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Born-Infeld problem with general nonlinearity ☆

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Abstract

In this paper, using variational methods, we look for non-trivial solutions to the following problem

$$\begin{cases} -\mathrm{div}\left(a(|\nabla u|^2)\nabla u\right) = g(u), & \text{in } \mathbb{R}^N,\ N \geq 3, \\ u(x) \to 0, & \text{as } |x| \to +\infty, \end{cases}$$

under general assumptions on the continuous nonlinearity g. We assume growth conditions of g at 0 and, in the zero mass case, growth conditions at infinity are imposed. If $a(s) = (1-s)^{-1/2}$, we obtain the well-known Born-Infeld operator, but we are able to study also a general class of a such that $a(s) \to +\infty$ as $s \to 1^-$. We find a radial solution to the problem with finite energy.

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1. Introduction

Almost a century ago, Born and Infeld introduced a new electromagnetic theory in a series of papers (see [16–19]) as a nonlinear alternative to the classical Maxwell theory. This theory was proposed to provide a model presenting a unitarian point of view to describe electrodynamics and had the notable feature to be a fine answer to the well-known *infinite-energy problem*. In the Born-Infeld model, indeed, the electromagnetic field generated by a point charge has finite energy. A crucial role is played by the following peculiar differential operator

$$\mathcal{Q}(u) = -\mathrm{div}\left(\frac{\nabla u}{\sqrt{1-|\nabla u|^2}}\right).$$

Such an operator is present also in classical relativity, where it represents the mean curvature operator in Lorentz-Minkowski space, see for instance [6,20].

In the last years many authors focused their attention on problems related to $\mathcal{Q}$ in the whole $\mathbb{R}^N$, with $N \geq 1$. In particular, some results for

$$-\mathrm{div}\left(\frac{\nabla u}{\sqrt{1-|\nabla u|^2}}\right) = \rho, \qquad \text{in } \mathbb{R}^N,$$

can be found in [10,12–15,24,27,28], under different assumptions about ρ. Here ρ can be considered as an assigned charge source. See also [5], where the Born-Infeld equation is coupled with the nonlinear Schrödinger one.

Little is still known, on the contrary, in the presence of a nonlinearity, namely, for equations of this type

$$-\mathrm{div}\left(\frac{\nabla u}{\sqrt{1-|\nabla u|^2}}\right) = g(u), \qquad \text{in } \mathbb{R}^N. \tag{1.1}$$

Let us observe that classical variational techniques do not work directly for this problem, due to the particular nature of the operator $\mathcal{Q}$. Indeed, at least formally, solutions of (1.1) are critical points of the functional

$$I(u) = \int_{\mathbb{R}^N} \left(1 - \sqrt{1-|\nabla u|^2}\right) - \int_{\mathbb{R}^N} G(u)\, dx,$$

where G is a primitive of g. However, since we have to impose the condition $|\nabla u| \leq 1$ a.e. in $\mathbb{R}^N$, the lack of regularity of the functional on the set $\{x \in \mathbb{R}^N : |\nabla u| = 1\}$ requires different and non-standard strategies.

One of the first paper dealing with this kind of problem using variational methods is [11], where $g(s) = |s|^{p-2}s$, for $p > 2^* = \frac{2N}{N-2}$ and $N \geq 3$. By means of suitable truncation arguments (that will be crucial in our approach, as we will see later), the existence of *finite energy* solutions is proved.

We mention, moreover, [2,3,33], where (1.1) was studied by means of ODE-techniques finding solutions which could have infinite energy. In particular, in [2,3], the existence of positive or sign-changing radial solutions is considered for a pure power nonlinearity or under suitable sign assumptions on g (a prototype of such nonlinearity is $g(s) = -\lambda s + s^p$, for $\lambda > 0$ and $p > 1$). In [33], instead, the existence of oscillating solutions of (1.1), namely, with an unbounded sequence of zeros, is proved for nonlinearities such that $g'(0) > 0$. Finally, in [7], a similar problem is considered in an exterior domain.

Our aim is to show existence of finite energy radial solutions involving a large class of operators and nonlinearities in the spirit of Berestycki and Lions [8,9] and we will present an adequate variational approach for the problem. More precisely we consider

$$\begin{cases} -\mathrm{div}\left(a(|\nabla u|^2)\nabla u\right) = g(u), & \text{in } \mathbb{R}^N,\ N \geq 3, \\ u(x) \to 0, & \text{as } |x| \to +\infty, \end{cases} \tag{1.2}$$

under the following assumptions about a:

(a0) $a : [0, 1) \to (0, +\infty)$ is continuous, of class $\mathcal{C}^1$ on $(0, 1)$, and $[0, 1) \ni s \mapsto a(s)s$ is strictly convex;
(a1) $\lim_{s \to 1^-} a(s) = +\infty$;

and g:

(g0) $g : \mathbb{R} \to \mathbb{R}$ is continuous and odd;
(g1) for some $\gamma \geq 2$, we have

$$-\infty < \liminf_{s \to 0} \frac{g(s)}{|s|^{\gamma-1}} \leq \limsup_{s \to 0} \frac{g(s)}{|s|^{\gamma-1}} = -m < 0;$$

(g2) there exists $\xi_0 > 0$ such that $G(\xi_0) > 0$, where

$$G(s) = \int_0^s g(t)\,dt, \quad \text{for } s \in \mathbb{R}.$$

Clearly, $a(s) = (1 - s)^\alpha$ with $\alpha < 0$ satisfies (a0), (a1), and we get the operator $\mathcal{Q}$ for $\alpha = -1/2$. Another important example is the following general mean curvature operator arising in the study of hypersurfaces in the Lorentz–Minkowski space $\mathbb{L}^{N+1}$ and in $\mathbb{R}^{N+1}$ given by

$$a(s) := \beta(1 - s)^{-1/2} - \gamma(1 + s)^{-1/2}, \quad \beta > 0, \gamma \geq 0, \tag{1.3}$$

see [20,23,29] and references therein.

With regard to g, by assumption (g1), the problem is in the so called *positive mass case*. We will consider also the *zero mass case*, namely, instead of (g1), we will assume

(g1′) for some $\gamma > 2^*$, we have

$$-\infty < \liminf_{s \to 0} \frac{g(s)}{|s|^{\gamma-1}} \leq \limsup_{s \to 0} \frac{g(s)}{|s|^{\gamma-1}} = 0.$$

If the constant γ in the assumption (g1′) is not greater than N, we need also a condition at infinity on g. More precisely, we require

(g1″) whenever $N \geq \gamma > 2^*$, $\limsup_{s\to+\infty} g(s)/|s|^{q^*-1} = 0$, for some $q \in \left(\frac{N\gamma}{N+\gamma}, N\right)$,

where $q^* = \frac{qN}{N-q}$. Observe that, clearly, we have $2^* < \gamma < q^*$ and it is easy to see that a pure power non-linearity $g(s) = |s|^{p-2}s$, with $p > 2^*$, satisfies assumptions (g1′) and (g1″). Therefore we generalize the existence results contained in [11].

We recall that these kinds of hypotheses about g were introduced for the first time in [8,9] for the study of

$$-\Delta u = g(u), \qquad \text{in } \mathbb{R}^N, \tag{1.4}$$

where $\gamma = 2$. However, we want to remark that, in contrast to what happens in these previous papers, in our case there is no assumption about the behaviour at infinity of g in the positive mass case or in the zero mass case if, in (g1′), $\gamma > N$. This is a direct consequence of the natural framework associated with (1.2), which has to take into account the condition $|\nabla u| \leq 1$ a.e. in $\mathbb{R}^N$: this ensures that each function is, actually, bounded. See Section 2 for more details.

An intermediate step for the study of (1.2), based on an approximation argument, has been widely studied in the literature, e.g., see [34] and references therein. Indeed, by the Taylor expansion of $\frac{1}{\sqrt{1-|u|}}$ to the k-th order, we arrive at the approximated problem

$$\mathcal{Q}(u) \approx -\Delta u - \frac{1}{2}\Delta_4 u - \frac{3}{2\cdot 2^2}\Delta_6 u - \cdots - \frac{(2k-3)!!}{(k-1)!\cdot 2^{k-1}}\Delta_{2k} u = g(u) \quad \text{in } \mathbb{R}^N. \tag{1.5}$$

Note that [34] deals precisely with (1.5), where g satisfying more restrictive Berestycki-Lions-type assumptions. In [34] (see also the references therein), it is not clear if one can solve (1.1) passing to the limit, as $k \to +\infty$. We would like to mention that some partial results using this approximation process have been obtained only in case of the fixed-charge source ρ on the right hand side instead of the nonlinear term $g(u)$, see, e.g., [12,13,27,28]. Therefore (1.1) requires a different variational approach presented in this work.

Our main result reads as follows.

Theorem 1.1. *Suppose that a satisfies* (a0), (a1) *and g satisfies* (g0) *and* (g2). *If, in addition,* (g1) *holds, or $\gamma > N$ and* (g1′) *holds, or $\gamma \leq N$ and both* (g1′), (g1″) *hold, then there exists a nontrivial radial solution u to* (1.2) *such that*

$$\int_{\mathbb{R}^N} A(|\nabla u|^2)\,dx,\ \int_{\mathbb{R}^N} a(|\nabla u|^2)|\nabla u|^2\,dx,\ \int_{\mathbb{R}^N} |G(u)|\,dx < +\infty,$$

where $A(s) = \int_0^s a(t)\,dt$.

We use a truncation argument applied to a in a similar way as in [11], but due to the lack of scaling of the nonlinearity, we use a different variational approach for (1.2). Inspired by [25,26] (see also [1,4,21,22]), we will adapt to our problem the method explored considering an auxiliary functional that allows to construct a suitable Palais-Smale sequence, which almost satisfies a Pohozaev type identity. The compactness properties of the general nonlinear term will be investigated in a similar way as in [31,32], see Sections 3 and 4 for more details.

