem (4) is invariant under gauge transformation,

$$\phi \to \phi e^{i\chi}, \quad A_\mu \to A_\mu - \partial_\mu \chi, \tag{5}$$

for any arbitrary C^∞ function χ.

This model was first proposed and studied in [14, 15, 16], and sometimes has received the name of Chern-Simons-Schrödinger equation. The initial value problem, well-posedness, global existence and blow-up, scattering, etc. have been addressed in [2, 11, 13, 20, 21] for the case $p = 3$. See also [19] for a global existence result in the defocusing case, and [5] for a uniqueness result to the infinite radial hierarchy.

The existence of stationary states for (4) and general $p > 1$ has been studied in [3] for the regular case (see also [6, 12, 22, 23]). Very recently, in [4] the case with a vortex point has been considered (with respect to that paper, our notation interchanges the indices 1 and 2). Consider the ansatz:

$$\begin{aligned} \phi &= u(r)e^{i(N\theta+\omega t)}, & A_0 &= A_0(r), \\ A_1 &= -\frac{x_2}{r^2}h(r), & A_2 &= \frac{x_1}{r^2}h(r) \end{aligned}$$

Here $(r,\ \theta)$ are the polar coordinates of $\mathbb{R}^2$, and $N \in \mathbb{N} \cup \{0\}$ is the order of the vortex at the origin ($N = 0$ corresponds to the regular case).

In [4] it is found that u solves the equation:

$$-\Delta u(x) + \omega\, u + \frac{(h_u(|x|) - N)^2}{|x|^2} u + A_0(|x|)u(x) = |u(x)|^{p-1}u(x), \quad x \in \mathbb{R}^2,$$

where

$$h_u(r) = \frac{1}{2}\int_0^r s u^2(s)\, ds, \tag{6}$$

and

$$A_0(r) = \xi + \int_r^{+\infty} \frac{h_u(s) - N}{s} u^2(s)\, ds, \ \ \xi \in \mathbb{R}.$$

The value ξ above appears as an integration constant. Without loss of generality, we can assume $\xi = 0$; otherwise it suffices to use the gauge invariance (5) with $\chi = \xi t$. Then, our problem becomes:

$$-\Delta u(x) + \omega\, u + \frac{(h_u(|x|) - N)^2}{|x|^2} u + \left(\int_{|x|}^{+\infty} \frac{h_u(s) - N}{s} u^2(s)\, ds \right) u(x) = |u(x)|^{p-1}u(x). \tag{7}$$

Observe that (7) is a nonlocal equation. In [4] it is shown that (7) is indeed the Euler-Lagrange equation of the energy functional $I_\omega : \mathcal{H} \to \mathbb{R}$,

$$\begin{aligned} I_\omega(u) = &\frac{1}{2}\int_{\mathbb{R}^2} \left(|\nabla u(x)|^2 + \omega u^2(x)\right)\, dx \\ &+ \frac{1}{2}\int_{\mathbb{R}^2} \frac{u^2(x)}{|x|^2}\left(h_u(r) - N\right)^2 dx - \frac{1}{p+1}\int_{\mathbb{R}^2} |u(x)|^{p+1}\, dx. \end{aligned}$$

The Hilbert space $\mathcal{H}$ is defined as:

$$\mathcal{H} = \{u \in H_r^1(\mathbb{R}^2) : \int_{\mathbb{R}} \frac{u^2(x)}{|x|^2} dx < +\infty\}, \tag{8}$$

endowed by the norm

$$\|u\|_{\mathcal{H}}^2 = \int_{\mathbb{R}^2} |\nabla u(x)|^2 + \left(1 + \frac{1}{|x|^2}\right) u^2(x)\, dx.$$

Let us observe that the energy functional I_ω presents a competition between the nonlocal term and the local nonlinearity of power-type. The study of the behavior of the functional under this competition is one of the main motivations of this paper. For $p > 3$, it is known that I_ω is unbounded from below, so it exhibits a mountain-pass geometry (see [3, 12] for the case $N = 0$ and [4, Section 5] for $N \in \mathbb{N}$). In a certain sense, in this case the local nonlinearity dominates the nonlocal term. However the existence of a solution is not so direct, since for $p \in (3, 5)$ the (PS) property is not known to hold. This problem is bypassed by combining the so-called monotonicity trick of Struwe ([25]) with a Pohozaev identity.

A special case in the above equation is $p = 3$: in this case, solutions have been explicitly found in [3, 4] as optimizers of a certain inequality. An alternative approach would be to pass to a self-dual equation, which leads to a Liouville equation in $\mathbb{R}^2$, singular if $N > 0$.

The situation is different if $p \in (1, 3)$; here the nonlocal term prevails over the local nonlinearity, in a certain sense. In [22], the second and third authors studied whether I_ω is bounded from below or not for $p \in (1, 3)$ and $N = 0$. The situation happened to be quite rich and unexpected, and very different from the usual nonlinear Schrödinger equation. Indeed, the boundedness of I_ω for $N = 0$ depends on the phase ω and the threshold value ω_0 is explicit, namely:

$$\omega_0 = \frac{3-p}{3+p}\, 3^{\frac{p-1}{2(3-p)}}\, 2^{\frac{2}{3-p}} \left(\frac{m^2(3+p)}{p-1}\right)^{-\frac{p-1}{2(3-p)}}, \tag{9}$$

with

$$m = \int_{-\infty}^{+\infty} \left(\frac{2}{p+1} \cosh^2\left(\frac{p-1}{2} r\right)\right)^{\frac{2}{1-p}} dr. \tag{10}$$

The purpose of this paper is to extend such result to the case $N > 0$, which is more relevant from the point of view of the applications. This study has been prompted by Remark 5.1 in [4].

Our main results are the following:

Theorem 1.1. *For ω_0 as given in (9), there holds:*

(i) if $\omega \in (0, \omega_0)$, then I_ω is unbounded from below;
(ii) if $\omega = \omega_0$, then I_{ω_0} is bounded from below, not coercive and $\inf I_{\omega_0} < 0$;
(iii) if $\omega > \omega_0$, then I_ω is bounded from below and coercive.

Regarding the existence of solutions, we obtain the following results:

Theorem 1.2. *There exist $\bar{\omega} > \tilde{\omega} > \omega_0$ such that:*

(i) if $\omega > \bar{\omega}$, then (7) has no solutions different from zero;
(ii) if $\omega \in (\omega_0, \tilde{\omega})$, then (7) admits at least two positive solutions: one of them is a global minimizer for I_ω and the other is a mountain-pass solution;
(iii) for almost every $\omega \in (0, \omega_0)$ (7) admits a positive solution.

The proofs follow the same ideas as in [22], and is related to a natural limit problem. Roughly speaking, this limit problem stems from the behavior of the map $\rho \mapsto I_\omega(u(\cdot - \rho))$ as $\rho \to +\infty$, and this does not depend on N. However, in our proofs the analysis made in Proposition 3.2 must be re-elaborated with respect to that of [22], and the new terms need new estimates in the asymptotic expansions that follow afterwards. Moreover, the non-existence result of Theorem 1.2 is

immediate for $N = 0$ but its proof becomes delicate for $N > 0$. Finally, the case $N > 0$ is more relevant from the point of view of the Physics model, since it includes a vortex at the origin. One of the main features of the Chern-Simons theory is the appearance of vortices in the model, see [7,26,27]).

The rest of the paper is organized as follows. Section 2 is devoted to some notations and preliminary results. In Section 3 we prove Theorems 1.1 and 1.2.

