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HESSIAN ESTIMATES IN WEIGHTED LEBESGUE SPACES FOR FULLY NONLINEAR ELLIPTIC EQUATIONS

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ABSTRACT. We prove global regularity in weighted Lebesgue spaces for the viscosity solutions to the Dirichlet problem for fully nonlinear elliptic equations. As a consequence, regularity in Morrey spaces of the Hessian is derived as well.

1. INTRODUCTION

The paper deals with the following Dirichlet problem for fully nonlinear elliptic equations

$$(1.1) \quad \begin{cases} F(D^2u, Du, u, x) = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in \mathbb{R}^n with $n \geq 2$. Here, $F = F(X, z, s, x)$ is a real valued Carathéodory function defined on $S(n) \times \mathbb{R}^n \times \mathbb{R} \times \Omega$, where $S(n)$ is the set of $n \times n$ real symmetric matrices ordered in the usual way: $X \geq 0$ when $\langle X\xi, \xi \rangle \geq 0$ for all $\xi \in \mathbb{R}^n$, where $\langle \cdot, \cdot \rangle$ is the Euclidean inner product, and $Y \geq X$ means $Y - X \geq 0$. We assume that F is uniformly elliptic with ellipticity constants λ and Λ , that is, there exist constants λ and Λ with $0 < \lambda \leq \Lambda < \infty$ such that

$$(1.2) \quad \lambda\|Y\| \leq F(X + Y, z, s, x) - F(X, z, s, x) \leq \Lambda\|Y\|,$$

for all $X, Y \in S(n), Y \geq 0, z \in \mathbb{R}^n, s \in \mathbb{R}$ and almost all $x \in \Omega$, and where $\|Y\| := \sup_{|x|=1} |Yx|$ that is equal to the maximum eigenvalue of Y whenever $Y \geq 0$.

Due to the discontinuous dependence on x of the nonlinear term F , the right notion of solution to the problem (1.1) would be that of function taken in a Sobolev space $W^{2,p}$ that satisfies the equation in a strong or viscosity sense and which vanishes identically on $\partial\Omega$. It was L. Caffarelli the first to derive in the seminal paper [2] interior *a priori* $W^{2,p}$ -estimates for the solutions of (1.1) for all $p \geq n$, and these led to significant progress in the general study of fully nonlinear elliptic equations. By adapting the approach of Caffarelli, L. Wang developed in [20] the $W^{2,p}$ -regularity theory of nonlinear parabolic equations. The restriction $p \geq n$ in [2] is due to the Aleksandrov–Bakel’man–Pucci maximum principle which turned out to be crucial in Caffarelli’s approach. By using weak reverse Hölder inequalities, L. Escauriaza extended in [10] the results from [2] to the range $p > n - \epsilon$ with a small $\epsilon > 0$ depending on the ellipticity constants of the nonlinear operator considered. Recently, employing the techniques from [2] and [10], N. Winter derived in [21] boundary (and thus also global) $W^{2,p}$ -*a priori* estimates for the solutions of (1.1),

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and proved $W^{2,p}$ -solvability results as well. In the works cited, it is supposed that the nonlinear term F supports linear growths with respect to D^2u , Du and u (see (2.1) below), while its behaviour in x is controlled in terms of small bounded mean oscillation (BMO) category. Just for the sake of completeness, let us note the papers [7, 15] where $W^{2,n}$ -solvability has been proved for Dirichlet and oblique derivative problems for quasilinear elliptic equations with quadratic gradient growths and where the discontinuity of the principal coefficients is measured in terms of vanishing mean oscillation (VMO).

The general aim of the present paper is to extend the results of Winter [21] to the settings of weighted Sobolev spaces. More precisely, the functional framework we are dealing with is the space $W_w^{2,p}(\Omega)$, with a weight w taken in an appropriate Muckenhoupt class. Our goal is to prove that, under appropriate hypotheses on the data, for each $f \in L_w^p(\Omega)$ there exists a unique strong solution $u \in W_w^{2,p}(\Omega)$ of (1.1) that satisfies the estimate

$$(1.3) \quad \|u\|_{W_w^{2,p}(\Omega)} \leq c \|f\|_{L_w^p(\Omega)}$$

with a positive constant c independent of u . Similar problem for *linear* second order elliptic operators has been already studied in [1].

It is worth noting at the very beginning that, thanks to the deep self-improving property of the Muckenhoupt weights, $f \in L_w^p(\Omega)$ implies $f \in L^{\tilde{p}}(\Omega)$ with appropriate \tilde{p} (cf. Remark 2.5) for which all the hypotheses of the Winter work [21] hold true. This ensures existence of a unique $W^{2,\tilde{p}}(\Omega)$ -viscosity solution of (1.1), that is also $W^{2,\tilde{p}}(\Omega)$ -strong solution as proved in [4]. This way, our task reduces to the proof of fine regularity of the Hessian, given by (1.3). Our approach to proving (1.3) is based on the suitable properties of the Hardy-Littlewood maximal operator and the Muckenhoupt weights. To be more concrete, we employ the reverse doubling property of the weights to estimate the power decay for the weighted measure of the upper level sets for the Hessian, and derive the interior and boundary $W^{2,p}$ estimates in the settings of weighted Sobolev spaces by applying the boundedness of the maximal operator on the weighted Lebesgue spaces. In particular, the reverse Hölder property of the weights plays a significant role in inducing the weighted L^p bound for the gradient of the solution u .

Indeed, taking the trivial weight $w \equiv 1$, our results reduce to that of Winter [21] in the unweighted case. Further on, an appropriate power of the characteristic function of a ball is a Muckenhoupt weight as known from [9]. We combine this fact with our main result in order to get regularity in Morrey spaces $L^{p,\mu}$ for the Hessian of the strong solution to (1.1). Thus, we prove that $f \in L^{p,\mu}(\Omega)$ implies $D^2u \in L^{p,\mu}(\Omega)$ which leads, by the known properties of functions with Morrey regular gradients, to better integrability and even Hölder continuity of the gradient of u .

The paper is organized as follows. In the next section we list the hypotheses on the nonlinearity F and the weight w , and state our main result (Theorem 2.4). Section 3 collects the basic tools employed in the proof of Theorem 2.4 with the corresponding auxiliary results. The bound (1.3) is then proved by establishing interior and boundary weighted estimates for the Hessian and using standard flattening and covering arguments. In Section 4 we state and prove the regularity in Morrey spaces of the second derivatives of solutions to (1.1), and the corresponding finer smoothness of the gradient.

2. ASSUMPTIONS AND MAIN RESULT

We start this section with some standard notations that will be used throughout the paper. For a point $y = (y_1, \dots, y_n) \in \mathbb{R}^n$ and a real number $r > 0$, let $B_r(y) = \{x \in \mathbb{R}^n : |x - y| < r\}$ and $B_r^+(y) = B_r(y) \cap \{x_n > 0\}$. We write $B_r = B_r(0)$ and $B_r^+ = B_r^+(0)$ for the sake of simplicity. For a function $u : \mathbb{R}^n \rightarrow \mathbb{R}$, we denote the gradient of u by $Du = (D_1u, \dots, D_nu)$, and its Hessian by $D^2u = (D_{ij}u)$, where $D_iu = D_{x_i}u = \frac{\partial u}{\partial x_i}$, $D_{ij}u = D_{x_i x_j}u = \frac{\partial^2 u}{\partial x_i \partial x_j}$ for $i, j = 1, \dots, n$. For a locally integrable function $g : U \rightarrow \mathbb{R}$ with a bounded set $U \subset \mathbb{R}^n$, we denote the mean value of g on U by

$$\bar{g}_U := \int_U g(x) dx = \frac{1}{|U|} \int_U g(x) dx.$$

2.1. Viscosity solution. Let us now discuss the structure conditions to be imposed on $F : S(n) \times \mathbb{R}^n \times \mathbb{R} \times \Omega \rightarrow \mathbb{R}$. Let $0 < \lambda \leq \Lambda$. We introduce the *Pucci extremal operators* $\mathcal{P}^-, \mathcal{P}^+$ associated with λ, Λ that are defined as follows: for $X \in S(n)$,

$$\mathcal{P}^-(X, \lambda, \Lambda) := \lambda \sum_{e_i > 0} e_i + \Lambda \sum_{e_i < 0} e_i \quad \text{and} \quad \mathcal{P}^+(X, \lambda, \Lambda) := \lambda \sum_{e_i < 0} e_i + \Lambda \sum_{e_i > 0} e_i,$$

where e_i are the eigenvalues of X .