The paper is organized as follows. In Section 2, we introduce our functional framework and some technical tools. Section 3 and Section 4 will deal, respectively, with the positive mass case and the zero mass one and, therein, we will prove our main result.

We conclude this introduction fixing some notations. For any $p \geq 1$, we denote by $L^p(\mathbb{R}^N)$ the usual Lebesgue spaces equipped by the standard norm $|\cdot|_p$. In our estimates, we will frequently denote by $C > 0$, $c > 0$ fixed constants, that may change from line to line, but never depend on the variable under consideration. We also use the notation $o_n(1)$ to indicate a quantity which goes to zero as $n \to +\infty$. Moreover, for any $R > 0$, we denote by B_R the ball of $\mathbb{R}^N$ centred at the origin with radius R. Finally, if u is a radial function of $\mathbb{R}^N$, with an abuse of notation, for any $x \in \mathbb{R}^N$, we denote $u(x) = u(r)$, with $r = |x|$.

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2. Functional framework

In this section we introduce the functional framework related to (1.2) with some useful continuous and compact embedding properties. Moreover, following [11], we present a truncated problem which will play a crucial role in our arguments.

Take any $q > 2$. Let $\mathcal{X}_0^{2,q}$ be the completion of $\mathcal{C}_0^\infty(\mathbb{R}^N)$ with respect to the following norm

$$\|u\|_0 = \left(|\nabla u|_2^2 + |\nabla u|_q^2\right)^{1/2}.$$

Recall that

$$\mathcal{X}_0^{2,q} \hookrightarrow L^p(\mathbb{R}^N), \qquad \text{for } p \in \begin{cases} [2^*, q^*] & \text{if } q < N, \\ [2^*, +\infty) & \text{if } q = N, \\ [2^*, +\infty] & \text{if } q > N, \end{cases}$$

and, denoting

$$\mathcal{X}_0 := \mathcal{X}_{0,\mathrm{rad}}^{2,q} = \left\{u \in \mathcal{X}_0^{2,q} : u \text{ radially symmetric}\right\},$$

we have

$$\mathcal{X}_0 \hookrightarrow\hookrightarrow L^p(\mathbb{R}^N), \qquad \text{for } p \in \begin{cases} (2^*, q^*) & \text{if } q < N, \\ (2^*, +\infty) & \text{if } q \geq N, \end{cases}$$

see e.g. [11,34]. Moreover, as in [11,35], we have the following

Lemma 2.1. *Let $p \in [2, q]$, if $q < N$, and $p \in [2, N)$, if $q \geq N$. Then there exists $C > 0$ (depending only on N and p) such that for all $u \in \mathcal{X}_0$, there holds*

$$|u(x)| \leq C|x|^{-\frac{N-p}{p}} |\nabla u|_p,$$

for almost every $x \in \mathbb{R}^N \setminus \{0\}$.

In the positive mass case we always assume that $q > N$ and let $\mathcal{X}^{2,q,\gamma}$ be the completion of $\mathcal{C}_0^\infty(\mathbb{R}^N)$ with respect to the following norm

$$\|u\| = \left(|\nabla u|_2^2 + |\nabla u|_q^2 + |u|_\gamma^2\right)^{1/2}$$

and, clearly, if $\gamma \geq 2^*$, then $\mathcal{X}^{2,q,\gamma}$ and $\mathcal{X}_0^{2,q}$ coincides. Moreover $\mathcal{X}^{2,q,\gamma}$ is continuously embedded into $L^p(\mathbb{R}^N)$ for $p \in [\min\{2^*, \gamma\}, +\infty]$ and

$$\mathcal{X} := \mathcal{X}_{\mathrm{rad}}^{2,q,\gamma} = \left\{u \in \mathcal{X}^{2,q,\gamma} : u \text{ radially symmetric}\right\}$$

embeds compactly into $L^p(\mathbb{R}^N)$, for $p \in (\min\{2^*, \gamma\}, +\infty)$.

The following lemma is an extension of the well-known Strauss Lemma [35] and the proof is standard, cf. [36].

Lemma 2.2. *Let $p \geq 2$. There exists $C = C(N, p) > 0$ such that for all $u \in W_{\mathrm{rad}}^{1,p}(\mathbb{R}^N)$, $N \geq 2$ there holds*

$$|u(x)| \leq C|x|^{-\frac{N-1}{p}} \|u\|_{W^{1,p}},$$

for all $|x| \geq 1$.

Lemma 2.3. *Let $N \geq 2$, $\gamma \geq 2$ and $q > \max\{N, \gamma\}$. There exists $C = C(N, \gamma, q) > 0$ such that for all $u \in \mathcal{X}$ there holds*

$$|u(x)| \leq C|x|^{-\frac{N-1}{\gamma}} \|u\|,$$

for all $|x| \geq 1$.

Proof. Since $\gamma \geq 2$ and $q > \max\{N, \gamma\}$, by interpolation arguments $\mathcal{X} \hookrightarrow W_{\mathrm{rad}}^{1,\gamma}(\mathbb{R}^N)$ and the conclusion follows from Lemma 2.2, where $p = \gamma$. □

In a similar way as in [11] for $\mathcal{Q}$ we introduce a truncated problem. Let us fix $\theta_1 \in (0,1)$. For any $\theta \in (0, \theta_1]$ we fix $q = q(\theta) > N$ such that

$$q \geq 2\frac{a'(1-\theta)(1-\theta) + a(1-\theta)}{a(1-\theta)}. \tag{2.1}$$

Then we define a continuous function $a_\theta : [0, +\infty) \to \mathbb{R}^+$ by

$$a_\theta(s) := \begin{cases} a(s) & \text{if } 0 \leq s \leq 1-\theta, \\ (1-\theta)^{-\frac{q-2}{2}} a(1-\theta) s^{\frac{q-2}{2}} & \text{if } s > 1-\theta. \end{cases}$$

The functions $a_\theta(s)$ and $\varphi(s) := a_\theta(s)s$ are differentiable in $[0, +\infty) \setminus \{1-\theta\}$ and, by (2.1) and (a0), we deduce that $\varphi'(s_1) < \varphi'_-(1-\theta) \leq \varphi'_+(1-\theta) < \varphi'(s_2)$, for any $s_1 < 1-\theta < s_2$.

Lemma 2.4. *The map $\varphi(s)$ is strictly convex.*

Proof. Clearly φ is strictly convex on $[0, 1-\theta]$ and on $[1-\theta, +\infty)$. Take $0 < s < 1-\theta < t$. If $\frac{s+t}{2} \leq 1-\theta$, then by the convexity we obtain

$$\varphi(s) - \varphi\Big(\frac{s+t}{2}\Big) > \varphi'\Big(\frac{s+t}{2}\Big)\Big(s - \frac{s+t}{2}\Big),$$
$$\varphi(1-\theta) - \varphi\Big(\frac{s+t}{2}\Big) > \varphi'\Big(\frac{s+t}{2}\Big)\Big(1-\theta - \frac{s+t}{2}\Big),$$
$$\varphi(t) - \varphi(1-\theta) > \varphi'_+(1-\theta)(t-1+\theta).$$

In view of (2.1) we get $\varphi'_+(1-\theta) \geq \varphi'\big(\frac{s+t}{2}\big)$ and we conclude

$$\frac{\varphi(s)+\varphi(t)}{2} > \varphi\Big(\frac{s+t}{2}\Big).$$

Similarly we argue if $\frac{s+t}{2} > 1-\theta$ and we conclude. □

For the positive mass case we will consider the following truncated problem

$$\begin{cases} -\operatorname{div}\big(a_\theta(|\nabla u|^2)\nabla u\big) = g(u) & \text{in } \mathbb{R}^N, \\ u \in \mathcal{X}. \end{cases} \tag{2.2}$$

For the zero mass case, instead, we will consider the following truncated problem

$$\begin{cases} -\operatorname{div}\big(a_\theta(|\nabla u|^2)\nabla u\big) = g(u) & \text{in } \mathbb{R}^N, \\ u \in \mathcal{X}_0. \end{cases} \tag{2.3}$$

Clearly, if u_θ is a solution of (2.2) or of (2.3) such that $|\nabla u_\theta| \leq 1-\theta$, then u_θ is a solution also of (1.2).

Observe that there exists $\bar{c}_\theta = \bar{c}_\theta(\theta) > 0$ such that

$$\bar{c}\left(s^2 + |s|^q\right) \le a_\theta(s^2)s^2 \le \bar{c}_\theta\left(s^2 + |s|^q\right), \qquad \text{for all } s \in \mathbb{R}, \tag{2.4}$$

$$\bar{c}\left(s^2 + |s|^q\right) \le A_\theta(s^2) \le \bar{c}_\theta\left(s^2 + |s|^q\right), \qquad \text{for all } s \in \mathbb{R}, \tag{2.5}$$

where $A_\theta(s) = \int_0^s a_\theta(t)\,dt$ and

$$\bar{c} := \frac{2}{q} \cdot \frac{(1-\theta_1)^{\frac{q-2}{2}}}{1 + (1-\theta_1)^{q-2}} \cdot \min_{s\in[0,1)} a(s)$$

is independent of θ.

We conclude this section with the following lemma, which will play a crucial role in our arguments. The proof of this result seems to be standard but we give the proof for the completeness.

Lemma 2.5. *Suppose that $u_n \rightharpoonup u_0$ in $\mathcal{X}_0$ and*

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} a_\theta(|\nabla u_n|^2)|\nabla u_n|^2\,dx = \int_{\mathbb{R}^N} a_\theta(|\nabla u_0|^2)|\nabla u_0|^2\,dx. \tag{2.6}$$

Then $u_n \to u_0$ strongly in $\mathcal{X}_0$.