2. Preliminaries

Let us first fix some notations. We denote by $H^1_r(\mathbb{R}^2)$ the Sobolev space of radially symmetric functions, and $\|\cdot\|$ its usual norm. We denote by $\|u\|_{L^p}$ the usual Lebesgue norm in $\mathbb{R}^2$. Moreover, we will write $\|\cdot\|_{H^1(\mathbb{R})}$, $\|\cdot\|_{H^1(a,b)}$ to indicate the norms of the Sobolev spaces of dimension 1.

However our functional I_ω is defined in the space $\mathcal{H}$, defined in (8). Its norm will be denoted by $\|\cdot\|_{\mathcal{H}}$. In [4, Proposition 3.1] it is shown that

$$\mathcal{H} \subset \{u \in C(\mathbb{R}^2) : u(0) = 0\} \cap L^\infty(\mathbb{R}^2).$$

If nothing is specified, strong and weak convergence of sequences of functions are assumed in the space $H^1(\mathbb{R}^2)$.

In our estimates, we will frequently denote by $C > 0$, $c > 0$ fixed constants, that may change from line to line, but are always independent of the variable under consideration. We also use the notations $O(1), o(1), O(\varepsilon), o(\varepsilon)$ to describe the asymptotic behaviors of quantities in a standard way. Finally the letters x, y indicate two-dimensional variables and r, s denote one-dimensional variables.

Let us start with the following proposition, proved in [3,4]:

Proposition 2.1. *I_ω is a C^1 functional, and its critical points correspond to classical solutions of (7).*

The next result is contained in [4, Proposition 3.4], and deals with the behavior of I_ω under weak limits.

Proposition 2.2. *Recalling the definition of h_u, (6), let us define:*

$$K(u) = \frac{1}{2}\int_{\mathbb{R}^2} h_u^2(x)\frac{u^2(x)}{|x|^2} - 2Nh_u(x)\frac{u^2(x)}{|x|^2}\,dx. \tag{11}$$

Then K and K' are weakly continuous in $\mathcal{H}$. As a consequence, I_ω is weak lower semicontinuous, and I'_ω is weakly continuous in $\mathcal{H}$.

Next lemma relates boundedness of sequences in $H^1(\mathbb{R}^2)$ and in $\mathcal{H}$, and will be very useful in Section 3.

Lemma 2.3. *The map K defined in (11) is actually well defined in $H^1(\mathbb{R}^2)$ and $K(u_n)$ is bounded if $\|u_n\|$ is bounded. As a consequence, for any sequence $u_n \in \mathcal{H}$ such that $I_\omega(u_n)$ is bounded from above, $\|u_n\|$ is bounded if and only if $\|u_n\|_{\mathcal{H}}$ is bounded.*

Proof. By [3], we only need to consider the term:

$$\int_{\mathbb{R}^2} h_u(x)\frac{u^2(x)}{|x|^2}\,dx = \pi\int_0^{+\infty}\frac{u^2(r)}{r}\left(\int_0^r su^2(s)\,ds\right)dr$$
$$\leqslant \pi\int_0^{+\infty} u^2(r)\left(\int_0^r u^2(s)\,ds\right)dr = \frac{\pi}{2}\left(\int_0^{+\infty} u^2(r)\,dr\right)^2$$

Observe now that:

$$2\pi\int_0^{+\infty} u^2(r)\,dr = \int_{\mathbb{R}^2}\frac{u^2(x)}{|x|}\,dx \leqslant \int_{B(0,1)}\frac{u^2(x)}{|x|}\,dx + \int_{\mathbb{R}^2\setminus B(0,1)} u^2(x)\,dx$$
$$\leqslant C(\|u\|_{L^2}^2 + \|u\|_{L^p}^2),\ p>4,$$

by Holder inequality. The first assertion of the Lemma follows then from the Sobolev embedding.

Suppose that u_n is bounded in $H^1(\mathbb{R}^2)$; then

$$I_\omega(u_n) = O(1) + \frac{N^2}{2}\int_{\mathbb{R}^2}\frac{u_n^2(x)}{|x|^2}\,dx,$$

and by hypothesis u_n is bounded in $\mathcal{H}$. The reverse is trivial.

□

The following is a Pohozaev-type identity for problem (7), see (2.11), (5.6) in [4]:

Proposition 2.4. *For any $u\in\mathcal{H}$ solution of (7), the following identity holds:*

$$\int_{\mathbb{R}^2}|\nabla u|^2dx + \int_{\mathbb{R}^2}\frac{u^2(x)}{|x|^2}\left(h_u(|x|)-N\right)^2dx - \frac{p-1}{p+1}\int_{\mathbb{R}^2}|u|^{p+1}dx = 0.$$

We now state an inequality which will prove to be fundamental in our analysis. This inequality is proved in [4, Proposition 3.5], where also the maximizers are found.

Proposition 2.5. *For any $u\in\mathcal{H}$,*

$$\int_{\mathbb{R}^2}|u(x)|^4\,dx \leqslant 4\left(\int_{\mathbb{R}^2}|\nabla u(x)|^2\,dx\right)^{1/2}\left(\int_{\mathbb{R}^2}\frac{u^2(x)}{|x|^2}\left(h_u(|x|)-N\right)^2dx\right)^{1/2}. \tag{12}$$

As commented in the introduction, this paper is concerned with the boundedness from below of I_ω. First of all, let us give a heuristic derivation of the limit energy functional. Consider $u(r)$ a fixed function, and define $u_\rho(r)=u(r-\rho)$. Let us now estimate $I_\omega(u_\rho)$ as $\rho\to+\infty$; after the change of variables $r\to r+\rho$, we obtain:

$$\frac{I_\omega(u_\rho)}{2\pi} = \frac{1}{2}\int_{-\rho}^{+\infty}(|u'|^2+\omega u^2)(r+\rho)\,dr$$
$$+\frac{1}{8}\int_{-\rho}^{\infty}\frac{u^2(r)}{r+\rho}\left(\int_{-\rho}^{r}(s+\rho)u^2(s)\,ds - 2N\right)^2dr - \frac{1}{p+1}\int_{-\rho}^{\infty}|u|^{p+1}(r+\rho)\,dr.$$

We estimate the above expression by simply replacing the expressions $(r+\rho)$, $(s+\rho)$ with the constant ρ; observe that the estimate is independent of N:

$$(2\pi)^{-1}I_\omega(u_\rho)$$
$$\sim\rho\left[\frac{1}{2}\int_{-\infty}^{+\infty}(|u|'^2+\omega u^2)\,dr + \frac{1}{8}\int_{-\infty}^{+\infty}u^2(r)\left(\int_{-\infty}^{r}u^2(s)\,ds\right)^2dr - \frac{1}{p+1}\int_{-\infty}^{+\infty}|u|^{p+1}\,dr\right]$$
$$=\rho\left[\frac{1}{2}\int_{-\infty}^{+\infty}(|u|'^2+\omega u^2)\,dr + \frac{1}{24}\left(\int_{-\infty}^{+\infty}u^2dr\right)^3 - \frac{1}{p+1}\int_{-\infty}^{+\infty}|u|^{p+1}\,dr\right].$$

Therefore, it is natural to consider the limit functional $J_\omega : H^1(\mathbb{R})\to\mathbb{R}$,

$$J_\omega(u) = \frac{1}{2}\int_{-\infty}^{+\infty}\left(|u'|^2+\omega u^2\right)dr + \frac{1}{24}\left(\int_{-\infty}^{+\infty}u^2\,dr\right)^3 - \frac{1}{p+1}\int_{-\infty}^{+\infty}|u|^{p+1}\,dr. \tag{13}$$