The basic structure conditions on $F(X, z, s, x)$ that we always assume in this paper are:

$$(2.1) \quad \begin{cases} F \text{ is nonincreasing in } s, \quad F(0, 0, 0, x) = 0, \\ \mathcal{P}^-(X - Y, \lambda, \Lambda) - \kappa_1 |z - \tilde{z}| - \kappa_2 |s - \tilde{s}| \\ \leq F(X, z, s, x) - F(Y, \tilde{z}, \tilde{s}, x) \\ \leq \mathcal{P}^+(X - Y, \lambda, \Lambda) + \kappa_1 |z - \tilde{z}| + \kappa_2 |s - \tilde{s}| \end{cases}$$

for all $X, Y \in S(n)$, $z, \tilde{z} \in \mathbb{R}^n$, $s, \tilde{s} \in \mathbb{R}$, $x \in \Omega$ and with constants $\kappa_1, \kappa_2 \geq 0$. It is obvious that the above condition (2.1) with $z = \tilde{z}$ and $s = \tilde{s}$ coincides with the uniform ellipticity of F as given in (1.2).

Now we recall the definition of viscosity solutions that will be treated throughout the paper. Let us consider the equation

$$(2.2) \quad F(D^2u, Du, u, x) = f \quad \text{in } \Omega.$$

We will always assume that F in (2.2) satisfies the structure conditions (2.1).

Definition 2.1. Let F be continuous in X, z, s and measurable in x . Suppose $q > \frac{n}{2}$ and $f \in L^q_{\text{loc}}(\Omega)$. A function $u \in C(\Omega)$ is called an *L^q -viscosity solution* of (2.2) if the following two conditions hold:

- (a) For all $\varphi \in W^{2,q}_{\text{loc}}(\Omega)$ whenever $\epsilon > 0$, $\mathcal{O} \subset \Omega$ is open and

$$F(D^2\varphi(x_0), D\varphi(x_0), u(x_0), x_0) \leq f(x_0) - \epsilon \quad \text{a.e. in } \mathcal{O},$$

$u - \varphi$ cannot attain a local maximum in \mathcal{O} .

- (b) For all $\varphi \in W^{2,q}_{\text{loc}}(\Omega)$ whenever $\epsilon > 0$, $\mathcal{O} \subset \Omega$ is open and

$$F(D^2\varphi(x_0), D\varphi(x_0), u(x_0), x_0) \geq f(x_0) + \epsilon \quad \text{a.e. in } \mathcal{O},$$

$u - \varphi$ cannot attain a local minimum in \mathcal{O} .

In the above definition, the function φ is called a test function. We note that the restriction on q in Definition 2.1, i.e. $q > \frac{n}{2}$, ensures that the test function $\varphi \in W_{\text{loc}}^{2,q}(\Omega)$ is continuous because $W_{\text{loc}}^{2,q}(\Omega)$ is imbedded into $C(\Omega)$. Moreover, it is pointwise twice differentiable almost everywhere by the classical result of Calderón and Zygmund [6]; see [4, Theorem 3.6] and [8] for more details.

If F and f are continuous in all variables and the test function $\varphi \in C^2(\Omega)$ in Definition 2.1, we say that u is a *C-viscosity solution* of (2.2). Note that whenever F and f are continuous in all variables, the *C-viscosity solutions* of (2.2) are L^q -viscosity solutions of (2.2); see [4, Proposition 2.9].

For $f \in L_{\text{loc}}^q(\Omega)$, we say that u is an *L^q -strong solution* of (2.2) if $u \in W_{\text{loc}}^{2,q}(\Omega)$ and the equation (2.2) holds almost everywhere in Ω . It is easy to see (cf. [4, Lemma 2.6, Remark 2.7]) that if u is an L^q -strong solution, then it is also L^q -viscosity solution and vice versa, if $u \in W_{\text{loc}}^{2,q}(\Omega)$ is an L^q -viscosity solution, then it is L^q -strong solution.

2.2. Muckenhoupt weights. We introduce the Muckenhoupt classes A_q , $1 \leq q < \infty$, and their basic properties, to be used in the sequel. Let w be a weight, that is, a locally integrable nonnegative function on \mathbb{R}^n that takes values in $(0, \infty)$ almost everywhere. We identify the weight w with the measure

$$w(E) = \int_E w(x) dx$$

for measurable sets $E \subset \mathbb{R}^n$. Given $1 \leq q < \infty$, a weight w is said to be of class A_q , $w \in A_q$, if there exists a constant $A \geq 1$ such that for all balls $B \subset \mathbb{R}^n$,

$$(2.3) \quad \left(\int_B w(x) dx \right) \left(\int_B w(x)^{\frac{-1}{q-1}} dx \right)^{q-1} \leq A$$

when $1 < q < \infty$, or

$$(2.4) \quad \int_B w(x) dx \leq A \operatorname{ess\,inf}_B w(x)$$

when $q = 1$. The smallest constant A for which (2.3) (or (2.4)) is fulfilled is denoted by $[w]_q$ and is called the A_q constant of w .

There is an alternate way of defining the A_q class. For any integrable function g and any ball $B \subset \mathbb{R}^n$, $w \in A_q$ with $1 \leq q < \infty$ if and only if there exists a constant $c \geq 1$ such that

$$(2.5) \quad (\bar{g}_B)^q \leq \frac{c}{w(B)} \int_B g^q w dx < \infty.$$

The smallest constant c for which (2.5) holds is the same as the A_q constant of w . The A_q condition is invariant under translation, dilation and multiplication by a positive scalar. Each A_q -weight satisfies the doubling property, i.e., there exists a constant $c > 0$ such that $w(B_{2r}(y)) \leq c w(B_r(y))$ for every $y \in \mathbb{R}^n$ and $r > 0$. Moreover, the classes A_q are monotone in q : $A_{q_1} \subset A_{q_2}$ for $q_1 \leq q_2$.

The next two results collect the most important properties of the A_q -weights.

Lemma 2.2 (Reverse doubling property). *Let $w \in A_q$ for some $q \in (1, \infty)$, and let D be a measurable subset of a ball $B \subset \mathbb{R}^n$. Then there exist two positive*

constants γ_1, γ_2 depending only on n, q and $[w]_q$ such that

$$\frac{1}{[w]_q} \left(\frac{|D|}{|B|} \right)^q \leq \frac{w(D)}{w(B)} \leq \gamma_1 \left(\frac{|D|}{|B|} \right)^{\gamma_2}.$$

Lemma 2.3 (Self-improving property). *Let $w \in A_q$ for some $q \in (1, \infty)$. Then there exists a sufficiently small constant $\tilde{\epsilon} = \tilde{\epsilon}(n, q, [w]_q) > 0$ such that $w \in A_{q-\tilde{\epsilon}}$.*

Suppose that $w \in A_q$ with $1 < q < \infty$ and Ω is a bounded domain in \mathbb{R}^n . The *weighted Lebesgue space* $L_w^q(\Omega)$ consists of all measurable functions g on Ω such that the norm

$$\|g\|_{L_w^q(\Omega)} := \left(\int_{\Omega} |g|^q w \, dx \right)^{1/q}$$

is finite. We define the *weighted Sobolev space* $W_w^{2,q}(\Omega)$, $1 < q < \infty$, as the set of functions $g \in L_w^q(\Omega)$ with weak derivatives $D^\alpha g \in L_w^q(\Omega)$ for $|\alpha| \leq 2$. The norm of g in $W_w^{2,q}(\Omega)$ is then given by

$$\|g\|_{W_w^{2,q}(\Omega)} := \left(\sum_{|\alpha| \leq 2} \int_{\Omega} |D^\alpha g|^q w \, dx \right)^{1/q}.$$

We refer the reader to [16, 18, 19] for the proofs of Lemmas 2.2 and 2.3 and also for further properties of the classes A_q and the relevant weighted Lebesgue and Sobolev spaces.