Proof. Let $\varphi : \mathbb{R}^N \to \mathbb{R}$ be given by $\varphi(v) := a_\theta(|v|^2)|v|^2$, for $v \in \mathbb{R}^N$. By Lemma 2.4, φ is strictly convex, hence the map $\Phi : \mathcal{X}_0 \to \mathbb{R}$, such that

$$\Phi(u) := \int_{\mathbb{R}^N} \varphi(\nabla u)\,dx, \qquad \text{for } u \in \mathcal{X}_0,$$

is well defined and strictly convex as well. So, since $\frac{1}{2}(\nabla u_n + \nabla u_0) \rightharpoonup \nabla u_0$, we obtain

$$\liminf_{n\to+\infty} \int_{\mathbb{R}^N} \varphi\Big(\frac{1}{2}(\nabla u_n + \nabla u_0)\Big)\,dx \ge \int_{\mathbb{R}^N} \varphi(\nabla u_0)\,dx. \tag{2.7}$$

Then, taking into account the convexity of φ, we know that, a.e. in $\mathbb{R}^N$,

$$\xi_n := \frac{1}{2}\big(\varphi(\nabla u_n) + \varphi(\nabla u_0)\big) - \varphi\Big(\frac{1}{2}(\nabla u_n + \nabla u_0)\Big) \ge 0,$$

hence, by (2.6) and (2.7),

$$\limsup_{n\to+\infty} \int_{\mathbb{R}^N} \xi_n\,dx = 0. \tag{2.8}$$

For any $k \geq 1$ we define

$$\mu_k := \inf\left\{\frac{1}{2}\big(\varphi(v_1)+\varphi(v_2)\big)-\varphi\Big(\frac{1}{2}(v_1+v_2)\Big) : v_1, v_2 \in \mathbb{R}^N \text{ s.t. } |v_1|, |v_2| \leq k, |v_1-v_2| \geq \frac{1}{k}\right\},$$

$$\Omega_{n,k} := \left\{x \in \mathbb{R}^N : |\nabla u_n|, |\nabla u_0| \leq k, |\nabla u_n - \nabla u_0| \geq \frac{1}{k}\right\}.$$

Since $\mu_k > 0$, by the strict convexity of φ, and (2.8) holds, we infer that the Lebesgue measure $|\Omega_{n,k}| \to 0$, as $n \to +\infty$. Take any $\varepsilon > 0$, we find a subsequence $\{n_k\}$ such that $|\bigcup_{k=1}^{\infty} \Omega_{n_k,k}| < \varepsilon$. Again letting $\varepsilon \to 0$ and passing to a subsequence we obtain that $\nabla u_n \to \nabla u_0$ a.e. on $\mathbb{R}^N$. Note that a_θ is of class $\mathcal{C}^1$ on $(0, 1-\theta)$ and $(1-\theta, +\infty)$, hence φ' exists almost everywhere. Now take $s \in [0, 1]$, by (2.4) we observe that the sequence $\{\varphi'(\nabla u_n - s\nabla u_0)\nabla u_0\}$ is uniformly integrable and tight and converges a.e. to $\varphi'\big((1-s)\nabla u_0\big)\nabla u_0$. In view of the Vitali Convergence Theorem we get

$$\begin{aligned}\int_{\mathbb{R}^N} \varphi(\nabla u_n)\,dx - \int_{\mathbb{R}^N} \varphi(\nabla u_n - \nabla u_0)\,dx &= \int_0^1 \int_{\mathbb{R}^N} \varphi'(\nabla u_n - s\nabla u_0)\nabla u_0\,dx\,ds \\ &\xrightarrow[n\to+\infty]{} \int_0^1 \int_{\mathbb{R}^N} \varphi'\big((1-s)\nabla u_0\big)\nabla u_0\,dx\,ds \\ &= \int_{\mathbb{R}^N} \varphi(\nabla u_0)\,dx.\end{aligned}$$

Since (2.6) holds, we get

$$\int_{\mathbb{R}^N} \varphi(\nabla u_n - \nabla u_0)\,dx \to 0,$$

as $n \to +\infty$, and by (2.4) we conclude. □

3. The positive mass case

In this section we deal with the positive mass case, namely, we will assume on g (g0), (g1) and (g2).

Let $g_1(s) := \max\{g(s) + ms^{\gamma-1}, 0\}$, for $s \geq 0$, and $g_2(s) = g_1(s) - g(s)$, for $s \geq 0$, and $g_i(s) = -g_i(-s)$ for $s < 0$. Then $g_1(s), g_2(s) \geq 0$, for $s \geq 0$,

$$\lim_{s\to 0} g_1(s)/s^{\gamma-1} = 0, \tag{3.1}$$

$$g_2(s) \geq ms^{\gamma-1}, \quad \text{for } s \geq 0. \tag{3.2}$$

If we set

$$G_i(s) = \int_0^s g_i(t)\,dt, \qquad \text{for } i = 1, 2,$$

then, by (3.2), we have

$$G_2(s) \geq \frac{m}{\gamma}|s|^\gamma, \qquad \text{for } s \in \mathbb{R}. \tag{3.3}$$

By (g1) and (3.1), we have that there exist two fixed positive constants, $\bar{c}_1, \bar{c}_2$ such that

$$|g(s)| \leq \bar{c}_1|s|^{\gamma-1}, \qquad \text{for all } |s| \leq \bar{c}_2, \tag{3.4}$$

$$|G(s)| \leq \bar{c}_1|s|^{\gamma}, \qquad \text{for all } |s| \leq \bar{c}_2, \tag{3.5}$$

$$|g_1(s)| \leq \bar{c}_1|s|^{\gamma-1}, \qquad \text{for all } |s| \leq \bar{c}_2, \tag{3.6}$$

$$|G_1(s)| \leq \bar{c}_1|s|^{\gamma}, \qquad \text{for all } |s| \leq \bar{c}_2. \tag{3.7}$$

Lemma 3.1. *For any $u \in \mathcal{X}$, $\int_{\mathbb{R}^N} G(u)\,dx$ and $\int_{\mathbb{R}^N} g(u)u\,dx$ are well defined. The same is true for $\int_{\mathbb{R}^N} G_i(u)\,dx$ and $\int_{\mathbb{R}^N} g_i(u)u\,dx$, for $1 = 1, 2$.*

Proof. Let $u \in \mathcal{X}$. Since $\mathcal{X}$ is embedded into $L^\gamma(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, we have that

$$\begin{aligned}
\int_{\mathbb{R}^N} |G(u)|\,dx &= \int_{\{|u|\leq\bar{c}_2\}} |G(u)|\,dx + \int_{\{|u|>\bar{c}_2\}} |G(u)|\,dx \\
&\leq \bar{c}_1 \int_{\{|u|\leq\bar{c}_2\}} |u|^\gamma\,dx + \operatorname{meas}\{|u| > \bar{c}_2\} \cdot \max_{\{s\leq\|u\|_\infty\}} |G(s)| \\
&\leq \bar{c}_1|u|_\gamma^\gamma + \operatorname{meas}\{|u| > \bar{c}_2\} \cdot \max_{\{s\leq\|u\|_\infty\}} |G(s)| < +\infty.
\end{aligned}$$

The arguments are similar for $\int_{\mathbb{R}^N} g(u)u\,dx$, $\int_{\mathbb{R}^N} G_i(u)\,dx$ and $\int_{\mathbb{R}^N} g_i(u)u\,dx$, $1 = 1, 2$. □

Lemma 3.2. *If $u_n \rightharpoonup u_0$ in $\mathcal{X}$, then*

$$\lim_n \int_{\mathbb{R}^N} g_1(u_n)u_n\,dx = \int_{\mathbb{R}^N} g_1(u_0)u_0\,dx \tag{3.8}$$

and

$$\lim_n \int_{\mathbb{R}^N} G_1(u_n)\,dx = \int_{\mathbb{R}^N} G_1(u_0)\,dx. \tag{3.9}$$

Proof. Here we follow some ideas of [31, Corollary 3.6] (cf. [32]) and we divide the proof into three intermediate steps by which the conclusion follows immediately.
STEP 1: We claim that

$$\lim_n \int_{\mathbb{R}^N} g_1(u_n)(u_n - u_0)\,dx = 0. \tag{3.10}$$

Since $\{u_n\}$ is bounded in $\mathcal{X}$ then, by the continuous embedding of $\mathcal{X}$ into $L^\infty(\mathbb{R}^N)$, we infer that there exists $M > 0$ such that $|u_n|_\infty \leq M$, for any $n \geq 1$. Take any $\varepsilon > 0$ and $\beta > 2^*$. Then, by (3.1), we find $0 < \delta < M$ and $c_\varepsilon > 0$ such that

$$\begin{aligned} |g_1(s)| &\leq \varepsilon |s|^{\gamma-1} \quad \text{if } |s| \in [0,\delta], \\ |g_1(s)| &\leq c_\varepsilon |s|^{\beta-1} \quad \text{if } |s| \in (\delta, M]. \end{aligned}$$

Therefore

$$\int_{\mathbb{R}^N} |g_1(u_n)(u_n-u_0)|\,dx \leq \varepsilon \int_{\mathbb{R}^N} |u_n|^{\gamma-1}|u_n-u_0|\,dx + c_\varepsilon \int_{\mathbb{R}^N} |u_n|^{\beta-1}|u_n-u_0|\,dx,$$

and, by the compact embedding of $\mathcal{X}$ into $L^\beta(\mathbb{R}^N)$, the boundedness of the sequence $\{u_n\}$ in $\mathcal{X}$, we infer that

$$\limsup_n \int_{\mathbb{R}^N} |g_1(u_n)(u_n-u_0)|\,dx \leq \varepsilon C$$

for some constant $C > 0$ and so (3.10) is proved.
STEP 2: We claim that

$$\lim_n \int_{\mathbb{R}^N} g_1(u_n)u_0\,dx = \int_{\mathbb{R}^N} g_1(u_0)u_0\,dx.$$