Clearly, the Euler-Lagrange equation of (13) is the following limit problem:

$$-u'' + \omega u + \frac{1}{4}\left(\int_{-\infty}^{+\infty} u^2(s)\,ds\right)^2 u = |u|^{p-1}u \quad \text{in } \mathbb{R}. \tag{14}$$

Let u be a positive solution of (14), and define $k = \omega + \frac{1}{4}\left(\int_{-\infty}^{+\infty} u^2\,dr\right)^2$. Then, it is well known that $u(r) = w_k(r-\xi)$ for some $\xi \in \mathbb{R}$, where

$$w_k(r) = k^{\frac{1}{p-1}} w_1(\sqrt{k}r), \quad \text{with} \quad w_1(r) = \left(\frac{2}{p+1}\cosh^2\left(\frac{p-1}{2}r\right)\right)^{\frac{1}{1-p}}.$$

We now recall the value of k:

$$k = \omega + \frac{1}{4}\left(\int_{-\infty}^{+\infty} w_k^2(r)\,dr\right)^2 = \omega + \frac{1}{4}k^{\frac{4}{p-1}}\left(\int_{-\infty}^{+\infty} w_1^2(\sqrt{k}r)\,dr\right)^2.$$

A change of variables leads us to the identity:

$$k = \omega + \frac{1}{4}m^2 k^{\frac{5-p}{p-1}},\ k > 0, \tag{15}$$

with m is given in (10). Therefore, the existence of solutions for (14) reduces to the existence of solutions of the algebraic equation (15). Moreover, we are also interested in the energy of those solutions, and whether it is positive or negative. Those questions have been treated in [22, Section 3], where the following results were obtained:

Proposition 2.6. *Assume $p \in (1,3)$ and take ω_0 as in (9). Then:*

(1) for any $\omega > 0$, J_ω is coercive and attain its infimum;
(2) There exists $\omega_1 > \omega_0$ such that for $\omega \in (0,\omega_1)$, equation (15) has two solutions $k_1(\omega) < k_2(\omega)$ and $w_{k_1}(r), w_{k_2}(r)$ are the only two positive solutions of (14) (apart from translations);
(3) if $\omega > \omega_0$, $\min J_\omega = 0$ and the unique minimizer is 0.
(4) if $\omega = \omega_0$, $\min J_\omega = 0$ and is attained at 0 and w_{k_2}.
(5) if $\omega \in (0,\omega_0)$, $\min J_\omega < 0$ and the minimizer is w_{k_2}, which is unique (up to change of sign and translation).

In this paper we are able to relate I_ω with the limit functional J_ω in the following way:

$$\inf I_\omega > -\infty \ \Leftrightarrow\ \inf J_\omega = 0.$$

That is the reason why the explicit value ω_0 comes as a threshold for I_ω.

We finish this section with a technical result from [22, Proposition 3.7], that will be of use later.

Proposition 2.7. *Assume $\omega \geqslant \omega_0$, and $u_n \in H^1(\mathbb{R})$ such that $J_\omega(u_n) \to 0$. There holds*

(1) if $\omega > \omega_0$, then $u_n \to 0$ in $H^1(\mathbb{R})$;
(2) if $\omega = \omega_0$, then, up to a subsequence, either $u_n \to 0$ or $u_n(\cdot - x_n) \to \pm w_{k_2}$ in $H^1(\mathbb{R})$, for some sequence $x_n \in \mathbb{R}$.

3. Proof of Theorems 1.1, 1.2

Our first lemma makes rigorous the heuristic derivation of the limit functional made in Section 2. Since the functions in $\mathcal{H}$ must vanish at 0, we need to truncate our sequence around the origin. For that purpose, take a Lipschitz continuous function $\phi_0 : \mathbb{R} \to \mathbb{R}$ such that

$$\phi_0(r) = \begin{cases} 0, & \text{if } |r| \leqslant 1, \\ 1, & \text{if } |r| \geqslant 2, \end{cases} \qquad |\phi_0'(r)| \leqslant 1. \tag{16}$$

Lemma 3.1. *Let $U \in H^1(\mathbb{R})$ be an even function which decays to zero exponentially at infinity, and $\phi_0(r)$ as in (16). Let us denote $U_\rho(r) = \phi_0(r)U(r-\rho)$. Then there exists $C > 0$ such that:*

$$I_\omega(U_\rho) = 2\pi\rho J_\omega(U(r)) - C + o_\rho(1).$$

Proof. This estimate has been accomplished in [22, Lemma 4.1] for $N = 0$, so we just need to estimate the extra terms:

$$N^2 \int_0^{+\infty} \frac{U_\rho^2(r)}{r} dr - N \int_0^{+\infty} \frac{U_\rho^2(r)}{r} \left(\int_0^r sU_\rho^2(s)\, ds \right) dr.$$

By using the properties of the cut-off function ϕ_0 we have

$$\int_0^{+\infty} \frac{U_\rho^2(r)}{r} dr = o_\rho(1)$$

and it is not difficult to see that

$$\int_0^{+\infty} \frac{U_\rho^2(r)}{r} \left(\int_0^r sU_\rho^2(s)\, ds \right) dr = \int_{-\infty}^{+\infty} U^2(r) \left(\int_{-\infty}^r U^2(s)\, ds \right) dr + o_\rho(1) = C + o_\rho(1),$$

with $C > 0$. Hence the conclusion follows.

□

In the next proposition we make use of the fundamental inequality (12) to study the behavior of unbounded sequences with energy bounded from above.

Proposition 3.2. *Assume $\omega > 0$, and $u_n \in \mathcal{H}$ such that $\|u_n\|$ is unbounded but $I_\omega(u_n)$ is bounded from above. Then, there exists a subsequence (still denoted by u_n) such that:*

i) for all $\varepsilon > 0$, $\displaystyle\int_{\varepsilon\|u_n\|^2}^{+\infty} \left(|u_n'|^2 + u_n^2\right) dr \leqslant C$;

ii) there exists $\delta \in (0,1)$ such that $\displaystyle\int_{\delta\|u_n\|^2}^{\delta^{-1}\|u_n\|^2} \left(|u_n'|^2 + u_n^2\right) dr \geqslant c > 0$;

iii) $\|u_n\|_{L^2(\mathbb{R}^2)} \to +\infty$.

Proof. The proof is quite similar to [22, Proposition 4.2], but there are some differences at certain points due to the presence of the singular term. For convenience of the reader, we reproduce it entirely here. By inequality (12) and Cauchy-Schwartz inequality, we can estimate:

$$I_\omega(u) \geqslant \frac{\pi}{2} \int_0^{+\infty} \left(|u'|^2 + \omega u^2\right) r\, dr + \frac{\pi}{8} \int_0^{+\infty} \frac{u^2(r)}{r} \left(\int_0^r su^2(s)\, ds - 2N \right)^2 dr$$

$$+ 2\pi \int_0^{+\infty} \left(\frac{\omega}{4} u^2 + \frac{1}{8} u^4 - \frac{1}{p+1} |u|^{p+1} \right) r\, dr. \tag{17}$$

Define

$$f : \mathbb{R}_+ \to \mathbb{R}, \qquad f(t) = \frac{\omega}{4} t^2 + \frac{1}{8} t^4 - \frac{1}{p+1} t^{p+1}.$$

Then, the set $\{t > 0 : f(t) < 0\}$ is of the form (α, β), where α, β are positive constants depending only on p, ω. Moreover, we denote by $-c_0 = \min f < 0$.

