A singular limit problem for conservation laws related to the Rosenau-Korteweg-de Vries equation☆

Giuseppe Maria Coclite

Dipartimento di Matematica, Università di Bari

via E. Orabona 4, 70125 Bari, Italy

Lorenzo di Ruvo

Dipartimento di Scienze e Metodi dell'Ingegneria, Università di Modena e Reggio Emilia

via G. Amendola 2, 42122 Reggio Emilia, Italy

Abstract

We consider the Rosenau-Korteweg-de Vries equation, which contains nonlinear dispersive effects. We prove that as the diffusion parameter tends to zero, the solutions of the dispersive equation converge to discontinuous weak solutions of the Burgers equation. The proof relies on deriving suitable a priori estimates together with an application of the L^p compensated compactness method.

Nous étudions l'équation de Rosenau-Korteweg-de Vries, qui présente des effets de dispersion non-linéaire. Nous montrons que si le paramètre de diffusion tends vers zéro les solutions de l'équation dispersive convergent vers des solutions faibles discontinues de l'équation de Burgers. La preuve repose sur la dérivation d'opportunes estimations "a priori" et sur une application de la méthode de compacité par compensation dans L^p.

Keywords: Singular limit, compensated compactness, Rosenau-KdV-equation, entropy condition.

☆The authors are members of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).

*Corresponding author: G. M. Coclite

Email addresses: giuseppemaria.coclite@uniba.it (Giuseppe Maria Coclite), lorenzo.diruvo@unimore.it (Lorenzo di Ruvo)

URL: http://www.dm.uniba.it/Members/coclitegm/ (Giuseppe Maria Coclite)

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1. Introduction

1.1. Motivations

Dynamics of shallow water waves that is observed along lake shores and beaches has been a research area for the past few decades in oceanography (see [1, 2]). Several models has been proposed: Boussinesq equation, Peregrine equation, regularized long wave (RLW) equation, Kawahara equation, Benjamin-Bona-Mahoney equation, Bona-Chen equation etc. These models were derived from first principles under various different hypothesis and approximations. They are all well studied and very well understood.

The first model developed to describe such phenomena was the Korteweg-de Vries equation

$$\partial_t u + \partial_x u^2 + \beta \partial^3_{xxx} u = \varepsilon \partial^2_{xx} u. \tag{1}$$

As β, $\varepsilon \to 0$, (1) becomes the Burgers equation

$$\partial_t u + \partial_x u^2 = 0. \tag{2}$$

Such singular limit has been rigorously studied in [3, 4, 5], where the convergence of the solution of (1) to the unique entropy solution of (2) is proven, under the assumption

$$u_0 \in L^2(\mathbf{R}) \cap L^4(\mathbf{R}), \quad \beta = o\left(\varepsilon^2\right). \tag{3}$$

[6, Appendixes A and B] show that it is possible to obtain the same convergence result, under the following assumptions

$$\begin{aligned} &u_0 \in L^2(\mathbf{R}), \quad -\infty < \textstyle\int_{\mathbf{R}} u_0(x)dx < \infty, \quad \beta = o\left(\varepsilon^3\right), \\ &u_0 \in L^2(\mathbf{R}), \quad \beta = o\left(\varepsilon^4\right). \end{aligned} \tag{4}$$

One generalization of (1) is the Ostrovsky equation (see [7]):

$$\partial_x(\partial_t u + \partial_x u^2 - \beta \partial^3_{xxx} u) = \gamma u, \quad \beta, \gamma \in \mathbf{R}, \tag{5}$$

that describes small-amplitude long waves in a rotating fluid of a finite depth thanks to the additional term γu induced by the Coriolis force. As $\beta \to 0$, (5) becomes the Ostrovsky-Hunter equation (see [8])

$$\partial_x(\partial_t u + \partial_x u^2) = \gamma u, \qquad t > 0, \quad x \in \mathbf{R}. \tag{6}$$

Entropy weak solution for (6) are defined as follows:

Definition 1.1. *We say that* $u \in L^\infty((0,T)\times\mathbf{R})$, $T > 0$, *is an entropy solution of (6) if*

i) u is a distributional solution of (6);

ii) for every convex function $\eta \in C^2(\mathbf{R})$ *the entropy inequality*

$$\partial_t \eta(u) + \partial_x q(u) - \gamma\eta'(u)P \leq 0, \qquad q(u) = \int^u f'(\xi)\eta'(\xi)\,d\xi, \tag{7}$$

holds in the sense of distributions in $(0,\infty)\times\mathbf{R}$.

The wellposedness of the entropy solutions of (6) was proved in [9, 10, 11, 12]. Moreover, the convergence of the solutions of (5) to the unique entropy solution of (6) was proved in [13], under the assumption (3).

The dynamics of dispersive shallow water waves, on the other hand, is captured with slightly different models, like the Rosenau-Kawahara equation and the Rosenau-KdV-RLW equation [14, 15, 16, 17, 18]

$$\partial_t u + a\partial_x u + k\partial_x u^n + b_1\partial^3_{xxx}u + b_2\partial^3_{txx}u + c\partial^5_{txxxx}u = 0, \quad a,\, k,\, b_1,\, b_2,\, c \in \mathbf{R}. \tag{8}$$

In [18], the authors analyzed (8) and showed the existence of solitary waves, shock waves, and singular solitons along with conservation laws.

In [19], we studied the two cases

$$\begin{aligned} &n = 2,\, a = 0,\, k = 1,\, b_1 = 1,\, b_2 = -1,\, c = 1,\\ &n = 2,\, a = 0,\, k = 1,\, b_1 = 0,\, b_2 = -1,\, c = 1, \end{aligned} \tag{9}$$

in which (8) reads

$$\begin{aligned} \partial_t u + \partial_x u^2 + \partial^3_{xxx}u - \partial^3_{txx}u + \partial^5_{txxxx}u = &\quad 0,\\ \partial_t u + \partial_x u^2 - \partial^3_{txx}u + \partial^5_{txxxx}u = &\quad 0, \end{aligned} \tag{10}$$

respectively. Equation (10) is known as Rosenau-RLW equation.

Arguing as in [20], we re-scaled the equations as follows

$$\begin{aligned} \partial_t u + \partial_x u^2 + \beta\partial^3_{xxx}u - \beta\partial^3_{txx}u + \beta^2\partial^5_{txxxx}u &= \varepsilon\partial^2_{xx}u, \\ \partial_t u + \partial_x u^2 - \beta\partial^3_{txx}u + \beta^2\partial^5_{txxxx}u_{\varepsilon,\beta} &= \varepsilon\partial^2_{xx}u, \end{aligned} \tag{11}$$

where β is the diffusion parameter.

In [19], the authors proved that the solutions of (11) and (11) converge to the unique entropy solution of (2), under the assumptions

$$u_0 \in L^2(\mathbf{R}) \cap L^4(\mathbf{R}), \quad \beta = \mathcal{O}\left(\varepsilon^4\right). \tag{12}$$

1.2. Main results of this paper

The KdV equation (1) has also been used in very wide applications and undergone research which can be used to describe wave propagation and spread interaction (see [21, 22, 23, 24]). In the study of the dynamics of dense discrete systems, the case of wave-wave and wave-wall interactions cannot be described using (1). To overcome this shortcoming of (1), Rosenau proposed the following equation (see [25, 26]):

$$\partial_t u + \partial_x u^2 + \partial^5_{txxxx}u = 0, \tag{13}$$

which may be obtained by (8) in correspondence of the choice $n = 2$, $a = 0$, $k = 1$, $b_1 = 0$, $b_2 = 0$, $c = 1$.

The existence and the uniqueness of the solution for (13) is proved in [27] and the numerical methods are studied in [28, 29, 30, 31, 32, 33].

On the other hand, for the further consideration of the nonlinear wave, the viscous term $\partial^3_{xxx}u$ needs to be included in (13) (see [34]). In this way we get

$$\partial_t u + \partial_x u^2 + \partial^3_{xxx}u + \partial^5_{txxxx}u = 0, \tag{14}$$

which is known as the Rosenau-Korteweg-de Vries equation. It may also be obtianed by (8), taking $n = 2$, $a = 0$, $k = 1$, $b_1 = 1$, $b_2 = 0$, $c = 1$.

The solitary wave solutions of (14) has been studied in [34]. In [16], a conservative linear finite difference scheme for the numerical solution for an initial-boundary value problem of Rosenau-KdV equation is developed. In [35, 36], authors discussed the solitary solutions for (14) with the solitary ansatz method. The authors also gave two invariants for (14). In particular, in [36], the authors studied the solitary wave and the singular soliton solutions. In [37], the authors proposed an average linear finite difference scheme for the numerical approximation of the initial-boundary value problem for (14).

Arguing as [20], we re-scale (13) as follows

$$\partial_t u + \partial_x u^2 + \beta^2 \partial^5_{txxxx} u_{\varepsilon,\beta} = \varepsilon \partial^2_{xx} u. \tag{15}$$

In [6], the authors proved that the solutions of (15) converge to the unique entropy solution of (2), choosing the initial datum in two different ways. The first one is:

$$u_0 \in L^2(\mathbf{R}), \quad \beta = o\left(\varepsilon^4\right). \tag{16}$$

The second choice is given by (12).

In this paper, we analyze (14). Arguing as [20], we re-scale the equation as follows

$$\partial_t u + \partial_x u^2 + \beta \partial^3_{xxx} u + \beta^2 \partial^5_{txxxx} u = \varepsilon \partial^2_{xx} u. \tag{17}$$

We are interested in the no high frequency limit, we send ε, $\beta \to 0$ in (17). In this way we pass from (17) to (2). We prove that, as ε, $\beta \to 0$, the solutions of (17) to the unique entropy solution of (2). In other to do this, we can choose the initial datum, β, and ε in two different ways. Following [38, Theorem 7.1], the first choice is given by (16) (see Theorem 2.1). Since $\|\cdot\|_{L^4}$ is a conserved quantity for (17), the second choice is given by (12) (see Theorem 3.1). It is interesting to observe that, while the summability on the initial datum in (12) is greater than the one in (16), the assumption on β in (12) is weaker than the one in (16).

From the mathematical point of view, the two assumptions require two different arguments for the $L^\infty-$estimate (see Lemmas 2.2 and 3.1). Indeed, the proof of Lemma 2.2, under the assumption (16), is more technical than the one of Lemma 3.1. Moreover, due to the presence of the third order term, Lemmas

2.2 and 3.2 are finer than [6, Lemmas 2.2 and 3.2]. Indeed, with respect to [6, Lemma 2.2], in Lemma 2.2 we need to prove the existence of two positive constants, while, with respect to [6, Lemma 3.2], in Lemma 3.2 we need to prove the existence of four positive constants.

The paper is organized in four sections. In Section 2, we prove the convergence of the solutions of (17) to the entropy ones of (2) in L^p, $1 \le p < 2$. In Section 3, we prove the convergence of the solutions of (17) to the entropy ones of (2) in L^p, $1 \le p < 4$. In Section 4 we prove that the solutions of the Benjamin-Bona-Mahony equation converge to discontinuous weak solutions of (2) in L^p, $1 \le p < 2$.

2. The Rosenau-KdV-equation. $u_0 \in L^2(\mathbf{R})$.

In this section, we consider (17), and assume (16) on the initial datum. We study the dispersion-diffusion limit for (17). Therefore, we fix two small numbers $0 < \varepsilon$, $\beta < 1$ and consider the following fifth-order problem

$$\begin{cases} \partial_t u_{\varepsilon,\beta} + \partial_x u^2_{\varepsilon,\beta} + \beta \partial^3_{xxx} u_{\varepsilon,\beta} + \beta^2 \partial^5_{txxxx} u_{\varepsilon,\beta} = \varepsilon \partial^2_{xx} u_{\varepsilon,\beta}, & t > 0, \ x \in \mathbf{R}, \\ u_{\varepsilon,\beta}(0,x) = u_{\varepsilon,\beta,0}(x), & x \in \mathbf{R}, \end{cases} \tag{18}$$

where $u_{\varepsilon,\beta,0}$ is a C^∞ approximation of u_0 such that

$$\begin{aligned} & u_{\varepsilon,\,\beta,\,0} \to u_0 \quad \text{in } L^p_{loc}(\mathbf{R}),\ 1 \le p < 2, \text{ as } \varepsilon,\ \beta \to 0, \\ & \|u_{\varepsilon,\beta,0}\|^2_{L^2(\mathbf{R})} + (\beta^{\frac{1}{2}} + \varepsilon^2) \|\partial_x u_{\varepsilon,\beta,0}\|^2_{L^2(\mathbf{R})} \le C_0, \quad \varepsilon, \beta > 0, \\ & \left(\beta^2 + \beta\varepsilon^2\right) \left\|\partial^2_{xx} u_{\varepsilon,\beta,0}\right\|^2_{L^2(\mathbf{R})} + \beta^{\frac{5}{2}} \left\|\partial^3_{xxx} u_{\varepsilon,\beta,0}\right\|^2_{L^2(\mathbf{R})} \le C_0, \quad \varepsilon, \beta > 0, \end{aligned} \tag{19}$$

and C_0 is a constant independent on ε and β.