Since the sequence $\{g_1(u_n)u_0\}$ is uniformly integrable and tight, then the conclusion follows by Vitali Convergence Theorem.
STEP 3: We claim that

$$\lim_n \left(\int_{\mathbb{R}^N} g_1(u_n)u_n\,dx - \int_{\mathbb{R}^N} g_1(u_n)(u_n-u_0)\,dx \right) = \int_{\mathbb{R}^N} g_1(u_0)u_0\,dx.$$

Indeed, if we set $\phi_n(s) = g_1(u_n)(u_n - su_0)$, for any $n \in \mathbb{N}$ and $s \in [0,1]$, taking in account Step 2, we have

$$\lim_n \left(\int_{\mathbb{R}^N} g_1(u_n)u_n\,dx - \int_{\mathbb{R}^N} g_1(u_n)(u_n-u_0)\,dx \right)$$

$$= \lim_n \int_{\mathbb{R}^N} \big(\phi_n(0)-\phi_n(1)\big)dx = -\lim_n \int_{\mathbb{R}^N} \left(\int_0^1 \phi_n'(s)\,ds \right) dx$$

$$= \int_0^1 \left(\lim_n \int_{\mathbb{R}^N} g_1(u_n)u_0\,dx \right) ds = \int_0^1 \left(\int_{\mathbb{R}^N} g_1(u_0)u_0\,dx \right) ds = -\int_0^1 \left(\int_{\mathbb{R}^N} \phi_0'(s)\,dx \right) ds$$

$$= \int_{\mathbb{R}^N} \big(\phi_0(0)-\phi_0(1)\big)dx = \int_{\mathbb{R}^N} g_1(u_0)u_0\,dx.$$

The proof of (3.9) is similar. □

Solutions of (2.2) will be found as critical points of the functional $I_\theta : \mathcal{X} \to \mathbb{R}$ defined as

$$I_\theta(u) = \frac{1}{2}\int_{\mathbb{R}^N} A_\theta(|\nabla u|^2)\,dx + \int_{\mathbb{R}^N} G_2(u)\,dx - \int_{\mathbb{R}^N} G_1(u)\,dx.$$

The functional is well defined in $\mathcal{X}$ by (2.5).

Lemma 3.3. *For any $\theta \in (0, \theta_1]$, the functional $I_\theta : \mathcal{X} \to \mathbb{R}$ verifies the mountain pass geometry. More precisely:*

(i) there are $\alpha, \rho > 0$ such that $I_\theta(u) \geq \alpha$, for $\|u\| = \rho$;
(ii) there is $\bar{u} \in \mathcal{X} \setminus \{0\}$, independent of $\theta \in (0, \theta_1]$, with $\|\bar{u}\| > \rho$ and $|\nabla \bar{u}| < 1 - \theta_1$, almost everywhere in $\mathbb{R}^N$, and such that $I_\theta(\bar{u}) < 0$.

Proof. (i) By the continuous embedding of $\mathcal{X}$ into $L^\infty(\mathbb{R}^N)$, and by (3.1), we can consider $\rho > 0$ sufficiently small such that

$$G_1(u(x)) \leq \frac{m}{2\gamma}|u(x)|^\gamma, \qquad \text{a.e. } x \in \mathbb{R}^N \text{ and for any } u \in \mathcal{X} \text{ with } \|u\| = \rho.$$

Hence, by (3.3) and (2.5), for any $u \in \mathcal{X}$ with $\|u\| = \rho$, we have

$$I_\theta(u) \geq \frac{\bar{c}}{2}\left(|\nabla u|_2^2 + |\nabla u|_q^q\right) + \frac{m}{2\gamma}|u|_\gamma^\gamma \geq c\|u\|^\beta \geq \alpha > 0,$$

where $\beta = \max\{2, q, \gamma\}$.

(ii) Let $u_R \in \mathcal{X}$ such that, for any $x \in \mathbb{R}^N$,

$$u_R(x) := \begin{cases} \xi_0 & \text{in } B_R, \\ -\frac{\xi_0}{\sqrt{R}}|x| + \xi_0(1+\sqrt{R}) & \text{in } B_{R+\sqrt{R}} \setminus B_R, \\ 0 & \text{in } \mathbb{R}^N \setminus B_{R+\sqrt{R}}. \end{cases}$$

Arguing as in [8], for R sufficiently large, we have $\int_{\mathbb{R}^N} G(u_R)\,dx > 0$ and, clearly, $|\nabla u_R| < 1-\theta_1$. Moreover, for any $t > 1$, we have also that $|\nabla u_R(\cdot/t)| \le 1-\theta_1$ and so, denoting $\bar{u} = u_R(\cdot/t)$, with R and t sufficiently large and independently by $\theta \in (0, \theta_1]$, we have $\|\bar{u}\| > \rho$ and

$$I_\theta(\bar{u}) \le c_1\left(t^{N-2}|\nabla u_R|_2^2 + t^{N-q}|\nabla u_R|_q^q\right) - t^N \int_{\mathbb{R}^N} G(u_R)\,dx < 0. \quad \square$$

Let us define the mountain pass level for the functional I_θ

$$m_\theta := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I_\theta(\gamma(t)),$$

where

$$\Gamma := \{\gamma \in \mathcal{C}([0,1], \mathcal{X}) \mid \gamma(0) = 0, \gamma(1) = \bar{u}\}.$$

By Lemma 3.3, we deduce that $m_\theta \ge \alpha$, for any $\theta \in (0, \theta_1]$.

Observe that, since $|\nabla \bar{u}| < 1-\theta_1$, we have that $I_{\theta_1}(t\bar{u}) = I_\theta(t\bar{u})$, for any $t \in [0,1]$ and for any $\theta \in (0, \theta_1]$. Hence we deduce that

$$m_\theta \le \max_{t\in[0,1]} I_\theta(t\bar{u}) = \max_{t\in[0,1]} I_{\theta_1}(t\bar{u}),$$

for any $\theta \in (0, \theta_1]$. Hence there exists $c > 0$ (independent of $\theta \in (0, \theta_1]$) such that

$$0 < m_\theta \le c, \qquad \text{for any } \theta \in (0, \theta_1]. \tag{3.11}$$

Following [25,26], we define the functional $J_\theta : \mathbb{R} \times \mathcal{X} \to \mathbb{R}$ as

$$J_\theta(\sigma, u) = I_\theta(u(e^{-\sigma}\cdot)) = \frac{e^{N\sigma}}{2}\int_{\mathbb{R}^N} A_\theta(e^{-2\sigma}|\nabla u|^2)\,dx + e^{N\sigma}\int_{\mathbb{R}^N} G_2(u)\,dx - e^{N\sigma}\int_{\mathbb{R}^N} G_1(u)\,dx.$$

With similar arguments of Lemma 3.3, also J_θ has a mountain pass geometry and we can define its mountain pass level as

$$\tilde{m}_\theta := \inf_{(\sigma,\gamma)\in\Sigma\times\Gamma} \max_{t\in[0,1]} J_\theta\big(\sigma(t), \gamma(t)\big),$$

where

$$\Sigma := \{\sigma \in \mathcal{C}([0,1], \mathbb{R}) \mid \sigma(0) = \sigma(1) = 0\}.$$

Observe that arguing as in [25, Lemma 3.1], we obtain

Lemma 3.4. *For any $\theta \in (0, \theta_1]$, the mountain pass levels of I_θ and J_θ coincide, namely $m_\theta = \tilde{m}_\theta$.*

Now, as an immediate consequence of Ekeland's variational principle [37, Theorem 2.8] (cf. [26, Lemma 2.3]) we obtain the following results.

Lemma 3.5. *Let $\theta \in (0, \theta_1]$ and $\varepsilon > 0$. Suppose that $\tilde{\gamma} \in \Sigma \times \Gamma$ satisfies*

$$\max_{t\in[0,1]} J_\theta(\tilde{\gamma}(t)) \le m_\theta + \varepsilon,$$

then there exists $(\sigma, u) \in \mathbb{R} \times \mathcal{X}$ such that

(1) $\mathrm{dist}_{\mathbb{R}\times\mathcal{X}}\big((\theta, u), \tilde{\gamma}([0, 1])\big) \le 2\sqrt{\varepsilon}$;
(2) $J_\theta(\sigma, u) \in [m_\theta - \varepsilon, m_\theta + \varepsilon]$;
(3) $\|DJ_\theta(\sigma, u)\|_{\mathbb{R}\times\mathcal{X}^*} \le 2\sqrt{\varepsilon}$.

Proposition 3.6. *For any $\theta \in (0, \theta_1]$, there exists a sequence $\{(\sigma_n, u_n)\} \subset \mathbb{R} \times \mathcal{X}$ such that, as $n \to +\infty$, we get*

(1) $\sigma_n \to 0$;
(2) $J_\theta(\sigma_n, u_n) \to m_\theta$;
(3) $\partial_\sigma J_\theta(\sigma_n, u_n) \to 0$;
(4) $\partial_u J_\theta(\sigma_n, u_n) \to 0$ *strongly in* $\mathcal{X}^*$.

Proof. In view of Lemma 3.5 we conclude by letting $\varepsilon \to 0$. □

Now we find a radial solution of the truncated problem (2.2).

Proposition 3.7. *For any $\theta \in (0, \theta_1]$, there exists $u_\theta \in \mathcal{X}$ a non-trivial solution of (2.2) such $I_\theta(u_\theta) = m_\theta$. Moreover there exists $C > 0$ such that*

$$\|u_\theta\| \le C, \quad \text{for any } \theta \in (0, \theta_1]. \tag{3.12}$$

Finally u_θ is a weak solution of

$$-\big(r^{N-1} a_\theta(|u_\theta'(r)|^2) u_\theta'(r)\big)' = r^{N-1} g(u_\theta(r)), \tag{3.13}$$

namely

$$\int_0^{+\infty} r^{N-1} a_\theta(|u_\theta'(r)|^2) u_\theta'(r) v'(r)\, dr = \int_0^{+\infty} r^{N-1} g(u_\theta(r)) v(r)\, dr,$$

for all $v \in \mathcal{X}$.