2.3. Main result. To measure the oscillation of the function F with respect to x , we define

$$\beta_F(x, y) := \sup_{X \in \mathcal{S}(n) \setminus \{0\}} \frac{|F(X, 0, 0, x) - F(X, 0, 0, y)|}{\|X\|},$$

and set $\beta(x, y) = \beta_F(x, y)$ for the sake of simplicity.

Theorem 2.4 (Main Theorem). *Assume that $F(X, z, s, x)$ satisfies the structure conditions (2.1) and that it is convex in X . Let $p \in (n_0, \infty)$ where $n_0 := n - \epsilon_0$ for some $\epsilon_0 = \epsilon_0(\frac{\Lambda}{\lambda}, n, \kappa_1, \text{diam}(\Omega)) > 0$ and $w \in A_{\frac{p}{n_0}}$. Suppose that $\partial\Omega \in C^{1,1}$ and $f \in L_w^p(\Omega)$. Then there exists a small $\delta = \delta(n, \lambda, \Lambda, p, w, \partial\Omega) > 0$ such that if*

$$(2.6) \quad \sup_{x_0 \in \bar{\Omega}, 0 < r \leq r_0} \left(\int_{B_r(x_0) \cap \Omega} \beta(x, x_0)^n \, dx \right)^{1/n} \leq \delta$$

for some $r_0 > 0$, then the problem (1.1) has a unique viscosity solution $u \in W_w^{2,p}(\Omega)$, satisfying the estimate

$$(2.7) \quad \|u\|_{W_w^{2,p}(\Omega)} \leq c \|f\|_{L_w^p(\Omega)}$$

with a positive constant $c = c(n, \lambda, \Lambda, \kappa_1, \kappa_2, p, w, \partial\Omega, \text{diam}(\Omega), r_0)$.

The small constant $\epsilon_0 \in (0, \frac{n}{2})$ which appears in the statement of Theorem 2.4 is the same as in [21], and it is related to fundamental estimates of Green's functions obtained by Fabes and Stroock [11]; see [4, 17] for more details. Hereafter, for simplicity, we denote $n_0 := n - \epsilon_0$.

Remark 2.5. It follows from Lemma 2.3 that if $w \in A_{\frac{p}{n_0}}$ then $w \in A_{\frac{p}{n_0} - \tilde{\epsilon}}$ for some small constant $\tilde{\epsilon} = \tilde{\epsilon}(n, \lambda, \Lambda, p, [w]_{\frac{p}{n_0}}) > 0$. This way, (2.5) yields

$$\left(\int_B |f|^{\frac{n_0 p}{p - n_0 \tilde{\epsilon}}} dx \right)^{\frac{p}{n_0} - \tilde{\epsilon}} \leq \frac{c}{w(B)} \int_B |f|^{\frac{n_0 p}{p - n_0 \tilde{\epsilon}} \cdot (\frac{p}{n_0} - \tilde{\epsilon})} w dx = \frac{c}{w(B)} \int_B |f|^p w dx,$$

whence

$$\int_B |f|^{\frac{n_0 p}{p - n_0 \tilde{\epsilon}}} dx \leq c (\|f\|_{L_w^p(B)})^{\frac{n_0 p}{p - n_0 \tilde{\epsilon}}} |B| w(B)^{\frac{-n_0}{p - n_0 \tilde{\epsilon}}}$$

for any ball $B \subset \mathbb{R}^n$. Therefore, standard covering arguments give

$$(2.8) \quad \|f\|_{L^{\frac{n_0 p}{p - n_0 \tilde{\epsilon}}}(\Omega)} \leq c \|f\|_{L_w^p(\Omega)}$$

for some $c = c(n, \lambda, \Lambda, p, [w]_{\frac{p}{n_0}}, \text{diam}(\Omega)) > 0$, that means $f \in L_w^p(\Omega)$ implies $f \in L^{\frac{n_0 p}{p - n_0 \tilde{\epsilon}}}(\Omega)$. It is clear that $\frac{n_0 p}{p - n_0 \tilde{\epsilon}} > n_0$ and hence, by virtue of [21, Theorem 4.6], there exists a unique viscosity (or strong) solution $u \in W^{2, \frac{n_0 p}{p - n_0 \tilde{\epsilon}}}(\Omega)$ of (1.1) with

$$\|u\|_{W^{2, \frac{n_0 p}{p - n_0 \tilde{\epsilon}}}(\Omega)} \leq c \|f\|_{L^{\frac{n_0 p}{p - n_0 \tilde{\epsilon}}}(\Omega)}.$$

Consequently, the existence of a unique $L^{\frac{n_0 p}{p - n_0 \tilde{\epsilon}}}$ -viscosity solution to (1.1) is already guaranteed by (2.8).

3. WEIGHTED $W^{2,p}$ -ESTIMATES

3.1. Preliminaries. We recall that for a locally integrable function $g: \mathbb{R}^n \rightarrow \mathbb{R}$, the Hardy-Littlewood maximal function of g is defined by

$$\mathcal{M}g(y) = \sup_{r>0} \frac{1}{|B_r(y)|} \int_{B_r(y)} |g(x)| dx,$$

at each point $y \in \mathbb{R}^n$. If g is defined on a bounded set $U \subset \mathbb{R}^n$, then $\mathcal{M}_U g = \mathcal{M}(\chi_U g)$, where χ_U stands for the characteristic function of U .

One of the central tools in proving Theorem 2.4 is the following result, known Muckenhoupt's theorem, which states that the Hardy-Littlewood maximal operator is bounded from L_w^q into itself, with $1 < q < \infty$, if and only if $w \in A_q$; see [13, 16, 19] for the proof and details. For $g \in L_w^q(\mathbb{R}^n)$ with $1 < q < \infty$, $\mathcal{M}g$ is meaningful from the fact that $L_w^q(\mathbb{R}^n) \subset L_{\text{loc}}^1(\mathbb{R}^n)$.

Lemma 3.1. *Suppose $w \in A_q$ with $1 < q < \infty$. Then there exists a constant $c = c(n, q, [w]_q) > 0$ such that*

$$(3.1) \quad \int_{\mathbb{R}^n} (\mathcal{M}g)^q w dx \leq c \int_{\mathbb{R}^n} |g|^q w dx$$

whenever $g \in L_w^q(\mathbb{R}^n)$. Conversely, if (3.1) holds for every $g \in L_w^q(\mathbb{R}^n)$, then $w \in A_q$.

We also need the following standard property, which comes from classical measure theory; see [3, 18].

Lemma 3.2. *Suppose that g is a nonnegative measurable function in a bounded domain $U \subset \mathbb{R}^n$. Let $\eta > 0$ and $M > 1$ be constants and w be a weight in \mathbb{R}^n . Then for $0 < q < \infty$,*

$$g \in L_w^q(U) \text{ if and only if } S := \sum_{k \geq 1} M^{qk} w(\{x \in U : g(x) > \eta M^k\}) < \infty$$

and moreover

$$c^{-1}S \leq \|g\|_{L_w^q(U)}^q \leq c(w(U) + S),$$

where $c > 0$ is a constant depending only on η , M and q .

We now introduce some objects that will be also employed. A function P is called a paraboloid with opening M if

$$(3.2) \quad P(x) = l_0 + l(x) \pm \frac{M}{2}|x|^2,$$

where M is a positive constant, l_0 is a constant and l is a linear function. Note that P is convex in the case $+$ of (3.2) and concave in the case $-$ of (3.2). Let U be a bounded domain in \mathbb{R}^n . Set $V \subset U$ and $M > 0$. For a continuous function u in U , we define

$$\underline{\mathcal{G}}_M(u, V) := \left\{ x_0 \in V : \begin{array}{l} \text{there is a concave paraboloid } P \text{ with opening } M \text{ such} \\ \text{that } P(x_0) = u(x_0) \text{ and } P(x) \leq u(x) \text{ for any } x \in V \end{array} \right\}$$

and $\underline{\mathcal{A}}_M(u, V) := V \setminus \underline{\mathcal{G}}_M(u, V)$. We analogously define $\overline{\mathcal{G}}_M(u, V)$ and $\overline{\mathcal{A}}_M(u, V)$ by using convex paraboloids. We also define $\mathcal{G}_M(u, V) := \underline{\mathcal{G}}_M(u, V) \cap \overline{\mathcal{G}}_M(u, V)$ and $\mathcal{A}_M(u, V) := \underline{\mathcal{A}}_M(u, V) \cap \overline{\mathcal{A}}_M(u, V)$. Set now

$$\begin{aligned} \underline{\Theta}(u, V)(x) &:= \inf\{M > 0 : x \in \underline{\mathcal{G}}_M(V)\}, \\ \overline{\Theta}(u, V)(x) &:= \inf\{M > 0 : x \in \overline{\mathcal{G}}_M(V)\}, \\ \Theta(u, V)(x) &:= \sup\{\underline{\Theta}(u, V)(x), \overline{\Theta}(u, V)(x)\}. \end{aligned}$$

The following lemma which will be used to prove Theorem 3.4 below gives that the L_w^p norm of the Hessian of u in U can be controlled by the L_w^p norm of $\Theta(u, U)$.