The main result of this section is the following theorem.

Theorem 2.1. *Assume that (16) and (19) hold. Fix $T > 0$, if*

$$\beta = o\left(\varepsilon^4\right), \tag{20}$$

then, there exist two sequences $\{\varepsilon_n\}_{n \in \mathbf{N}}$, $\{\beta_n\}_{n \in \mathbf{N}}$, with $\varepsilon_n, \beta_n \to 0$, and a limit function

$$u \in L^\infty((0,T); L^2(\mathbf{R})), \tag{21}$$

such that

i) $u_{\varepsilon_n,\beta_n} \to u$ *strongly in* $L^p_{loc}(\mathbf{R}^+ \times \mathbf{R})$, *for each* $1 \le p < 2$,

ii) u *is the unique entropy solution of (2).*

Let us prove some a priori estimates on $u_{\varepsilon,\beta}$, denoting with C_0 the constants which depend only on the initial data.

Lemma 2.1. *For each* $t > 0$,

$$\begin{aligned} \|u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbf{R})} & +\beta^2 \left\|\partial^2_{xx} u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})} \\ & +2\varepsilon \int_0^t \|\partial_x u_{\varepsilon,\beta}(s,\cdot)\|^2_{L^2(\mathbf{R})} ds \le C_0. \end{aligned} \tag{22}$$

Proof. We begin by observing that

$$\int_{\mathbf{R}} u_{\varepsilon,\beta} \partial^3_{xxx} u_{\varepsilon,\beta} dx = 0. \tag{23}$$

Therefore, arguing as [6, Lemma 2.1], we have (22). ⋄

Lemma 2.2. *Fix* $T > 0$. *Assume (20). There exists* $C_0 > 0$, *independent on* ε, β *such that*

$$\|u_{\varepsilon,\beta}\|_{L^\infty((0,T)\times\mathbf{R})} \le C_0 \beta^{-\frac{1}{4}}. \tag{24}$$

Moreover,

i) the families $\{\beta^{\frac{1}{2}} \partial_x u_{\varepsilon,\beta}\}_{\varepsilon,\,\beta}$, $\{\beta^{\frac{1}{4}} \varepsilon \partial_x u_{\varepsilon,\beta}\}_{\varepsilon,\,\beta}$, $\{\beta^{\frac{3}{4}} \varepsilon \partial^2_{xx} u_{\varepsilon,\beta}\}_{\varepsilon,\,\beta}$, $\{\beta^{\frac{3}{2}} \partial^3_{xxx} u_{\varepsilon,\beta}\}_{\varepsilon,\,\beta}$, *are bounded in* $L^\infty((0,T); L^2(\mathbf{R}))$;

ii) the families $\{\beta^{\frac{3}{4}} \varepsilon^{\frac{1}{2}} \partial^2_{tx} u_{\varepsilon,\beta}\}_{\varepsilon,\,\beta}$, $\{\beta^{\frac{7}{4}} \varepsilon^{\frac{1}{2}} \partial^4_{txxx} u_{\varepsilon,\beta}\}_{\varepsilon,\,\beta}$, $\{\beta^{\frac{1}{4}} \varepsilon \partial_t u_{\varepsilon,\beta}\}_{\varepsilon,\,\beta}$, $\{\beta^{\frac{5}{4}} \varepsilon^{\frac{1}{2}} \partial^3_{txx} u_{\varepsilon,\beta}\}_{\varepsilon,\,\beta}$, $\{\beta^{\frac{1}{2}} \varepsilon^{\frac{1}{2}} \partial^2_{xx} u_{\varepsilon,\beta}\}_{\varepsilon,\,\beta}$ *are bounded in* $L^2((0,T) \times \mathbf{R})$.

Proof. Let $0 < t < T$. Let A, B be some positive constants which will be specified later. Multiplying (18) by $-\beta^{\frac{1}{2}} \partial^2_{xx} u_{\varepsilon,\beta} - A\beta\varepsilon \partial^3_{txx} u_{\varepsilon,\beta} + B\varepsilon \partial_t u_{\varepsilon,\beta}$, we have

$$\begin{aligned} &\left(-\beta^{\frac{1}{2}} \partial^2_{xx} u_{\varepsilon,\beta} - A\beta\varepsilon \partial^3_{txx} u_{\varepsilon,\beta} + B\varepsilon \partial_t u_{\varepsilon,\beta}\right) \partial_t u_{\varepsilon,\beta} \\ &\qquad +2\left(-\beta^{\frac{1}{2}} \partial^2_{xx} u_{\varepsilon,\beta} - A\beta\varepsilon \partial^3_{txx} u_{\varepsilon,\beta} + B\varepsilon \partial_t u_{\varepsilon,\beta}\right) u_{\varepsilon,\beta} \partial_x u_{\varepsilon,\beta} \\ &\qquad +\beta\left(-\beta^{\frac{1}{2}} \partial^2_{xx} u_{\varepsilon,\beta} - A\beta\varepsilon \partial^3_{txx} u_{\varepsilon,\beta} + B\varepsilon \partial_t u_{\varepsilon,\beta}\right) \partial^3_{xxx} u_{\varepsilon,\beta} \\ &\qquad +\beta^2\left(-\beta^{\frac{1}{2}} \partial^2_{xx} u_{\varepsilon,\beta} - A\beta\varepsilon \partial^3_{txx} u_{\varepsilon,\beta} + B\varepsilon \partial_t u_{\varepsilon,\beta}\right) \partial^5_{txxxx} u_{\varepsilon,\beta} \\ &\quad = \varepsilon\left(-\beta^{\frac{1}{2}} \partial^2_{xx} u_{\varepsilon,\beta} - A\beta\varepsilon \partial^3_{txx} u_{\varepsilon,\beta} + B\varepsilon \partial_t u_{\varepsilon,\beta}\right) \partial^2_{xx} u_{\varepsilon,\beta}. \end{aligned} \tag{25}$$

We observe that

$$\begin{aligned}
&\int_{\mathbf{R}} \left(-\beta^{\frac{1}{2}}\partial_{xx}^2 u_{\varepsilon,\beta} - A\beta\varepsilon\partial_{txx}^3 u_{\varepsilon,\beta} + B\varepsilon\partial_t u_{\varepsilon,\beta}\right)\partial_t u_{\varepsilon,\beta}dx\\
&\quad = \tfrac{\beta^{\frac{1}{2}}}{2}\tfrac{d}{dt}\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})} + \beta\varepsilon A\left\|\partial_{tx}^2 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\
&\qquad + B\varepsilon\left\|\partial_t u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}.
\end{aligned} \tag{26}$$

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$$\begin{aligned}
&2\int_{\mathbf{R}} \left(-\beta^{\frac{1}{2}}\partial_{xx}^2 u_{\varepsilon,\beta} - A\beta\varepsilon\partial_{txx}^3 u_{\varepsilon,\beta} + B\varepsilon\partial_t u_{\varepsilon,\beta}\right)u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}dx\\
&\quad = -2\beta^{\frac{1}{2}}\int_{\mathbf{R}} u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}\partial_{xx}^2 u_{\varepsilon,\beta}dx - 2A\beta\varepsilon\int_{\mathbf{R}} u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}\partial_{txx}^3 u_{\varepsilon,\beta}dx\\
&\qquad + 2B\varepsilon\int_{\mathbf{R}} u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}\partial_t u_{\varepsilon,\beta}dx,\\
&\beta\int_{\mathbf{R}} \left(-\beta^{\frac{1}{2}}\partial_{xx}^2 u_{\varepsilon,\beta} - A\beta\varepsilon\partial_{txx}^3 u_{\varepsilon,\beta} + B\varepsilon\partial_t u_{\varepsilon,\beta}\right)\partial_{xxx}^3 u_{\varepsilon,\beta}dx\\
&\quad = A\beta^2\varepsilon\int_{\mathbf{R}}\partial_{xx}^2 u_{\varepsilon,\beta}\partial_{txxx}^4 u_{\varepsilon,\beta}dx + B\beta\varepsilon\int_{\mathbf{R}}\partial_x u_{\varepsilon,\beta}\partial_{txx}^3 u_{\varepsilon,\beta}dx,\\
&\beta^2\int_{\mathbf{R}} \left(-\beta^{\frac{1}{2}}\partial_{xx}^2 u_{\varepsilon,\beta} - A\beta\varepsilon\partial_{txx}^3 u_{\varepsilon,\beta} + B\varepsilon\partial_t u_{\varepsilon,\beta}\right)\partial_{txxxx}^5 u_{\varepsilon,\beta}dx\\
&\quad = \tfrac{\beta^{\frac{5}{2}}}{2}\tfrac{d}{dt}\left\|\partial_{xxx}^3 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})} + A\beta^3\varepsilon\left\|\partial_{txxx}^4 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\
&\qquad + B\beta^2\varepsilon\left\|\partial_{txx}^3 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})},\\
&\varepsilon\int_{\mathbf{R}} \left(-\beta^{\frac{1}{2}}\partial_{xx}^2 u_{\varepsilon,\beta} - A\beta\varepsilon\partial_{txx}^3 u_{\varepsilon,\beta} + B\varepsilon\partial_t u_{\varepsilon,\beta}\right)\partial_{xx}^2 u_{\varepsilon,\beta}dx\\
&\quad = -\beta^{\frac{1}{2}}\varepsilon\left\|\partial_{xx}^2 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})} - \tfrac{A\beta\varepsilon^2}{2}\tfrac{d}{dt}\left\|\partial_{xx}^2 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\
&\qquad - \tfrac{B\varepsilon^2}{2}\tfrac{d}{dt}\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})},
\end{aligned} \tag{27}$$

an integration on $\mathbf{R}$ of (26) gives

$$\begin{aligned}
&\tfrac{d}{dt}\left(\tfrac{\beta^{\frac{1}{2}}+B\varepsilon^2}{2}\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})} + \tfrac{A\beta\varepsilon^2}{2}\left\|\partial_{xx}^2 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\right)\\
&\qquad + \tfrac{\beta^{\frac{5}{2}}}{2}\tfrac{d}{dt}\left\|\partial_{xxx}^3 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})} + \beta\varepsilon A\left\|\partial_{tx}^2 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\
&\qquad + B\varepsilon\left\|\partial_t u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})} + A\beta^3\varepsilon\left\|\partial_{txxx}^4 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\
&\qquad + B\beta^2\varepsilon\left\|\partial_{txx}^3 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})} + \beta^{\frac{1}{2}}\varepsilon\left\|\partial_{xx}^2 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\
&\quad = 2\beta^{\frac{1}{2}}\int_{\mathbf{R}} u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}\partial_{xx}^2 u_{\varepsilon,\beta}dx + 2A\beta\varepsilon\int_{\mathbf{R}} u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}\partial_{txx}^3 u_{\varepsilon,\beta}dx\\
&\qquad - 2B\varepsilon\int_{\mathbf{R}} u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}\partial_t u_{\varepsilon,\beta}dx - A\beta^2\varepsilon\int_{\mathbf{R}}\partial_{xx}^2 u_{\varepsilon,\beta}\partial_{txxx}^4 u_{\varepsilon,\beta}dx\\
&\qquad - B\beta\varepsilon\int_{\mathbf{R}}\partial_x u_{\varepsilon,\beta}\partial_{txx}^3 u_{\varepsilon,\beta}dx.
\end{aligned} \tag{28}$$