Proof. Fix $\theta \in (0, \theta_1]$. By Proposition 3.6, there exists a sequence $\{(\sigma_n, u_n)\} \subset \mathbb{R} \times \mathcal{X}$ such that

$$\begin{cases} \frac{e^{N\sigma_n}}{2} \int\limits_{\mathbb{R}^N} A_\theta(e^{-2\sigma_n}|\nabla u_n|^2)\,dx + e^{N\sigma_n} \int\limits_{\mathbb{R}^N} G_2(u_n)\,dx - e^{N\sigma_n} \int\limits_{\mathbb{R}^N} G_1(u_n)\,dx = m_\theta + o_n(1), \\ \frac{Ne^{N\sigma_n}}{2} \int\limits_{\mathbb{R}^N} A_\theta(e^{-2\sigma_n}|\nabla u_n|^2)\,dx - e^{(N-2)\sigma_n} \int\limits_{\mathbb{R}^N} a_\theta(e^{-2\sigma_n}|\nabla u_n|^2)|\nabla u_n|^2\,dx \\ \qquad + Ne^{N\sigma_n} \int\limits_{\mathbb{R}^N} G_2(u_n)\,dx - Ne^{N\sigma_n} \int\limits_{\mathbb{R}^N} G_1(u_n)\,dx = o_n(1), \\ e^{(N-2)\sigma_n} \int\limits_{\mathbb{R}^N} a_\theta(e^{-2\sigma_n}|\nabla u_n|^2)|\nabla u_n|^2\,dx + e^{N\sigma_n} \int\limits_{\mathbb{R}^N} g_2(u_n)u_n\,dx \\ \qquad - e^{N\sigma_n} \int\limits_{\mathbb{R}^N} g_1(u_n)u_n\,dx = o_n(1)\|u_n\|. \end{cases} \tag{3.14}$$

From the first and the second equation of the previous system we get

$$e^{(N-2)\sigma_n} \int\limits_{\mathbb{R}^N} a_\theta(e^{-2\sigma_n}|\nabla u_n|^2)|\nabla u_n|^2\,dx = Nm_\theta + o_n(1).$$

Therefore, since $\sigma_n \to 0$, as $n \to +\infty$, by (2.4) we deduce that $\{u_n\}$ is a bounded sequence in $\mathcal{X}_0$ and so also in $L^\infty(\mathbb{R}^N)$, namely there exists $\bar{C} > 0$ such that $|u_n|_\infty \le \bar{C}$, for any $n \ge 1$. This implies that, by (3.1) and Lemma 2.1, there exists $R > 1$ such that

$$G_1(u_n(x)) \le \frac{m}{2\gamma}|u_n(x)|^\gamma, \qquad \text{a.e. } x \in \mathbb{R}^N \text{ with } |x| \ge R \text{ and for any } n \ge 1.$$

Hence

$$\int\limits_{\mathbb{R}^N} G_1(u_n)\,dx = \int\limits_{B_R} G_1(u_n)\,dx + \int\limits_{\mathbb{R}^N\setminus B_R} G_1(u_n)\,dx \le C \max_{\{s \le \bar{C}\}} |G_1(s)| + \frac{m}{2\gamma} \int\limits_{\mathbb{R}^N} |u_n(x)|^\gamma\,dx.$$

By this, by (3.3) and by the first equation of (3.14), we infer that $\{u_n\}$ is a bounded sequence also in $\mathcal{X}$. Then there exists $u_\theta \in \mathcal{X}$ such that $u_n \rightharpoonup u_\theta$ in $\mathcal{X}$. Since $\partial_u J_\theta(\sigma_n, u_n) \to 0$ strongly in $\mathcal{X}^*$ and $\sigma_n \to 0$, we have that u_θ is a weak (possibly trivial) solution of (2.2) and so it satisfies

$$\int\limits_{\mathbb{R}^N} a_\theta(|\nabla u_\theta|^2)|\nabla u_\theta|^2\,dx + \int\limits_{\mathbb{R}^N} g_2(u_\theta)u_\theta\,dx = \int\limits_{\mathbb{R}^N} g_1(u_\theta)u_\theta\,dx.$$

Since $u_n \rightharpoonup u_\theta$ in $\mathcal{X}$, by the weak lower semicontinuity and the Fatou's Lemma we have that

$$\int_{\mathbb{R}^N} a_\theta(|\nabla u_\theta|^2)|\nabla u_\theta|^2\,dx \le \liminf_{n\to+\infty} \int_{\mathbb{R}^N} a_\theta(|\nabla u_n|^2)|\nabla u_n|^2\,dx,$$

$$\int_{\mathbb{R}^N} g_2(u_\theta)u_\theta\,dx \le \liminf_{n\to+\infty} \int_{\mathbb{R}^N} g_2(u_n)u_n\,dx;$$

while, by Lemma 3.2, we have

$$\int_{\mathbb{R}^N} g_1(u_\theta)u_\theta\,dx = \lim_{n\to+\infty} \int_{\mathbb{R}^N} g_1(u_n)u_n\,dx.$$

Therefore, by the third equation of (3.14),

$$\begin{aligned}
&\int_{\mathbb{R}^N} a_\theta(|\nabla u_\theta|^2)|\nabla u_\theta|^2\,dx + \int_{\mathbb{R}^N} g_2(u_\theta)u_\theta\,dx \\
&\le \liminf_{n\to+\infty}\left[\int_{\mathbb{R}^N} a_\theta(|\nabla u_n|^2)|\nabla u_n|^2\,dx + \int_{\mathbb{R}^N} g_2(u_n)u_n\,dx\right] \\
&= \liminf_{n\to+\infty}\left[e^{(N-2)\sigma_n}\int_{\mathbb{R}^N} a_\theta(e^{-2\sigma_n}|\nabla u_n|^2)|\nabla u_n|^2\,dx + e^{N\sigma_n}\int_{\mathbb{R}^N} g_2(u_n)u_n\,dx\right] \\
&= \liminf_{n\to+\infty}\left[e^{N\sigma_n}\int_{\mathbb{R}^N} g_1(u_n)u_n\,dx + o_n(1)\|u_n\|\right] \\
&= \int_{\mathbb{R}^N} g_1(u_\theta)u_\theta\,dx = \int_{\mathbb{R}^N} a_\theta(|\nabla u_\theta|^2)|\nabla u_\theta|^2\,dx + \int_{\mathbb{R}^N} g_2(u_\theta)u_\theta\,dx
\end{aligned}$$

and so

$$\int_{\mathbb{R}^N} a_\theta(|\nabla u_\theta|^2)|\nabla u_\theta|^2\,dx = \lim_{n\to+\infty} \int_{\mathbb{R}^N} a_\theta(|\nabla u_n|^2)|\nabla u_n|^2\,dx, \tag{3.15}$$

$$\int_{\mathbb{R}^N} g_2(u_\theta)u_\theta\,dx = \lim_{n\to+\infty} \int_{\mathbb{R}^N} g_2(u_n)u_n\,dx. \tag{3.16}$$

In view of Lemma 2.5 equation (3.15) implies that $u_n \to u_\theta$ strongly in $\mathcal{X}_0$.
Moreover, since, by (3.2), we know that for any $s \in \mathbb{R}$ we can write $g_2(s)s = m|s|^\gamma + h(s)$, where h is a non-negative continuous function, by Fatou's Lemma we deduce that

$$\int_{\mathbb{R}^N} |u_\theta|^\gamma\,dx \le \liminf_{n\to+\infty} \int_{\mathbb{R}^N} |u_n|^\gamma\,dx,$$

$$\int_{\mathbb{R}^N} h(u_\theta)\,dx \le \liminf_{n\to+\infty} \int_{\mathbb{R}^N} h(u_n)\,dx.$$

These last two inequalities and (3.16) imply that

$$\int_{\mathbb{R}^N} |u_\theta|^\gamma\,dx = \lim_{n\to+\infty} \int_{\mathbb{R}^N} |u_n|^\gamma\,dx$$

and so, actually, $u_n \to u_\theta$ strongly in $\mathcal{X}$ and so $I_\theta(u_\theta) = m_\theta$.
Finally, since

$$\int_{\mathbb{R}^N} a_\theta(|\nabla u_\theta|^2)|\nabla u_\theta|^2\,dx = N m_\theta,$$

by (3.11) and (2.4), we prove that there exists $C > 0$ such that $\|u_\theta\|_0 \le C$, for any $\theta \in (0, \theta_1]$. Since $\{u_\theta\}$ are uniformly bounded in $\mathcal{X}_0$ and so also in $L^\infty(\mathbb{R}^N)$, there exists $\bar{C} > 0$ such that $|u_\theta|_\infty \le \bar{C}$, for any $\theta \in (0, \theta_1]$. This implies that, by (3.1) and Lemma 2.1, there exists $R > 1$ such that

$$G_1(u_\theta(x)) \le \frac{m}{2\gamma}|u_\theta(x)|^\gamma, \quad \text{a.e. } x \in \mathbb{R}^N \text{ with } |x| \ge R \text{ and for any } \theta \in (0, \theta_1].$$

Hence

$$\int_{\mathbb{R}^N} G_1(u_\theta)\,dx = \int_{B_R} G_1(u_\theta)\,dx + \int_{\mathbb{R}^N\setminus B_R} G_1(u_\theta)\,dx \le C \max_{\{s\le\bar{C}\}} |G_1(s)| + \frac{m}{2\gamma}\int_{\mathbb{R}^N} |u_\theta(x)|^\gamma\,dx.$$

By this, by (3.3), since $I_\theta(u_\theta) = m_\theta$ and by (3.11), we infer that exists $C > 0$ such that $\|u_\theta\| \le C$ for any $\theta \in (0, \theta_1]$. □

We are now able to conclude the proof of our main theorem in the positive mass case.