For each function u_n, we define:

$$A_n = \{x \in \mathbb{R}^2 : u_n(x) \in (\alpha, \beta)\}, \ \rho_n = \sup\{|x| : x \in A_n\}.$$

With these definitions, we can rewrite (17) in the form

$$I_\omega(u_n) \geqslant \frac{\pi}{2} \int_0^{+\infty} \left(|u_n'|^2 + \omega u_n^2\right) r\, dr + \frac{\pi}{8} \int_0^{+\infty} \frac{u_n^2(r)}{r} \left(\int_0^r su_n^2(s)\, ds - 2N \right)^2 dr - c_0 |A_n|. \tag{18}$$

In particular this implies that $|A_n|$ must diverge, and hence ρ_n. This already proves (iii).

By Strauss Lemma [24], we have

$$\alpha \leqslant u_n(\rho_n) \leqslant \frac{\|u_n\|}{\sqrt{\rho_n}}, \Rightarrow \|u_n\|^2 \geqslant \alpha^2 \rho_n. \tag{19}$$

We now estimate the nonlocal term. For that, define

$$B_n = A_n \cap B(0, \gamma_n), \text{ for } \gamma_n \in (0, \rho_n) \text{ such that } |B_n| = \frac{1}{2}|A_n|. \tag{20}$$

Then $\int_{B_n} u_n^2(x)\,dx \geqslant \alpha^2 |B_n|$ diverges, indeed

$$\int_{B_n} u_n^2(x)\,dx - 2N > c|A_n|.$$

We now estimate:

$$\begin{aligned}
\int_0^{+\infty} \frac{u_n^2(r)}{r}\left(\int_0^r s u_n^2(s)\,ds - 2N\right)^2 dr &\geqslant \int_{\gamma_n}^{+\infty} \frac{u_n^2(r)}{r}\left(\int_0^r s u_n^2(s)\,ds - 2N\right)^2 dr \\
&\geqslant \frac{1}{4\pi^2}\int_{\gamma_n}^{+\infty} \frac{u_n^2(r)}{r}\left(\int_{B_n} u_n^2(x)\,dx - 2N\right)^2 dr \\
&\geqslant c|A_n|^2 \int_{\gamma_n}^{+\infty} \frac{u_n^2(r)}{r}\,dr \\
&\geqslant c|A_n|^2 \int_{A_n \setminus B_n} \frac{u_n^2(x)}{|x|^2}\,dx \\
&\geqslant c\frac{|A_n|^2}{\rho_n^2}\int_{A_n\setminus B_n} u_n^2(x)\,dx \\
&\geqslant c\frac{|A_n|^3}{\rho_n^2}. \qquad (21)
\end{aligned}$$

Hence, by (17), (19) and (21), we get

$$I_\omega(u_n) \geqslant c\rho_n + c\frac{|A_n|^3}{\rho_n^2} - c_0|A_n| = \rho_n\left(c + c\frac{|A_n|^3}{\rho_n^3} - c_0\frac{|A_n|}{\rho_n}\right).$$

Observe that $t \mapsto c + ct^3 - c_0 t$ is strictly positive near zero and goes to $+\infty$, as $t \to +\infty$. Then we can assume, passing to a subsequence, that $|A_n| \sim \rho_n$. In other words, there exists $m > 0$ such that $\rho_n |A_n|^{-1} \to m$ as $n \to +\infty$.
Taking into account (18) and (19), we conclude that up to a subsequence, $\|u_n\|^2 \sim \rho_n$. Moreover, for any fixed $\varepsilon > 0$, we have:

$$C\rho_n \geqslant \|u_n\|_{L^2}^2 \geqslant \int_{\varepsilon\rho_n}^{+\infty} u_n^2 r\,dr \geqslant \varepsilon\rho_n \int_{\varepsilon\rho_n}^{+\infty} u_n^2\,dr.$$

An analogous estimate works also for $\int_{\varepsilon\rho_n}^{+\infty} |u_n'|^2 dr$. This proves (i).

We now show that for some $\delta > 0$, $\|u_n\|_{H^1(\delta\rho_n, \rho_n)} \nrightarrow 0$, which implies assertion (ii).
First, recall the definition of B_n and γ_n in (20). Then,

$$\int_{\gamma_n}^{\rho_n} u_n^2(r)\,dr \geqslant \rho_n^{-1}\int_{\gamma_n}^{\rho_n} u_n^2(r) r\,dr \geqslant \rho_n^{-1}\int_{A_n\setminus B_n} u_n^2(x)dx \geqslant \rho_n^{-1}|A_n\setminus B_n|\alpha^2 > c > 0.$$

To conclude it suffices to show that $\gamma_n \sim \rho_n$. Define

$$C_n = B_n \cap B(0, \tau_n), \text{ for } \tau_n \in (0, \gamma_n) \text{ such that } |C_n| = \frac{1}{2}|B_n|.$$

We can repeat the estimate (21) with A_n, B_n replaced with B_n, C_n respectively, to obtain that

$$\int_0^{+\infty} \frac{u_n^2(r)}{r} \left(\int_0^r s u_n^2(s)\, ds - 2N \right)^2 dr \geqslant c \frac{|B_n|^3}{\gamma_n^2}.$$

Hence,

$$I_\omega(u_n) \geqslant c\rho_n + c\frac{|A_n|^3}{\gamma_n^2} - c_0|A_n| = \gamma_n \left(c\frac{\rho_n}{\gamma_n} + c\frac{|A_n|^3}{\gamma_n^3} - c_0\frac{|A_n|}{\gamma_n} \right).$$

And we are done since $I_\omega(u_n)$ is bounded from above.

□

Proof of Theorem 1.1. If $\omega \in (0, \omega_0)$, then $J_\omega(w_{k_2}) < 0$ (see Proposition 2.6): applying Lemma 3.1 with $U = w_{k_2}$ we conclude assertion (i).

We now prove (ii) and (iii). We denote by $H^1_{0,r}(B(0,R))$ the Sobolev space of radial functions with zero boundary value and

$$\mathcal{H}(B(0,R)) = \left\{ u \in H^1_{0,r}(B(0,R)) : \int_{B(0,R)} \frac{u^2(x)}{|x|^2} dx < +\infty \right\},$$

endowed by the norm $\|\cdot\|_{\mathcal{H}}$.
Fixed $n \in \mathbb{N}$ and given a sequence $v_i \in \mathcal{H}(B(0,n))$ unbounded with respect to the norm $\|\cdot\|$, (18) implies that $I_\omega(v_i) \to +\infty$. By Lemma 2.3, we conclude that $I_\omega|_{\mathcal{H}(B(0,n))}$ is coercive.
So, there exists u_n a minimizer for $I_\omega|_{\mathcal{H}(B(0,n))}$. By taking absolute value, we can assume that $u_n \geqslant 0$. Moreover,

$$I_\omega(u_n) \to \inf I_\omega, \text{ as } n \to +\infty.$$

In the following, u_n may be extended as functions in $\mathcal{H}$ by setting $u_n(x) = 0$ for $x \in \mathbb{R}^2 \setminus B(0,n)$. If u_n is bounded in $H^1(\mathbb{R}^2)$, Lemma 2.3 implies that u_n is bounded in $\mathcal{H}$ and then $I_\omega(u_n)$ is bounded. In such case we conclude that $\inf I_\omega$ is finite. In what follows we assume that u_n is an unbounded sequence in $H^1(\mathbb{R}^2)$, and we shall show that $I_\omega(u_n)$ is still bounded for $\omega \geqslant \omega_0$.