Lemma 3.3. *Let $1 < p < \infty$ and u be a continuous function in a bounded domain $U \subset \mathbb{R}^n$. For a positive constant r , we define*

$$\Theta(u, r)(x) := \Theta(u, U \cap B_r(x))(x) \text{ for } x \in U.$$

If $\Theta(u, r) \in L_w^p(U)$, then we have $D^2u \in L_w^p(U)$ and

$$(3.3) \quad \|D^2u\|_{L_w^p(U)} \leq 2\|\Theta(u, r)\|_{L_w^p(U)}.$$

Proof. Using the same approach as in the proof of Proposition 1.1 in [3], we deduce that

$$\begin{aligned} \left| \int_U u \varphi_{ij} dx \right| &\leq 2 \int_U |\Theta(u, r)| |\varphi| dx = 2 \int_U |\Theta(u, r)| |\varphi| w^{\frac{1}{p}} w^{-\frac{1}{p}} dx \\ &\leq 2\|\Theta(u, r)\|_{L_w^p(U)} \|\varphi\|_{L_{w^*}^{p'}(U)} \end{aligned}$$

for any $\varphi \in C_c^\infty(U)$ and any indices i, j . Here p' is the conjugate exponent of p , $p' = \frac{p}{p-1}$, and $w^* = w^{\frac{-1}{p-1}}$. Therefore, we obtain $D^2u \in L_w^p(U)$ with the bound (3.3). \square

3.2. Interior and boundary weighted estimates. In this subsection, we establish interior and boundary $W^{2,p}$ estimates for the solutions of the fully nonlinear elliptic equation (1.1) in the settings of weighted Sobolev spaces. For the sake of simplicity, we still denote $n_0 := n - \epsilon_0$. Here, the constant $\epsilon_0 > 0$ depends only on $\frac{\Lambda}{\lambda}$, n , and κ_1 , since we are dealing with the local estimates.

Theorem 3.4. *Let $p \in (n_0, \infty)$ and $w \in A_{\frac{p}{n_0}}$. Set $\tilde{p} := \frac{n_0 p}{p - n_0 \tilde{\epsilon}}$ for some $\tilde{\epsilon} = \tilde{\epsilon}(n, \lambda, \Lambda, p, [w]_{\frac{p}{n_0}}) > 0$.*

- (i) *Assume that $F(X, z, s, x)$ satisfies the structure conditions (2.1) in B_1 and it is convex in X . Suppose $f \in L_w^p(B_1)$. Then there exists a small $\delta = \delta(n, \lambda, \Lambda, p, w) > 0$ such that if*

$$\sup_{x_0 \in B_1, 0 < r \leq r_0} \left(\int_{B_r(x_0)} \beta(x, x_0)^n dx \right)^{1/n} \leq \delta$$

for some $r_0 > 0$, then for any bounded $L^{\tilde{p}}$ -viscosity solution u of

$$F(D^2u, Du, u, x) = f(x) \quad \text{in } B_1,$$

we have that $u \in W_w^{2,p}(B_{\frac{1}{2}})$ with the estimate

$$(3.4) \quad \|u\|_{W_w^{2,p}(B_{\frac{1}{2}})} \leq c \left(\|f\|_{L_w^p(B_1)} + \|u\|_{L^\infty(B_1)} \right),$$

for some positive constant $c = c(n, \lambda, \Lambda, \kappa_1, \kappa_2, p, w, r_0)$.

- (ii) *Assume that $F(X, z, s, x)$ satisfies the structure conditions (2.1) in B_1^+ and it is convex in X . Suppose $f \in L_w^p(B_1^+)$. Then there exists a small $\delta = \delta(n, \lambda, \Lambda, p, w) > 0$ such that if*

$$\sup_{x_0 \in B_1^+, 0 < r \leq r_0} \left(\int_{B_r(x_0) \cap B_1^+} \beta(x, x_0)^n dx \right)^{1/n} \leq \delta$$

for some $r_0 > 0$, then for any bounded $L^{\tilde{p}}$ -viscosity solution u of

$$(3.5) \quad \begin{cases} F(D^2u, Du, u, x) = f & \text{in } B_1^+, \\ u = 0 & \text{on } B_1 \cap \{x_n = 0\}, \end{cases}$$

we have that $u \in W_w^{2,p}(B_{\frac{1}{2}}^+)$ with the estimate

$$(3.6) \quad \|u\|_{W_w^{2,p}(B_{\frac{1}{2}}^+)} \leq c \left(\|f\|_{L_w^p(B_1^+)} + \|u\|_{L^\infty(B_1^+)} \right),$$

for some positive constant $c = c(n, \lambda, \Lambda, \kappa_1, \kappa_2, p, w, r_0)$.

Since the proof of the interior weighted estimate (3.4) is analogous to that of (3.6), we only show the boundary estimate (3.6) in Theorem 3.4.

In order to derive (3.6), we need power decay estimates for the weighted measure of \mathcal{A}_t ; see (3.8) in Lemma 3.7. Hereafter, for simplicity, we write $Q_1^+ := Q_1^{n-1} \times (0, 1)$, where

$$Q_1^{n-1} := \{x \in \mathbb{R}^{n-1} : \max_{1 \leq i \leq n-1} |x_i| < 1\}.$$

We first recall the following lemma that can be found in [21]; see [21, Lemma 2.15] for its proof and more details.

Lemma 3.5. *Assume that $F(X, 0, 0, x)$ satisfies the structure conditions (2.1) in $B_{14\sqrt{n}}^+$, and it is convex in X . Furthermore, F and f are supposed to be continuous in all variables. Let $u \in C^0(\Omega)$ be a C -viscosity solution of*

$$(3.7) \quad \begin{cases} F(D^2u, 0, 0, x) = f & \text{in } B_{14\sqrt{n}}^+, \\ u = 0 & \text{on } B_{14\sqrt{n}} \cap \{x_n = 0\}, \end{cases}$$

with $\|u\|_{L^\infty(B_{14\sqrt{n}}^+)} \leq 1$. Then there is a constant $M = M(n, \lambda, \Lambda) > 1$ such that for any $\epsilon \in (0, 1)$, there exists $\delta = \delta(n, \lambda, \Lambda, \epsilon) \in (0, 1)$ such that if $\|f\|_{L^{n_0}(B_{14\sqrt{n}}^+)} \leq \delta$ and

$$\left(\int_{B_r(x_0) \cap B_{14\sqrt{n}}^+} \beta(x, x_0)^n dx \right)^{1/n} \leq \delta$$

for any $x_0 \in B_{14\sqrt{n}}^+$ and $r > 0$, then for $k = 0, 1, 2, \dots$ we have

$$\begin{aligned} & \left| \mathcal{A}_{M^{k+1}}(u, B_{14\sqrt{n}}^+) \cap Q_1^+ \right| \\ & \leq \epsilon \left| \left(\mathcal{A}_{M^k}(u, B_{14\sqrt{n}}^+) \cap Q_1^+ \right) \cup \{x \in Q_1^+ : \mathcal{M}(f^{n_0})(x) \geq (c_0 M^i)^{n_0}\} \right| \end{aligned}$$

for some constant $c_0 = c_0(n, \lambda, \Lambda, \epsilon) > 0$.