Proof of Theorem 1.1. By Proposition 3.7, for any $\theta \in (0, \theta_1]$, there exists $u_\theta \in \mathcal{X}$ a nontrivial solution of (2.2) such $I_\theta(u_\theta) = m_\theta$. Since $q > N$, $u_\theta \in L^\infty(\mathbb{R}^N)$ and since u_θ is a solution of (3.13) in $(0, +\infty)$, it is easy to check that u_θ is regular for $r > 0$.
Claim 1: $u_\theta \in \mathcal{C}^{1,\alpha}$ in a neighbourhood of 0 for some $\alpha \in (0, 1)$.
Integrating the equation (3.13), for any $r_2 > r_1 > 0$, we have

$$-r_2^{N-1} a_\theta(|u_\theta'(r_2)|^2)u_\theta'(r_2) + r_1^{N-1} a_\theta(|u_\theta'(r_1)|^2)u_\theta'(r_1) = \int_{r_1}^{r_2} s^{N-1} g(u_\theta(s))\,ds.$$

Observe that

$$\int_{r_1}^{r_2} s^{N-1}|g(u_\theta(s))|\,ds \le C(r_2^N - r_1^N),$$

for some constant $C > 0$. Thus $\mathcal{A} := \lim_{r\to 0} r^{N-1} a_\theta(|u_\theta'(r)|^2)u_\theta'(r)$ exists and it is finite. If $\mathcal{A} \neq 0$, then $\lim_{r\to 0} |u_\theta'(r)| = +\infty$. Since we can find constants $c_1, c_2, \rho > 0$ such that

$$c_1|s|^q \le a_\theta(s^2)s^2 \le c_2|s|^q, \qquad \text{for } |s| > \rho,$$

and u_θ is constant on a sphere centred at 0, in view of Lieberman's result [30], $u_\theta \in \mathcal{C}^{1,\alpha}$ in a neighbourhood of 0 for some $\alpha \in (0, 1)$. This contradicts $\lim_{r\to 0} |u_\theta'(r)| = +\infty$. Therefore $\mathcal{A} = 0$. Furthermore, since for any $r_2 > r_1 > 0$

$$-a_\theta(|u_\theta'(r_2)|^2)u_\theta'(r_2) + \frac{r_1^{N-1}}{r_2^{N-1}} a_\theta(|u_\theta'(r_1)|^2)u_\theta'(r_1), = \frac{1}{r_2^{N-1}} \int_{r_1}^{r_2} s^{N-1} g(u_\theta(s))\, ds,$$

and letting $r_1 \to 0$, we deduce that

$$\left| a_\theta(|u_\theta'(r_2)|^2)u_\theta'(r_2) \right| \le \frac{1}{r_2^{N-1}} \int_0^{r_2} s^{N-1} |g(u_\theta(s))|\, ds \le C r_2.$$

Therefore

$$\lim_{r\to 0} a_\theta(|u_\theta'(r)|^2)u_\theta'(r) = 0,$$

hence

$$\lim_{r\to 0} u_\theta'(r) = 0.$$

Since, for some constants $c_1, c_2, \rho > 0$, we also have

$$c_1 s^2 \le a_\theta(s^2)s^2 \le c_2 s^2, \qquad \text{for } |s| < \rho,$$

in view of [30], we conclude the claim.
Claim 2: There exists $C > 0$ such that

$$|a_\theta(|u_\theta'(r)|^2)u_\theta'(r)| \le C, \qquad \text{for any } r \ge 0 \text{ and } \theta \in (0, \theta_1]. \tag{3.17}$$

By the regularity of u_θ, we infer that $u_\theta'(0) = 0$ and so also

$$a_\theta(|u_\theta'(0)|^2)u_\theta'(0) = 0.$$

Now, integrating the equation (3.13), for any $r > 0$, we have

$$-a_\theta(|u_\theta'(r)|^2)u_\theta'(r) = \frac{1}{r^{N-1}} \int_0^r s^{N-1} g(u_\theta(s))\, ds.$$

By Lemma 2.3 and by (3.12), we deduce that there exists $R > 1$, such that

$$|u_\theta(r)| \le \bar{c}_2, \quad \text{for any } \theta \in (0, \theta_1] \text{ and for any } r > R, \tag{3.18}$$

where $\bar{c}_2$ is defined in (3.4).
By the continuous embedding of $\mathcal{X}$ in $L^\infty(\mathbb{R}^N)$ and by (3.12), there exists $C > 0$ such that $|u_\theta|_\infty \le C\|u_\theta\| \le C$, for any $\theta \in (0, \theta_1]$, and so we have that, for any $0 < r \le R$ and $\theta \in (0, \theta_1]$,

$$|a_\theta(|u'_\theta(r)|^2)u'_\theta(r)| \le \frac{1}{r^{N-1}} \int_0^r s^{N-1}|g(u_\theta(s))|\,ds \le C.$$

If $r > R$, then

$$\begin{aligned} |a_\theta(|u'_\theta(r)|^2)u'_\theta(r)| &\le \frac{1}{r^{N-1}} \int_0^r s^{N-1}|g(u_\theta(s))|\,ds \\ &\le \frac{1}{r^{N-1}} \left(\int_0^R s^{N-1}|g(u_\theta(s))|\,ds + \int_R^r s^{N-1}|g(u_\theta(s))|\,ds \right) \\ &\le \frac{C}{r^{N-1}} + \underbrace{\frac{c_1}{r^{N-1}} \int_1^r s^{N-1}|g(u_\theta(s))|\,ds}_{(A)}. \end{aligned}$$

We have to estimate (A). First of all, Lemma 2.3 and (3.12), for $r > 1$, we have that

$$|u_\theta(r)| \le C r^{-\frac{N-1}{\gamma}} \|u_\theta\| \le \bar{C} r^{-\frac{N-1}{\gamma}}.$$

From (3.18) and (3.4), and since $\gamma \ge 2$, we get

$$(A) \le \frac{C}{r^{N-1}} \int_1^r s^{N-1}|u_\theta(s)|^{\gamma-1}\,ds \le \frac{C}{r^{N-1}} \int_1^r s^{N-1-\frac{N-1}{\gamma}(\gamma-1)}\,ds \le C\left(r^{1-\frac{N-1}{\gamma}(\gamma-1)} + 1\right) \le C.$$

Therefore the claim is proved.
Claim 3: There exists $\bar{\theta} \in (0, \theta_1]$ such that

$$|u'_{\bar{\theta}}(r)| \le 1 - \bar{\theta}, \qquad \text{for any } r \ge 0. \tag{3.19}$$

Suppose by contradiction that (3.19) does not hold, then there exists a sequence $\{\theta_n\} \subset (0, \theta_1]$ which tends to zero and a sequence $\{r_n\} \subset \mathbb{R}_+$ such that

$$\lim_n |u'_{\theta_n}(r_n)| = 1,$$

which implies, by (a1), that

$$\lim_n a_{\theta_n}(|u'_{\theta_n}(r_n)|)|u'_{\theta_n}(r_n)| = +\infty.$$

Thus we obtain a contradiction with (3.17).

Finally, observe that $u_{\bar{\theta}}$ solves (1.2). Moreover, taking into account (2.4), (2.5) and Lemma 3.1, we get

$$\int_{\mathbb{R}^N} A(|\nabla u_{\bar{\theta}}|^2)\,dx,\ \int_{\mathbb{R}^N} a(|\nabla u_{\bar{\theta}}|^2)|\nabla u_{\bar{\theta}}|^2\,dx,\ \int_{\mathbb{R}^N} |G(u_{\bar{\theta}})|\,dx < +\infty. \qquad \square$$

4. The zero mass case

In this section we deal with the zero mass case, namely, we will assume that g satisfies (g0) and (g2). Moreover $\gamma > N$ and (g1$'$) holds, or $\gamma \le N$ and both (g1$'$), (g1$''$) hold. In the former case, for the definition of $\mathcal{X}_0$, we fix $q \in (N, \gamma)$, while in the latter, q is given by (g1$''$).

Let $g_1(s) := \max\{g(s), 0\}$ and $g_2(s) := g_1(s) - g(s)$ for $s \ge 0$ and then we can extend them as odd functions for $s < 0$. Then $g_1(s), g_2(s) \ge 0$, for $s \ge 0$ and

$$\lim_{s\to 0} g_1(s)/|s|^{\gamma-1} = 0, \quad \text{for some } \gamma > 2^*. \tag{4.1}$$

Moreover, whenever $\gamma \in (2^*, N]$, we have

$$\lim_{s\to+\infty} g_1(s)/|s|^{q^*-1} = 0. \tag{4.2}$$

For $i = 1, 2$ we set

$$G_i(s) = \int_0^s g_i(t)\,dt$$

and note that $G_i(s) \ge 0$ for $s \in \mathbb{R}$.