Our sequence u_n satisfies the hypotheses of Proposition 3.2, so let $\delta > 0$ be given by that proposition.

The proof will be divided in several steps.

Step 1:
$\displaystyle\int_{\frac{\delta}{2}\|u_n\|^2}^{\frac{2}{\delta}\|u_n\|^2} |u_n|^{p+1}\, dr \nrightarrow 0.$

By Proposition 3.2, i), we have that:

$$\sum_{k=1}^{[\frac{\delta}{2}\|u_n\|^2]} \int_{\frac{\delta}{2}\|u_n\|^2+k-1}^{\frac{\delta}{2}\|u_n\|^2+k} \left(|u_n'|^2 + u_n^2\right) dr \leqslant \int_{\frac{\delta}{2}\|u_n\|^2}^{\delta\|u_n\|^2} \left(|u_n'|^2 + u_n^2\right) dr \leqslant C.$$

Taking the smaller summand in the left hand side we find x_n,

$$\frac{\delta}{2}\|u_n\|^2 \leqslant x_n \leqslant \delta\|u_n\|^2 - 1 \text{ such that } \|u_n\|^2_{H^1(x_n, x_n+1)} \leqslant \frac{C}{\|u_n\|^2}.$$

Reasoning in an analogous way, we can choose y_n,

$$\delta^{-1}\|u_n\|^2 + 1 \leqslant y_n \leqslant 2\delta^{-1}\|u_n\|^2 \text{ such that } \|u_n\|^2_{H^1(y_n, y_n+1)} \leqslant \frac{C}{\|u_n\|^2}.$$

Observe that if $\delta^{-1}\|u_n\|^2 \geqslant n$, the choice of y_n can be arbitrary, but it is unnecessary. Take $\phi_n : [0,+\infty] \to [0,1]$ be a C^∞-function such that

$$\phi_n(r) = \begin{cases} 0, & \text{if } r \leqslant x_n, \\ 1, & \text{if } x_n + 1 \leqslant r \leqslant y_n, \\ 0, & \text{if } r \geqslant y_n + 1. \end{cases} \quad |\phi_n'(r)| \leqslant 2.$$

Let

$$F(u) = \int_0^{+\infty} \frac{u^2(r)}{r} \left(\int_0^r s u^2(s) ds - 2N \right)^2 dr.$$

By the choice of x_n, y_n and Proposition 3.2, i), we have

$$\begin{aligned} F'(u_n)[\phi_n u_n] &\geqslant 4 \int_0^{+\infty} \frac{u_n^2(r)}{r} \left(\int_0^r s u_n^2(s) ds - 2N \right) \left(\int_0^r s u_n^2(s) \phi_n(s) ds \right) dr \\ &\geqslant -8N \left(\int_{x_n}^{+\infty} u_n^2(r) dr \right)^2 > -C. \end{aligned}$$

It follows that

$$\begin{aligned} 0 = I_\omega'(u_n)[\phi_n u_n] &\geqslant 2\pi \int_{x_n}^{y_n} \left(|u_n'|^2 + \omega u_n^2 \right) r\, dr - 2\pi \int_{x_n}^{y_n} |u_n|^{p+1} r\, dr + O(1) \\ &\geqslant \|u_n\|^2 \left(\frac{\delta}{2} \int_{x_n}^{y_n} \left(|u_n'|^2 + \omega u_n^2 \right) dr - \frac{2}{\delta} \int_{x_n}^{y_n} |u_n|^{p+1}\, dr \right) + O(1). \end{aligned}$$

This, together with the fact that $\|u_n\|_{H^1(x_n,y_n)}$ does not tend to zero, allows us to conclude the proof of Step 1.

Step 2: Exponential decay.

At this point we can apply the concentration-compactness principle (see [18, Lemma 1.1]); there exists $\sigma > 0$ such that

$$\sup_{\xi \in [x_n,\, y_n]} \int_{\xi-1}^{\xi+1} u_n^2\, dr \geqslant 2\sigma > 0.$$

Let us define:

(22)
$$D_n = \left\{ \xi > 0 : \int_{\xi-1}^{\xi+1} \left(|u_n'|^2 + u_n^2 \right) dr \geqslant \sigma \right\} \neq \emptyset, \text{ and } \xi_n = \max D_n \in [x_n, n+1).$$

Let us observe that $\xi_n \sim \|u_n\|^2$; indeed $\xi_n \geqslant x_n \geqslant c\|u_n\|^2$ and, moreover,

$$\|u_n\|^2 \geqslant c \int_{\xi_n-1}^{\xi_n+1} \left(|u_n'|^2 + u_n^2 \right) r\, dr \geqslant c(\xi_n - 1) \int_{\xi_n-1}^{\xi_n+1} \left(|u_n'|^2 + u_n^2 \right) dr \geqslant c(\xi_n - 1).$$

By definition, $\int_{\zeta-1}^{\zeta+1} (|u_n'|^2 + u_n^2)\, dr < \sigma$ for all $\zeta > \xi_n$. By embedding of $H^1(\zeta - 1, \zeta + 1)$ in L^∞, $0 \leqslant u_n(\zeta) < C\sigma$ for any $\zeta > \xi_n$. From this we will get exponential decay of u_n. Indeed, u_n is a solution of

$$-u_n''(r) - \frac{u_n'(r)}{r} + \omega u_n(r) + f_n(r) u_n(r) = u_n^p(r),$$

with

$$f_n(r) = \frac{(h_n(r) - N)^2}{r^2} + \int_r^n \frac{h_n(s) - N}{s} u_n^2(s)\, ds, \quad h_n(r) = \frac{1}{2} \int_0^r s u_n^2(s)\, ds.$$

If $r > \delta\|u_n\|^2$, again by Proposition 3.2, i), we see that $\int_r^n \frac{N}{s} u_n^2(s)\,ds = o(1)$. Then, by taking smaller σ, if necessary, we can conclude that there exists $C > 0$ such that

$$|u_n(r)| < C\exp\left(-\sqrt{\frac{\omega}{2}}(r-\xi_n)\right), \qquad \text{for all } r > \xi_n.$$

The local C^1 regularity theory for the Laplace operator (see [8, Section 3.4]) implies a similar estimate for $u_n'(r)$. In other words,

$$|u_n(r)| + |u_n'(r)| < C\exp\left(-\sqrt{\frac{\omega}{2}}(r-\xi_n)\right), \qquad \text{for all } r > \xi_n.$$

Step 3: Splitting of $I_\omega(u_n)$.
Reasoning as in the beginning of Step 1, we can take z_n:

$$\xi_n - 3\|u_n\| \leqslant z_n \leqslant \xi_n - 2\|u_n\| \ \text{ with } \|u_n\|^2_{H^1(z_n,z_n+1)} \leqslant \frac{C}{\|u_n\|}.$$

Define $\psi_n : [0,+\infty] \to [0,1]$ be a smooth function such that

$$\psi_n(r) = \begin{cases} 0, & \text{if } r \leqslant z_n, \\ 1, & \text{if } r \geqslant z_n + 1, \end{cases} \qquad |\psi_n'(r)| \leqslant 2.$$

