The Brezis–Nirenberg type problem for the p-laplacian (1<p<2): Multiple positive solutions

THE BREZIS-NIRENBERG TYPE PROBLEM FOR THE p-LAPLACIAN $(1 < p < 2)$: MULTIPLE POSITIVE SOLUTIONS

SILVIA CINGOLANI AND GIUSEPPINA VANNELLA

Abstract. In this paper we consider the quasilinear critical problem

$$(P_\lambda) \quad \begin{cases} -\Delta_p u = \lambda u^{q-1} + u^{p^*-1} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary, $N \geq p^2$, $1 < p < 2$ and $p \leq q < p^*$, $p^* = Np/(N-p)$, $\lambda > 0$ is a parameter. In spite of the lack of C^2 regularity of the energy functional associated to problem (P_λ), we employ new Morse techniques to derive a multiplicity result of solutions. We show that there exists $\lambda^* > 0$ such that, for each $\lambda \in (0, \lambda^*)$, either (P_λ) has $\mathcal{P}_1(\Omega)$ distinct solutions or there exists a sequence of quasilinear problems approximating (P_λ), each of them having at least $\mathcal{P}_1(\Omega)$ distinct solutions. These results complete those obtained in [23] for the case $p \geq 2$.

1. Introduction

Since the pioneer work of Brezis and Nirenberg [8], there was a large amount of results dealing with semilinear problems involving critical Sobolev exponent. We mention the well known papers [11, 12, 5, 30, 7]. In the last twenty years, a lot of efforts have been made to obtain similar results for the quasilinear critical problem

$$(S_\lambda) \quad \begin{cases} -\Delta_p u = \lambda u^{q-1} + u^{p^*-1} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary, $N \geq p^2$, $1 < p \leq q < p^*$, $p^* = Np/(N-p)$, $\lambda > 0$ is a parameter.

By using the concentration compactness principle, the results of [8] were extended to the quasilinear cases by Azorero and Peral [3, 4] and Guedda Veron [27], independently. Precisely they proved that if $N \geq p^2$, $p = q$, then (S_λ) has a positive solution if $\lambda \in (0, \lambda_1)$ where λ_1 is the first eigenvalue of $-\Delta_p$ with Dirichlet boundary condition and no positive solution for $\lambda \geq \lambda_1$ or $\lambda \leq 0$ and Ω starshaped. The case $q \in (p, p^*)$ has been studied by Azorero and Peral, who proved the existence of a positive solution for any $\lambda > 0$.

Only a few years ago, Degiovanni and Lancelotti [24] extended the results in [11] by proving the existence of a nontrivial solution for (S_λ) with $p = q$ when $N \geq p^2$ and $\lambda > \lambda_1$ is not an eigenvalue, and when $N^2/(N+1) > p^2$ and $\lambda \geq \lambda_1$.

Recently, in [10], Cao, Peng and Yan extended the results [12] to the quasilinear case (S_λ) with $p = q$ proving that if $1 < p < N$ there exist infinitely many solutions to (S_λ)

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The authors are members of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).

for $\lambda > 0$ and $N > p^2 + p$. By using the Picone identity, it can be proved that every nonzero solution of (S_λ) is sign-changing for $\lambda > \lambda_1$.

Concerning the effect of the domain topology on the existence of positive solutions for the quasilinear critical problem, the first multiplicity result is due to Alves and Ding [2]. They proved that if $N \geq p^2$ and $2 \leq p \leq q < p^*$, then there exists $\lambda^* > 0$ such that for each $\lambda \in (0, \lambda^*)$ the problem

$$(P_\lambda) \quad \begin{cases} -\Delta_p u = \lambda u^{q-1} + u^{p^*-1} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has at least $cat(\Omega)$ solutions, where $cat(\Omega)$ denotes the Ljusternik- Schnirelmann category of Ω in itself.

From [5, 7], it is well known that if Ω is topologically rich, one can derive better information on the number of solutions by means of Morse theory and domain topology instead of the Lusternick-Schrnirelmann category. However the extension of the results in [5, 7] to quasilinear problems remained for a long time an open field of investigation, since it leads to the applicability of Morse techniques in a Banach (not Hilbert) variational framework.

Some years ago, in [23], we obtained a first result, which correlates the topological properties of the domain and the number of solutions of (P_λ), counted with their multiplicities.

Let us introduce the classical topological definition.

Definition 1.1. Let $\mathbb{G}$ be an abelian group. For any $A \subset \mathbb{R}^n$, we denote $\mathcal{P}_t(A)$ the Poincaré polynomial of A, defined by

$$\mathcal{P}_t(A) = \sum_{k=0}^{+\infty} \dim H^k(A, \emptyset)\, t^k,$$

where $H^k(A, \emptyset)$ stands for the k-th Alexander-Spanier relative cohomology group of the pair $(A, \emptyset)$, with coefficients in $\mathbb{G}$.

In [23, Theorem 1.4] we proved the following topological result:

Theorem 1.2. *Assume that $N \geq p^2$, $1 < p \leq q < p^*$, $p^* = Np/(N-p)$. There exists $\lambda^* > 0$ such that, for any $\lambda \in (0, \lambda^*)$, (P_λ) has at least $\mathcal{P}_1(\Omega)$ solutions, possibly counted with their multiplicities.*

Clearly Theorem 1.2 does not guarantee an effective multiplicity result, since all the solutions could *collapse* into the same critical point, having multiplicity $\mathcal{P}_1(\Omega)$. Hence the interpretation of the multiplicity of the solutions is a crucial and challenging issue in a Banach space. If $p \neq 2$, a lot of conceptual difficulties arise in the passage from semilinear elliptic equations to quasilinear elliptic ones. Firstly we notice that the solutions of (P_λ) correspond to critical points of the energy functional $I_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by setting

$$I_\lambda(u) = \frac{1}{p}\int_\Omega |\nabla u|^p\, dx - \frac{\lambda}{q}\int_\Omega (u^+)^q\, dx - \frac{1}{p^*}\int_\Omega (u^+)^{p^*}\, dx \tag{1.1}$$

and every critical point of I_λ is degenerate in the classical sense given in a Hilbert space, namely the second derivative is never an isomorphism between the space $W_0^{1,p}(\Omega)$ and its dual one. Moreover in a Banach framework we also lose the Fredholm properties of the second derivatives at the critical points.

In [23], for $p \geq 2$, we furnished an interpretation of the multiplicity of a critical point of I_λ in terms of distinct critical points of functionals approximating I_λ, and having nondegenerate critical points in a weak sense, namely the second derivative in the critical point has a trivial nullspace. For each nondegenerate critical point of the approximating functional we can apply the estimates in [19, 20, 22] and compute the critical groups of such approximating functional in the nondegenerate critical point by means of its Morse index, so that its multiplicity is exactly one. We underline that for $p \geq 2$ we can take advantage of the fact that the energy functional I_λ is C^2 and the Morse index at each critical point u_0, denoted by $m(f, u_0)$, is well defined.

In the present paper we are interested to derive an interpretation of the multiplicity of a solution of (P_λ) when $1 < p < 2$, which remained an open and difficult problem in literature. In this case we emphasize that the energy functional I_λ is never C^2, since $u \in W_0^{1,p}(\Omega) \mapsto \int_\Omega |\nabla u|^p \, dx \in \mathbb{R}$ is not C^2. The lack of regularity of I_λ is even greater when in addition $q \leq 2$, as the function $G_0 : \mathbb{R} \to \mathbb{R}$ defined by $G_0(s) = \frac{\lambda}{q} (s^+)^q + \frac{1}{p^*} (s^+)^{p^*}$ is not C^2.

We will face this problem via suitable approximations of the functional I_λ, as in [23]. Unfortunately when $1 < p < 2$, the approximating functionals continue to be C^1, not C^2, so that the results in [19, 22] cannot be applied. The main idea of this work consists in the introduction of some bilinear forms definited on a Hilbert space, which are inspired by the formal second derivatives of the approximating functionals. In this way we are able to compute the critical groups of the approximating functionals, by means of some differential notions, which have the same roles of the Morse indices in a regular setting, so that the multiplicity of a nondegenerate critical point is exactly one. This can be done, exploiting some recent results contained in the paper [16]. Despite these difficulties, we obtain the following multiplicity result of solutions for problem (P_λ) with $1 < p < 2$, for small value of parameter λ, via the Morse relations and the domain topology. We extend the multiplicity result in [23, Theorem 1.6] to $1 < p < 2$, showing that the problem (P_λ) has, for λ small, at least $\mathcal{P}_1(\Omega)$ distinct solutions or is *near*, in a suitable sense, to a quasilinear elliptic critical problem having at least $\mathcal{P}_1(\Omega)$ distinct solutions.

In what follows, we say that $\partial\Omega$ satisfies the interior sphere condition if for each $x_0 \in \partial\Omega$ there exists a ball $B_R(x_1) \subset \Omega$ such that $\overline{B_R(x_1)} \cap \partial\Omega = \{x_0\}$.