In view of (g1$'$), there exist two positive constants, $\bar{c}_1$ and $\bar{c}_2$, such that

$$|g(s)| \le \bar{c}_1 |s|^{\gamma-1}, \qquad \text{for all } |s| \le \bar{c}_2, \tag{4.3}$$

$$|G(s)| \le \bar{c}_1 |s|^{\gamma}, \qquad \text{for all } |s| \le \bar{c}_2, \tag{4.4}$$

$$|g_1(s)| \le \bar{c}_1 |s|^{\gamma-1}, \qquad \text{for all } |s| \le \bar{c}_2, \tag{4.5}$$

$$|G_1(s)| \le \bar{c}_1 |s|^{\gamma}, \qquad \text{for all } |s| \le \bar{c}_2. \tag{4.6}$$

Moreover, in the case $\gamma \in (2^*, N]$, by (g1$'$) and (g1$''$), there exists a positive constant $\bar{c}_3$ such that

$$|g(s)| \le \bar{c}_3 \left(|s|^{\gamma-1} + |s|^{q^*-1}\right), \qquad \text{for all } s \in \mathbb{R}, \tag{4.7}$$

$$|G(s)| \le \bar{c}_3 \left(|s|^{\gamma} + |s|^{q^*}\right), \qquad \text{for all } s \in \mathbb{R}, \tag{4.8}$$

$$|g_1(s)| \le \bar{c}_3 \left(|s|^{\gamma-1} + |s|^{q^*-1}\right), \qquad \text{for all } s \in \mathbb{R}, \tag{4.9}$$

$$|G_1(s)| \le \bar{c}_3 \left(|s|^{\gamma} + |s|^{q^*}\right), \qquad \text{for all } s \in \mathbb{R}. \tag{4.10}$$

Arguing as in the proof of Lemma 3.1, we have

Lemma 4.1. *For any $u \in \mathcal{X}_0$, $\int_{\mathbb{R}^N} G(u)\,dx$ and $\int_{\mathbb{R}^N} g(u)u\,dx$ are well defined. The same is true for $\int_{\mathbb{R}^N} G_i(u)\,dx$ and $\int_{\mathbb{R}^N} g_i(u)u\,dx$, for $1 = 1, 2$.*

The following compactness results hold.

Lemma 4.2. *If $u_n \rightharpoonup u_0$ in $\mathcal{X}_0$, then*

$$\lim_n \int_{\mathbb{R}^N} g_1(u_n)u_n\,dx = \int_{\mathbb{R}^N} g_1(u_0)u_0\,dx$$

and

$$\lim_n \int_{\mathbb{R}^N} G_1(u_n)\,dx = \int_{\mathbb{R}^N} G_1(u_0)\,dx.$$

Proof. In the case $\gamma > N$, the arguments are similar to those of the proof of Lemma 3.2. Here we treat only the case $\gamma \in (2^*, N]$, enlightening the main differences.
By (4.1) and (4.2), take any $\varepsilon > 0$ and $\beta \in (2^*, q^*)$, then we find $\delta > 0$ and $c_\varepsilon > 0$ such that

$$\begin{aligned} &|g_1(s)| \le \varepsilon |s|^{\gamma-1} && \text{if } |s| \in [0, \delta], \\ &|g_1(s)| \le c_\varepsilon |s|^{\beta-1} && \text{if } |s| \in (\delta, 1/\delta), \\ &|g_1(s)| \le \varepsilon |s|^{q^*-1} && \text{if } |s| \in [1/\delta, +\infty). \end{aligned}$$

Therefore

$$\begin{aligned} \int_{\mathbb{R}^N} |g_1(u_n)(u_n - u_0)|\,dx \le{}& \varepsilon \int_{\mathbb{R}^N} |u_n|^{\gamma-1}|u_n - u_0|\,dx + c_\varepsilon \int_{\mathbb{R}^N} |u_n|^{\beta-1}|u_n - u_0|\,dx \\ &+ \varepsilon \int_{\mathbb{R}^N} |u_n|^{q^*-1}|u_n - u_0|\,dx, \end{aligned}$$

and, by the compact embedding of $\mathcal{X}_0$ into $L^\beta(\mathbb{R}^N)$, the boundedness of the sequence $\{u_n\}$ in $\mathcal{X}_0$, we infer that

$$\limsup_n \int_{\mathbb{R}^N} |g_1(u_n)(u_n - u_0)|\, dx \le \varepsilon C$$

for some constant $C > 0$. Now the proof goes on in a similar way as in Lemma 3.2. □

Solutions of (2.3) will be found as critical points of the functional $I_\theta : \mathcal{X}_0 \to \mathbb{R}$ defined as

$$I_\theta(u) = \frac{1}{2}\int_{\mathbb{R}^N} A_\theta(|\nabla u|^2)\, dx + \int_{\mathbb{R}^N} G_2(u)\, dx - \int_{\mathbb{R}^N} G_1(u)\, dx,$$

which is well defined in $\mathcal{X}_0$. Here and in what follows, with an abuse of notation, we use I_θ, J_θ, m_θ, $\tilde{m}_\theta$, Γ, and Σ in the zero mass setting, as well.

We show that I_θ satisfies the mountain pass geometry.

Lemma 4.3. *For any $\theta \in (0, \theta_1]$, the functional $I_\theta : \mathcal{X}_0 \to \mathbb{R}$ verifies the mountain pass geometry. More precisely:*

(i) there are $\alpha, \rho > 0$ such that $I_\theta(u) \ge \alpha$, for $\|u\|_0 = \rho$;
(ii) there is $\bar{u} \in \mathcal{X}_0 \setminus \{0\}$, independent of $\theta \in (0, \theta_1]$, with $\|\bar{u}\|_0 > \rho$ and $|\nabla \bar{u}| < 1 - \theta_1$, almost everywhere in $\mathbb{R}^N$, and such that $I_\theta(\bar{u}) < 0$.

Proof. (i) We start with the case $\gamma > N$. Since $q \in (N, \gamma)$, by the continuous embedding of $\mathcal{X}_0$ into $L^\infty(\mathbb{R}^N)$, and by (4.4), we can consider $\rho > 0$ sufficiently small such that

$$G(u(x)) \le \bar{c}_1 |u(x)|^\gamma, \qquad \text{a.e. } x \in \mathbb{R}^N \text{ and for any } u \in \mathcal{X}_0 \text{ with } \|u\|_0 = \rho.$$

Hence, by (2.5) and since $\mathcal{X}_0$ is embedded into $L^\gamma(\mathbb{R}^N)$, for any $u \in \mathcal{X}_0$ with $\|u\|_0 = \rho$, we have

$$I_\theta(u) \ge c\left(|\nabla u|_2^2 + |\nabla u|_q^q - |u|_\gamma^\gamma\right) \ge c\left(|\nabla u|_2^2 + |\nabla u|_q^q - |\nabla u|_2^\gamma - |\nabla u|_q^\gamma\right) \ge \alpha > 0.$$

Let us consider now the case $\gamma \in (2^*, N]$. By (4.1) and (4.2), take any $\varepsilon > 0$ and $\beta \in (\max\{2^*, q\}, q^*)$, then we find $c_\varepsilon > 0$ such that

$$0 \le G_1(s) \le \varepsilon\left(|s|^\gamma + |s|^{q^*}\right) + c_\varepsilon |s|^\beta, \qquad \text{for all } s \in \mathbb{R}.$$

Hence, if $\rho < 1$, we have

$$\begin{aligned} I_\theta(u) &\ge c\left(|\nabla u|_2^2 + |\nabla u|_q^q\right) - \varepsilon\left(|u|_\gamma^\gamma + |u|_{q^*}^{q^*}\right) - c_\varepsilon |u|_\beta^\beta \\ &\ge c\left[|\nabla u|_2^2 + |\nabla u|_q^q - \varepsilon\left(|\nabla u|_2^\gamma + |\nabla u|_q^\gamma + |\nabla u|_2^{q^*} + |\nabla u|_q^{q^*}\right) - \left(|\nabla u|_2^\beta + |\nabla u|_q^\beta\right)\right] \\ &\ge c\left[\|u\|_0^q - \|u\|_0^\beta - \varepsilon\left(\|u\|_0^\gamma + \|u\|_0^{q^*}\right)\right] \ge \alpha > 0. \end{aligned}$$

(ii) As in the proof of Lemma 3.3. □

Let us define the mountain pass level for the functional I_θ

$$m_\theta := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I_\theta(\gamma(t)),$$

where

$$\Gamma := \{\gamma \in \mathcal{C}([0,1], \mathcal{X}_0) \mid \gamma(0) = 0, \gamma(1) = \bar{u}\}.$$

By Lemma 4.3, we deduce that $m_\theta \geq \alpha$, for any $\theta \in (0, \theta_1]$.

Observe that, since $|\nabla \bar{u}| < 1 - \theta_1$, we have that $I_{\theta_1}(t\bar{u}) = I_\theta(t\bar{u})$, for any $t \in [0,1]$ and for any $\theta \in (0, \theta_1]$. Hence we deduce that

$$m_\theta \leq \max_{t\in[0,1]} I_\theta(t\bar{u}) = \max_{t\in[0,1]} I_{\theta_1}(t\bar{u}),$$

for any $\theta \in (0, \theta_1]$. Hence there exists $c > 0$ (independent of $\theta \in (0, \theta_1]$) such that

$$0 < m_\theta \leq c_2, \qquad \text{for any } \theta \in (0, \theta_1]. \tag{4.11}$$

As done in Section 3, we define the functional $J_\theta : \mathbb{R} \times \mathcal{X}_0 \to \mathbb{R}$ as

$$J_\theta(\sigma, u) = I_\theta(u(e^{-\sigma}\cdot)) = \frac{e^{N\sigma}}{2} \int_{\mathbb{R}^N} A_\theta(e^{-2\sigma}|\nabla u|^2)\,dx + e^{N\sigma} \int_{\mathbb{R}^N} G_2(u)\,dx - e^{N\sigma} \int_{\mathbb{R}^N} G_1(u)\,dx.$$

The functional J_θ has a mountain pass geometry and we can define its mountain pass level as

$$\tilde{m}_\theta := \inf_{(\sigma,\gamma)\in\Sigma\times\Gamma} \max_{t\in[0,1]} J_\theta\big(\sigma(t), \gamma(t)\big),$$

where

$$\Sigma := \{\sigma \in \mathcal{C}([0,1], \mathbb{R}) \mid \sigma(0) = \sigma(1) = 0\}.$$

The following holds

Lemma 4.4. *For any $\theta \in (0, \theta_1]$, the mountain pass levels of I_θ and J_θ coincide, namely $m_\theta = \tilde{m}_\theta$.*

Lemma 4.5. *Let $\theta \in (0, \theta_1]$ and $\varepsilon > 0$. Suppose that $\tilde{\gamma} \in \Sigma \times \Gamma$ satisfies*

$$\max_{t\in[0,1]} J_\theta(\tilde{\gamma}(t)) \leq m_\theta + \varepsilon,$$

then there exists $(\sigma, u) \in \mathbb{R} \times \mathcal{X}_0$ such that

(1) $\mathrm{dist}_{\mathbb{R}\times\mathcal{X}_0}\big((\theta, u), \tilde{\gamma}([0,1])\big) \leq 2\sqrt{\varepsilon}$;

(2) $J_\theta(\sigma, u) \in [m_\theta - \varepsilon, m_\theta + \varepsilon]$;

(3) $\|DJ_\theta(\sigma,u)\|_{\mathbb{R}\times\mathcal{X}^*}\le 2\sqrt{\varepsilon}$.