Using (19), $0<\beta<1$, and the Young inequality,

$$\begin{aligned}
&2\beta^{\frac{1}{2}}\int_{\mathbf{R}}|u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}||\partial_{xx}^2 u_{\varepsilon,\beta}|dx = \beta^{\frac{1}{2}}\int_{\mathbf{R}}\left|\frac{2u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}}{\varepsilon^{\frac{1}{2}}}\right|\left|\varepsilon^{\frac{1}{2}}\partial_{xx}^2 u_{\varepsilon,\beta}(t,\cdot)\right|dx\\
&\quad\le \frac{2\beta^{\frac{1}{2}}}{\varepsilon}\int_{\mathbf{R}}u_{\varepsilon,\beta}^2(\partial_x u_{\varepsilon,\beta})^2dx + \frac{\beta^{\frac{1}{2}}\varepsilon}{2}\left\|\partial_{xx}^2 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\
&\quad\le C_0\varepsilon\left\|u_{\varepsilon,\beta}\right\|^2_{L^2((0,T)\times\mathbf{R})}\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})} + \frac{\beta^{\frac{1}{2}}\varepsilon}{2}\left\|\partial_{xx}^2 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})},\\
&2A\beta\varepsilon\int_{\mathbf{R}}|u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}||\partial_{txx}^3 u_{\varepsilon,\beta}|dx = \varepsilon\int_{\mathbf{R}}\left|\frac{2Au_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}}{\sqrt{B}}\right|\left|\sqrt{B}\beta\partial_{txx}^3 u_{\varepsilon,\beta}\right|dx\\
&\quad\le \frac{2A^2\varepsilon}{B}\int_{\mathbf{R}}u_{\varepsilon,\beta}^2(\partial_x u_{\varepsilon,\beta})^2dx + \frac{B\beta^2\varepsilon}{2}\left\|\partial_{txx}^3 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\
&\quad\le \frac{2A^2\varepsilon}{B}\left\|u_{\varepsilon,\beta}\right\|^2_{L^2((0,T)\times\mathbf{R})}\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})} + \frac{B\beta^2\varepsilon}{2}\left\|\partial_{txx}^3 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})},\\
&2B\varepsilon\int_{\mathbf{R}}|u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}||\partial_t u_{\varepsilon,\beta}|dx = B\varepsilon\int_{\mathbf{R}}|2u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}|\left|\partial_t u_{\varepsilon,\beta}\right|dx\\
&\quad\le 2B\varepsilon\int_{\mathbf{R}}u_{\varepsilon,\beta}^2(\partial_x u_{\varepsilon,\beta})^2dx + \frac{B\varepsilon}{2}\left\|\partial_t u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\
&\quad\le 2B\varepsilon\left\|u_{\varepsilon,\beta}\right\|^2_{L^2((0,T)\times\mathbf{R})}\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})} + \frac{B\varepsilon}{2}\left\|\partial_t u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})},\\
&A\beta^2\varepsilon\int_{\mathbf{R}}|\partial_{xx}^2 u_{\varepsilon,\beta}||\partial_{txxx}^4 u_{\varepsilon,\beta}|dx = A\varepsilon\int_{\mathbf{R}}\left|\beta^{\frac{1}{2}}\partial_{xx}^2 u_{\varepsilon,\beta}\right|\left|\beta^{\frac{3}{2}}\partial_{txxx}^4 u_{\varepsilon,\beta}\right|dx\\
&\quad\le \frac{A\beta\varepsilon}{2}\left\|\partial_{xx}^2 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})} + \frac{A\beta^3\varepsilon}{2}\left\|\partial_{txxx}^4 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\
&\quad\le \frac{A\beta^{\frac{1}{2}}\varepsilon}{2}\left\|\partial_{xx}^2 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})} + \frac{A\beta^3\varepsilon}{2}\left\|\partial_{txxx}^4 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})},\\
&B\beta\varepsilon\int_{\mathbf{R}}\partial_x u_{\varepsilon,\beta}\partial_{txx}^3 u_{\varepsilon,\beta}dx = \varepsilon\int_{\mathbf{R}}|\partial_x u_{\varepsilon,\beta}|\left|B\beta\partial_{txx}^3 u_{\varepsilon,\beta}\right|dx\\
&\quad\le \frac{\varepsilon}{2}\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})} + \frac{B^2\beta^2\varepsilon}{2}\left\|\partial_{txx}^3 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}.
\end{aligned}\tag{29}$$

Therefore, (28) gives

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{\beta^{\frac{1}{2}}+B\varepsilon^2}{2}\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})} + \frac{A\beta\varepsilon^2}{2}\left\|\partial_{xx}^2 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\right)\\
&\qquad+\frac{\beta^{\frac{5}{2}}}{2}\frac{d}{dt}\left\|\partial_{xxx}^3 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})} + \beta\varepsilon A\left\|\partial_{tx}^2 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\
&\qquad+\frac{B\varepsilon}{2}\left\|\partial_t u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})} + \frac{A\beta^3\varepsilon}{2}\left\|\partial_{txxx}^4 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\
&\qquad+\frac{B}{2}\beta^2\varepsilon\left(1-B\right)\left\|\partial_{txx}^3 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})} + \frac{\beta^{\frac{1}{2}}\varepsilon}{2}\left(1-A\right)\left\|\partial_{xx}^2 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\
&\quad\le C_0\varepsilon\left\|u_{\varepsilon,\beta}\right\|^2_{L^2((0,T)\times\mathbf{R})}\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})} + \frac{\varepsilon}{2}\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\
&\qquad+\frac{2A^2\varepsilon}{B}\left\|u_{\varepsilon,\beta}\right\|^2_{L^2((0,T)\times\mathbf{R})}\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\
&\qquad+2B\varepsilon\left\|u_{\varepsilon,\beta}\right\|^2_{L^2((0,T)\times\mathbf{R})}\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}.
\end{aligned}\tag{30}$$

Choosing $A=\dfrac{1}{2}$, $B=\dfrac{1}{2}$, from (30), we have

$$\begin{split}
&\tfrac{d}{dt}\left(\tfrac{2\beta^{\frac{1}{2}}+\varepsilon^2}{4}\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}+\tfrac{\beta\varepsilon^2}{4}\left\|\partial^2_{xx} u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\right)\\
&\qquad+\tfrac{\beta^{\frac{5}{2}}}{2}\tfrac{d}{dt}\left\|\partial^3_{xxx} u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}+\tfrac{\beta\varepsilon}{2}\left\|\partial^2_{tx} u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\
&\qquad+\tfrac{\varepsilon}{4}\left\|\partial_t u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}+\tfrac{\beta^3\varepsilon}{4}\left\|\partial^4_{txxx} u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\
&\qquad+\tfrac{\beta^2\varepsilon}{8}\left\|\partial^3_{txx} u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}+\tfrac{\beta^{\frac{1}{2}}\varepsilon}{4}\left\|\partial^2_{xx} u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\
&\quad\le C_0\varepsilon\left\|u_{\varepsilon,\beta}\right\|^2_{L^2((0,T)\times\mathbf{R})}\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}+\tfrac{\varepsilon}{2}\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}.
\end{split}\tag{31}$$

(19), (22), and an integration on $(0,t)$ give

$$\begin{split}
&\tfrac{2\beta^{\frac{1}{2}}+\varepsilon^2}{4}\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}+\tfrac{\beta\varepsilon^2}{4}\left\|\partial^2_{xx} u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\
&\qquad+\tfrac{\beta^{\frac{5}{2}}}{2}\left\|\partial^3_{xxx} u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}+\tfrac{\beta\varepsilon}{2}\int_0^t\left\|\partial^2_{tx} u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbf{R})}ds\\
&\qquad+\tfrac{\varepsilon}{4}\int_0^t\left\|\partial_t u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbf{R})}ds+\tfrac{\beta^3\varepsilon}{4}\int_0^t\left\|\partial^4_{txxx} u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbf{R})}ds\\
&\qquad+\tfrac{\beta^2\varepsilon}{8}\int_0^t\left\|\partial^3_{txx} u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbf{R})}ds+\tfrac{\beta^{\frac{1}{2}}\varepsilon}{4}\int_0^t\left\|\partial^2_{xx} u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbf{R})}ds\\
&\quad\le C_0+C_0\varepsilon\left\|u_{\varepsilon,\beta}\right\|^2_{L^2((0,T)\times\mathbf{R})}\int_0^t\left\|\partial_x u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbf{R})}ds\\
&\qquad+\tfrac{\varepsilon}{2}\int_0^t\left\|\partial_x u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbf{R})}ds\\
&\quad\le C_0\left(1+\left\|u_{\varepsilon,\beta}\right\|^2_{L^2((0,T)\times\mathbf{R})}\right).
\end{split}\tag{32}$$

We prove (24). Due to (22), (32), and the Hölder inequality,

$$\begin{split}
u^2_{\varepsilon,\beta}(t,x)&=2\int_{-\infty}^x u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}dx\le 2\int_{\mathbf{R}}|u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}|dx\\
&\le\left\|u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbf{R})}\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbf{R})}\\
&\le\tfrac{C_0}{\beta^{\frac{1}{4}}}\sqrt{1+\left\|u_{\varepsilon,\beta}\right\|^2_{L^2((0,T)\times\mathbf{R})}},
\end{split}\tag{33}$$

that is

$$\left\|u_{\varepsilon,\beta}\right\|^4_{L^\infty((0,T)\times\mathbf{R})}\le\frac{C_0}{\beta^{\frac{1}{2}}}\left(1+\left\|u_{\varepsilon,\beta}\right\|^2_{L^2((0,T)\times\mathbf{R})}\right).\tag{34}$$

Arguing as [6, Lemma 2.2], we have (24).

It follows from (24) and (32) that

$$\begin{aligned}
&\tfrac{2\beta^{\frac{1}{2}}+\varepsilon^2}{4}\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}+\tfrac{\beta\varepsilon^2}{4}\left\|\partial^2_{xx} u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\
&\quad+\tfrac{\beta^{\frac{5}{2}}}{2}\left\|\partial^3_{xxx} u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}+\tfrac{\beta\varepsilon}{2}\int_0^t\left\|\partial^2_{tx} u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbf{R})}ds\\
&\quad+\tfrac{\varepsilon}{4}\int_0^t\left\|\partial_t u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbf{R})}ds+\tfrac{\beta^3\varepsilon}{4}\int_{\mathbf{R}}\left\|\partial^4_{txxx} u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbf{R})}ds\\
&\quad+\tfrac{\beta^2\varepsilon}{8}\int_0^t\left\|\partial^3_{txx} u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbf{R})}ds+\tfrac{\beta^{\frac{1}{2}}\varepsilon}{4}\int_0^t\left\|\partial^2_{xx} u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}ds\\
&\quad\le C_0\beta^{-\frac{1}{2}},
\end{aligned}\tag{35}$$

that is,

$$\begin{aligned}
&\tfrac{2\beta+\beta^{\frac{1}{2}}\varepsilon^2}{4}\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}+\tfrac{\beta^{\frac{3}{2}}\varepsilon^2}{4}\left\|\partial^2_{xx} u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\
&\quad+\tfrac{\beta^3}{2}\left\|\partial^3_{xxx} u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}+\tfrac{\beta^{\frac{3}{2}}\varepsilon}{2}\int_0^t\left\|\partial^2_{tx} u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbf{R})}ds\\
&\quad+\tfrac{\beta^{\frac{1}{2}}\varepsilon}{4}\int_0^t\left\|\partial_t u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbf{R})}ds+\tfrac{\beta^{\frac{7}{2}}\varepsilon}{4}\int_0^t\left\|\partial^4_{txxx} u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbf{R})}ds\\
&\quad+\tfrac{\beta^{\frac{5}{2}}\varepsilon}{8}\int_0^t\left\|\partial^3_{txx} u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbf{R})}ds+\tfrac{\beta\varepsilon}{4}\int_0^t\left\|\partial^2_{xx} u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}ds\le C_0.
\end{aligned}\tag{36}$$

Hence,

$$\begin{aligned}
\beta^{\frac{1}{2}}\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbf{R})}\le&\quad C_0,\\
\beta^{\frac{1}{4}}\varepsilon\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbf{R})}\le&\quad C_0,\\
\beta^{\frac{3}{4}}\varepsilon\left\|\partial^2_{xx} u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbf{R})}\le&\quad C_0,\\
\beta^{\frac{3}{2}}\left\|\partial^3_{xxx} u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbf{R})}\le&\quad C_0,\\
\beta^{\frac{3}{2}}\varepsilon\int_0^t\left\|\partial^2_{tx} u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbf{R})}ds\le&\quad C_0,\\
\beta^{\frac{1}{2}}\varepsilon\int_0^t\left\|\partial_t u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbf{R})}ds\le&\quad C_0,\\
\beta^{\frac{7}{2}}\varepsilon\int_{\mathbf{R}}\left\|\partial^4_{txxx} u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbf{R})}ds\le&\quad C_0,\\
\beta^{\frac{5}{2}}\varepsilon\int_0^t\left\|\partial^3_{txx} u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbf{R})}ds\le&\quad C_0,\\
\beta\varepsilon\int_0^t\left\|\partial^2_{xx} u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}ds\le&\quad C_0,
\end{aligned}\tag{37}$$

for every $0\le t\le T$. $\diamond$ To prove Theorem 2.1, the following technical lemma is needed [39].