Precisely, we prove the following main result.

Theorem 1.3. *Assume that $\partial\Omega$ satisfies the interior sphere condition and that $N \geq p^2$, $1 < p < 2$, $p \leq q < p^*$, $p^* = Np/(N-p)$. There exists $\lambda^* > 0$ such that, for any $\lambda \in (0, \lambda^*)$, either (P_λ) has at least $\mathcal{P}_1(\Omega)$ distinct solutions or, if not, for any sequence (α_n), with $\alpha_n > 0$, $\alpha_n \to 0$, there exists a sequence (f_n) with $f_n \in C^1(\overline{\Omega})$, $\|f_n\|_{C^1} \to 0$, such that problem*

$$(P_n) \quad \begin{cases} -div\big((|\nabla u|^2 + \alpha_n)^{\frac{p-2}{2}} \nabla u\big) & \\ = \lambda u^{1+\alpha_n} (\alpha_n + u^{2+\alpha_n})^{\frac{q-2-\alpha_n}{2+\alpha_n}} + u^{1+\alpha_n} (\alpha_n + u^{2+\alpha_n})^{\frac{p^*-2-\alpha_n}{2+\alpha_n}} + f_n & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has at least $\mathcal{P}_1(\Omega)$ distinct solutions, for n large enough.

Theorem 1.3 is quantitative and variational in nature. We remark that if Ω is obtained by an open contractible domain cutting k holes we derive that the number of solutions of (P_λ) is affected by k, whereas the category of Ω is 2.

The previous result has been announced, without proof, in [35].

Remark 1.4. If we add the assumption $q > 2$ to the previous ones, Theorem 1.3 still holds replacing (P_n) with the simpler

$$(\bar{P}_n) \quad \begin{cases} -div\big((|\nabla u|^2 + \alpha_n)^{\frac{p-2}{2}} \nabla u\big) = \lambda u^{q-1} + u^{p^*-1} + f_n & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Throughout the paper we use the following notations:

(1) $\|\cdot\|$ denotes the usual norm both in $W_0^{1,p}(\Omega)$ and in $W^{-1,p'}(\Omega)$;
(2) $\|\cdot\|_{1,2}$ denotes the usual norm in $W_0^{1,2}(\Omega)$;
(3) $\|\cdot\|_\infty$ denotes the usual norm in $L^\infty(\Omega)$;
(4) $|\cdot|_r$ denotes the usual norm in $L^r(\Omega)$;
(5) $\|\cdot\|_{C^1(A)}$ denotes the usual norm in $C^1(A, \mathbb{R})$, where $A \subseteq X$ and X is a Banach space;
(6) $\langle \cdot, \cdot \rangle : W^{-1,p'}(\Omega) \times W_0^{1,p}(\Omega) \to \mathbb{R}$ denotes the duality pairing;
(7) $B_r(u) = \{v \in W_0^{1,p}(\Omega) : \|v - u\| < r\}$, where $u \in W_0^{1,p}(\Omega)$ and $r > 0$.

2. Critical groups estimates via approximating functionals

In this section we introduce a class of functionals approximating the C^1 energy functional $I_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by setting

$$I_\lambda(u) = \frac{1}{p}\int_\Omega |\nabla u|^p\,dx - \frac{\lambda}{q}\int_\Omega (u^+)^q\,dx - \frac{1}{p^*}\int_\Omega (u^+)^{p^*}\,dx \tag{2.1}$$

associated to problem (P_λ).

For any $\alpha > 0$ and $f \in C^1(\overline{\Omega})$ we define

$$\begin{aligned} J_\alpha(u) = \; & \frac{1}{p}\int_\Omega \Big((\alpha + |\nabla u|^2)^{\frac{p}{2}}\Big)dx \\ & -\frac{\lambda}{q}\int_\Omega \big(\alpha + (u^+)^{2+\alpha}\big)^{\frac{q}{2+\alpha}}dx - \frac{1}{p^*}\int_\Omega \big(\alpha + (u^+)^{2+\alpha}\big)^{\frac{p^*}{2+\alpha}}dx, \end{aligned} \tag{2.2}$$

$$J_{\alpha,f}(u) = J_\alpha(u) - \int_\Omega fu\,dx.$$

For any $\alpha \geq 0$ let us introduce $\Psi_\alpha : \mathbb{R}^N \to \mathbb{R}$, $G_\alpha : \mathbb{R} \to \mathbb{R}$ and $g_\alpha : \mathbb{R} \to \mathbb{R}$ defined by

$$\begin{gathered} \Psi_\alpha(\xi) = \tfrac{1}{p}\,(\alpha + |\xi|^2)^{\frac{p}{2}} \\ G_\alpha(t) = \tfrac{\lambda}{q}\big(\alpha + (t^+)^{2+\alpha}\big)^{\frac{q}{2+\alpha}} + \tfrac{1}{p^*}\big(\alpha + (t^+)^{2+\alpha}\big)^{\frac{p^*}{2+\alpha}} \\ g_\alpha(t) = G'_\alpha(t). \end{gathered} \tag{2.3}$$

Note that $\Psi_\alpha \in C^2(\mathbb{R}^N, \mathbb{R})$ and $G_\alpha \in C^2(\mathbb{R}, \mathbb{R})$ when $\alpha > 0$, while ψ_0 is just C^1 when $p \in (1, 2)$ and G_0 is C^1 when $q \leq 2$.

It is immediate that

$$I_\lambda(u) = \int_\Omega \Psi_0(\nabla u)dx - \int_\Omega G_0(u)dx, \qquad J_\alpha(u) = \int_\Omega \Psi_\alpha(\nabla u)dx - \int_\Omega G_\alpha(u)dx.$$

By basic computations we infer the following lemma.

Lemma 2.1. *If $p > 1$, $r \in [p, p^*]$ and $\alpha > 0$, let us denote by P_α, P, A_α and A the functionals defined by*

$$P_\alpha(u) = \int_\Omega \Psi_\alpha(\nabla u)\, dx \qquad\qquad P(u) = \int_\Omega \Psi_0(\nabla u)\, dx$$

$$A_\alpha(u) = \frac{1}{r}\int_\Omega \left(\alpha + \left(u^+\right)^{2+\alpha}\right)^{\frac{r}{2+\alpha}} dx \qquad\qquad A(u) = \frac{1}{r}\int_\Omega \left(u^+\right)^r dx.$$

Then P_α, P and A are in $C^1(W_0^{1,p}(\Omega), \mathbb{R})$, A_α is in $C^2(W_0^{1,p}(\Omega), \mathbb{R})$, and, for any bounded $B \subset W_0^{1,p}(\Omega)$, we have

$$\lim_{\alpha\to 0} \|P_\alpha - P\|_{C^1(B)} = 0, \qquad \lim_{\alpha\to 0} \|A_\alpha - A\|_{C^1(B)} = 0.$$

Through the previous Lemma, we obtain the following result.

Theorem 2.2. *Let $\lambda^* > 0$ be defined by Theorem 1.2. If $\lambda \in (0, \lambda^*)$, $p > 1$, $q \in [p, p^*)$, $f \in C^1(\bar\Omega)$ and $\alpha > 0$, then J_α and $J_{\alpha,f}$ are $C^1(W_0^{1,p}(\Omega))$ functionals and, for any bounded $B \subset W_0^{1,p}(\Omega)$, we have*

$$\lim_{\alpha\to 0} \|J_\alpha - I_\lambda\|_{C^1(B)} = 0,$$

$$\lim_{\|f\|_\infty\to 0} \|J_{\alpha,f} - J_\alpha\|_{C^1(B)} = 0.$$

We now aim to prove that, for any $\alpha \in [0, 1]$ and $f \in C^1(\bar\Omega)$, $J_{\alpha,f}$ satisfies a local compactness condition. We begin to recall a classical definition in a reflexive Banach space, taken from [9, 31] and a recent result, established in [17, Proposition 3.5].

Definition 2.3. Let X be a reflexive Banach space and $D \subset X$. A map $H : D \to X'$ is said to be of class (S_+), if, for every sequence (u_k) in D weakly convergent to u in X with

$$\limsup_{k\to\infty} \langle H(u_k), u_k - u\rangle \le 0,$$

we have $\|u_k - u\| \to 0$.

Proposition 2.4. *Let $f : X \to \mathbb{R}$ be a function of class C^1. Assume that f' is of class $(S)_+$ on $C \subset X$, then*

(a) *if C is bounded, then f satisfies (P.S.) condition on C;*
(b) *if C is closed and convex, then f is sequentially lower semicontinuos on C with respect to the weak topology;*
(c) *if C is closed and convex and if (u_k) is a sequence in C weakly convergent to u with*

$$\limsup_{k\to\infty} f(u_k) \le f(u)\,,$$

then $\|u_k - u\| \to 0$.