We claim that

$$I_\omega(u_n) \geqslant I_\omega(u_n\psi_n) + I_\omega\left(u_n(1-\psi_n)\right) + c\|u_n(1-\psi_n)\|^2_{L^2(\mathbb{R}^2)} + O(\|u_n\|). \tag{23}$$

This estimate has been accomplished in [22] for $N = 0$. Therefore we just need to estimate the two new terms; it is easy to get that

$$\int_0^n \frac{u_n^2(r)}{r}dr = \int_0^n \frac{u_n^2(r)\psi_n^2(r)}{r}dr + \int_0^n \frac{u_n^2(r)(1-\psi_n(r))^2}{r}dr + o(1).$$

Moreover,

$$\begin{aligned}
\int_0^n \frac{u_n^2(r)}{r}\left(\int_0^r s u_n^2(s)\,ds\right)dr &= \int_0^n \frac{u_n^2(r)\psi_n^2(r)}{r}\left(\int_0^r s\psi_n^2(s)u_n^2(s)\,ds\right)dr \\
&+ \int_0^n \frac{u_n^2(r)(1-\psi_n(r))^2}{r}\left(\int_0^r s(1-\psi_n(s))^2 u_n^2(s)\,ds\right)dr \\
&+ \underbrace{\int_0^n \frac{u_n^2(r)\psi_n^2(r)}{r}\left(\int_0^r s u_n^2(s)(1-\psi_n^2(s))\,ds\right)dr}_{(I)} \\
&+ \underbrace{2\int_0^n \frac{u_n^2(r)\psi_n^2(r)}{r}\left(\int_0^r s u_n^2(s)\psi_n(s)(1-\psi_n(s))\,ds\right)dr}_{(II)} \\
&+ \underbrace{\int_0^n \frac{u_n^2(r)(1-\psi_n(r))^2}{r}\left(\int_0^r s u_n^2(s)\psi_n^2(s)\,ds\right)dr}_{(III)} \\
&+ \underbrace{2\int_0^n \frac{u_n^2(r)(1-\psi_n(r))^2}{r}\left(\int_0^r s u_n^2(s)\psi_n(s)(1-\psi_n(s))\,ds\right)dr}_{(IV)} \\
&+ \underbrace{2\int_0^n \frac{u_n^2(r)(1-\psi_n(r))\psi_n(r)}{r}\left(\int_0^r s u_n^2(s)\,ds\right)dr}_{(V)}.
\end{aligned}$$

We now observe that (I) ... (V) are bounded, as follows:

$$(I) \leqslant \int_{z_n}^{n} \frac{u_n^2(r)}{r} \left(\int_0^{z_n+1} s u_n^2(s)\, ds \right) dr \leqslant \frac{C\|u_n\|^2}{z_n} = O(1),$$
$$(II) \leqslant 2 \int_{z_n}^{n} \frac{u_n^2(r)}{r} \left(\int_{z_n}^{z_n+1} s u_n^2(s)\, ds \right) dr = O(1),$$

and the other terms can be estimated similarly. Therefore, we conclude the proof of (23).

Step 4: The following estimate holds:

$$I_\omega(u_n\psi_n) = 2\pi\xi_n J_\omega(u_n\psi_n) + O(\|u_n\|). \tag{24}$$

In [22] this estimate was made for $N = 0$. So we just need to check the new nonlocal terms

$$\int_0^n \frac{(u_n\psi_n)^2(r)}{r} dr \leqslant \frac{C}{z_n} = o(1),$$
$$\int_0^n \frac{(u_n\psi_n)^2(r)}{r} \left(\int_0^r s(u_n\psi_n)^2(s)\, ds \right) dr \leqslant \int_{z_n}^n \frac{u_n^2(r)}{r} \left(\int_{z_n}^r s u_n^2(s)\, ds \right) dr$$
$$\leqslant \left(\int_{z_n}^n u_n^2(r)\, dr \right)^2 = O(1).$$

Step 5: Conclusion for $\omega > \omega_0$.

By (23) and (24), we have

$$I_\omega(u_n) \geqslant 2\pi\xi_n J_\omega(u_n\psi_n) + I_\omega(u_n(1-\psi_n)) + c\|u_n(1-\psi_n)\|^2_{L^2(\mathbb{R}^2)} + O(\|u_n\|). \tag{25}$$

Recall that $\|u_n\psi_n\|^2_{H^1(\mathbb{R})} \geqslant \sigma > 0$. By Proposition 2.7, we have that $J_\omega(u_n\psi_n) \to c > 0$, up to a subsequence. Since $\xi_n \sim \|u_n\|^2$, it turns out from (25) that $I_\omega(u_n) > I_\omega(u_n(1-\psi_n))$, which is a contradiction with the definition of u_n. Therefore, u_n needs to be a bounded sequence and, in particular, $\inf I_\omega > -\infty$.

Let us now show that I_ω is coercive. Indeed, take $u_n \in \mathcal{H}$ an unbounded sequence, and assume that $I_\omega(u_n)$ is bounded from above. By Lemma 2.3, $\|u_n\|$ is unbounded, so that Proposition 3.2, (iii), shows us that $I_{\hat\omega}(u_n) \to -\infty$ for any $\omega_0 < \hat\omega < \omega$, a contradiction.

Step 6: Conclusion for $\omega = \omega_0$.

As above, (25) gives a contradiction unless $J_\omega(u_n\psi_n) \to 0$. Proposition 2.7 now implies that $\psi_n u_n(\cdot - t_n) \to w_{k_2}$ up to a subsequence, for some $t_n \in (0, +\infty)$. Since $\xi_n \in D_n$ (recall its definition in (22)), we have that $|t_n - \xi_n|$ is bounded. With this extra information, we have a better estimate of the decay of the solutions: indeed,

$$|u_n(r)| + |u_n'(r)| < C\exp\left(-\sqrt{\frac{\omega}{2}}|r - \xi_n| \right), \quad \text{for all } r > \xi_n - 2\|u_n\|. \tag{26}$$

This allows us to do the cut-off procedure in a much more accurate way. Indeed, take $\tilde z_n = \xi_n - \|u_n\|$. Then, (26) implies that

$$\|u_n\|^2_{H^1(\tilde{z}_n,\tilde{z}_n+1)} \leqslant C\exp(-\sqrt{\frac{\omega}{2}}\|u_n\|). \tag{27}$$

Define $\tilde{\psi}_n : [0,+\infty] \to [0,1]$ accordingly:

$$\tilde{\psi}_n(r) = \begin{cases} 0, & \text{if } r \leqslant \tilde{z}_n, \\ 1, & \text{if } r \geqslant \tilde{z}_n + 1, \end{cases} \quad |\tilde{\psi}_n'(r)| \leqslant 2.$$

The advantage is that, in the estimate of $I_\omega(u_n)$, now the errors are exponentially small. Indeed, by repeating the estimates of Step 3 with the new information (27), we obtain:

$$I_\omega(u_n) \geqslant I_\omega(u_n\tilde{\psi}_n) + I_\omega(u_n(1-\tilde{\psi}_n)) + c\|u_n(1-\tilde{\psi}_n)\|^2_{L^2(\mathbb{R}^2)} + O(1),$$

Then,

$$\begin{aligned} I_\omega(u_n) &\geqslant I_\omega(u_n\tilde{\psi}_n) + I_\omega\big(u_n(1-\tilde{\psi}_n)\big) + c\|u_n(1-\tilde{\psi}_n)\|^2_{L^2(\mathbb{R}^2)} + O(1) \\ &= 2\pi\xi_n J_\omega(u_n\tilde{\psi}_n) + I_\omega\big(u_n(1-\tilde{\psi}_n)\big) + c\|u_n(1-\tilde{\psi}_n)\|^2_{L^2(\mathbb{R}^2)} + O(1) \\ &\geqslant I_{(\omega+2c)}\big(u_n(1-\tilde{\psi}_n)\big) + O(1). \end{aligned}$$

But, by Step 5, we already know that $I_{(\omega+2c)}$ is bounded from below, and hence $\inf I_{\omega_0} > -\infty$.