Lemma 2.3. *Let Ω be a bounded open subset of $\mathbf{R}^2$. Suppose that the sequence $\{\mathcal{L}_n\}_{n\in\mathbf{N}}$ of distributions is bounded in $W^{-1,\infty}(\Omega)$. Suppose also that*

$$\mathcal{L}_n=\mathcal{L}_{1,n}+\mathcal{L}_{2,n},\tag{38}$$

where $\{\mathcal{L}_{1,n}\}_{n\in\mathbf{N}}$ lies in a compact subset of $H^{-1}_{loc}(\Omega)$ and $\{\mathcal{L}_{2,n}\}_{n\in\mathbf{N}}$ lies in a bounded subset of $\mathcal{M}_{loc}(\Omega)$. Then $\{\mathcal{L}_n\}_{n\in\mathbf{N}}$ lies in a compact subset of $H^{-1}_{loc}(\Omega)$.

Moreover, we consider the following definition.

Definition 2.1. *A pair of functions (η, q) is called an entropy–entropy flux pair if*
$\eta : \mathbf{R} \to \mathbf{R}$ is a C^2 function and $q : \mathbf{R} \to \mathbf{R}$ is defined by

$$q(u) = 2\int_0^u \xi\eta'(\xi)d\xi. \tag{39}$$

An entropy-entropy flux pair $(\eta,\, q)$ is called convex/compactly supported if, in addition, η is convex/compactly supported.

Following [4], we prove Theorem 2.1. **Proof.**[Proof of Theorem 2.1.] Let us consider a compactly supported entropy–entropy flux pair (η, q). Multiplying (18) by $\eta'(u_{\varepsilon,\beta})$, we have

$$\begin{aligned}
&\partial_t\eta(u_{\varepsilon,\beta}) + \partial_x q(u_{\varepsilon,\beta})\\
&\quad = \varepsilon\eta'(u_{\varepsilon,\beta})\partial^2_{xx}u_{\varepsilon,\beta} - \beta\partial^3_{xxx}u_{\varepsilon,\beta} - \beta^2\eta'(u_{\varepsilon,\beta})\partial^5_{txxxx}u_{\varepsilon,\beta}\\
&\quad = I_{1,\,\varepsilon,\,\beta} + I_{2,\,\varepsilon,\,\beta} + I_{3,\,\varepsilon,\,\beta} + I_{4,\,\varepsilon,\,\beta} + I_{5,\,\varepsilon,\,\beta} + I_{6,\,\varepsilon,\,\beta},
\end{aligned} \tag{40}$$

where

$$\begin{aligned}
I_{1,\,\varepsilon,\,\beta} &= \partial_x(\varepsilon\eta'(u_{\varepsilon,\beta})\partial_x u_{\varepsilon,\beta}),\\
I_{2,\,\varepsilon,\,\beta} &= -\varepsilon\eta''(u_{\varepsilon,\beta})(\partial_x u_{\varepsilon,\beta})^2,\\
I_{3,\,\varepsilon,\,\beta} &= -\partial_x(\beta\eta'(u_{\varepsilon,\beta})\partial^2_{xx}u_{\varepsilon,\beta}),\\
I_{4,\,\varepsilon,\,\beta} &= \beta\eta''(u_{\varepsilon,\beta})\partial_x u_{\varepsilon,\beta}\partial^2_{xx}u_{\varepsilon,\beta},\\
I_{5,\,\varepsilon,\,\beta} &= -\partial_x(\beta^2\eta'(u_{\varepsilon,\beta})\partial^4_{txxx}u_{\varepsilon,\beta}),\\
I_{6,\,\varepsilon,\,\beta} &= \beta^2\eta''(u_{\varepsilon,\beta})\partial_x u_{\varepsilon,\beta}\partial^4_{txxx}u_{\varepsilon,\beta}.
\end{aligned} \tag{41}$$

Fix $T > 0$. Arguing as in [13, Lemma 3.2], we have that $I_{1,\,\varepsilon,\,\beta} \to 0$ in $H^{-1}((0,T)\times\mathbf{R})$, and $\{I_{2,\,\varepsilon,\,\beta}\}_{\varepsilon,\beta>0}$ is bounded in $L^1((0,T)\times\mathbf{R})$. Arguing as in [6, Theorem B.1], $I_{3,\,\varepsilon,\,\beta} \to 0$ in $H^{-1}((0,T)\times\mathbf{R})$, and $I_{4,\,\varepsilon,\,\beta} \to 0$ in $L^1((0,T)\times\mathbf{R})$, while Arguing as in [6, Lemma 2.4], $I_{5,\,\varepsilon,\,\beta} \to 0$ in $H^{-1}((0,T)\times\mathbf{R})$, and $\{I_{6,\,\varepsilon,\,\beta}\}_{\varepsilon,\beta>0}$ is bounded in $L^1((0,T)\times\mathbf{R})$.

Therefore, $i)$ follows from Lemmas 2.1, 2.3 and the L^p compensated compactness of [5].

Arguing as in [19, Theorem 2.1], we have $ii)$. $\diamond$

3. The Rosenau-KdV equation: $u_0 \in L^2(\mathbf{R}) \cap L^4(\mathbf{R})$.

In this section, we study the convergence of (17), and assume (12) on the initial datum. More precisely, we consider (18), where $u_{\varepsilon,\beta,0}$ is a C^∞ approximation of u_0 such that

$$\begin{aligned}
&u_{\varepsilon,\,\beta,\,0} \to u_0 \quad \text{in } L^p_{loc}(\mathbf{R}),\ 1 \le p < 2, \text{ as } \varepsilon,\ \beta \to 0,\\
&\|u_{\varepsilon,\beta,0}\|^4_{L^4(\mathbf{R})} + \|u_{\varepsilon,\beta,0}\|^2_{L^2(\mathbf{R})} + \left(\beta^{\frac{1}{2}} + \varepsilon^2\right)\|\partial_x u_{\varepsilon,\beta,0}\|^2_{L^2(\mathbf{R})} \le C_0,\\
&\left(\beta^2 + \beta\varepsilon^2\right)\left\|\partial^2_{xx} u_{\varepsilon,\beta,0}\right\|^2_{L^2(\mathbf{R})} + \left(\beta^{\frac{5}{2}} + \beta^2\varepsilon^2\right)\left\|\partial^3_{xxx} u_{\varepsilon,\beta,0}\right\|^2_{L^2(\mathbf{R})} \le C_0,\\
&\beta^4\left\|\partial^4_{xxxx} u_{\varepsilon,\beta,0}\right\|^2_{L^2(\mathbf{R})} \le C_0,
\end{aligned} \tag{42}$$

for every $\varepsilon, \beta > 0$ and C_0 is a constant independent on ε and β.

The main result of this section is the following theorem.

Theorem 3.1. *Assume that (12) and (42) hold. Fix $T > 0$, if*

$$\beta = \mathcal{O}\left(\varepsilon^4\right), \tag{43}$$

there exist two sequences $\{\varepsilon_n\}_{n\in\mathbf{N}}$, $\{\beta_n\}_{n\in\mathbf{N}}$, with $\varepsilon_n, \beta_n \to 0$, and a limit function

$$u \in L^\infty((0,T); L^2(\mathbf{R}) \cap L^4(\mathbf{R})), \tag{44}$$

such that

i) $u_{\varepsilon_n,\beta_n} \to u$ strongly in $L^p_{loc}((0,T)\times\mathbf{R})$, for each $1 \le p < 4$,

ii) u is the unique entropy solution of (2).

Let us prove some a priori estimates on $u_{\varepsilon,\beta}$, denoting with C_0 the constants which depend only on the initial data.

Lemma 3.1. *Fix $T > 0$. Assume (43) holds. There exists $C_0 > 0$, independent on ε, β such that (24) holds. In particular, we have*

$$\begin{aligned}
\beta\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbf{R})} \quad &+\beta^3\left\|\partial^3_{xxx} u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\
&+\frac{3\beta\varepsilon}{2}\int_0^t \left\|\partial^2_{xx} u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbf{R})} ds \le C_0,
\end{aligned} \tag{45}$$

for every $0 < t < T$. Moreover,

$$\|\partial_x u_{\varepsilon,\beta}\|_{L^\infty((0,T)\times\mathbf{R})} \le C_0\beta^{-\frac{3}{4}}. \tag{46}$$

Remark 3.1. *Observe that the proof of Lemma 3.1 is simpler than the one of Lemma 2.2. Indeed, we only need to prove (24).*

Proof.[Proof of Lemma 3.1.] Let $0 < t < T$. Multiplying (18) by $-\beta^{\frac{1}{2}}\partial_{xx}^2 u_{\varepsilon,\beta}$, we have

$$\begin{aligned} -\beta^{\frac{1}{2}}\partial_{xx}^2 u_{\varepsilon,\beta}\partial_t u_{\varepsilon,\beta} & -2\beta^{\frac{1}{2}} u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}\partial_{xx}^2 u_{\varepsilon,\beta} \\ & +\beta^{\frac{3}{2}}\partial_{xx}^2 u_{\varepsilon,\beta}\partial_{xxx}^3 u_{\varepsilon,\beta} - \beta^{\frac{5}{2}}\partial_{txxxx}^5 u_{\varepsilon,\beta}\partial_{xx}^2 u_{\varepsilon,\beta} \\ = & -\beta^{\frac{1}{2}}\varepsilon(\partial_{xx}^2 u_{\varepsilon,\beta})^2. \end{aligned} \tag{47}$$

We note that

$$\beta^{\frac{3}{2}}\int_{\mathbf{R}}\partial_{xx}^2 u_{\varepsilon,\beta}\partial_{xxx}^3 u_{\varepsilon,\beta} dx = 0. \tag{48}$$

Therefore, arguing as [6, Lemma 3.1], we have (24), (45) and (46). ⋄ Following [40, Lemma 2.2], or [41, Lemma 4.2], we prove the following result.

Lemma 3.2. *Fix $T > 0$. Assume (43) holds. Then:*

i) the family $\{u_{\varepsilon,\beta}\}_{\varepsilon,\,\beta}$ is bounded in $L^\infty((0,T);L^4(\mathbf{R}))$;

ii) the families $\{\varepsilon\partial_x u_{\varepsilon,\beta}\}_{\varepsilon,\beta}$, $\{\beta^{\frac{1}{2}}\varepsilon\partial_{xx}^2 u_{\varepsilon,\beta}\}_{\varepsilon,\beta}$, $\{\beta\partial_{xx}^2 u_{\varepsilon,\beta}\}_{\varepsilon,\beta}$, $\{\beta\varepsilon\partial_{xxx}^3 u_{\varepsilon,\beta}\}_{\varepsilon,\beta}$, $\{\beta\partial_{xxxx}^4 u_{\varepsilon,\beta}\}_{\varepsilon,\beta}$are bounded in $L^\infty((0,T);L^2(\mathbf{R}))$;

iii) the families $\{\beta^{\frac{1}{2}}\varepsilon^{\frac{1}{2}}\partial_{tx}^2 u_{\varepsilon,\beta}\}_{\varepsilon,\,\beta}$, $\{\varepsilon^{\frac{1}{2}}\partial_t u_{\varepsilon,\beta}\}_{\varepsilon,\,\beta}$, $\{\beta^{\frac{3}{2}}\varepsilon^{\frac{1}{2}}\partial_{txxx}^4 u_{\varepsilon,\beta}\}_{\varepsilon,\,\beta}$, $\{\beta\varepsilon^{\frac{1}{2}}\partial_{txx}^3 u_{\varepsilon,\beta}\}_{\varepsilon,\,\beta}$, $\{\varepsilon^{\frac{1}{2}} u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}\}_{\varepsilon,\,\beta}\{\varepsilon^{\frac{3}{2}}\partial_{xx}^2 u_{\varepsilon,\beta}\}_{\varepsilon,\,\beta}$, $\{\beta\varepsilon^{\frac{1}{2}}\partial_{xxx}^3 u_{\varepsilon,\beta}\}_{\varepsilon,\,\beta}$, are bounded in $L^2((0,T)\times\mathbf{R})$;