Next result provides a uniform local compactness property of the approximating functionals $J_{\alpha,f}$. It is based on [1, Theorem 3.4] (see also [15, Theorem 2.1]). For reader's convenience, we sketch the proof.

Theorem 2.5. *For any $p > 1$, there exists $R > 0$ such that, for any $\alpha \in [0, 1]$, $f \in C^1(\bar{\Omega})$, $\bar{u} \in W_0^{1,p}(\Omega)$, the functional $J'_{\alpha,f}$ is of class (S_+) on $\overline{B_R(\bar{u})}$.*

Proof. Once fixed $\alpha \in [0, 1]$, let $H_\alpha : W_0^{1,p}(\Omega) \to W^{-1,p'}$ denote the map

$$H_\alpha(u) = \langle J'_\alpha(u), \ \cdot \ \rangle$$

so that $H_\alpha(u) = -\operatorname{div}\,(\nabla\Psi_\alpha(\nabla u)) - g_\alpha(u)$, where Ψ_α and g_α are defined by (2.3). It is easy to see there exists $C > 0$ such that

$$\begin{array}{ll} |\nabla\Psi_\alpha(\xi)| \le |\xi|^{p-1} & |g_\alpha(s)| \le C + C|s|^{p^*-1} \\ \nabla\Psi_\alpha(\xi)\cdot\xi \ge 1/2|\xi|^p - C & -g_\alpha(s)s \ge -C - C|s|^{p^*} \end{array}$$

for any $\alpha \in [0, 1]$, $\xi \in \mathbb{R}^N$, $s \in \mathbb{R}$.

Moreover, by the monotonicity of the real function $t \in \mathbb{R} \mapsto t\,(\alpha + t^2)^{\frac{p-2}{2}}$, we infer that

$$\big(\nabla\Psi_\alpha(\xi) - \nabla\Psi_\alpha(\eta)\big)\cdot\big(\xi - \eta\big) > 0$$

for any $\xi, \eta \in \mathbb{R}^N$ with $\eta \neq \xi$.

This means that $\nabla\Psi_\alpha$ and $-g_\alpha$ satisfy the assumptions required by Theorem 3.4 in [1], so there exists $R > 0$ such that H_α is of class (S_+) on $\overline{B_R(\bar{u})}$. Hence, for any $f \in C^1(\overline{\Omega})$, it is immediate that $J_{\alpha,f}$ is of class (S_+) on $\overline{B_R(\bar{u})}$. □

Remark 2.6. Taking account of (a) in Proposition 2.4, the previous theorem assures there exists $R > 0$ such that $J_{\alpha,f}$ satisfies (P.S.) condition in $\overline{B_R(\bar{u})}$, for any $\alpha \in [0, 1]$, $f \in C^1(\overline{\Omega})$ and $\bar{u} \in W_0^{1,p}(\Omega)$.

Now let us consider a critical point u_0 of $J_{\alpha,f}$, with $\alpha \ge 0$ and $f \in C^1(\bar{\Omega})$. According to [16, 17] and references therein, $u_0 \in C^{1,\delta}(\bar{\Omega})$, for some $\delta \in (0, 1]$. It is crucial to give a notion of Morse index, which is not standard, as $J_{\alpha,f}$ is not of class C^2.

If $p \in (1, 2)$, $\alpha > 0$ and $u \in W^{1,\infty}(\Omega)$, let us denote by $B_\alpha(u)$ the bilinear form on $W_0^{1,2}(\Omega)$ defined by

$$B_\alpha(u)(z_1, z_2) = \int_\Omega \Psi''_\alpha(\nabla u)[\nabla z_1, \nabla z_2]\,dx - \int_\Omega g'_\alpha(u) z_1 z_2\,dx,$$

so that

$$\begin{aligned} B_\alpha(u)(z_1, z_2) = &\int_\Omega (\alpha + |\nabla u|^2)^{\frac{p-2}{2}} (\nabla z_1|\nabla z_2)\,dx \\ &+(p-2)\int_\Omega (\alpha + |\nabla u|^2)^{\frac{p-4}{2}} (\nabla u|\nabla z_1)(\nabla u|\nabla z_2)\,dx \\ &-\lambda\int_\Omega \left(\alpha + \alpha^2 + (q-1)(u^+)^{2+\alpha}\right)\left(\alpha + (u^+)^{2+\alpha}\right)^{\frac{q-4-2\alpha}{2+\alpha}} (u^+)^\alpha z_1 z_2\,dx \\ &-\int_\Omega \left(\alpha + \alpha^2 + (p^*-1)(u^+)^{2+\alpha}\right)\left(\alpha + (u^+)^{2+\alpha}\right)^{\frac{p^*-4-2\alpha}{2+\alpha}} (u^+)^\alpha z_1 z_2\,dx. \end{aligned}$$

In addition, we introduce $Q_u^\alpha : W_0^{1,2}(\Omega) \to \mathbb{R}$ defined by

$$Q_u^\alpha(z) = B_\alpha(u)(z,z)\,.$$

The definition of $B_\alpha(u)$ is inspired by the formal second derivative of $J_{\alpha,f}$ in u.
Let us point out that, as $p < 2$, for any $u \in W^{1,\infty}(\Omega)$, $B_\alpha(u)$ and Q_u^α are well defined on $W_0^{1,2}(\Omega)$, but not on $W_0^{1,p}(\Omega)$. In particular, $Q_{u_0}^\alpha$ is a smooth quadratic form on $W_0^{1,2}(\Omega)$ and we define the *Morse index of* $J_{\alpha,f}$ *at* u_0 (denoted by $m(J_{\alpha,f}, u_0)$) as the supremum of the dimensions of the linear subspaces of $W_0^{1,2}(\Omega)$ where $Q_{u_0}^\alpha$ is negative definite and the *large Morse index of* $J_{\alpha,f}$ *at* u_0 (denoted by $m^*(J_{\alpha,f}, u_0)$) as the supremum of the dimensions of the linear subspaces of $W_0^{1,2}(\Omega)$ where $Q_{u_0}^\alpha$ is negative semidefinite. We clearly have $m(J_{\alpha,f}, u_0) \leq m^*(J_{\alpha,f}, u_0) < +\infty$.

The introduction of a suitable notion of Morse index will be crucial to derive estimates of some topological objects, like the critical groups. For reader's convenience, we recall some definitions (see [13, 14]).

Definition 2.7. Let $\mathbb{G}$ be an abelian group, X be a Banach space, $f \in C^1(X,\mathbb{R})$ and u a critical point of f. The *k-th critical group of f at u* is defined by

$$C_k(f,u) = H^k(f^c, f^c \setminus \{u\})$$

where $k \in \mathbb{N}$, $c = f(u)$, $f^c = \{v \in X : f(v) \leq c\}$ and $H^k(f^c, f^c \setminus \{u\})$ stands for the k-th Alexander-Spanier cohomology group of the pair $(f^c, f^c \setminus \{u\})$ with coefficients in $\mathbb{G}$ (see [32]).

Definition 2.8. We denote $i(f,u)(t)$ the Morse polynomial of f in u, defined by

$$i(f,u)(t) = \sum_{k=0}^{+\infty} \dim C_k(f,u)\, t^k.$$

We call *multiplicity* of u the number $i(f,u)(1) \in \mathbb{N} \cup \{+\infty\}$.

Next result gives a description of the critical groups of the functional $J_{\alpha,f}$ at u_0 in terms of the Morse index. The proof derives directly from [17, Theorem 2.3] (see also [16, Theorem 1.3]).

Theorem 2.9. *Let $p \in (1,2)$, $q \in [p,p^*)$, $\lambda > 0$, $f \in C^1(\overline{\Omega})$ and $\alpha > 0$. If u_0 is a critical point of $J_{\alpha,f}$ and*

$$m(J_{\alpha,f}, u_0) = m^*(J_{\alpha,f}, u_0),$$

then u_0 is an isolated critical point of $J_{\alpha,f}$ and we have

$$\begin{cases} C_m(J_{\alpha,f}, u_0) \approx \mathbb{G} & \text{if } m = m(J_{\alpha,f}, u_0)\,,\\ C_m(J_{\alpha,f}, u_0) = \{0\} & \text{if } m \neq m(J_{\alpha,f}, u_0)\,. \end{cases}$$

Remark 2.10. If the assumptions of the previous theorem are satisfied, then the multiplicity of u_0 is 1, namely $i(J_{\alpha,f}, u_0)(1) = 1$.

In order to prove Theorem 1.3, we recall an abstract theorem, proved in [18] (see also [6] and [13]).