Finally, by applying Lemma 3.1 to $U = w_{k_2}$ we readily get that I_{ω_0} is not coercive.

□

Proof of Theorem 1.2. We shall prove each assessment separately.

Proof of (ii). First, we observe that since $\inf I_{\omega_0} < 0$, there exists $\tilde{\omega} > \omega_0$ such that $\inf I_\omega < 0$ if and only if $\omega \in (\omega_0, \tilde{\omega})$. Since, by Theorem 1.1 and Proposition 2.2, I_ω is coercive and weakly lower semicontinuous, we infer that the infimum is attained at a negative value. This gives the first solution u_1.
Clearly, 0 is a local minimum for I_ω, and $I_\omega(u_1) < 0$. Then, the functional satisfies the geometrical assumptions of the Mountain Pass Theorem, see [1]. Since I_ω is coercive, (PS) sequences are bounded. By the compact embedding of $H^1_r(\mathbb{R}^2)$ into $L^{p+1}(\mathbb{R}^2)$ and Proposition 2.2, standard arguments show that I_ω satisfies the Palais-Smale condition and so we find a second solution which is at a positive energy level.

Proof of (iii). Let now consider $\omega \in (0, \omega_0)$. Performing the rescaling $u \mapsto u_\omega = \sqrt{\omega}\, u(\sqrt{\omega}\,\cdot)$, we get

$$\begin{aligned} I_\omega(u_\omega) = \omega \Bigg[&\frac{1}{2}\int_{\mathbb{R}^2} \left(|\nabla u|^2 + u^2\right) dx + \frac{1}{8}\int_{\mathbb{R}^2} \frac{u^2(x)}{|x|^2}\left(\int_0^{|x|} s u^2(s)\, ds - 2N\right)^2 dx \\ &- \frac{\omega^{\frac{p-3}{2}}}{p+1}\int_{\mathbb{R}^2} |u|^{p+1}\, dx \Bigg]. \end{aligned}$$

Define $\lambda = \omega^{\frac{p-3}{2}}$ and $\mathcal{I}_\lambda : H(\mathbb{R}^2) \to \mathbb{R}$ as

$$\mathcal{I}_\lambda(u) = \Phi(u) - \frac{\lambda}{p+1}\int_{\mathbb{R}^2} |u|^{p+1}\, dx,$$

with

$$\Phi(u) = \frac{1}{2}\|u\|^2 + \frac{1}{2}\int_{\mathbb{R}^2} \frac{u^2(x)}{|x|^2}(h_u(|x|) - N)^2 dx$$
$$= \frac{1}{2}\|u\|^2 + K(u) + \frac{N^2}{2}\int_{\mathbb{R}^2} \frac{u^2(x)}{|x|^2}\, dx,$$

where K is as defined in (11). Then $\mathcal{I}_\lambda$ satisfies the geometrical assumptions of the Mountain Pass Theorem. The main problem here is that we do not know whether a (PS) sequence could be unbounded.

By Lemma 2.3, the functional $\Phi : \mathcal{H} \to \mathbb{R}$ is coercive. Then we can use [17, Theorem 1.1] to obtain a bounded Palais-Smale sequence $u_n \in \mathcal{H}$ for almost every λ. Passing to a subsequence, we can assume that $u_n \rightharpoonup u$; Proposition 2.2 and standard arguments imply that u is a critical point of $\mathcal{I}_\lambda$. Making the change of variables back we obtain a solution of (7) for almost every $\omega \in (0, \omega_0)$.

Finally, in order to find positive solutions of (7), we simply observe that the above arguments apply to the functional $I_\omega^+ : \mathcal{H} \to \mathbb{R}$

$$I_\omega^+(u) = \frac{1}{2}\int_{\mathbb{R}^2} \left(|\nabla u|^2 + \omega u^2\right) dx + \frac{1}{8}\int_{\mathbb{R}^2} \frac{u^2(x)}{|x|^2}\left(\int_0^{|x|} s u^2(s)\, ds - 2N\right)^2 dx$$
$$- \frac{1}{p+1}\int_{\mathbb{R}^2} (u^+)^{p+1}\, dx.$$

Due to the maximum principle, the critical points of I_ω^+ are positive solutions of (7).

Proof of (i). This part happens to be quite delicate, compared to the case $N = 0$ studied in [22]. Let u be a solution of (7). If we multiply (7) by u and integrate, we get

$$0 = \int_{\mathbb{R}^2} \left(|\nabla u|^2 + \omega u^2\right) dx + 3\int_{\mathbb{R}^2} \frac{u^2(x)}{|x|^2}\left(h_u(|x|) - N\right)^2 dx$$
$$+ 2N\int_{\mathbb{R}^2} \frac{u^2(x)}{|x|^2}\left(h_u(|x|) - N\right) dx - \int_{\mathbb{R}^2} |u|^{p+1} dx. \tag{28}$$

From (28) and the Pohozaev identity (Proposition 2.4), we obtain that, for any $l > 0$,

$$0 = (l+1)\int_{\mathbb{R}^2} |\nabla u|^2 dx + \omega\int_{\mathbb{R}^2} u^2 dx + (l+3)\int_{\mathbb{R}^2} \frac{u^2(x)}{|x|^2}\left(h_u(|x|) - N\right)^2 dx$$
$$+ 2N\int_{\mathbb{R}^2} \frac{u^2(x)}{|x|^2}\left(h_u(|x|) - N\right) dx - \left(\frac{p-1}{p+1}l + 1\right)\int_{\mathbb{R}^2} |u|^{p+1} dx. \tag{29}$$

By using (12) in (29),

$$0 \geqslant \int_{\mathbb{R}^2} \left(|\nabla u|^2 + \omega u^2\right) dx + 3\int_{\mathbb{R}^2} \frac{u^2(x)}{|x|^2}\left(h_u(|x|) - N\right)^2 dx$$
$$+ 2N\int_{\mathbb{R}^2} \frac{u^2(x)}{|x|^2}\left(h_u(|x|) - N\right) dx - \left(\frac{p-1}{p+1}l + 1\right)\int_{\mathbb{R}^2} |u|^{p+1} dx + \frac{l}{2}\int_{\mathbb{R}^2} |u|^4 dx. \tag{30}$$

We can estimate

$$
\begin{aligned}
&3\int_{\mathbb{R}^2}\frac{u^2(x)}{|x|^2}\left(h_u(|x|)-N\right)^2+2N\int_{\mathbb{R}^2}\frac{u^2(x)}{|x|^2}\left(h_u(|x|)-N\right)dx\\
&=\int_{\mathbb{R}^2}\frac{u^2(x)}{|x|^2}(h_u(|x|)-N)(3h_u(|x|)-N)dx\\
&\geqslant -c_1\int_{\{N/3\leqslant h_u\leqslant N\}}\frac{u^2(x)}{|x|^2}dx, \qquad (31)
\end{aligned}
$$

where $\{N/3\leqslant h_u\leqslant N\}=\{r_0\leqslant |x|\leqslant r_1\}$ with $h_u(r_0)=N/3$ and $h_u(r_1)=N$ (here we have used that h_u is increasing in r). For any $r>0$, by the definition of h_u we have

$$
4\pi h_u(r)=\int_{B_r}u^2(x)dx\leqslant Cr\left(\int_{B_r}u^4(x)dx\right)^{1/2}.
$$