Proof. Let $0 < t < T$. Let A, B, C, E be some positive constants which will be specified later. Multiplying (18) by

$$u_{\varepsilon,\beta}^3 - A\varepsilon^2\partial_{xx}^2 u_{\varepsilon,\beta} - B\beta\varepsilon\partial_{txx}^3 u_{\varepsilon,\beta} + C\varepsilon\partial_t u_{\varepsilon,\beta} + E\beta^2\partial_{xxxx}^4 u_{\varepsilon,\beta}, \tag{49}$$

we have

$$
\begin{aligned}
&\left(u^3_{\varepsilon,\beta} - A\varepsilon^2\partial^2_{xx}u_{\varepsilon,\beta} - B\beta\varepsilon\partial^3_{txx}u_{\varepsilon,\beta}\right)\partial_t u_{\varepsilon,\beta}\\
&\qquad + \left(C\varepsilon\partial_t u_{\varepsilon,\beta} + E\beta^2\partial^4_{xxxx}u_{\varepsilon,\beta}\right)\partial_t u_{\varepsilon,\beta}\\
&\qquad +2\left(u^3_{\varepsilon,\beta} - A\varepsilon^2\partial^2_{xx}u_{\varepsilon,\beta} - B\beta\varepsilon\partial^3_{txx}u_{\varepsilon,\beta}\right)u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}\\
&\qquad +2\left(C\varepsilon\partial_t u_{\varepsilon,\beta} + E\beta^2\partial^4_{xxxx}u_{\varepsilon,\beta}\right)u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}\\
&\qquad +\beta\left(u^3_{\varepsilon,\beta} - A\varepsilon^2\partial^2_{xx}u_{\varepsilon,\beta} - B\beta\varepsilon\partial^3_{txx}u_{\varepsilon,\beta}\right)\partial^3_{xxx}u_{\varepsilon,\beta}\\
&\qquad +\beta\left(C\varepsilon\partial_t u_{\varepsilon,\beta} + E\beta^2\partial^4_{xxxx}u_{\varepsilon,\beta}\right)\partial^3_{xxx}u_{\varepsilon,\beta}\\
&\qquad +\beta^2\left(u^3_{\varepsilon,\beta} - A\varepsilon^2\partial^2_{xx}u_{\varepsilon,\beta} - B\beta\varepsilon\partial^3_{txx}u_{\varepsilon,\beta}\right)\partial^5_{txxxx}u_{\varepsilon,\beta}\\
&\qquad +\beta^2\left(C\varepsilon\partial_t u_{\varepsilon,\beta} + E\beta^2\partial^4_{xxxx}u_{\varepsilon,\beta}\right)\partial^5_{txxxx}u_{\varepsilon,\beta}\\
&\quad = \varepsilon\left(u^3_{\varepsilon,\beta} - A\varepsilon^2\partial^2_{xx}u_{\varepsilon,\beta} - B\beta\varepsilon\partial^3_{txx}u_{\varepsilon,\beta}\right)\partial^2_{xx}u_{\varepsilon,\beta}\\
&\qquad +\varepsilon\left(C\varepsilon\partial_t u_{\varepsilon,\beta} + E\beta^2\partial^4_{xxxx}u_{\varepsilon,\beta}\right)\partial^2_{xx}u_{\varepsilon,\beta}.
\end{aligned}
\tag{50}
$$

Since

$$
\begin{aligned}
\int_{\mathbf{R}} &\left(u^3_{\varepsilon,\beta} - A\varepsilon^2\partial^2_{xx}u_{\varepsilon,\beta} - B\beta\varepsilon\partial^3_{txx}u_{\varepsilon,\beta}\right)\partial_t u_{\varepsilon,\beta}dx\\
= &\ \tfrac{1}{4}\tfrac{d}{dt}\left\|u_{\varepsilon,\beta}(t,\cdot)\right\|^4_{L^4(\mathbf{R})} + \tfrac{A\varepsilon^2}{2}\tfrac{d}{dt}\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\
&+B\beta\varepsilon\left\|\partial^2_{tx}u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})},\\
\int_{\mathbf{R}} &\left(C\varepsilon\partial_t u_{\varepsilon,\beta} + E\beta^2\partial^4_{xxxx}u_{\varepsilon,\beta}\right)\partial_t u_{\varepsilon,\beta}dx\\
= &\ C\varepsilon\left\|\partial_t u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})} + \tfrac{E\beta^2}{2}\tfrac{d}{dt}\left\|\partial^2_{xx}u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})},\\
2\int_{\mathbf{R}} &\left(u^3_{\varepsilon,\beta} - A\varepsilon^2\partial^2_{xx}u_{\varepsilon,\beta} - B\beta\varepsilon\partial^3_{txx}u_{\varepsilon,\beta}\right)u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}dx\\
= &\ -2A\varepsilon^2\int_{\mathbf{R}} u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}\partial^2_{xx}u_{\varepsilon,\beta}dx - 2B\beta\varepsilon\int_{\mathbf{R}} u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}\partial^3_{txx}u_{\varepsilon,\beta}dx,\\
2\int_{\mathbf{R}} &\left(C\varepsilon\partial_t u_{\varepsilon,\beta} + E\beta^2\partial^4_{xxxx}u_{\varepsilon,\beta}\right)u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}dx\\
= &\ 2C\int_{\mathbf{R}} u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}\partial_t u_{\varepsilon,\beta}dx - 2E\beta^2\int_{\mathbf{R}}(\partial_x u_{\varepsilon,\beta})^2\partial^3_{xxx}u_{\varepsilon,\beta}dx\\
&-2E\beta^2\int_{\mathbf{R}} u_{\varepsilon,\beta}\partial^2_{xx}u_{\varepsilon,\beta}\partial^3_{xxx}u_{\varepsilon,\beta}dx,\\
-2E\beta^2\int_{\mathbf{R}} &(\partial_x u_{\varepsilon,\beta})^2\partial^3_{xxx}u_{\varepsilon,\beta}dx - 2E\beta^2\int_{\mathbf{R}} u_{\varepsilon,\beta}\partial^2_{xx}u_{\varepsilon,\beta}\partial^3_{xxx}u_{\varepsilon,\beta}dx\\
= &\ 5E\beta^2\int_{\mathbf{R}}(\partial^2_{xx}u_{\varepsilon,\beta})^2\partial_x u_{\varepsilon,\beta}dx = -\tfrac{5E\beta^2}{2}\int_{\mathbf{R}}(\partial_x u_{\varepsilon,\beta})^2\partial^3_{xxx}u_{\varepsilon,\beta}dx,\\
2\int_{\mathbf{R}} &\left(C\varepsilon\partial_t u_{\varepsilon,\beta} + E\beta^2\partial^4_{xxxx}u_{\varepsilon,\beta}\right)u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}dx\\
= &\ 2C\varepsilon\int_{\mathbf{R}} u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}\partial_t u_{\varepsilon,\beta}dx - \tfrac{5E\beta^2}{2}\int_{\mathbf{R}}(\partial_x u_{\varepsilon,\beta})^2\partial^3_{xxx}u_{\varepsilon,\beta}dx,\\
\beta\int_{\mathbf{R}} &\left(u^3_{\varepsilon,\beta} - A\varepsilon^2\partial^2_{xx}u_{\varepsilon,\beta} - B\beta\varepsilon\partial^3_{txx}u_{\varepsilon,\beta}\right)\partial^3_{xxx}u_{\varepsilon,\beta}dx\\
= &\ -3\beta\int_{\mathbf{R}} u^2_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}\partial^2_{xx}u_{\varepsilon,\beta}dx - B\beta^2\varepsilon\int_{\mathbf{R}}\partial^3_{txx}u_{\varepsilon,\beta}\partial^3_{xxx}u_{\varepsilon,\beta}dx,\\
\beta\int_{\mathbf{R}} &\left(C\varepsilon\partial_t u_{\varepsilon,\beta} + E\beta^2\partial^4_{xxxx}u_{\varepsilon,\beta}\right)\partial^3_{xxx}u_{\varepsilon,\beta} = C\beta\varepsilon\int_{\mathbf{R}}\partial^3_{txx}u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}dx,
\end{aligned}
\tag{51}
$$

$$
\begin{aligned}
\beta^2\int_{\mathbf{R}} & \left(u^3_{\varepsilon,\beta}-A\varepsilon^2\partial^2_{xx}u_{\varepsilon,\beta}-B\beta\varepsilon\partial^3_{txx}u_{\varepsilon,\beta}\right)\partial^5_{txxxx}u_{\varepsilon,\beta}dx\\
= & -3\beta^2\int_{\mathbf{R}}u^2_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}\partial^4_{txxx}u_{\varepsilon,\beta}dx+\tfrac{A\beta^2\varepsilon^2}{2}\tfrac{d}{dt}\left\|\partial^3_{xxx}u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\
& +B\beta^3\varepsilon\left\|\partial^4_{txxx}u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})},\\
\beta^2\int_{\mathbf{R}} & \left(C\varepsilon\partial_t u_{\varepsilon,\beta}+E\beta^2\partial^4_{xxxx}u_{\varepsilon,\beta}\right)\partial^5_{txxxx}u_{\varepsilon,\beta}dx\\
= & \ C\beta^2\varepsilon\left\|\partial^3_{txx}u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}+\tfrac{E\beta^4}{2}\tfrac{d}{dt}\left\|\partial^4_{xxxx}u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})},\\
\varepsilon\int_{\mathbf{R}} & \left(u^3_{\varepsilon,\beta}-A\varepsilon^2\partial^2_{xx}u_{\varepsilon,\beta}-B\beta\varepsilon\partial^3_{txx}u_{\varepsilon,\beta}\right)\partial^2_{xx}u_{\varepsilon,\beta}dx\\
= & -3\varepsilon\left\|u_{\varepsilon,\beta}(t,\cdot)\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}-A\varepsilon^3\left\|\partial^2_{xx}u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\
& -\tfrac{B\beta\varepsilon^2}{2}\tfrac{d}{dt}\left\|\partial^2_{xx}u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})},\\
\varepsilon\int_{\mathbf{R}} & \left(C\varepsilon\partial_t u_{\varepsilon,\beta}+E\beta^2\partial^4_{xxxx}u_{\varepsilon,\beta}\right)\partial^2_{xx}u_{\varepsilon,\beta}dx\\
= & -\tfrac{C\varepsilon^2}{2}\tfrac{d}{dt}\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}-E\beta^2\varepsilon\left\|\partial^3_{xxx}u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})},
\end{aligned}
\tag{52}
$$

an integration on $\mathbf{R}$ of (50) gives

$$
\begin{aligned}
&\tfrac{d}{dt}\left(\tfrac{1}{4}\left\|u_{\varepsilon,\beta}(t,\cdot)\right\|^4_{L^4(\mathbf{R})}+\tfrac{(A+C)\varepsilon^2}{2}\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\right)\\
&\quad+\tfrac{d}{dt}\left(\tfrac{A\beta^2\varepsilon^2}{2}\left\|\partial^3_{xxx}u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}+\tfrac{E\beta^4}{2}\left\|\partial^4_{xxxx}u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\right)\\
&\quad+\tfrac{B\beta\varepsilon^2+E\beta^2}{2}\tfrac{d}{dt}\left\|\partial^2_{xx}u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}+B\beta\varepsilon\left\|\partial^2_{tx}u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\
&\quad+C\varepsilon\left\|\partial_t u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}+B\beta^3\varepsilon\left\|\partial^4_{txxx}u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\
&\quad+C\beta^2\varepsilon\left\|\partial^3_{txx}u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}+3\varepsilon\left\|u_{\varepsilon,\beta}(t,\cdot)\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\
&\quad+A\varepsilon^3\left\|\partial^2_{xx}u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}+E\beta^2\varepsilon\left\|\partial^3_{xxx}u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\
&=2A\varepsilon^2\int_{\mathbf{R}}u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}\partial^2_{xx}u_{\varepsilon,\beta}dx+2B\beta\varepsilon\int_{\mathbf{R}}u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}\partial^3_{txx}u_{\varepsilon,\beta}dx\\
&\quad+2C\varepsilon\int_{\mathbf{R}}u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}\partial_t u_{\varepsilon,\beta}dx+\tfrac{5E\beta^2}{2}\int_{\mathbf{R}}(\partial_x u_{\varepsilon,\beta})^2\partial^3_{xxx}u_{\varepsilon,\beta}dx\\
&\quad+3\beta\int_{\mathbf{R}}u^2_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}\partial^2_{xx}u_{\varepsilon,\beta}dx+B\beta^2\varepsilon\int_{\mathbf{R}}\partial^3_{txx}u_{\varepsilon,\beta}\partial^3_{xxx}u_{\varepsilon,\beta}dx\\
&\quad-C\beta\varepsilon\int_{\mathbf{R}}\partial^3_{txx}u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}dx+3\beta^2\int_{\mathbf{R}}u^2_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}\partial^4_{txxx}u_{\varepsilon,\beta}dx.
\end{aligned}
\tag{53}
$$