Remark 3.6. Since the convexity of the function F in X guarantees $C^{1,1}$ interior and boundary estimates for solutions of the homogeneous equations from [3, Theorem 6.6] and [14, Proposition 2.4], we assume the convexity of F in X instead of the $C^{1,1}$ estimates as the hypothesis of Lemma 3.5 above; see [3, Remark 1 in Chapter 7] for more details.

By means of the properties of the Muckenhoupt weights, especially Lemma 2.2, we can obtain the next result from Lemma 3.5.

Lemma 3.7. *Under the same assumptions as in Lemma 3.5, we further suppose $w \in A_{\frac{p}{n_0}}$. Then there is a constant $M = M(n, \lambda, \Lambda) > 1$ such that for any $\epsilon \in (0, 1)$, there exists $\delta = \delta(n, \lambda, \Lambda, p, w, \epsilon) \in (0, 1)$ such that if $\|f\|_{L^{n_0}(B_{14\sqrt{n}}^+)} \leq \delta$ and*

$$\left(\int_{B_r(x_0) \cap B_{14\sqrt{n}}^+} \beta(x, x_0)^n dx \right)^{1/n} \leq \delta$$

for any $x_0 \in B_{14\sqrt{n}}^+$ and $r > 0$, then for $k = 0, 1, 2, \dots$ we have

$$(3.8) \quad \begin{aligned} & w \left(\mathcal{A}_{M^k}(u, B_{14\sqrt{n}}^+) \cap Q_1^+ \right) \\ & \leq \epsilon^k w(Q_1) + \sum_{i=0}^{k-1} \epsilon^{k-i} w \left(\{x \in Q_1^+ : \mathcal{M}(f^{n_0})(x) \geq (c_0 M^i)^{n_0}\} \right) \end{aligned}$$

for some constant $c_0 = c_0(n, \lambda, \Lambda, \epsilon) > 0$.

Proof. Let $\epsilon \in (0, 1)$ be given and choose $\delta = \delta(n, \lambda, \Lambda, \gamma_1, \gamma_2, \epsilon)$ as in Lemma 3.5, with ϵ replaced by $\left(\frac{\epsilon}{\gamma_1}\right)^{\frac{1}{\gamma_2}}$, where γ_1, γ_2 are the constants depending on $n, \frac{p}{n_0}$, and

$[w]_{\frac{p}{n_0}}$ from Lemma 2.2. We set

$$\begin{aligned} D &:= \mathcal{A}_{M^{k+1}}(u, B_{14\sqrt{n}}^+) \cap Q_1^+, \\ E &:= \left(\mathcal{A}_{M^k}(u, B_{14\sqrt{n}}^+) \cap Q_1^+ \right) \cup \{x \in Q_1^+ : \mathcal{M}(f^{n_0})(x) \geq (c_0 M^i)^{n_0}\} \end{aligned}$$

for $k = 0, 1, 2, \dots$. Then Lemma 3.5 gives

$$|D| \leq \left(\frac{\epsilon}{\gamma_1} \right)^{\frac{1}{\gamma_2}} |E|.$$

It follows from Lemma 2.2 that

$$\frac{w(D)}{w(E)} \leq \gamma_1 \left(\frac{|D|}{|E|} \right)^{\gamma_2} \leq \gamma_1 \left(\frac{\epsilon}{\gamma_1} \right)^{\frac{1}{\gamma_2} \cdot \gamma_2} = \epsilon,$$

and hence, we find that

$$\begin{aligned} &w \left(\mathcal{A}_{M^{k+1}}(u, B_{14\sqrt{n}}^+) \cap Q_1^+ \right) \\ &\leq \epsilon w \left(\mathcal{A}_{M^k}(u, B_{14\sqrt{n}}^+) \cap Q_1^+ \right) + \epsilon w \left(\{x \in Q_1^+ : \mathcal{M}(f^{n_0})(x) \geq (c_0 M^i)^{n_0}\} \right) \end{aligned}$$

for $k = 0, 1, 2, \dots$. Iterating this estimate, we finally obtain the desired estimate (3.8). \square

We are ready now to prove the main result of this subsection. As mentioned before, we only derive the boundary weighted $W^{2,p}$ estimates (3.6) in Theorem 3.4.

To do this, by virtue of Lemma 3.3, it suffices to show that $\Theta(u, B_{\frac{1}{2}}^+) \in L_w^p(B_{\frac{1}{2}}^+)$ with

$$(3.9) \quad \|\Theta(u, B_{\frac{1}{2}}^+)\|_{L_w^p(B_{\frac{1}{2}}^+)} \leq c,$$

for some positive constant c depending only on n, λ, Λ , and w , via the standard normalization procedure. It is clear that

$$w \left(\{x \in B_{\frac{1}{2}}^+ : \Theta(u, B_{\frac{1}{2}}^+)(x) > t\} \right) \leq w \left(\mathcal{A}_t(u, B_{\frac{1}{2}}^+) \right),$$

for $t > 0$, and so we see that

$$S_\Theta := \sum_{k \geq 1} M^{pk} w \left(\{x \in B_{\frac{1}{2}}^+ : \Theta(u, B_{\frac{1}{2}}^+)(x) > t\} \right) \leq \sum_{k \geq 1} M^{pk} w \left(\mathcal{A}_t(u, B_{\frac{1}{2}}^+) \right).$$

for some universal constant $M > 1$ given in Lemma 3.7. Therefore, if we show that

$$(3.10) \quad \sum_{k \geq 1} M^{pk} w \left(\mathcal{A}_{M^k}(u, B_{\frac{1}{2}}^+) \right) < \infty,$$

then Lemma 3.2 would imply $\Theta(u, B_{\frac{1}{2}}^+) \in L_w^p(B_{\frac{1}{2}}^+)$ with

$$\int_{B_{\frac{1}{2}}^+} |\Theta(u, B_{\frac{1}{2}}^+)(x)|^p w(x) dx \leq c \left(w(B_{\frac{1}{2}}^+) + S_\Theta \right),$$

where the constant c depends only on M and w . This way, it will be sufficient to get (3.10) in order to prove (3.6) in Theorem 3.4.

Proof of Theorem 3.4 (ii). We only need to derive the desired estimate (3.6) for the problem (3.5) without dependence on the terms of Du and u , thanks to the structure conditions of the operator F and the properties of the weight w . More precisely, we first consider $\tilde{f}(x) := F(D^2u, 0, 0, x)$. From the structure conditions (2.1), we have that $|\tilde{f}(x)| \leq \kappa_1|Du(x)| + \kappa_2|u(x)| + |f(x)|$ for a.e. $x \in B_1^+$.

Since $w \in A_{\frac{p}{n_0}}$, we use the reverse Hölder property of w in [12, Theorem 9.25], which gives that there exists $\gamma = \gamma(n, \lambda, \Lambda, p, [w]_{\frac{p}{n_0}}) > 0$ such that $w^{1+\gamma} \in A_{\frac{p}{n_0}}$. Then, given $g \in L^{p(1+\frac{1}{\gamma})}(U)$, one has

$$\int_U |g|^p w \, dx \leq \left(\int_U |g|^{p(1+\frac{1}{\gamma})} \, dx \right)^{\frac{\gamma}{1+\gamma}} \left(\int_U w^{1+\gamma} \, dx \right)^{\frac{1}{1+\gamma}},$$

for any bounded subset U of \mathbb{R}^n , which implies that $L^{p(1+\frac{1}{\gamma})}(U) \hookrightarrow L_w^p(U)$. Therefore, applying interior $W^{1,\bar{p}}$ estimates in [17, Theorem 2.1] and boundary $W^{1,\bar{p}}$ estimates in [21, Theorem 3.1] with $q := p(1 + \frac{1}{\gamma})$ by choosing δ sufficiently small, the standard covering argument allows us to discover that $Du \in L_w^p(B_{\frac{2}{3}}^+)$ with

$$\begin{aligned} \|Du\|_{L_w^p(B_{\frac{2}{3}}^+)} &\leq c \|Du\|_{L^{p(1+\frac{1}{\gamma})}(B_{\frac{2}{3}}^+)} \\ &\leq c \left(\|f\|_{L^{\bar{p}}(B_1^+)} + \|u\|_{L^\infty(B_1^+)} \right) \\ &\leq c \left(\|f\|_{L_w^p(B_1^+)} + \|u\|_{L^\infty(B_1^+)} \right). \end{aligned}$$

Recalling [17, Corollary 1.6], we observe that u is an $L^{\bar{p}}$ -viscosity solution of $F(D^2u, 0, 0, x) = \tilde{f}(x)$ in B_1^+ .