Theorem 2.11. *Let A be a open subset of a Banach space X. Let f be a C^1 functional on A and $u \in A$ be an isolated critical point of f. Assume that there exists an open neighborhood U of u such that $\overline{U} \subset A$, u is the only critical point of f in $\overline{U}$ and f satisfies the Palais–Smale condition in $\overline{U}$.*
Then there exists $\bar{\mu} > 0$ such that, for any $g \in C^1(A, \mathbb{R})$ such that

- $\|f - g\|_{C^1(A)} < \bar{\mu}$,
- *g satisfies the Palais–Smale condition in $\overline{U}$,*
- *g has a finite number $\{u_1, u_2, \dots, u_m\}$ of critical points in U,*

we have

$$\sum_{j=1}^{m} i(g, u_j)(t) = i(f, u)(t) + (1+t)z(t),$$

where $z(t)$ is a formal series with coefficients in $\mathbb{N} \cup \{+\infty\}$.

3. The finite dimensional reduction

From now on, we assume that $p \in (1, 2)$. Let $\lambda \in (0, \lambda^*)$, where $\lambda^* > 0$ is defined by Theorem 1.2. If (P_λ) has at least $\mathcal{P}_1(\Omega)$ distinct solutions, then the assert of Theorem 1.3 is proved, otherwise I_λ has a finite number of isolated critical points $\bar{u}_1, \dots \bar{u}_k$ having multiplicities $\bar{m}_1, \dots \bar{m}_k$ where

$$1 \le k < \mathcal{P}_1(\Omega) \qquad \text{and} \qquad \sum_{j=1}^{k} \bar{m}_j \ge \mathcal{P}_1(\Omega).$$

Let (α_n) be a sequence such that $\alpha_n \to 0$ and J_{α_n} be defined by (2.2). If J_{α_n} has at least $\mathcal{P}_1(\Omega)$ distinct critical points, then we just choose $f_n = 0$, otherwise J_{α_n} has $h_n < \mathcal{P}_1(\Omega)$ isolated critical points $u_1, \ \dots u_{h_n}$, having multiplicities $m_1, \ \dots m_{h_n}$.

For simplicity, we will often omit the dependence from n of u_i and their related objects. Let R be defined by Theorem 2.5, $R_1 \in (0, R]$ be such that, the sets $\overline{B_{R_1}(\bar{u}_j)}$ are pairwise disjoint, and denote by

$$A = \bigcup_{j=1}^{k} B_{R_1}(\bar{u}_j). \tag{3.1}$$

Theorems 2.2, 2.11 and Remark 2.6 assure that, if n is sufficiently large, then $h_n \ge k$, any $u_i \in A$ and

$$\sum_{i=1}^{h_n} m_i \ge \sum_{j=1}^{k} \bar{m}_j \ge \mathcal{P}_1(\Omega). \tag{3.2}$$

For any $i \in \{1, \dots h_n\}$, the derivative of the smooth quadratic form $Q_{u_i}^{\alpha_n} : W_0^{1,2}(\Omega) \to \mathbb{R}$ is a compact perturbation of the Riesz isomorphism, so it is standard that there exists a direct sum decomposition

$$W_0^{1,2}(\Omega) = V_i \oplus \widehat{W}_i$$

such that

$$
\begin{aligned}
&\dim V_i = m_i^* = m^*(J_{\alpha_n}, u_i) < +\infty, \\
&\widehat{W}_i = \left\{ w \in W_0^{1,2}(\Omega) : \int_\Omega vw\,dx = 0 \quad \text{for any } v \in V_i \right\}, \\
&Q_{u_i}^{\alpha_n}(v+w) = Q_{u_i}^{\alpha_n}(v) + Q_{u_i}^{\alpha_n}(w) && \text{for any } v \in V_i \text{ and } w \in \widehat{W}_i\,, \\
&Q_{u_i}^{\alpha_n}(v) \le 0 && \text{for any } v \in V_i\,, \\
&Q_{u_i}^{\alpha_n}(w) > 0 && \text{for any } w \in \widehat{W}_i \setminus \{0\}\,.
\end{aligned}
$$

Moreover, either $V_i = \{0\}$ or $V_i = \operatorname{span}\left\{e_1, \dots, e_{m_i^*}\right\}$ and, for any $j = 1, \dots m_i^*$, there is $\lambda_j \le 0$ such that $e_j \in W_0^{1,2}(\Omega) \setminus \{0\}$ is a solution of

$$B_{\alpha_n}(u_i)(z, e_j) = \lambda_j \int_\Omega z e_j\; dx$$

for any $z \in W_0^{1,2}(\Omega)$. Hence e_j weakly solves the equation

$$-\operatorname{div}[\Psi''_{\alpha_n}(\nabla u_i)\nabla e_j] - g'_{\alpha_n}(u_i)e_j = \lambda_j e_j \qquad \text{in } W^{-1,p'}(\Omega)\,,$$

where Ψ_{α_n} and g_{α_n} are defined according to (2.3).

Therefore we have

$$\int_\Omega \Psi''_{\alpha_n}(\nabla u_i)[\nabla e_j]^2\,dx - \int_\Omega g'_{\alpha_n}(u_i)e_j^2\,dx = \lambda_j \int_\Omega e_j^2\,dx \le 0\,.$$

On the other hand

(3.3)
$$\Psi''_{\alpha_n}(\nabla u(x))[\xi]^2 \ge \frac{(p-1)}{\left(\alpha_n + \|\nabla u\|_\infty^2\right)^{\frac{2-p}{2}}}\,|\xi|^2 \quad \text{for a.e. } x \in \Omega \text{ and } \forall\, \xi \in \mathbb{R}^N, \quad \text{if } \|\nabla u\|_\infty < +\infty,$$

whence

$$\frac{(p-1)}{\left(\alpha_n + \|\nabla u_i\|_\infty^2\right)^{\frac{2-p}{2}}} \int_\Omega |\nabla e_j|^2\,dx - \int_\Omega g'_{\alpha_n}(u_i)e_j^2\,dx \le 0\,.$$

Since $g'_{\alpha_n}(u_i) \in L^\infty(\Omega)$, it is standard (see e.g. [28]) that $e_j \in L^\infty(\Omega)$, whence $V_i \subseteq W_0^{1,2}(\Omega) \cap L^\infty(\Omega) \subseteq W_0^{1,p}(\Omega)$, as $p < 2$.

Now we introduce

$$\widetilde{W}_i = \left\{ w \in L^1(\Omega) : \int_\Omega vw\,dx = 0 \quad \text{for any } v \in V_i \right\},$$

then any $\widetilde{W}_i$ is a closed linear subspace of $L^1(\Omega)$ and

$$L^1(\Omega) = V_i \oplus \widetilde{W}_i\,.$$

Set $W_i = \tilde{W}_i \cap W_0^{1,p}(\Omega)$ which is a closed linear subspace of $W_0^{1,p}(\Omega)$, we infer

$$W_0^{1,p}(\Omega) = V_i \oplus W_i.$$

Therefore for any $i = 1, \dots h_n$, there are V_i and W_i subspaces of $W_0^{1,p}(\Omega)$ such that

- $W_0^{1,p}(\Omega) = V_i \oplus W_i\,;$
- $V_i \subset L^\infty(\Omega) \cap W_0^{1,2}(\Omega) \subset W_0^{1,p}(\Omega)$ with $\dim(V_i) = m^*(J_{\alpha_n}, u_i)$;
- $\int_\Omega vw = 0$ for any $v \in V_i$, $w \in W_i$.

According to [27, 25, 29, 33, 34], $u_i \in C^{1,\beta_i}(\overline{\Omega})$ for some $\beta_i \in (0,1]$ (see also Theorems 3.1 and 3.2 in [23]), then we infer that $e_j \in C^1(\overline{\Omega})$ (see Theorem 8.8 and Theorem 8.10 in [26]), so that $V_i \subset C^1(\overline{\Omega})$.

Setting

$$V^n = V_1 + V_2 + \cdots + V_{h_n} \quad \text{and} \quad W^n = \bigcap_{i=1}^{h_n} W_i, \tag{3.4}$$

we still have:

- $W_0^{1,p}(\Omega) = V^n \oplus W^n$;
- $V^n \subset C^1(\overline{\Omega})$ is finite dimensional and $W^n \subset W_i$ for any $i = 1, \dots h_n$;
- $\int_\Omega vw = 0$ for any $v \in V^n$, $w \in W^n$.

We recall the following regularity results (see Theorems 3.1 and 3.2 in [17]).