Due to the Young inequality,

$$\begin{aligned}
&2A\varepsilon^2\int_{\mathbf{R}}|u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}||\partial_{xx}^2 u_{\varepsilon,\beta}|dx=\int_{\mathbf{R}}\left|\varepsilon^{\frac{1}{2}}u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}\right|\left|2A\varepsilon^{\frac{3}{2}}\partial_{xx}^2 u_{\varepsilon,\beta}\right|dx\\
&\qquad\le\frac{\varepsilon}{2}\left\|u_{\varepsilon,\beta}(t,\cdot)\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}+2A^2\varepsilon^3\left\|\partial_{xx}^2 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})},\\
&2B\beta\varepsilon\int_{\mathbf{R}}u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}\partial_{txx}^3 u_{\varepsilon,\beta}dx=\varepsilon\int_{\mathbf{R}}|u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}|\left|2B\beta\partial_{txx}^3 u_{\varepsilon,\beta}\right|dx\\
&\qquad\frac{\varepsilon}{2}\left\|u_{\varepsilon,\beta}(t,\cdot)\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}+4B^2\beta^2\varepsilon\left\|\partial_{txx}^3 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})},\\
&2C\varepsilon\int_{\mathbf{R}}|u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}|\partial_t u_{\varepsilon,\beta}dx=\varepsilon\int_{\mathbf{R}}|u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}|\left|2C\partial_t u_{\varepsilon,\beta}\right|dx\\
&\qquad\le\frac{\varepsilon}{2}\left\|u_{\varepsilon,\beta}(t,\cdot)\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}+2C^2\varepsilon\left\|\partial_t u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})},\\
&B\beta^2\varepsilon\int_{\mathbf{R}}|\partial_{txx}^3 u_{\varepsilon,\beta}||\partial_{xxx}^3 u_{\varepsilon,\beta}|dx=\beta^2\varepsilon\int_{\mathbf{R}}\left|2B\partial_{txx}^3 u_{\varepsilon,\beta}\right|\left|\frac{\partial_{xxx}^3 u_{\varepsilon,\beta}}{2}\right|dx\\
&\qquad\le 4B^2\beta^2\varepsilon\left\|\partial_{txx}^3 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}+\frac{\beta^2\varepsilon}{2}\left\|\partial_{xxx}^3 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})},\\
&C\beta\varepsilon\int_{\mathbf{R}}|\partial_{txx}^3 u_{\varepsilon,\beta}||\partial_x u_{\varepsilon,\beta}|dx=C\varepsilon\int_{\mathbf{R}}\left|\beta\partial_{txx}^3 u_{\varepsilon,\beta}\right||\partial_x u_{\varepsilon,\beta}|dx\\
&\qquad\le\frac{C\beta^2\varepsilon}{2}\left\|\partial_{txx}^3 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}+\frac{C\varepsilon}{2}\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}.
\end{aligned}\tag{54}$$

Therefore, from (53), we have

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{4}\left\|u_{\varepsilon,\beta}(t,\cdot)\right\|^4_{L^4(\mathbf{R})}+\frac{(A+C)\varepsilon^2}{2}\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\right)\\
&\qquad+\frac{d}{dt}\left(\frac{A\beta^2\varepsilon^2}{2}\left\|\partial_{xxx}^3 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}+\frac{E\beta^4}{2}\left\|\partial_{xxxx}^4 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\right)\\
&\qquad+\frac{B\beta\varepsilon^2+E\beta^2}{2}\frac{d}{dt}\left\|\partial_{xx}^2 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}+B\beta\varepsilon\left\|\partial_{tx}^2 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\
&\qquad+(1-2C)\,C\varepsilon\left\|\partial_t u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}+B\beta^3\varepsilon\left\|\partial_{txxx}^4 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\
&\qquad+\left(\frac{C}{2}-8B^2\right)\beta^2\varepsilon\left\|\partial_{txx}^3 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}+\frac{3\varepsilon}{2}\left\|u_{\varepsilon,\beta}(t,\cdot)\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\
&\qquad+\left(A-2A^2\right)\varepsilon^3\left\|\partial_{xx}^2 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}+\left(E-\frac{1}{2}\right)\beta^2\varepsilon\left\|\partial_{xxx}^3 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\
&\quad\le\frac{5E\beta^2}{2}\int_{\mathbf{R}}(\partial_x u_{\varepsilon,\beta})^2|\partial_{xxx}^3 u_{\varepsilon,\beta}|dx+3\beta\int_{\mathbf{R}}u^2_{\varepsilon,\beta}|\partial_x u_{\varepsilon,\beta}||\partial_{xx}^2 u_{\varepsilon,\beta}|dx\\
&\qquad\le 3\beta^2\int_{\mathbf{R}}u^2_{\varepsilon,\beta}|\partial_x u_{\varepsilon,\beta}||\partial_{txxx}^4 u_{\varepsilon,\beta}|dx+\frac{C\varepsilon}{2}\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}.
\end{aligned}\tag{55}$$

From (43), we get

$$\beta\le D^2\varepsilon^4,\tag{56}$$

where D is a positive constant that will be specified later. It follows from (46), (56), and the Young inequality that

$$
\begin{aligned}
&\tfrac{5E\beta^2}{2}\int_{\mathbf{R}}(\partial_x u_{\varepsilon,\beta})^2|\partial_{xxx}^3 u_{\varepsilon,\beta}|dx = E\beta^2\int_{\mathbf{R}}\tfrac{5}{2\varepsilon^{\frac{1}{2}}}(\partial_x u_{\varepsilon,\beta})^2\left|\varepsilon^{\frac{1}{2}}\partial_{xxx}^3 u_{\varepsilon,\beta}\right|dx\\
&\quad\le \tfrac{25E\beta^2}{8}\varepsilon\int_{\mathbf{R}}(\partial_x u_{\varepsilon,\beta})^4dx + \tfrac{E\beta^2\varepsilon}{2}\left\|\partial_{xxx}^3 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\
&\quad\le \tfrac{25E\beta^2}{8\varepsilon}\left\|\partial_x u_{\varepsilon,\beta}\right\|^2_{L^\infty((0,T)\times\mathbf{R})}\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\
&\qquad+\tfrac{E\beta^2\varepsilon}{2}\left\|\partial_{xxx}^3 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\
&\quad\le \tfrac{C_0\beta^{\frac{1}{2}}}{\varepsilon}\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}+\tfrac{E\beta^2\varepsilon}{2}\left\|\partial_{xxx}^3 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\
&\quad\le C_0D\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}+\tfrac{E\beta^2\varepsilon}{2}\left\|\partial_{xxx}^3 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})},\\
&3\beta\int_{\mathbf{R}}u_{\varepsilon,\beta}^2|\partial_x u_{\varepsilon,\beta}||\partial_{xx}^2 u_{\varepsilon,\beta}|dx\le 3\beta\left\|u_{\varepsilon,\beta}\right\|^2_{L^\infty((0,T)\times\mathbf{R})}\int_{\mathbf{R}}|\partial_x u_{\varepsilon,\beta}||\partial_{xx}^2 u_{\varepsilon,\beta}|dx\\
&\quad\le 3C_0D\varepsilon^2\int_{\mathbf{R}}|\partial_x u_{\varepsilon,\beta}||\partial_{xx}^2 u_{\varepsilon,\beta}|dx = 3\int_{\mathbf{R}}\left|\varepsilon^{\frac{1}{2}}\partial_x u_{\varepsilon,\beta}\right|\left|C_0D\varepsilon^{\frac{3}{2}}\partial_{xx}^2 u_{\varepsilon,\beta}\right|dx\\
&\quad\le \tfrac{3\varepsilon}{2}\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}+C_0^2D^2\varepsilon^3\left\|\partial_{xx}^2 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})},\\
&3\beta^2\int_{\mathbf{R}}u_{\varepsilon,\beta}^2|\partial_x u_{\varepsilon,\beta}||\partial_{txxx}^4 u_{\varepsilon,\beta}|dx=\int_{\mathbf{R}}\left|\tfrac{3\beta^{\frac{1}{2}}u_{\varepsilon,\beta}^2\partial_x u_{\varepsilon,\beta}}{\sqrt{B}\varepsilon^{\frac{1}{2}}}\right|\left|\sqrt{B}\beta^{\frac{3}{2}}\varepsilon^{\frac{1}{2}}\partial_{txxx}^4 u_{\varepsilon,\beta}\right|dx\\
&\quad\le \tfrac{3\beta}{2B\varepsilon}\int_{\mathbf{R}}u_{\varepsilon,\beta}^4(\partial_x u_{\varepsilon,\beta})^2dx+\tfrac{B\beta^3\varepsilon}{2}\left\|\partial_{txxx}^4 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\
&\quad\le \tfrac{3\beta}{2B\varepsilon}\left\|u_{\varepsilon,\beta}\right\|^2_{L^\infty((0,T)\times\mathbf{R})}\left\|u_{\varepsilon,\beta}(t,\cdot)\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\
&\qquad+\tfrac{B\beta^3\varepsilon}{2}\left\|\partial_{txxx}^4 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\
&\quad\le \tfrac{C_0D\varepsilon}{B}\left\|u_{\varepsilon,\beta}(t,\cdot)\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}+\tfrac{B\beta^3\varepsilon}{2}\left\|\partial_{txxx}^4 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}.
\end{aligned}
\tag{57}
$$

Then, using (55)

$$
\begin{aligned}
&\tfrac{d}{dt}\left(\tfrac{1}{4}\left\|u_{\varepsilon,\beta}(t,\cdot)\right\|^4_{L^4(\mathbf{R})}+\tfrac{(A+C)\varepsilon^2}{2}\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\right)\\
&\qquad+\tfrac{d}{dt}\left(\tfrac{A\beta^2\varepsilon^2}{2}\left\|\partial_{xxx}^3 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}+\tfrac{E\beta^4}{2}\left\|\partial_{xxxx}^4 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\right)\\
&\qquad+\tfrac{B\beta\varepsilon^2+E\beta^2}{2}\tfrac{d}{dt}\left\|\partial_{xx}^2 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}+B\beta\varepsilon\left\|\partial_{tx}^2 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\
&\qquad+(1-2C)\,C\varepsilon\left\|\partial_t u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}+\tfrac{B\beta^3\varepsilon}{2}\left\|\partial_{txxx}^4 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\
&\qquad+\left(\tfrac{C}{2}-8B^2\right)\beta^2\varepsilon\left\|\partial_{txx}^3 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}+(E-1)\,\tfrac{\beta^2\varepsilon}{2}\left\|\partial_{xxx}^3 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\
&\qquad+\left(A-2A^2-C_0^2D^2\right)\varepsilon^3\left\|\partial_{xx}^2 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\
&\qquad+\left(\tfrac{3}{2}-\tfrac{C_0D}{B}\right)\varepsilon\left\|u_{\varepsilon,\beta}(t,\cdot)\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\le C_0\varepsilon\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}.
\end{aligned}
\tag{58}
$$

We search A, B, C, E such that

$$\{\ 1-2C>0,\frac{C}{2}-8B^2>0,E-1>0,A-2A^2-C_0^2D^2>0,\frac{3}{2}-\frac{C_0D}{B}>0, \tag{59}$$

that is

$$\{\ C<\frac{1}{2},B^2<\frac{C}{16},E>1,2A^2-A+C_0^2D^2<0,D<\frac{3B}{2C_0}. \tag{60}$$

We choose

$$C = \frac{1}{4}, \quad E = 2. \tag{61}$$

It follows from the second inequality of (60), and (61) that

$$B < \frac{1}{8}. \tag{62}$$

Hence, we can choose

$$B = \frac{1}{9}. \tag{63}$$

Substituting (63) in the fifth inequality of (60), we have

$$D < \frac{1}{6C_0}. \tag{64}$$

The fourth inequality admits solution when

$$D < \frac{2\sqrt{2}}{8C_0}. \tag{65}$$

It follows from (64) and (65) that

$$D < \min\left\{\frac{1}{6C_0}, \frac{2\sqrt{2}}{8C_0}\right\} = \frac{1}{6C_0}. \tag{66}$$

Therefore, from (60) and (66), there exist $0 < A_1 < A_2$ such that

$$0 < A_1 < A < A_2. \tag{67}$$

Substituting (61), (63), and (66) in (58), from (67), we get

$$\begin{aligned}
&\tfrac{d}{dt}\left(\tfrac{1}{4}\left\|u_{\varepsilon,\beta}(t,\cdot)\right\|^4_{L^4(\mathbf{R})} + \tfrac{(4A+1)\varepsilon^2}{8}\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\right)\\
&\qquad + \tfrac{d}{dt}\left(\tfrac{A\beta^2\varepsilon^2}{2}\left\|\partial^3_{xxx} u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})} + \beta^4\left\|\partial^4_{xxxx} u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\right)\\
&\qquad + \tfrac{\beta\varepsilon^2+18\beta^2}{18}\tfrac{d}{dt}\left\|\partial^2_{xx} u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})} + \tfrac{\beta\varepsilon}{9}\left\|\partial^2_{tx} u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\
&\qquad + \tfrac{\varepsilon}{8}\left\|\partial_t u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})} + \tfrac{\beta^3\varepsilon}{18}\left\|\partial^4_{txxx} u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\
&\qquad + \tfrac{73\beta^2\varepsilon}{648}\left\|\partial^3_{txx} u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})} + \tfrac{\beta^2\varepsilon}{2}\left\|\partial^3_{xxx} u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\
&\qquad + K_2\varepsilon^3\left\|\partial^2_{xx} u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})} + K_2\varepsilon\left\|u_{\varepsilon,\beta}(t,\cdot)\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\
&\quad \leq C_0\varepsilon\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})},
\end{aligned} \tag{68}$$

for some K_1, $K_2 > 0$.