Accordingly, if the resulting estimate (3.6) is derived for equation (3.5) without dependence on the terms of Du and u , we infer that

$$\begin{aligned} (3.11) \quad \|u\|_{W_w^{2,p}(B_{\frac{1}{2}}^+)} &\leq c \left(\|\tilde{f}\|_{L_w^p(B_{\frac{2}{3}}^+)} + \|u\|_{L^\infty(B_{\frac{2}{3}}^+)} \right) \\ &\leq c \left(\|f\|_{L_w^p(B_{\frac{2}{3}}^+)} + \|Du\|_{L_w^p(B_{\frac{2}{3}}^+)} + \|u\|_{L^\infty(B_{\frac{2}{3}}^+)} \right) \\ &\leq c \left(\|f\|_{L_w^p(B_1^+)} + \|u\|_{L^\infty(B_1^+)} \right). \end{aligned}$$

Furthermore, it suffices to establish the estimate (3.11) for a C -viscosity solution u of

$$\begin{cases} F(D^2u, 0, 0, x) = f & \text{in } B_1^+, \\ u = 0 & \text{on } B_1 \cap \{x_n = 0\}, \end{cases}$$

under the additional assumption that F and f are continuous in x , using the same approximation procedure as in the proof of [21, Theorem 4.3].

In order to obtain the estimate (3.11), let us first fix $x_0 \in B_{\frac{1}{2}} \cap \{x_n = 0\}$ and choose a small constant r such that $0 < r < \frac{1}{28\sqrt{n}}$, which will be determined later. Then we define $\tilde{u}(x) := \frac{1}{K r^2} u(rx + x_0)$ and $\tilde{w}(x) = w(rx + x_0)$, where

$$K := r^{2-\frac{n}{p}} \delta^{-1} \|f\|_{L_w^p(B_{14r\sqrt{n}}^+(x_0))} + \|u\|_{L^\infty(B_{14r\sqrt{n}}^+(x_0))}$$

and $\delta = \delta(n, \lambda, \Lambda, p, w, \epsilon) \in (0, 1)$ is the same as in Lemma 3.7 and ϵ will be taken later. Then we observe that \tilde{u} is a viscosity solution of $\tilde{F}(D^2\tilde{u}, 0, 0, x) = \tilde{f}$ in $B_{8\sqrt{n}}$,

where

$$\tilde{F}(X, x) := \frac{1}{K}F(KX, rx + x_0) \quad \text{and} \quad \tilde{f}(x) := \frac{1}{K}f(rx + x_0).$$

It is clear that $\tilde{w} \in A_{\frac{p}{n_0}}$, \tilde{F} has the same structure conditions as F , $\beta_{\tilde{F}} = \beta_F$, and $\tilde{F}(X, x)$ is also convex in X . Moreover, it is easy to see that

$$\|\tilde{u}\|_{L^\infty(B_{14\sqrt{n}}^+)} \leq 1.$$

As in (2.8), we obtain that

$$(3.12) \quad \|\tilde{f}\|_{L^{n_0}(B_{14\sqrt{n}}^+)} \leq c\|\tilde{f}\|_{L_{\tilde{w}}^p(B_{14\sqrt{n}}^+)} \leq c\delta,$$

for some $c = c(n, p, w, r) > 0$. Hence, all the hypotheses of Lemma 3.7 are satisfied.

Let $M = M(n, \lambda, \Lambda)$ and $c_0 = c_0(n, \lambda, \Lambda, \epsilon)$ be the same constants as in Lemma 3.7, and take ϵ such that $M^p\epsilon = \frac{1}{2}$. Then Lemma 3.7 leads to

$$(3.13) \quad \begin{aligned} & \tilde{w} \left(\mathcal{A}_{M^k}(\tilde{u}, B_{14\sqrt{n}}^+) \cap Q_1^+ \right) \\ & \leq \epsilon^k \tilde{w}(Q_1) + \sum_{i=0}^{k-1} \epsilon^{k-i} \tilde{w} \left(\{x \in Q_1^+ : \mathcal{M}(\tilde{f}^{n_0})(x) \geq (c_0 M^i)^{n_0}\} \right). \end{aligned}$$

Note that $\tilde{w} \in A_{\frac{p}{n_0}}$ and $|\tilde{f}|^{n_0} \in L_{\tilde{w}}^{\frac{p}{n_0}}(B_{14\sqrt{n}}^+)$. Then it follows from (3.1) and (3.12) that $\mathcal{M}(|\tilde{f}|^{n_0}) \in L_{\tilde{w}}^{\frac{p}{n_0}}(B_{14\sqrt{n}}^+)$ with

$$\|\mathcal{M}(|\tilde{f}|^{n_0})\|_{L_{\tilde{w}}^{\frac{p}{n_0}}(B_{14\sqrt{n}}^+)} \leq c\|\tilde{f}\|_{L_{\tilde{w}}^p(B_{14\sqrt{n}}^+)}^{n_0} = c\|\tilde{f}\|_{L_{\tilde{w}}^p(B_{14\sqrt{n}}^+)}^{n_0} \leq c$$

for some constant $c = c(n, p, w, r) > 0$. Therefore, Lemma 3.2 yields that

$$\sum_{k \geq 1} M^{pk} \tilde{w} \left(\{x \in Q_1^+ : \mathcal{M}(\tilde{f}^{n_0})(x) \geq (c_0 M^i)^{n_0}\} \right) \leq c.$$

Accordingly, we obtain from (3.13) that

$$\begin{aligned} & \sum_{k \geq 1} M^{pk} \tilde{w} \left(\mathcal{A}_{M^k}(\tilde{u}, B_{14\sqrt{n}}^+) \cap Q_1^+ \right) \\ & \leq \sum_{k \geq 1} M^{pk} \epsilon^k \tilde{w}(Q_1) \\ & \quad + \sum_{k \geq 1} M^{pk} \sum_{i=0}^{k-1} \epsilon^{k-i} \tilde{w} \left(\{x \in Q_1^+ : \mathcal{M}(\tilde{f}^{n_0})(x) \geq (c_0 M^i)^{n_0}\} \right) \\ & \leq \tilde{w}(Q_1) \sum_{k \geq 1} (M^p \epsilon)^k \\ & \quad + \left(\sum_{j \geq 1} (M^p \epsilon)^j \right) \left(\sum_{i \geq 0} M^{pi} \tilde{w} \left(\{x \in Q_1^+ : \mathcal{M}(\tilde{f}^{n_0})(x) \geq (c_0 M^i)^{n_0}\} \right) \right) \\ & \leq (\tilde{w}(Q_1) + c) \sum_{k \geq 1} \left(\frac{1}{2} \right)^k \leq c, \end{aligned}$$

by the choice of ϵ , and hence, we see $\|D^2\tilde{u}\|_{L_w^p(B_{\frac{1}{2}}^+)} \leq c$, which implies

$$(3.14) \quad \|D^2u\|_{L_w^p(B_{\frac{1}{2}}^+(x_0))} \leq c \left(r^{2-\frac{n}{p}} \delta^{-1} \|f\|_{L_w^p(B_{14r\sqrt{n}}^+(x_0))} + \|u\|_{L^\infty(B_{14r\sqrt{n}}^+(x_0))} \right).$$

On the other hand, we can also establish the interior estimate

$$(3.15) \quad \|D^2u\|_{L_w^p(B_{\frac{r}{2}})} \leq c \left(r^{2-\frac{n}{p}} \delta^{-1} \|f\|_{L_w^p(B_{8r\sqrt{n}})} + \|u\|_{L^\infty(B_{8r\sqrt{n}})} \right)$$

in a similar way that we have derived (3.14) applying the weighted version of [3, Lemma 7.12] instead of Lemma 3.7. Indeed, this weighted version can be discovered from [3, Lemma 7.12] using Lemma 2.2 in the same way as in the proof of Lemma 3.7.