Theorem 3.1. *For every $u_0 \in W_0^{1,p}(\Omega)$, there exists $r > 0$ such that, for any $u \in W_0^{1,p}(\Omega)$ and $f \in L^\infty(\Omega)$ satisfying*

$$\begin{cases} \displaystyle\int_\Omega [\nabla \Psi_{\alpha_n}(\nabla u) \cdot \nabla v - g_{\alpha_n}(u) v] \, dx = \int_\Omega f v \, dx \qquad \forall v \in W_0^{1,p}(\Omega) \\ \\ \|\nabla u - \nabla u_0\|_p \le r \,, \end{cases}$$

we have $u \in L^\infty(\Omega)$ and

$$\|u\|_\infty \le C\left(\|f\|_\infty\right) .$$

Theorem 3.2. *Assume that $\partial\Omega$ is of class $C^{1,\delta}$ for some $\delta \in (0,1]$. Then there exists $\beta \in (0,1]$ such that any solution u of*

$$\begin{cases} u \in W_0^{1,p}(\Omega) \,, \\ - \operatorname{div} [\nabla \Psi_{\alpha_n}(\nabla u)] = w_0 - \operatorname{div}\, w_1 \qquad \text{in } W^{-1,p'}(\Omega) \,, \end{cases}$$

with $w_0 \in L^\infty(\Omega)$ and $w_1 \in C^{0,\delta}(\overline{\Omega}; \mathbb{R}^N)$, belongs also to $C^{1,\beta}(\overline{\Omega})$ and we have

$$\|u\|_{C^{1,\beta}} \le C\left(\|w_0\|_\infty, \|w_1\|_{C^{0,\delta}}\right) .$$

Lemma 3.3. *There are $\tilde{R} > 0$, $M > 0$ and $\beta \in (0,1]$ such that for any $i \in \{1, \dots h_n\}$ and $v \in V^n \cap \overline{B_{\tilde{R}}(0)}$, the derivative of the functional $F_{n,i,v} : W^n \to \mathbb{R}$ defined by*

$$F_{n,i,v}(w) = J_{\alpha_n}(u_i + v + w)$$

is of class $(S)_+$ in $W^n \cap \overline{B_{\tilde{R}}(0)}$ and if $w \in W^n \cap B_{\tilde{R}}(0)$ is a critical point of $F_{n,i,v}$, then $v + w \in C^{1,\beta}(\overline{\Omega})$, with $\|v + w\|_{C^{1,\beta}} \le M$.
Moreover the sets $\overline{B_{2\tilde{R}}(u_i)}$ are disjointed and $\bigcup_{i=1}^{h_n} \overline{B_{2\tilde{R}}(u_i)} \subset A$, where A is introduced by (3.1).

Proof. Let $R > 0$ be introduced by Theorem 2.5 and $R_2 \in (0, R/2]$ be such that $\bigcup_{i=1}^{h_n} \overline{B_{2R_2}(u_i)} \subset A$ and

$$i_1 \ne i_2 \quad \Rightarrow \qquad \overline{B_{2R_2}(u_{i_1})} \cap \overline{B_{2R_2}(u_{i_2})} = \emptyset.$$

As $2R_2 \le R$, Theorem 2.5 assures that J'_{α_n} is of class $(S)_+$ in any $\overline{B_{2R_2}(u_i)}$. Hence, for any $i \in \{1, \dots h_n\}$ and for any $v \in V \cap \overline{B_{R_2}(0)}$, the derivative of the functional

$$w \in W \quad \mapsto \quad J_{\alpha_n}(u_i + v + w) \in \mathbb{R}$$

is of class $(S)_+$ in $W \cap \overline{B_{R_2}(0)}$.

Let $\bar{n} = \dim V^n$ and $(\bar{e}_j)_{1 \le j \le \bar{n}}$ be an orthonormal basis of V^n according to the L^2 norm. There is $K > 0$ such that

$$\| \sum_{j=1}^{\bar{n}} \left\langle J'_{\alpha_n}(u), \bar{e}_j \right\rangle \bar{e}_j \|_\infty \le K \qquad \forall u \in \bigcup_{i=1}^{h_n} \overline{B_{2R_2}(u_i)}. \tag{3.5}$$

If $i \in \{1, \dots h_n\}$, $v \in V^n \cap \overline{B_{R_2}(0)}$ and $w \in W^n \cap B_{R_2}(0)$ is a critical point of $F_{n,i,v}$, then for any $u \in W_0^{1,p}(\Omega)$

$$\left\langle J'_{\alpha_n}(u_i + v + w), u \right\rangle = \int_\Omega f_i(v, w)\, u \, dx$$

where $f_i(v, w) = \sum_{j=1}^{\bar{n}} \left\langle J'_{\alpha_n}(u_i + v + w), \bar{e}_j \right\rangle \bar{e}_j \ \in L^\infty(\Omega)$ and by (3.5) $\|f_i(v, w)\|_\infty \le K$. Hence, by Theorem 3.1, there is $\tilde{R} \in (0, R_2]$ such that if $\bar{v} \in V^n \cap \overline{B_{\tilde{R}}(0)}$ and $\bar{w} \in W^n \cap B_{\tilde{R}}(0)$ is a critical point of $F_{n,i,\bar{v}}$, then $u_i + \bar{v} + \bar{w} \in L^\infty(\Omega)$ and $\|u_i + \bar{v} + \bar{w}\|_\infty \le C(K)$. So, applying Theorem 3.2 with $w_1 = 0$, the proof is completed. □

Lemma 3.4. *For any $M > 0$, there exist $r, \delta > 0$ such that*

$$B_{\alpha_n}(u)(w, w) \ge \delta \int_\Omega |\nabla w|^2 \, dx$$

for every $u \in \bigcup_{i=1}^{h_n} \overline{B_r(u_i)} \cap W^{1,\infty}(\Omega)$ such that $\|u\|_\infty + \|\nabla u\|_\infty \le M$ and every $w \in W^n \cap W_0^{1,2}(\Omega)$.

Proof. By contradiction, let $i \in \{1, \dots h_n\}$, $M > 0$, (v_k) in $W_0^{1,p}(\Omega) \cap W^{1,\infty}(\Omega)$ and $(w_k) \subset W^n \cap W_0^{1,2}(\Omega)$ be such that $v_k \to u_i$, $\|v_k\|_\infty + \|\nabla v_k\|_\infty \le M$ and

$$B_{\alpha_n}(v_k)(w_k, w_k) < \frac{1}{k} \int_\Omega |\nabla w_k|^2 \, dx\,. \tag{3.6}$$

Without loss of generality, we may assume that $|\nabla w_k|_2 = 1$. Then, up to a subsequence, (w_k) is weakly convergent to some w in $W_0^{1,2}(\Omega) \cap W^n$. Since $\|v_k\|_\infty \le M$ and (w_k) strongly converges to w in $L^2(\Omega)$, by Lebesgue's dominated convergence theorem, we infer

$$\int_\Omega g'_{\alpha_n}(u_i) w^2 \, dx = \lim_{k \to \infty} \int_\Omega g'_{\alpha_n}(v_k) w_k^2 \, dx.$$

Combining with Fatou's Lemma and (3.6), we get

$$0 \le B_{\alpha_n}(u_i)(w, w) \le \liminf_{k \to \infty} B_{\alpha_n}(v_k)(w_k, w_k) \le \limsup_{k \to \infty} B_{\alpha_n}(v_k)(w_k, w_k) \le 0$$

whence, in particular, $w = 0$.

As $\|\nabla v_k\|_\infty \le M$, taking account of (3.3), there is $c = c(\alpha_n, p, M) > 0$ such that

$$c \, |\nabla w_k(x)|^2 \le \Psi''_{\alpha_n}(\nabla v_k(x))[\nabla w_k(x)]^2 \qquad \text{a.e. in } \Omega.$$

Since $|\nabla w_k|_2 = 1$, a contradiction follows as

$$c \le \lim_{k\to\infty} \int_\Omega \Psi''_{\alpha_n}(\nabla v_k)[\nabla w_k]^2\,dx = \lim_{k\to\infty} \big(B_{\alpha_n}(v_k)(w_k, w_k) + \int_\Omega g'_{\alpha_n}(v_k) w_k^2\,dx\big) = 0.$$

□

We recall a result relating the minimality in the C^1-topology and that in the $W_0^{1,p}$-topology. For the proof, involving Theorem 3.2, see Theorem 3.6 in [17] and references therein.

Theorem 3.5. *Assume that $\partial\Omega$ of class $C^{1,\beta}$ and that $u_0 \in W_0^{1,p}(\Omega) \cap C^{1,\beta}(\overline{\Omega})$ for some $\beta \in (0,1]$. Suppose also that $W_0^{1,p}(\Omega) = V \oplus W$, where V is a finite dimensional subspace of $W_0^{1,p}(\Omega)$, W is closed in $W_0^{1,p}(\Omega)$ and the projection $P_V : W_0^{1,p}(\Omega) \to V$, associated with the direct sum decomposition, is continuous from the topology of $L^1(\Omega)$ to that of V.*

If u_0 is a strict local minimum for the functional J_{α_n} along $u_0 + (W \cap C^1(\overline{\Omega}))$ for the $C^1(\overline{\Omega})$-topology, then u_0 is a strict local minimum of J_{α_n} along $u_0 + W$ for the $W_0^{1,p}(\Omega)$-topology.