An integration on $(0,t)$, (22), and (42) give

$$\begin{aligned}
&\tfrac{1}{4}\left\|u_{\varepsilon,\beta}(t,\cdot)\right\|^4_{L^4(\mathbf{R})}+\tfrac{(4A+1)\varepsilon^2}{8}\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\
&\quad+\tfrac{A\beta^2\varepsilon^2}{2}\left\|\partial^3_{xxx}u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}+\beta^4\left\|\partial^4_{xxxx}u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\
&\quad+\tfrac{\beta\varepsilon^2+18\beta^2}{18}\left\|\partial^2_{xx}u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}+\tfrac{\beta\varepsilon}{9}\int_0^t\left\|\partial^2_{tx}u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbf{R})}ds\\
&\quad+\tfrac{\varepsilon}{8}\int_0^t\left\|\partial_t u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbf{R})}ds+\tfrac{\beta^3\varepsilon}{18}\int_0^t\left\|\partial^4_{txxx}u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbf{R})}ds\\
&\quad+\tfrac{73\beta^2\varepsilon}{648}\int_0^t\left\|\partial^3_{txx}u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbf{R})}ds+\tfrac{\beta^2\varepsilon}{2}\int_0^t\left\|\partial^3_{xxx}u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbf{R})}ds\\
&\quad+K_2\varepsilon^3\int_0^t\left\|\partial^2_{xx}u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}ds+K_2\varepsilon\int_0^t\left\|u_{\varepsilon,\beta}(s,\cdot)\partial_x u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbf{R})}ds\\
&\le C_0+C_0\varepsilon\int_0^t\left\|\partial_x u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbf{R})}ds\le C_0.
\end{aligned}\tag{69}$$

Hence,

$$\begin{aligned}
&\left\|u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^4(\mathbf{R})}\le C_0,\\
&\varepsilon\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbf{R})}\le C_0,\\
&\beta\varepsilon\left\|\partial^3_{xxx}u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbf{R})}\le C_0,\\
&\beta^2\left\|\partial^4_{xxxx}u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbf{R})}\le C_0,\\
&\beta^{\frac{1}{2}}\varepsilon\left\|\partial^2_{xx}u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbf{R})}\le C_0,\\
&\beta\left\|\partial^2_{xx}u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbf{R})}\le C_0,\\
&\beta\varepsilon\int_0^t\left\|\partial^2_{tx}u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbf{R})}ds\le C_0,\\
&\varepsilon\int_0^t\left\|\partial_t u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbf{R})}ds\le C_0,\\
&\beta^3\varepsilon\int_0^t\left\|\partial^4_{txxx}u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbf{R})}ds\le C_0,\\
&\beta^2\varepsilon\int_0^t\left\|\partial^3_{txx}u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbf{R})}ds\le C_0,\\
&\beta^2\varepsilon\int_0^t\left\|\partial^3_{xxx}u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbf{R})}ds\le C_0,\\
&\varepsilon^3\int_0^t\left\|\partial^2_{xx}u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}ds\le C_0,\\
&\varepsilon\int_0^t\left\|u_{\varepsilon,\beta}(s,\cdot)\partial_x u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbf{R})}ds\le C_0,
\end{aligned}\tag{70}$$

for every $0<t<T$. $\diamond$ We are ready for the proof of Theorem 3.1. **Proof.**[Proof of Theorem 3.1.] Let us consider a compactly supported entropy–entropy flux pair $(\eta,\,q)$. Multiplying (18) by $\eta'(u_{\varepsilon,\beta})$, we have

$$\begin{aligned}
&\partial_t\eta(u_{\varepsilon,\beta})+\partial_x q(u_{\varepsilon,\beta})\\
&\quad=\varepsilon\eta'(u_{\varepsilon,\beta})\partial^2_{xx}u_{\varepsilon,\beta}-\beta\partial^3_{xxx}u_{\varepsilon,\beta}-\beta^2\eta'(u_{\varepsilon,\beta})\partial^5_{txxxx}u_{\varepsilon,\beta}\\
&\quad=I_{1,\,\varepsilon,\,\beta}+I_{2,\,\varepsilon,\,\beta}+I_{3,\,\varepsilon,\,\beta}+I_{4,\,\varepsilon,\,\beta}+I_{5,\,\varepsilon,\,\beta}+I_{6,\,\varepsilon,\,\beta},
\end{aligned}\tag{71}$$

where $I_{1,\,\varepsilon,\,\beta}$, $I_{2,\,\varepsilon,\,\beta}$, $I_{3,\,\varepsilon,\,\beta}$, $I_{4,\,\varepsilon,\,\beta}$, $I_{5,\,\varepsilon,\,\beta}$, $I_{6,\,\varepsilon,\,\beta}$ are defined in (41).

As in [6, Theorem 3.1], we obtain that $I_{1,\,\varepsilon,\,\beta} \to 0$ in $H^{-1}((0,T)\times\mathbf{R})$, $\{I_{2,\,\varepsilon,\,\beta}\}_{\varepsilon,\beta>0}$ is bounded in $L^1((0,T)\times\mathbf{R})$, $I_{4,\,\varepsilon,\,\beta} \to 0$ in $H^{-1}((0,T)\times\mathbf{R})$, $I_{5,\,\varepsilon,\,\beta} \to 0$ in $L^1((0,T)\times\mathbf{R})$, while as in [19, Theorem 2.1] $I_{3,\,\varepsilon,\,\beta} \to 0$ in $H^{-1}((0,T)\times\mathbf{R})$, and, $I_{4,\,\varepsilon,\,\beta} \to 0$ in $L^1((0,T)\times\mathbf{R})$

Arguing as in [19, Theorem 2.1], we have *ii*). $\diamond$

4. The Benjamin-Bona-Mahony equation

In this section, we consider the Benjamin-Bona-Mahony equation

$$\partial_t u + u\partial_x u - \beta\partial_{txx}^3 u = 0. \tag{72}$$

We augment (72) with the initial condition

$$u(0,x) = u_0(x), \tag{73}$$

on which we assume (16). We study the dispersion-diffusion limit for (72). Therefore, we fix two small parameters ε, β and consider the following third order problem

$$\begin{cases} \partial_t u_{\varepsilon,\beta} + u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta} - \beta\partial_{txx}^3 u_{\varepsilon,\beta} = \varepsilon\partial_{xx}^2 u_{\varepsilon,\beta}, & t>0,\ x\in\mathbf{R},\\ u_{\varepsilon,\beta}(0,x) = u_{\varepsilon,\beta,0}(x), & x\in\mathbf{R}, \end{cases} \tag{74}$$

where $u_{\varepsilon,\beta,0}$ is a C^∞ approximation of u_0 such that

$$\begin{aligned} & u_{\varepsilon,\,\beta,\,0} \to u_0 \quad \text{in } L^p_{loc}(\mathbf{R}),\ 1\le p<2, \text{ as } \varepsilon,\ \beta\to 0,\\ & \|u_{\varepsilon,\beta,0}\|^2_{L^2(\mathbf{R})} + \left(\beta+\beta^{\frac{1}{2}}\right)\|\partial_x u_{\varepsilon,\beta,0}\|^2_{L^2(\mathbf{R})} \le C_0, \quad \varepsilon,\beta>0,\\ & \left(\beta^{\frac{3}{2}}+\beta\varepsilon^2\right)\left\|\partial_{xx}^2 u_{\varepsilon,\beta,0}\right\|^2_{L^2(\mathbf{R})} \le C_0, \quad \varepsilon,\beta>0, \end{aligned} \tag{75}$$

and C_0 is a constant independent on ε and β.

The main result of this section is the following theorem.

Theorem 4.1. *Assume that (16) and (75) hold. If (20) holds, then, there exist two sequences $\{\varepsilon_n\}_{n\in\mathbf{N}}$, $\{\beta_n\}_{n\in\mathbf{N}}$, with $\varepsilon_n, \beta_n \to 0$, and a limit function*

$$u \in L^\infty(\mathbf{R}^+; L^2(\mathbf{R})), \tag{76}$$

such that

i) $u_{\varepsilon_n,\beta_n} \to u$ *strongly in* $L^p_{loc}(\mathbf{R}^+ \times \mathbf{R})$*, for each* $1 \le p < 2$*,*

ii) u *is the unique entropy solution of (2).*

Let us prove some a priori estimates on $u_{\varepsilon,\beta}$, denoting with C_0 the constants which depend only on the initial data.

Arguing as [5], we have the following result

Lemma 4.1. *For each* $t > 0$*,*

$$\|u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbf{R})} + \beta\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbf{R})} + 2\varepsilon\int_0^t \|\partial_x u_{\varepsilon,\beta}(s,\cdot)\|^2_{L^2(\mathbf{R})}ds \le C_0. \quad (77)$$

Moreover,

$$\|u_{\varepsilon,\beta}(t,\cdot)\|_{L^\infty(\mathbf{R})} \le C_0\beta^{-\frac{1}{4}}. \quad (78)$$

Lemma 4.2. *Assume (43). For each* $t > 0$*,*

$$\begin{aligned}&\beta\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbf{R})} + \tfrac{2\beta^2 + \beta^{\frac{3}{2}}\varepsilon^2}{2}\left\|\partial^2_{xx}u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\ &\quad + \tfrac{3\beta\varepsilon}{2}\int_0^t\left\|\partial^2_{xx}u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbf{R})}ds + \tfrac{\beta^{\frac{5}{2}}\varepsilon}{2}\int_0^t\left\|\partial^3_{txx}u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbf{R})}ds\\ &\quad + \beta^{\frac{3}{2}}\varepsilon\left\|\partial^2_{tx}u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})} \le C_0.\end{aligned} \quad (79)$$

Proof. Let $t > 0$. Multiplying (74) by $-2\beta^{\frac{1}{2}}\partial^2_{xx}u_{\varepsilon,\beta} - \beta\varepsilon\partial^3_{txx}u_{\varepsilon,\beta}$, we have

$$\begin{aligned}&\left(-2\beta^{\frac{1}{2}}\partial^2_{xx}u_{\varepsilon,\beta} - \beta\varepsilon\partial^3_{txx}u_{\varepsilon,\beta}\right)\partial_t u_{\varepsilon,\beta}\\ &\quad + \left(-2\beta^{\frac{1}{2}}\partial^2_{xx}u_{\varepsilon,\beta} - \beta\varepsilon\partial^3_{txx}u_{\varepsilon,\beta}\right)u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}\\ &\quad - \beta\left(-2\beta^{\frac{1}{2}}\partial^2_{xx}u_{\varepsilon,\beta} - \beta\varepsilon\partial^3_{txx}u_{\varepsilon,\beta}\right)\partial^3_{txx}u_{\varepsilon,\beta}\\ &= \varepsilon\left(-2\beta^{\frac{1}{2}}\partial^2_{xx}u_{\varepsilon,\beta} - \beta\varepsilon\partial^3_{txx}u_{\varepsilon,\beta}\right)\partial^2_{xx}u_{\varepsilon,\beta}.\end{aligned} \quad (80)$$