Take r sufficiently small so that $B_{\frac{1}{2}}^+$ can be covered by finite number of half balls $B_r^+(x_0)$ for $x_0 \in B_{\frac{1}{2}} \cap \{x_n = 0\}$ and balls $B_r(x_0)$ for $x_0 \in B_{\frac{1}{2}}^+$. Then, from the boundary estimate (3.14), along with the interior estimate (3.15), we finally obtain the desired estimate (3.11). \square

3.3. Global weighted estimates. We now prove the main result, Theorem 2.4, via the standard flattening and covering arguments applying the interior and boundary estimates from Theorem 3.4.

Proof of Theorem 2.4. We first recall Remark 2.5 to see that there exists a unique $L^{\tilde{p}}$ -viscosity solution u of (1.1), where $\tilde{p} := \frac{n_0 p}{p - n_0 \tilde{\epsilon}}$ for some small constant $\tilde{\epsilon} = \tilde{\epsilon}(n, \lambda, \Lambda, p, [w]_{\frac{p}{n_0}}) > 0$. We can also obtain the existence of $L^{\tilde{p}}$ -viscosity solutions of (1.1) by using the same approximation argument as in the proof of [21, Theorem 4.6]. Moreover, Theorem 2.10 in [4] gives the uniqueness of solutions to (1.1). Therefore, we only need to derive the estimate (2.7).

Let us fix a point $x_0 \in \partial\Omega$. We now flatten the boundary near x_0 and apply the boundary estimate (3.6). From the assumption $\partial\Omega \in C^{1,1}$, there exist a neighborhood \mathcal{N} of x_0 and a $C^{1,1}$ -diffeomorphism $\Phi: \mathcal{N} \rightarrow B_1$ such that $\Phi(x_0) = 0$ and $\Phi(\mathcal{N} \cap \Omega) = B_1^+$. We set $\Psi := \Phi^{-1}$ and then $x = \Psi(y)$. Now we define $\tilde{u}(y) = u(\Psi(y))$ and $\tilde{w}(y) = w(\Psi(y))$. Then it is clear that $\tilde{w} \in A_{\frac{p}{n_0}}$ and we observe that \tilde{u} is an $L^{\tilde{p}}$ -viscosity solution of

$$\begin{cases} \tilde{F}(D^2\tilde{u}, D\tilde{u}, \tilde{u}, y) = \tilde{f} & \text{in } B_1^+, \\ \tilde{u} = 0 & \text{on } B_1 \cap \{x_n = 0\}, \end{cases}$$

where

$$\tilde{F}(X, z, s, y) := F \left(D\Phi^T \circ \Psi X D\Phi \circ \Psi + (z D_{ij}\Phi \circ \Psi)_{1 \leq i, j \leq n}, z D\Phi \circ \Psi, s, \Psi(y) \right)$$

and $\tilde{f}(y) := f(\Psi(y))$. Note that \tilde{F} is convex in X and $\tilde{F}(0, 0, 0, y) = 0$. Moreover, we see that $\beta_{\tilde{F}}(x, x_0) \leq c(\Phi)\beta_F(\Psi(x), \Psi(x_0))$ and \tilde{F} is uniformly elliptic with ellipticity constants $\lambda c(\Phi), \Lambda c(\Phi)$. Therefore all the hypotheses of Theorem 3.4 (ii) are satisfied, and so we apply Theorem 3.4 (ii) to obtain the estimate

$$\|\tilde{u}\|_{W_w^{2,p}(B_{\frac{1}{2}}^+)} \leq c \left(\|\tilde{f}\|_{L_w^p(B_1^+)} + \|\tilde{u}\|_{L^\infty(B_1^+)} \right).$$

Turning back to the x -variables, we then deduce that

$$\begin{aligned} \|u\|_{W_w^{2,p}(\Psi(B_{\frac{1}{2}}^+))} &\leq c \left(\|f\|_{L_w^p(\Psi(B_1^+))} + \|u\|_{L^\infty(\Psi(B_1^+))} \right) \\ &\leq c \left(\|f\|_{L_w^p(\Omega)} + \|u\|_{L^\infty(\Omega)} \right). \end{aligned}$$

From this estimate, along with the interior bound (3.4) in Theorem 3.4, the standard covering arguments lead to

$$(3.16) \quad \|u\|_{W_w^{2,p}(\Omega)} \leq c \left(\|f\|_{L_w^p(\Omega)} + \|u\|_{L^\infty(\Omega)} \right).$$

At this point, the desired estimate (2.7) follows from the uniqueness property of the homogeneous equation. Indeed, if (2.7) is not true, there exist sequences $\{u_k\}_{k=1}^\infty$ and $\{f_k\}_{k=1}^\infty$ such that u_k is a $L^{\tilde{p}}$ -viscosity solution of

$$\begin{cases} F(D^2u_k, Du_k, u_k, x) = f_k & \text{in } \Omega, \\ u_k = 0 & \text{on } \partial\Omega, \end{cases}$$

satisfying

$$(3.17) \quad \|u_k\|_{W_w^{2,p}(\Omega)} > k \|f_k\|_{L_w^p(\Omega)} \quad \text{for any } k \geq 1.$$

Without loss of generality, we may suppose that $\|u_k\|_{W_w^{2,p}(\Omega)} = 1$. Then it follows from (3.17) that

$$\|f_k\|_{L_w^p(\Omega)} < \frac{1}{k} \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

Moreover, there exist a subsequence of $\{u_k\}_{k=1}^\infty$, which is still denoted by $\{u_k\}_{k=1}^\infty$ and a function $v \in W_w^{2,p}(\Omega)$ such that $u_k \rightharpoonup v$ weakly in $W_w^{2,p}(\Omega)$ as $k \rightarrow \infty$. Note that $W_w^{2,p}(\Omega) \hookrightarrow W^{2,\tilde{p}}(\Omega) \hookrightarrow C^0(\Omega)$ from (2.8) and the fact that $\tilde{p} > \frac{n}{2}$. Then u_k converges strongly to v in $C^0(\Omega)$, and hence we observe that v is a $L^{\tilde{p}}$ -viscosity solution of

$$(3.18) \quad \begin{cases} F(D^2v, Dv, v, x) = 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

by applying [21, Proposition 1.5]. Accordingly, we have $v \equiv 0$ by the uniqueness of strong solutions to (3.18) from [4, Theorem 2.10]. However, (3.16) implies

$$1 = \|u_k\|_{W_w^{2,p}(\Omega)} \leq c \left(\|f_k\|_{L_w^p(\Omega)} + \|u_k\|_{L^\infty(\Omega)} \right) \rightarrow 0 \quad \text{as } k \rightarrow \infty,$$

which is a contradiction. This completes the proof. \square

4. REGULARITY IN MORREY SPACES AND HÖLDER CONTINUITY OF THE GRADIENT

We first recall the definition of the Morrey spaces. The *Morrey space* $L^{q,\mu}(\Omega)$ with $1 < q < \infty$ and $0 < \mu < n$ is defined as the set of all measurable functions $g \in L^q(\Omega)$ for which the norm

$$\|g\|_{L^{q,\mu}(\Omega)} := \left(\sup_{y \in \Omega, \rho > 0} \frac{1}{\rho^\mu} \int_{B_\rho(y) \cap \Omega} |g(x)|^q dx \right)^{1/q}$$

is finite. Moreover, we define the space $W^{2,q,\mu}(\Omega)$ as the Banach space of functions belonging to the classical Sobolev space $W^{2,q}(\Omega)$ and having second order derivatives lying in the Morrey space $L^{q,\mu}(\Omega)$. A natural norm in this space is given by

$$\|g\|_{W^{2,q,\mu}(\Omega)} := \|g\|_{L^q(\Omega)} + \|D^2g\|_{L^{q,\mu}(\Omega)}.$$

The following is an outgrowth of our main result, Theorem 2.4.