Lemma 3.6. *If $u, v \in W^{1,\infty}(\Omega)$, then*

$$J_{\alpha_n}(v) = J_{\alpha_n}(u) + \langle J'_{\alpha_n}(u), v-u\rangle + \int_0^1 (1-t) B_{\alpha_n}(u + t(v-u))(v-u, v-u)dt \tag{3.7}$$

$$\langle J'_{\alpha_n}(v) - J'_{\alpha_n}(u), z\rangle = \int_0^1 B_{\alpha_n}(u + t(v-u))(v-u, z)dt \qquad \forall z \in W_0^{1,2}(\Omega). \tag{3.8}$$

Proof. Let Ψ_{α_n} and G_{α_n} be defined according to (2.3), by Lemma 2.1

$$z \in W_0^{1,p}(\Omega) \mapsto \int_\Omega G_{\alpha_n}(z(x))\,dx \quad \text{ belongs to } C^2(W_0^{1,p}(\Omega), \mathbb{R}). \tag{3.9}$$

As $\Psi_{\alpha_n} \in C^2(\mathbb{R}^N, \mathbb{R})$, for any ξ, $\eta \in \mathbb{R}^N$

$$\Psi_{\alpha_n}(\eta) = \Psi_{\alpha_n}(\xi) + (\nabla\Psi_{\alpha_n}(\xi)|(\eta-\xi)) + \int_0^1 (1-t)\Psi''_{\alpha_n}(\xi + t(\eta-\xi))[\eta-\xi]^2 dt$$

so, a.e.in Ω

$$\begin{aligned}\Psi_{\alpha_n}(\nabla v(x)) &= \Psi_{\alpha_n}(\nabla u(x)) + (\nabla\Psi_{\alpha_n}(\nabla u(x))|(\nabla v(x) - \nabla u(x)))\\ &+ \int_0^1 (1-t)\Psi''_{\alpha_n}(\nabla u(x) + t(\nabla v(x) - \nabla u(x)))[\nabla v(x) - \nabla u(x)]^2 dt.\end{aligned}$$

As $(x,t) \mapsto (1-t)\Psi''_{\alpha_n}(\nabla u(x) + t(\nabla v(x) - \nabla u(x)))[\nabla v(x) - \nabla u(x)]^2$ belongs to $L^1([0,1]\times\Omega)$, Fubini's Theorem gives that

$$\begin{aligned}\int_\Omega \Psi_{\alpha_n}(\nabla v(x))dx &= \int_\Omega \Psi_{\alpha_n}(\nabla u(x))dx + \int_\Omega (\nabla\Psi_{\alpha_n}(\nabla u(x))|(\nabla v(x) - \nabla u(x)))\,dx\\ &+ \int_0^1 \left(\int_\Omega \Psi''_{\alpha_n}(\nabla u(x) + t(\nabla v(x) - \nabla u(x)))[\nabla v(x) - \nabla u(x)]^2 dx\right) dt\end{aligned}$$

so, by (3.9), we infer (3.7).

Moreover there is $C = C(\alpha_n) > 0$ such that

$$|\Psi''_{\alpha_n}(\eta)[\xi_1, \xi_2]| \le C|\xi_1||\xi_2| \qquad \forall\, \eta, \xi_1, \xi_2 \in \mathbb{R}^N.$$

Hence, for any $u, v \in W^{1,\infty}(\Omega)$ and $z \in W_0^{1,2}(\Omega)$,
$(1-t)\Psi''_{\alpha_n}(\nabla u(x)+t(\nabla v(x)-\nabla u(x)))[\nabla v(x)-\nabla u(x), \nabla z(x)]$ belongs to $L^1([0,1]\times\Omega)$, so (3.8) follows by Fubini's Theorem together with (3.9). □

Theorem 3.7. *For any $i \in \{1, \dots h_n\}$, u_i is a local strict minimum point of J_{α_n} along $u_i + W^n$, according to the topology of $W_0^{1,p}(\Omega)$.*

Proof. Let us fix $i \in \{1, \dots h_n\}$ and $M_i > \|u_i\|_{C^1}$. According to Lemma 3.4, there are $r_i, \delta_i > 0$ such that

$$B_{\alpha_n}(u)(w,w) \ge \delta_i \int_\Omega |\nabla w|^2 dx \qquad \forall u \in B_{r_i}(u_i) \cap C^1(\overline{\Omega}),\ \|u\|_{C^1} \le M_i. \tag{3.10}$$

Let us choose $k_i > 0$ so that

$$\|u_i\|_{C^1} + k_i < M_i \quad \text{ and } \quad \{u \in C^1(\overline{\Omega}) \ :\ \|u\|_{C^1} \le k_i\} \subset B_{r_i}(0)\,.$$

If $w \in W^n \cap C^1(\overline{\Omega})$ and $\|w\|_{C^1} \le k_i$, by (3.10) and (3.7) we infer

$$J_{\alpha_n}(u_i + w) = J_{\alpha_n}(u_i) + \int_0^1 (1-t) B_{\alpha_n}(u_i + tw)(w,w)\, dt \ge J_{\alpha_n}(u_i) + \frac{\delta_i}{2}\int_\Omega |\nabla w|^2 dx$$

so u_i is a local strict minimum point of J_{α_n} along $u_i + W^n$, according to the topology of $C^1(\overline{\Omega})$. Finally we apply Theorem 3.5.

□

Theorem 3.8. *There exist $M, r > 0$, $\beta \in (0,1]$ and $\varrho \in (0, r]$ such that for any $i \in \{1, \dots h_n\}$ and $v \in V^n \cap \overline{B_\varrho(0)}$ there exists one and only one $\psi_i(v) \in W^n \cap B_r(0)$ such that*

$$J_{\alpha_n}(u_i + v + \psi_i(v)) \le J_{\alpha_n}(u_i + v + w) \qquad \forall w \in W^n \cap \overline{B_r(0)},$$

moreover $v + \psi_i(v) \in C^{1,\beta}(\overline{\Omega})$, $\|v + \psi_i(v)\|_{C^{1,\beta}(\overline{\Omega})} \le M$ and $\psi_i(v)$ is the only element of $W^n \cap \overline{B_r(0)}$ such that

$$\langle J'_{\alpha_n}(u_i + v + \psi_i(v)), w\rangle = 0 \qquad \forall w \in W^n.$$

Furthermore, denoting by

$$U_i = u_i + (V^n \cap B_\varrho(0)) + (W^n \cap B_r(0)), \tag{3.11}$$

we have $\overline{U_i} \cap \overline{U_j} = \emptyset$ if $i \ne j$ and $\bigcup_{i=1}^{h_n} \overline{U_i} \subset A$, where A is the open bounded set defined by (3.1).
Finally, there exists $\delta > 0$ such that

$$B_{\alpha_n}(u_i + u)(w,w) \ge \delta \int_\Omega |\nabla w|^2 dx \tag{3.12}$$

for every $(u_i + u) \in \bigcup_{i=1}^{h_n} \overline{U}_i \cap C^1(\overline{\Omega})$ such that $\|u\|_{C^1} \le M$ and every $w \in W^n \cap W_0^{1,2}(\Omega)$.

Proof. Let $M, \tilde{R}$ and β be as in Lemma 3.3. By Lemma 3.4 there are $r \in (0, \tilde{R})$ and $\delta > 0$ such that

$$B_{\alpha_n}(u_i + u)(w,w) \ge \delta \int_\Omega |\nabla w|^2 dx$$

for every $u \in \overline{B_{2r}(0)} \cap C^1(\overline{\Omega})$ such that $\|u\|_{C^1} \leq M$, every $w \in W^n \cap W_0^{1,2}(\Omega)$ and every $i \in \{1, \dots h_n\}$. By Theorem 3.7 we can also assume that $J_{\alpha_n}(u_i) < J_{\alpha_n}(u_i + w)$ for every $w \in \overline{B_r(0)} \cap W^n$ with $w \neq 0$ and every $i \in \{1, \dots h_n\}$. In particular, we get (3.11) and (3.12), for any arbitrary $\varrho \in (0, r]$.

We claim that there exists a suitable $\varrho \in (0, r]$ such that

$$J_{\alpha_n}(u_i + v) < J_{\alpha_n}(u_i + v + w) \tag{3.13}$$

for every $i \in \{1, \dots h_n\}$, $v \in V^n \cap \overline{B_\varrho(0)}$ and $w \in W^n$ such that $\|\nabla w\|_p = r$.