Then

$$
\begin{aligned}
&\int_{B_r}h_u^2(|x|)\frac{u^2(x)}{|x|^2}dx\leqslant C\int_{B_r}u^2(x)\left(\int_{B_{|x|}}u^4(y)dy\right)dx \qquad (32)\\
&\leqslant C\int_{B_r}u^2(x)dx\int_{B_r}u^4(x)dx\leqslant Ch_u(r)\int_{B_r}u^4(x)dx.
\end{aligned}
$$

We now apply (32) to estimate

$$
\begin{aligned}
&\int_{\{N/3\leqslant h_u\leqslant N\}}\frac{u^2(x)}{|x|^2}dx\leqslant C\int_{\{N/3\leqslant h_u\leqslant N\}}h_u^2(|x|)\frac{u^2(x)}{|x|^2}dx\leqslant C\int_{B_{r_1}}h_u^2(|x|)\frac{u^2(x)}{|x|^2}dx\\
&\leqslant Ch_u(r_1)\int_{B_{r_1}}u^4(x)dx\leqslant C\int_{\mathbb{R}^2}u^4dx. \qquad (33)
\end{aligned}
$$

We apply (31) and (33) in (30):

$$
0\geqslant\int_{\mathbb{R}^2}|\nabla u|^2dx+\omega\int_{\mathbb{R}^2}u^2dx-c_0\int_{\mathbb{R}^2}u^4dx+\frac{l}{2}\int_{\mathbb{R}^2}u^4-\left(\frac{p-1}{p+1}l+1\right)\int_{\mathbb{R}^2}|u|^{p+1}dx.
$$

Therefore it suffices to take l so that $-c_0+\frac{l}{2}=1$, and then to take ω so that the function

$$
s\to\omega s^2+s^4-\left(\frac{p-1}{p+1}l+1\right)|s|^{p+1}
$$

is non-negative for any s. Therefore u must be identically equal to zero.

□

REFERENCES

[1] A. Ambrosetti and P. H. Rabinowitz, Dual variational methods in critical point theory and applications, J. Functional Analysis 14 (1973), 349-381.

[2] L. Bergé, A. de Bouard and J. C. Saut, Blowing up time-dependent solutions of the planar Chern-Simons gauged nonlinear Schrödinger equation, Nonlinearity 8 (1995), 235-253.

[3] J. Byeon, H. Huh and J. Seok, Standing waves of nonlinear Schrödinger equations with the gauge field, J. Funct. Anal. 263 (2012), no. 6, 1575-1608.

[4] J. Byeon, H. Huh and J. Seok, On standing waves with a vortex point of order N for the nonlinear Chern-Simons-Schrödinger equations, preprint.

[5] X. Chen and P. Smith, On the unconditional uniqueness of solutions to the infinite radial Chern-Simons-Schrödinger hierarchy, preprint arXiv:1406.2649, to appear in Analysis and PDE's.

[6] P. L. Cunha, P. D'Avenia, A. Pomponio and G. Siciliano, A multiplicity result for Chern-Simons-Schrödinger equation with a general nonlinearity, preprint arXiv:1407.6629.

[7] V. Dunne, Self-dual Chern-Simons Theories, Springer, 1995.

[8] D. Gilbarg and N. S. Trudinger, Elliptic Partial Differential Equations of Second Order, Springer, 3rd edition, 1998.

[9] C. Hagen, A new gauge theory without an elementary photon, Ann. of Phys. 157 (1984), 342-359.

[10] C. Hagen, Rotational anomalies without anyons, Phys. Review D 31 (1985), 2135-2136.
[11] H. Huh, Blow-up solutions of the Chern-Simons-Schrödinger equations, Nonlinearity 22 (2009) 967-974.
[12] H. Huh, Standing waves of the Schrödinger equation coupled with the Chern-Simons gauge field. J. Math. Phys. 53 (2012), no. 6, 063702, 8 pp.
[13] H. Huh, Energy Solution to the Chern-Simons-Schrödinger Equations, Abstract and Applied Analysis Volume 2013, Article ID 590653, 7 pages.
[14] R. Jackiw and S.-Y. Pi, Soliton solutions to the gauged nonlinear Schrödinger Equations, Phys. Review Letters 64 (1990), 2969-2972.
[15] R. Jackiw and S.-Y. Pi, Classical and quantal nonrelativistic Chern-Simons theory, Phys. Rev. D 42 (1990) 3500-3513.
[16] R. Jackiw and S.-Y. Pi, Self-dual Chern-Simons solitons, Progr. Theoret. Phys. Suppl. 107 (1992) 1-40.
[17] L. Jeanjean, On the existence of bounded Palais-Smale sequences and applications to a Landesman-Lazer type problem set on R^N, Proc. Roy. Soc. Edinburgh A 129 (1999) 787-809.
[18] P.-L. Lions, The concentration-compactness principle in the calculus of variations. The locally compact case, part 1 and 2, Ann. Inst. Henri. Poincaré Anal. Non linéaire 1 (1984), 109-145 and 223-283.
[19] B. Liu and P. Smith, Global wellposedness of the equivariant Chern-Simons-Schrödinger equation, preprint arXiv:1312.5567.
[20] B. Liu, P. Smith and D. Tataru, Local wellposedness of Chern-Simons-Schrödinger, Int. Math. Res. Notices, doi: 10.1093/imrn/rnt161.
[21] S.-J. Oh and F. Pusateri, Decay and Scattering for the Chern-Simons-Schrödinger Equations, preprint arXiv:1311.2088.
[22] A. Pomponio and D. Ruiz, A Variational Analysis of a Gauged Nonlinear Schrödinger Equation, preprint arXiv:1306.2051, to appear in J. Eur. Math. Soc.
[23] A. Pomponio and D. Ruiz, Boundary concentration of a Gauged Nonlinear Schrodinger Equation, to appear in Calc. Var. PDE's, d.o.i 10.1007/s00526-014-0749-2.
[24] W.-A. Strauss, Existence of solitary waves in higher dimensions, Comm. Math. Phys. 55 (1977), 149-162.
[25] M. Struwe, On the evolution of harmonic mappings of Riemannian surfaces, Comment. Math. Helv. 60 (1985) 558-581.
[26] G. Tarantello, Self-Dual Gauge Field Vortices: An Analytical Approach, PNLDE 72, Birkhäuser Boston, Inc., Boston, MA, 2007.
[27] Y. Yang, Solitons in Field Theory and Nonlinear Analysis, Springer Monographs in Mathematics, Springer, New York, 2001.

[1]School of Statistic and Mathematics, Zhongnan University of Economics and Law, 430073 Wuhan, P.R.China

[2]Dipartimento di Meccanica, Matematica e Management, Politecnico di Bari, Via E. Orabona 4, 70125 Bari, Italy.

[3]Dpto. Análisis Matemático, Avda. Fuentenueva s/n, 18071 Granada (Spain).
E-mail address: jiangys@znufe.edu.cn, alessio.pomponio@poliba.it, daruiz@ugr.es