Since

$$\begin{aligned}&\int_{\mathbf{R}}\left(-2\beta^{\frac{1}{2}}\partial^2_{xx}u_{\varepsilon,\beta} - \beta\varepsilon\partial^3_{txx}u_{\varepsilon,\beta}\right)\partial_t u_{\varepsilon,\beta}dx\\ &\quad = \beta^{\frac{1}{2}}\tfrac{d}{dt}\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbf{R})} + \beta\varepsilon\left\|\partial^2_{tx}u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})},\\ &-\beta\int_{\mathbf{R}}\left(-2\beta^{\frac{1}{2}}\partial^2_{xx}u_{\varepsilon,\beta} - \beta\varepsilon\partial^3_{txx}u_{\varepsilon,\beta}\right)\partial^3_{txx}u_{\varepsilon,\beta}dx\\ &\quad = \beta^{\frac{3}{2}}\tfrac{d}{dt}\left\|\partial^2_{xx}u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})} + \beta^2\varepsilon\left\|\partial^3_{txx}u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})},\\ &\varepsilon\int_{\mathbf{R}}\left(-2\beta^{\frac{1}{2}}\partial^2_{xx}u_{\varepsilon,\beta} - \beta\varepsilon\partial^3_{txx}u_{\varepsilon,\beta}\right)\partial^2_{xx}u_{\varepsilon,\beta}dx\\ &\quad = -2\beta^{\frac{1}{2}}\varepsilon\left\|\partial^2_{xx}u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})} - \tfrac{\beta\varepsilon^2}{2}\tfrac{d}{dt}\left\|\partial^2_{xx}u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})},\end{aligned} \quad (81)$$

integrating (80) on $\mathbf{R}$, we get

$$\begin{aligned}
&\frac{d}{dt}\left(\beta^{\frac{1}{2}}\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}+\frac{2\beta^{\frac{3}{2}}+\beta\varepsilon^2}{2}\left\|\partial_{xx}^2 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\right)\\
&\qquad+2\beta^{\frac{1}{2}}\varepsilon\left\|\partial_{xx}^2 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}+\beta^2\varepsilon\left\|\partial_{txx}^3 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\
&\qquad+\beta\varepsilon\left\|\partial_{tx}^2 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\
&\qquad=2\beta^{\frac{1}{2}}\int_{\mathbf{R}} u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}\partial_{xx}^2 u_{\varepsilon,\beta}dx-\beta\varepsilon\int_{\mathbf{R}} u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}\partial_{txx}^3 u_{\varepsilon,\beta}dx.
\end{aligned}\tag{82}$$

Due to (43), (78), and the Young inequality,

$$\begin{aligned}
&2\beta^{\frac{1}{2}}\int_{\mathbf{R}}|u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}||\partial_{xx}^2 u_{\varepsilon,\beta}|dx=\beta^{\frac{1}{2}}\int_{\mathbf{R}}\left|\frac{2u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}}{\varepsilon^{\frac{1}{2}}}\right|\left|\varepsilon^{\frac{1}{2}}\partial_{xx}^2 u_{\varepsilon,\beta}\right|dx\\
&\quad\le\frac{2\beta^{\frac{1}{2}}}{\varepsilon}\int_{\mathbf{R}}u_{\varepsilon,\beta}^2(\partial_x u_{\varepsilon,\beta})^2dx+\frac{\beta^{\frac{1}{2}}\varepsilon}{2}\left\|\partial_{xx}^2 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\
&\quad\le C_0\varepsilon\int_{\mathbf{R}}u_{\varepsilon,\beta}^2(\partial_x u_{\varepsilon,\beta})^2dx+\frac{\beta^{\frac{1}{2}}\varepsilon}{2}\left\|\partial_{xx}^2 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\
&\quad\le C_0\varepsilon\left\|u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^\infty(\mathbf{R})}\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}+\frac{\beta^{\frac{1}{2}}\varepsilon}{2}\left\|\partial_{xx}^2 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\
&\quad\le\frac{C_0\varepsilon}{\beta^{\frac{1}{2}}}\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}+\frac{\beta^{\frac{1}{2}}\varepsilon}{2}\left\|\partial_{xx}^2 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}.
\end{aligned}\tag{83}$$

Thanks to (78), and the Young inequality,

$$\begin{aligned}
&\beta\varepsilon\int_{\mathbf{R}}|u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}||\partial_{txx}^3 u_{\varepsilon,\beta}|dx=\varepsilon\int_{\mathbf{R}}|u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}|\left|\beta\partial_{txx}^3 u_{\varepsilon,\beta}\right|dx\\
&\quad\le\frac{\varepsilon}{2}\int_{\mathbf{R}}u_{\varepsilon,\beta}^2(\partial_x u_{\varepsilon,\beta})^2dx+\frac{\beta\varepsilon}{2}\left\|\partial_{txx}^3 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\
&\quad\le\frac{\varepsilon}{2}\left\|u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^\infty(\mathbf{R})}\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}+\frac{\beta\varepsilon}{2}\left\|\partial_{txx}^3 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\
&\quad\le\frac{\varepsilon}{2\beta^{\frac{1}{2}}}\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}+\frac{\beta\varepsilon}{2}\left\|\partial_{txx}^3 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}.
\end{aligned}\tag{84}$$

It follows from (82), (83), and (84) that

$$\begin{aligned}
&\frac{d}{dt}\left(\beta^{\frac{1}{2}}\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}+\frac{2\beta^{\frac{3}{2}}+\beta\varepsilon^2}{2}\left\|\partial_{xx}^2 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\right)\\
&\qquad+\frac{3\beta^{\frac{1}{2}}\varepsilon}{2}\left\|\partial_{xx}^2 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}+\frac{\beta^2\varepsilon}{2}\left\|\partial_{txx}^3 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\
&\qquad+\beta\varepsilon\left\|\partial_{tx}^2 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\le\frac{C_0\varepsilon}{\beta^{\frac{1}{2}}}\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}.
\end{aligned}\tag{85}$$

Hence,

$$\begin{aligned}
&\frac{d}{dt}\left(\beta\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}+\frac{2\beta^2+\beta^{\frac{3}{2}}\varepsilon^2}{2}\left\|\partial_{xx}^2 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\right)\\
&\qquad+\frac{3\beta\varepsilon}{2}\left\|\partial_{xx}^2 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}+\frac{\beta^{\frac{5}{2}}\varepsilon}{2}\left\|\partial_{txx}^3 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\
&\qquad+\beta^{\frac{3}{2}}\varepsilon\left\|\partial_{tx}^2 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\le C_0\varepsilon\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}.
\end{aligned}\tag{86}$$

An integration on $(0,t)$ and (77) give

$$\begin{aligned}\beta\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}+\tfrac{2\beta^2+\beta^{\frac{3}{2}}\varepsilon^2}{2}\left\|\partial^2_{xx}u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\ +\tfrac{3\beta\varepsilon}{2}\int_0^t\left\|\partial^2_{xx}u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbf{R})}+\tfrac{\beta^{\frac{5}{2}}\varepsilon}{2}\int_0^t\left\|\partial^3_{txx}u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbf{R})}ds\\ +\beta^{\frac{3}{2}}\varepsilon\left\|\partial^2_{tx}u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\le C_0+C_0\varepsilon\int_0^t\left\|\partial_x u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbf{R})}ds\le C_0,\end{aligned}\tag{87}$$

that is (79). $\diamond$ **Proof.**[Proof of Theorem 4.1.] Let us consider a compactly supported entropy–entropy flux pair (η,q). Multiplying (74) by $\eta'(u_{\varepsilon,\beta})$, we have

$$\begin{aligned}\partial_t\eta(u_{\varepsilon,\beta})+\partial_x q(u_{\varepsilon,\beta})=&\ \ \varepsilon\eta'(u_{\varepsilon,\beta})\partial^2_{xx}u_{\varepsilon,\beta}+\beta\eta'(u_{\varepsilon,\beta})\partial^3_{txx}u_{\varepsilon,\beta}\\ =&\ \ I_{1,\,\varepsilon,\,\beta}+I_{2,\,\varepsilon,\,\beta}+I_{3,\,\varepsilon,\,\beta}+I_{4,\,\varepsilon,\,\beta},\end{aligned}\tag{88}$$

where

$$\begin{aligned}I_{1,\,\varepsilon,\,\beta}&=\partial_x(\varepsilon\eta'(u_{\varepsilon,\beta})\partial_x u_{\varepsilon,\beta}),\\ I_{2,\,\varepsilon,\,\beta}&=-\varepsilon\eta''(u_{\varepsilon,\beta})(\partial_x u_{\varepsilon,\beta})^2,\\ I_{3,\,\varepsilon,\,\beta}&=\partial_x(\beta\eta'(u_{\varepsilon,\beta})\partial^2_{tx}u_{\varepsilon,\beta}),\\ I_{4,\,\varepsilon,\,\beta}&=-\beta\eta''(u_{\varepsilon,\beta})\partial_x u_{\varepsilon,\beta}\partial^2_{tx}u_{\varepsilon,\beta}.\end{aligned}\tag{89}$$

Fix $T>0$. Arguing as in [13, Lemma 3.2], we have that $I_{1,\,\varepsilon,\,\beta}\to 0$ in $H^{-1}((0,T)\times\mathbf{R})$, and $\{I_{2,\,\varepsilon,\,\beta}\}_{\varepsilon,\beta>0}$ is bounded in $L^1((0,T)\times\mathbf{R})$.

We claim that

$$I_{3,\,\varepsilon,\,\beta}\to 0\quad\text{in } H^{-1}((0,T)\times\mathbf{R}),\ T>0,\ \text{as } \varepsilon\to 0.\tag{90}$$

By (43) and (79),

$$\begin{aligned}&\left\|\beta\eta'(u_{\varepsilon,\beta})\partial^2_{tx}u_{\varepsilon,\beta}\right\|^2_{L^2((0,T)\times\mathbf{R})}\\ &\quad\le\beta^2\left\|\eta'\right\|_{L^\infty(\mathbf{R})}\left\|\partial^2_{tx}u_{\varepsilon,\beta}\right\|^2_{L^2((0,T)\times\mathbf{R})}\\ &\quad=\left\|\eta'\right\|_{L^\infty(\mathbf{R})}\tfrac{\beta^2\varepsilon}{\varepsilon}\left\|\partial^2_{tx}u_{\varepsilon,\beta}\right\|^2_{L^2((0,T)\times\mathbf{R})}\\ &\quad=\left\|\eta'\right\|_{L^\infty(\mathbf{R})}\tfrac{\beta^{\frac{1}{2}}\beta^{\frac{3}{2}}\varepsilon}{\varepsilon}\left\|\partial^2_{tx}u_{\varepsilon,\beta}\right\|^2_{L^2((0,T)\times\mathbf{R})}\le C_0\left\|\eta'\right\|_{L^\infty(\mathbf{R})}\varepsilon\to 0.\end{aligned}\tag{91}$$

Let us show that

$$I_{4,\,\varepsilon,\,\beta}\to 0\quad\text{in}L^1((0,T)\times\mathbf{R}),\ T>0.\tag{92}$$

Thanks to (20), (77), (79), and the Hölder inequality,

$$
\begin{aligned}
&\left\|\beta\eta''(u_{\varepsilon,\beta})\partial_x u_{\varepsilon,\beta}\partial^2_{tx}u_{\varepsilon,\beta}\right\|_{L^1((0,T)\times\mathbf{R})}\\
&\quad\le \beta\left\|\eta''\right\|_{L^\infty(\mathbf{R})}\int_0^T\!\!\int_{\mathbf{R}}\left|\partial_x u_{\varepsilon,\beta}\partial^2_{tx}u_{\varepsilon,\beta}\right|dtdx\\
&\quad= \left\|\eta''\right\|_{L^\infty(\mathbf{R})}\frac{\beta^{\frac{1}{4}}\beta^{\frac{3}{4}}\varepsilon}{\varepsilon}\left\|\partial_x u_{\varepsilon,\beta}\right\|_{L^2((0,T)\times\mathbf{R})}\left\|\partial^2_{tx}u_{\varepsilon,\beta}\right\|_{L^2((0,T)\times\mathbf{R})}\\
&\quad\le C_0\left\|\eta''\right\|_{L^\infty(\mathbf{R})}\frac{\beta^{\frac{1}{4}}}{\varepsilon}\to 0.
\end{aligned}
\tag{93}
$$

Arguing as in [4, 5], we have ii). ⋄

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