By contradiction, let $i \in \{1, \dots h_n\}$, $(v_k) \subset V^n$ and $(w_k) \subset W^n$ such that $v_k \to 0$, $\|\nabla w_k\|_p = r$ and $J_{\alpha_n}(u_i + v_k) \geq J_{\alpha_n}(u_i + v_k + w_k)$. Up to a subsequence, (w_k) is weakly convergent to some $\bar{w} \in W^n \cap \overline{B_r(0)}$. Then $(u_i + v_k + w_k)$ is weakly convergent to $u_i + \bar{w}$ with

$$\limsup_k J_{\alpha_n}(u_i + v_k + w_k) \leq \lim_k J_{\alpha_n}(u_i + v_k) = J_{\alpha_n}(u_i) \leq J_{\alpha_n}(u_i + \bar{w}) \,.$$

Combining Proposition 2.4 with Lemma 3.3, we deduce that $(u_i + v_k + w_k)$ is strongly convergent to $u_i + \bar{w}$, whence $J_{\alpha_n}(u_i + \bar{w}) = J_{\alpha_n}(u_i)$ with $\|\nabla \bar{w}\|_p = r$, and a contradiction follows.

Again from Proposition 2.4 and Lemma 3.3 we know that $\{w \mapsto J_{\alpha_n}(u_i + v + w)\}$ is weakly lower semicontinuous on $W^n \cap \overline{B_r(0)}$ for any $v \in V^n \cap \overline{B_\varrho(0)}$. Therefore there exists a minimum point $\overline{w} \in W^n \cap \overline{B_r(0)}$ and by (3.13) $\overline{w} \in B_r$. So, in particular

$$\langle J'_{\alpha_n}(u_i + v + \overline{w}), w \rangle = 0 \qquad \text{for any } w \in W^n \,. \tag{3.14}$$

Let us assume $w_1, w_2 \in B_r(0)$ be such that $\langle J'_{\alpha_n}(u_i+v+w_1), w \rangle = \langle J'_{\alpha_n}(u_i+v+w_2), w \rangle = 0$ for any $w \in W^n$. Then, from Lemma 3.3, we infer that $v + w_1$, $v + w_2 \in C^{1,\beta}(\overline{\Omega})$ with $\|v + w_1\|_{C^{1,\beta}}$, $\|v + w_2\|_{C^{1,\beta}} \leq M$. Hence, by (3.12) and (3.8), we get

$$\begin{aligned} \delta \int_\Omega |\nabla(w_2 - w_1)|^2 dx &\leq \int_0^1 B_{\alpha_n}\left(u_i + v + w_1 + t\,(w_2 - w_1)\right)(w_2 - w_1, w_2 - w_1)\, dt \\ &= \langle J'_{\alpha_n}(u_i + v + w_2) - J'_{\alpha_n}(u_i + v + w_1), w_2 - w_1 \rangle = 0, \end{aligned}$$

so there is only one $\psi_i(v) = \overline{w}$ satisfying (3.14), moreover $v + \psi_i(v) \in C^{1,\beta}(\overline{\Omega})$ and $\|v + \psi_i(v)\|_{C^{1,\beta}(\overline{\Omega})} \leq M$. $\square$

Now we introduce the functionals $\varphi_i : v \in V^n \cap \overline{B_\varrho(0)} \to \mathbb{R}$ defined by

$$\varphi_i(v) = J_{\alpha_n}(u_i + v + \psi_i(v)).$$

Since we aim to apply the Sard's Lemma to the reduction map φ_i, it becomes crucial to investigate the C^2 regularity of φ_i. In the case $p \geq 2$, this fact is sharp, as the energy functional is C^2 (see [21, Section 2]). Unfortunately, when $1 < p < 2$, the functionals I_λ and $J_{\alpha,f}$ are only C^1. Despite this fact, we derive the C^2 regularity result of the reduction map φ_i.

Lemma 3.9. *For any $i = 1, \cdots h_n$, ψ_i is continuous from $V^n \cap \overline{B_\varrho(0)}$ in $W^n \cap C^1(\overline{\Omega})$ and of class C^1 into $W_0^{1,2}(\Omega)$. In addition,*

$$B_{\alpha_n}(u_i + z + \psi_i(z))(h + \langle \psi'_i(z), h \rangle, w) = 0 \tag{3.15}$$

for any $z \in V^n \cap \overline{B_\varrho(0)}$, $h \in V^n$ and $w \in W^n \cap W_0^{1,2}(\Omega)$.

Moreover, for any $i = 1, \cdots h_n$, the function φ_i is of class C^2 and, for any $z \in V^n \cap \overline{B_\varrho(0)}$ and $h, v \in V^n$

$$\langle \varphi_i'(z), h\rangle = \langle J'_{\alpha_n}(u_i + z + \psi_i(z)), h\rangle \tag{3.16}$$

$$\langle \varphi_i''(z)h, v\rangle = B_{\alpha_n}\big(u_i + z + \psi_i(z)\big)(h + \psi_i'(z)h, v). \tag{3.17}$$

Proof. Using the notations introduced in the previous Theorem, for any $i = 1, \ldots h_n$, the map $v \mapsto v + \psi_i(v)$ is defined from $V^n \cap \overline{B_\varrho(0)}$ into

$$K = \Big\{ u \in C^{1,\beta}(\overline{\Omega}) \ : \ \|u\|_{C^{1,\beta}(\overline{\Omega})} \leq M \Big\}$$

which is a compact subset of $C^1(\overline{\Omega})$. As J_{α_n} is continuous, we infer that any ψ_i is continuous from $V^n \cap \overline{B_\varrho(0)}$ in $W^n \cap C^1(\overline{\Omega})$.

In the remainder of this proof we will refer to any $i \in \{1, \ldots h_n\}$, $z, z + h \in V^n \cap \overline{B_\varrho(0)}$ and $z_1, z_2 \in W_0^{1,2}(\Omega)$.

Let us denote by $u_z^i = u_i + z + \psi_i(z)$ and $\omega_h = h + \psi_i(z + h) - \psi_i(z)$.

There exists $C > 0$ such that, for any $\tau \in [0, 1]$,

$$\big|B_{\alpha_n}(u_z^i + \tau\omega_h)(z_1, z_2)\big| \leq C \, \|z_1\|_{1,2} \, \|z_2\|_{1,2} \, . \tag{3.18}$$

As the map $z \mapsto \psi_i(z)$ is continuos from $V^n \cap \overline{B_\varrho(0)}$ into $C^1(\overline{\Omega})$, for any $\tau \in [0, 1]$ we have

$$\lim_{\|h\| \to 0} \frac{|(B_{\alpha_n}(u_z^i + \tau\omega_h) - B_{\alpha_n}(u_z^i))\,(z_1, z_2)|}{\|z_1\|_{1,2} \cdot \|z_2\|_{1,2}} = 0. \tag{3.19}$$

From (3.18) and (3.12) we infer that the bilinear form

$$(z_1, z_2) \in W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega) \ \mapsto \ B_{\alpha_n}(u_z^i)(z_1, z_2)$$

is continuous and positive definite on $W^n \cap W_0^{1,2}(\Omega)$.

Therefore, for any $h \in V^n$, the functional

$$w \mapsto B_{\alpha_n}(u_z^i)(w/2 + h, w)$$

admits one and only one minimum point $L_z h \in W^n \cap W_0^{1,2}(\Omega)$, which satisfies

$$B_{\alpha_n}(u_z^i)(L_z h + h, w) = 0 \qquad \forall w \in W^n \cap W_0^{1,2}(\Omega). \tag{3.20}$$

Moreover the map $L_z h : V^n \to W_0^{1,2}(\Omega)$ is linear and continuous, as V^n is finite dimensional.

For every $w \in W^n \cap W_0^{1,2}(\Omega)$, (3.8) gives

$$0 = \langle J'_{\alpha_n}(u_i + z + h + \psi_i(z + h)) - J'_{\alpha_n}(u_z^i), w\rangle = \int_0^1 B_{\alpha_n}\big(u_z^i + t\omega_h\big)(\omega_h, w)\, dt,$$

while (3.20) gives

$$B_{\alpha_n}(u_z^i)(L_z h + h, w),$$

so that

$$\int_0^1 B_{\alpha_n}\big(u_z^i+t\omega_h\big)(\psi_i(z+h)-\psi_i(z)-L_zh,\psi_i(z+h)-\psi_i(z)-L_zh)\,dt$$
$$=\int_0^1 B_{\alpha_n}\big(u_z^i+t\omega_h\big)(\omega_h,\psi_i(z+h)-\psi_i(z)-L_zh)\,dt$$
$$+\int_0^1\Big(B_{\alpha_n}(u_z^i)-B_{\alpha_n}\big(u_z^i+t\omega_h\big)\Big)(L_zh+h,\psi_i(z+h)-\psi_i(z)-L_zh)\,dt.$$

Hence from (3.12) and (3.19) we infer

$$\lim_{\|h\|\to 0}\frac{\|\psi_i(z+h)-\psi_i(z)-L_zh\|_{1,2}}{\|h\|}=0.$$

Therefore ψ_i is C^1 from $V^n\cap\overline{B_\varrho}$ into $W_0^{1,2}(\Omega)$ and $\psi_i(z)=L_z$.