Theorem 4.1. *Assume the hypotheses on F and $\partial\Omega$ given in Theorem 2.4. In addition, suppose that $f \in L^{p,\mu}$ with $p \in (n_0, \infty)$ and $\mu \in (0, n)$. There exists a small constant $\delta = \delta(n, \lambda, \Lambda, p, \mu, \Omega, \partial\Omega) > 0$ so that if (2.6) is satisfied for some $r_0 > 0$, then the second derivative of the solution u to the problem (1.1) belongs to $L^{p,\mu}(\Omega)$ with the estimate*

$$(4.1) \quad \|u\|_{W^{2,p,\mu}(\Omega)} \leq c \|f\|_{L^{p,\mu}(\Omega)},$$

for some positive constant $c = c(n, \lambda, \Lambda, \kappa_1, \kappa_2, p, \mu, \partial\Omega, \text{diam}(\Omega), r_0)$.

Proof. We first extend f by zero outside Ω and fix arbitrary $x_0 \in \Omega$ and $r > 0$. Let $\chi_{B_r(x_0)}$ denote a characteristic function of $B_r(x_0)$. It follows from [9, Proposition 2] that if $\sigma \in (0, 1)$ then

$$(\mathcal{M}\chi_{B_r(x_0)}(x))^\sigma \in A_1.$$

Therefore, since $\frac{p}{n_0} > 1$, we have by the monotonicity of the classes A_q that

$$(\mathcal{M}\chi_{B_r(x_0)}(x))^\sigma \in A_1 \subset A_{\frac{p}{n_0}}$$

with $[(\mathcal{M}\chi_{B_r(x_0)}(x))^\sigma]_{\frac{p}{n_0}} = c(n, n_0, p, \sigma)$.

Let us fix an arbitrary $\sigma \in (\frac{p}{n}, 1)$. We apply Theorem 2.4 to discover that there exists a constant $\delta = \delta(n, \lambda, \Lambda, p, \partial\Omega) > 0$ such that if (2.6) is satisfied, then

$$(4.2) \quad \begin{aligned} \int_{B_r(x_0) \cap \Omega} |D^2 u|^p dx &= \int_{\Omega} |D^2 u|^p (\chi_{B_r(x_0)})^\sigma dx \\ &\leq \int_{\Omega} |D^2 u|^p (\mathcal{M}\chi_{B_r(x_0)})^\sigma dx \\ &\leq c \int_{\Omega} |f|^p (\mathcal{M}\chi_{B_r(x_0)})^\sigma dx \end{aligned}$$

for some constant $c = c(n, \lambda, \Lambda, \kappa_1, \kappa_2, p, \partial\Omega, \text{diam}(\Omega), r_0) > 0$. We use now the dyadic decomposition of \mathbb{R}^n related to $B_r(x_0)$,

$$\mathbb{R}^n = B_{2r}(x_0) \cup \left(\bigcup_{k=1}^{\infty} B_{2^{k+1}r}(x_0) \setminus B_{2^k r}(x_0) \right),$$

in order to obtain that

$$(4.3) \quad \int_{\Omega} |f|^p (\mathcal{M}\chi_{B_r(x_0)})^\sigma dx = \int_{\mathbb{R}^n} |f|^p (\mathcal{M}\chi_{B_r(x_0)})^\sigma dx = I_0 + \sum_{k=1}^{\infty} I_k,$$

where

$$I_0 := \int_{B_{2r}(x_0)} |f|^p (\mathcal{M}\chi_{B_r(x_0)})^\sigma dx$$

and

$$I_k := \int_{B_{2^{k+1}r}(x_0) \setminus B_{2^k r}(x_0)} |f|^p (\mathcal{M}\chi_{B_r(x_0)})^\sigma dx.$$

It is clear that $\mathcal{M}\chi_{B_r(x_0)}(x) \leq 1$ for a.a. $x \in \mathbb{R}^n$ and thus we have

$$(4.4) \quad I_0 \leq \int_{B_{2r}(x_0)} |f|^p dx \leq c(n)r^\mu \|f\|_{L^{p,\mu}(\Omega)}^p.$$

Now we estimate I_k for $k = 1, 2, \dots$. Note that

$$0 < \int_{B_\rho(x)} \chi_{B_r(x_0)}(y) dy \leq \frac{|B_r(x_0)|}{|B_\rho(x)|} = \left(\frac{r}{\rho}\right)^n$$

for each $x \in B_{2^{k+1}r}(x_0) \setminus B_{2^k r}(x_0)$ and for each $\rho > (2^{k+1} - 1)r$. Then from the fact that $2^{k+1} - 1 \geq 2^k - 1 \geq 2^{k-1}$ for any $k \geq 1$, it follows that

$$\int_{B_\rho(x)} \chi_{B_r(x_0)}(y) dy \leq \left(\frac{r}{2^{k-1}r}\right)^n = \frac{1}{2^{n(k-1)}},$$

which implies

$$(\mathcal{M}\chi_{B_r(x_0)}(x))^\sigma = \left(\sup_{\rho>0} \int_{B_\rho(x)} \chi_{B_r(x_0)}(y) dy\right)^\sigma \leq \frac{1}{2^{\sigma n(k-1)}}.$$

Accordingly, we deduce that

(4.5)

$$\begin{aligned} I_k &\leq \frac{1}{2^{\sigma n(k-1)}} \int_{B_{2^{k+1}r}(x_0) \setminus B_{2^k r}(x_0)} |f|^p dx \\ &\leq \frac{1}{2^{\sigma n(k-1)}} \int_{B_{2^{k+1}r}(x_0)} |f|^p dx = \frac{(2^{k+1}r)^\mu}{2^{\sigma n(k-1)}} \left(\frac{1}{(2^{k+1}r)^\mu} \int_{B_{2^{k+1}r}(x_0)} |f|^p dx\right) \\ &\leq 2^{(\mu+\sigma n)+(\mu-\sigma n)k} r^\mu \|f\|_{L^{p,\mu}(\Omega)}^p. \end{aligned}$$

Having in mind (4.4) and (4.5), and remembering the choice of $\sigma \in (\frac{\mu}{n}, 1)$, (4.3) becomes

$$\begin{aligned} \int_{\Omega} |f|^p (\mathcal{M}\chi_{B_r(x_0)})^\sigma dx &= I_0 + \sum_{k=1}^{\infty} I_k \\ &\leq c(n)r^\mu \|f\|_{L^{p,\mu}(\Omega)}^p + 2^{(\mu+\sigma n)r^\mu} \|f\|_{L^{p,\mu}(\Omega)}^p \sum_{k=1}^{\infty} 2^{(\mu-\sigma n)k} \\ &\leq cr^\mu \left(\sum_{k=0}^{\infty} 2^{(\mu-\sigma n)k}\right) \|f\|_{L^{p,\mu}(\Omega)}^p \\ &\leq cr^\mu \|f\|_{L^{p,\mu}(\Omega)}^p, \end{aligned}$$

whence

$$\int_{B_r(x_0) \cap \Omega} |D^2 u|^p dx \leq c \int_{\Omega} |f|^p (\mathcal{M}\chi_{B_r(x_0)})^\sigma dx \leq cr^\mu \|f\|_{L^{p,\mu}(\Omega)}^p.$$

By dividing the both side above by r^μ and taking the supremum with respect to $x_0 \in \Omega$ and $r > 0$, we obtain $D^2 u \in L^{p,\mu}(\Omega)$ with the desired estimate (4.1). \square

Since $p > \frac{n}{2}$, we have Hölder continuity of the solution to problem (1.1) by the Sobolev imbedding theorem. However, employing the known properties of functions with Morrey regular gradient (cf. [5, Lemmas 3.III and 3.IV]), Theorem 4.1 allows to get better integrability and even Hölder continuity of the gradient for appropriate values of p and μ . Precisely,

Corollary 4.2. *Under the assumptions of Theorem 4.1, let $u \in W^{2,p,\mu}(\Omega)$ be the viscosity solution of the problem (1.1). Then*

- (1) $Du \in L^{\frac{np}{n-p}, \frac{n\mu}{n-p}}(\Omega; \mathbb{R}^n) \subset L^{p, p+\mu}(\Omega; \mathbb{R}^n)$ if $p + \mu < n$;
- (2) $Du \in L^{p', \mu'}(\Omega; \mathbb{R}^n)$ for each $p' < \infty$ and each $\mu' < n$, if $p + \mu = n$;
- (3) $Du \in C^{0, 1 - \frac{n-\mu}{p}}(\bar{\Omega}; \mathbb{R}^n)$ if $p + \mu > n$.

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