From (3.8) we infer that, for a suitable $s\in(0,1)$,

$$\varphi_i(z+h)-\varphi_i(z)-\langle J'_{\alpha_n}(u_z^i),h\rangle$$
$$=J_{\alpha_n}(u_i+z+h+\psi_i(z+h))-J_{\alpha_n}(u_z^i)-\langle J'_{\alpha_n}(u_z^i),h\rangle$$
$$=\langle J'_{\alpha_n}(u_z^i+s\omega_h),\omega_h\rangle-\langle J'_{\alpha_n}(u_z^i),h\rangle$$
$$=\langle J'_{\alpha_n}(u_z^i+s\omega_h)-J'_{\alpha_n}(u_z^i),h\rangle+\langle J'_{\alpha_n}(u_z^i+s\omega_h)-J'_{\alpha_n}(u_z^i),\psi_i(z+h)-\psi_i(z)\rangle$$
$$=\int_0^1 B_{\alpha_n}\left(u_z^i+ts\omega_h\right)(s\omega_h,h)\,dt+\int_0^1 B_{\alpha_n}\left(u_z^i+ts\omega_h\right)(s\omega_h,\psi_i(z+h)-\psi_i(z))\,dt.$$

As the differentiability of ψ_i assures that $\|\psi_i(z+h)-\psi_i(z)\|_{1,2}\le c\|h\|$, by (3.18) we get

$$\lim_{\|h\|\to 0}\frac{\left|\varphi_i(z+h)-\varphi_i(z)-\langle J'_{\alpha_n}(u_z^i),h\rangle\right|}{\|h\|}=0$$

which proves (3.16).

Again by (3.8), for any $v\in V^n$

$$\langle\varphi_i'(z+h),v\rangle-\langle\varphi_i'(z),v\rangle-B_{\alpha_n}(u_z^i)(h+\langle\psi_i'(z),h\rangle,v)$$
$$=\langle J'_{\alpha_n}(u_i+z+h+\psi_i(z+h))-J'_{\alpha_n}(u_z^i),v\rangle-B_{\alpha_n}(u_z^i)(h+\langle\psi_i'(z),h\rangle,v)$$
$$=\int_0^1 B_{\alpha_n}\left(u_z^i+t\omega_h\right)(\omega_h,v)\,dt-B_{\alpha_n}(u_z^i)(h+\langle\psi_i'(z),h\rangle,v)$$
$$=\int_0^1\left(B_{\alpha_n}\left(u_z^i+t\omega_h\right)-B_{\alpha_n}(u_z^i)\right)(\omega_h,v)\,dt+B_{\alpha_n}(u_z^i)(\psi_i(z+h)-\psi_i(z)-\langle\psi_i'(z),h\rangle,v).$$

Finally, from (3.18), (3.19) and the differentiability of ψ_i, it follows that

$$\lim_{\|h\|\to 0}\frac{|\langle\varphi_i'(z+h),v\rangle-\langle\varphi_i'(z),v\rangle-B_{\alpha_n}(u_z^i)(h+\langle\psi_i'(z),h\rangle,v)|}{\|h\|}=0$$

which proves (3.17). $\square$

4. Proof of Theorem 1.3

Let us denote by $V = V^n$ and $W = W^n$, the spaces introduced in (3.4) and let $\{\bar{e}_1, \dots \bar{e}_{\bar{n}}\}$ be an L^2-orthonormal basis of V, where $\bar{n} = \dim V$. For any $v' \in V'$ we introduce the functional $L_{v'} : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$L_{v'}(u) = \int_\Omega \Big(\sum_{j=1}^{\bar{n}} \langle v', \bar{e}_j \rangle \bar{e}_j \Big) u \, dx.$$

For any $i = 1, \dots, h_n$, let μ_i be defined by Theorem 2.11 relatively to $J_{\alpha_n},\ u_i,\ A$ and U_i, where A is introduced in (3.1) and U_i in (3.11). Setting $\mu = \min\{\mu_1, \dots \mu_{h_n}\}$, let $\varepsilon > 0$ be such that, if $\|v'\|_{V'} \le \varepsilon$, then $\|L_{v'}\|_{C^1(A)} < \mu/h_n$.

Denoting by $\varepsilon_1 = \min\{\varepsilon, 1/n\}$, by Sard's Lemma there exists $v'_1 \in V'$ such that if $\|v'_1\|_{V'} < \varepsilon_1$ and $\varphi'_1(v) = v'_1$, then $\varphi''_1(v)$ is an isomorphism. Moreover there is $\beta_1 > 0$ such that if $v' \in V'$, $\|v'\|_{V'} \le \beta_1$ and $\varphi'_1(v) = v'_1 + v'$, then $\varphi''_1(v)$ is an isomorphism. Analogously, for $i = 2, \dots h_n$, there exist $\beta_i > 0$, $\varepsilon_i = \min\{\varepsilon_{i-1},\ \beta_{i-1}/(h_n - i + 1)\}$ and $v'_i \in V'$ such that $\|v'_i\|_{V'} < \varepsilon_i$ and if $v' \in V'$, $\|v'\|_{V'} \le \beta_i$ and $\varphi'_i(v) = v'_1 + \dots v'_i + v'$, then $\varphi''_i(v)$ is an isomorphism.

So we choose

$$f_n = \sum_{i=1}^{h_n} \sum_{j=1}^{\bar{n}} \langle v'_i, \bar{e}_j \rangle \bar{e}_j.$$

Let $J_n : W_0^{1,p}(\Omega) \to \mathbb{R}$ be defined by

$$J_n(u) = J_{\alpha_n, f_n}(u) = J_{\alpha_n}(u) - \int_\Omega f_n u$$

and $K_n = \{u \in \bigcup_{i=1}^{h_n} U_i \ : \ J'_n(u) = 0\}$.

Claim If there are $\tilde{u} \in K_n$ and $\bar{z} \in W_0^{1,2}(\Omega)$ such that $B(\tilde{u})(\,\cdot\,, \bar{z}) = 0$ in $W_0^{1,2}(\Omega)$, then $\bar{z} = 0$.

As $f_n \in V$, for any $w \in W$ we have $\langle J'_{\alpha_n}(\tilde{u}), w \rangle = \langle J'_n(\tilde{u}), w \rangle + \int_\Omega f_n w = 0$, so there are $i \in \{1, \dots h_n\}$ and $\tilde{v} \in V \cap B_\varrho(0)$ such that $\tilde{u} = u_i + \tilde{v} + \psi_i(\tilde{v})$.

Recalling (3.16), for any $v \in V$

$$\langle \varphi'_i(\tilde{v}), v \rangle = \langle J'_{\alpha_n}(\tilde{u}), v \rangle = \langle J'_n(\tilde{u}), v \rangle + \int_\Omega f_n v = \sum_{i=1}^{h_n} \langle v'_i, v \rangle$$

so that $\varphi'_i(\tilde{v}) = v'_1 + \dots v'_i + (v'_{i+1} + \dots v'_{h_n})$. As, by construction, $\|v'_{i+1} + \dots v'_{h_n}\| < \beta_i$, we get that

$$\varphi''_i(\tilde{v}) \text{ is an isomorphism.} \tag{4.1}$$

Let $\bar{z} = \bar{v} + \bar{w}$, where $\bar{v} \in V$ and $\bar{w} \in W$. By (3.15) we infer

$$B_{\alpha_n}(\tilde{u})\big(v + \langle \psi'_i(\tilde{v}), v \rangle, \bar{w}\big) = 0 \qquad \forall v \in V.$$

Combining with (3.17), for any $v \in V$ we get

$$\langle \varphi''_i(\tilde{v})v, \bar{v} \rangle = B_{\alpha_n}(\tilde{u})\big(v + \langle \psi'_i(\tilde{v}), v \rangle, \bar{v}\big) = B_{\alpha_n}(\tilde{u})\big(v + \psi'_i(\tilde{v})v, \bar{z}\big) = 0$$

thus (4.1) gives that $\bar{v} = 0$, hence $\bar{z} = \bar{w} \in W$.

Moreover, from (3.12),

$$\delta \int_\Omega |\nabla \bar{w}|^2 \leq B_{\alpha_n}(\tilde{u})(\bar{w}, \bar{w}) = 0$$

thus $\bar{w} = 0$ and the claim is proved.

This assures that, if $u \in K_n$, then $m^*(J_n, u) = m(J_n, u)$, so multiplicity of any $u \in K_n$ is 1 (see Remark 2.10).

By Theorem 2.11 and (3.2), J_n has at least $\mathcal{P}_1(\Omega)$ distinct critical points.

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Dipartimento di Matematica, Università degli Studi di Bari Aldo Moro, Via Orabona 4, 70125 Bari, Italy,

E-mail address: `silvia.cingolani@uniba.it`

Dipartimento di Meccanica, Matematica e Management, Politecnico di Bari, Via Orabona 4, 70125 Bari, Italy,

E-mail address: `giuseppina.vannella@poliba.it`