Proposition 4.6. *For any $\theta\in(0,\theta_1]$, there exists a sequence $\{(\sigma_n,u_n)\}\subset\mathbb{R}\times\mathcal{X}_0$ such that, as $n\to+\infty$, we get*

(1) $\sigma_n\to 0$;
(2) $J_\theta(\sigma_n,u_n)\to m_\theta$;
(3) $\partial_\sigma J_\theta(\sigma_n,u_n)\to 0$;
(4) $\partial_u J_\theta(\sigma_n,u_n)\to 0$ *strongly in* $\mathcal{X}_0^*$.

Proposition 4.7. *For any $\theta\in(0,\theta_1]$, there exists $u_\theta\in\mathcal{X}_0$ a non-trivial solution of (2.2) such $I_\theta(u_\theta)=m_\theta$. Moreover there exists $C>0$ such that*

$$\|u_\theta\|_0\le C,\qquad \text{for any } \theta\in(0,\theta_1]. \tag{4.12}$$

Finally u_θ is a weak solution of

$$-\big(r^{N-1}a_\theta(|u_\theta'(r)|^2)u_\theta'(r)\big)'=r^{N-1}g(u_\theta(r)), \tag{4.13}$$

namely

$$\int_0^{+\infty} r^{N-1}a_\theta(|u_\theta'(r)|^2)u_\theta'(r)v'(r)\,dr=\int_0^{+\infty} r^{N-1}g(u_\theta(r))v(r)\,dr,$$

for all $v\in\mathcal{X}_0$.

Proof. Fix $\theta\in(0,\theta_1]$. By Proposition 4.6, there exists a sequence $\{(\sigma_n,u_n)\}\subset\mathbb{R}\times\mathcal{X}_0$ such that

$$\begin{cases}\dfrac{e^{N\sigma_n}}{2}\displaystyle\int_{\mathbb{R}^N}A_\theta(e^{-2\sigma_n}|\nabla u_n|^2)\,dx+e^{N\sigma_n}\int_{\mathbb{R}^N}G_2(u_n)\,dx-e^{N\sigma_n}\int_{\mathbb{R}^N}G_1(u_n)\,dx=m_\theta+o_n(1),\\ \dfrac{Ne^{N\sigma_n}}{2}\displaystyle\int_{\mathbb{R}^N}A_\theta(e^{-2\sigma_n}|\nabla u_n|^2)\,dx-e^{(N-2)\sigma_n}\int_{\mathbb{R}^N}a_\theta(e^{-2\sigma_n}|\nabla u_n|^2)|\nabla u_n|^2\,dx\\ \qquad\qquad+Ne^{N\sigma_n}\displaystyle\int_{\mathbb{R}^N}G_2(u_n)\,dx-Ne^{N\sigma_n}\int_{\mathbb{R}^N}G_1(u_n)\,dx=o_n(1),\\ e^{(N-2)\sigma_n}\displaystyle\int_{\mathbb{R}^N}a_\theta(e^{-2\sigma_n}|\nabla u_n|^2)|\nabla u_n|^2\,dx+e^{N\sigma_n}\int_{\mathbb{R}^N}g_2(u_n)u_n\,dx\\ \qquad\qquad-e^{N\sigma_n}\displaystyle\int_{\mathbb{R}^N}g_1(u_n)u_n\,dx=o_n(1)\|u_n\|.\end{cases}$$

From the first and the second equation of the previous system we get

$$e^{(N-2)\sigma_n}\int_{\mathbb{R}^N} a_\theta(e^{-2\sigma_n}|\nabla u_n|^2)|\nabla u_n|^2\,dx = Nm_\theta + o_n(1).$$

Therefore, since $\sigma_n \to 0$, as $n \to +\infty$, by (2.4) we deduce that $\{u_n\}$ is a bounded sequence in $\mathcal{X}_0$. Then there exists $u_\theta \in \mathcal{X}_0$ such that $u_n \rightharpoonup u_\theta$ in $\mathcal{X}_0$. Since $\partial_u J_\theta(\sigma_n, u_n) \to 0$ strongly in $\mathcal{X}_0^*$ and $\sigma_n \to 0$, we have that u_θ is a weak (possibly trivial) solution of (2.3) and so it satisfies

$$\int_{\mathbb{R}^N} a_\theta(|\nabla u_\theta|^2)|\nabla u_\theta|^2\,dx + \int_{\mathbb{R}^N} g_2(u_\theta)u_\theta\,dx = \int_{\mathbb{R}^N} g_1(u_\theta)u_\theta\,dx.$$

Arguing as in proof of Proposition 3.7 we can show that

$$\int_{\mathbb{R}^N} a_\theta(|\nabla u_\theta|^2)|\nabla u_\theta|^2\,dx = \lim_{n\to+\infty}\int_{\mathbb{R}^N} a_\theta(|\nabla u_n|^2)|\nabla u_n|^2\,dx.$$

In view of Lemma 2.5, we have that $u_n \to u_\theta$ strongly in $\mathcal{X}_0$ and so $I_\theta(u_\theta) = m_\theta$.
Finally, since

$$\int_{\mathbb{R}^N} a_\theta(|\nabla u_\theta|^2)|\nabla u_\theta|^2\,dx = Nm_\theta,$$

by (4.11) and (2.4), we prove that there exists $C > 0$ such that $\|u_\theta\|_0 \le C$, for any $\theta \in (0, \theta_1]$. □

We are now able to conclude the proof of Theorem 1.1.

Proof of Theorem 1.1. By Proposition 4.7, for any $\theta \in (0, \theta_1]$, there exists $u_\theta \in \mathcal{X}_0$ a nontrivial solution of (2.3) such $I_\theta(u_\theta) = m_\theta$. When $\gamma > N$, since $q > N$, the space $\mathcal{X}_0$ is embedded into $L^\infty(\mathbb{R}^N)$ and the regularity arguments and the estimates of Section 3 can be adapted with slight changes. Therefore, here we deal just with the case $2^* < \gamma \le N$ and so we have to assume, in addition, (g1$''$). Being $q < N$, we cannot repeat the arguments of the previous section and now we follow some ideas of [11, Lemma 3.2]. Since u_θ is a solution of (4.13) in $(0, +\infty)$, it is easy to check that u_θ is regular for $r > 0$. Moreover, $r^{N-1}a_\theta(|u_\theta'(r)|^2)u_\theta'(r)$ satisfies the Cauchy condition at the origin so that it has a finite limit as $r \to 0$. We claim that

$$\lim_{r\to 0} r^{N-1}a_\theta(|u_\theta'(r)|^2)u_\theta'(r) = 0. \tag{4.14}$$

Suppose, by contradiction, that it is different from zero and then there should exist $r_0 > 0$ such that $|u_\theta'(r)| > 1 - \theta$, for $r \in (0, r_0]$. Therefore, for r sufficiently small,

$$C \le \left| r^{N-1}a_\theta(|u_\theta'(r)|^2)u_\theta'(r)\right| = r^{N-1}|u_\theta'(r)|^{q-1},$$

namely

$$|u_\theta'(r)| \ge Cr^{-\frac{N-1}{q-1}}.$$

By this we have

$$r^{N-1}a_\theta(|u_\theta'(r)|^2)|u_\theta'(r)|^2 = r^{N-1}|u_\theta'(r)|^q \geq Cr^{-\frac{N-1}{q-1}}$$

near 0, which is not integrable since $q < N$. Since u_θ is a solution of (4.13), we get a contradiction.

Let us prove the following

Claim: there exists $C > 0$ such that

$$|a_\theta(|u_\theta'(r)|^2)u_\theta'(r)| \leq C, \qquad \text{for any } r \geq 0 \text{ and } \theta \in (0, \theta_1].$$

By the regularity of u_θ, we infer that $u_\theta'(0) = 0$ and so also

$$a_\theta(|u_\theta'(0)|^2)u_\theta'(0) = 0.$$

We now consider the case $r > 0$. Integrating the equation (4.13), for any $r > 0$, we have

$$-a_\theta(|u_\theta'(r)|^2)u_\theta'(r) = \frac{1}{r^{N-1}}\int_0^r s^{N-1}g(u_\theta(s))\,ds.$$

By Lemma 2.1 and by (4.12), we deduce that there exists $R > 1$, such that

$$|u_\theta(r)| \leq \bar{c}_2, \quad \text{for any } \theta \in (0, \theta_1] \text{ and for any } r > R, \tag{4.15}$$

where $\bar{c}_2$ is given in (4.3).

By the continuous embedding of $\mathcal{X}_0$ in $L^p(\mathbb{R}^N)$, for $p \in [2^*, q^*]$, and (4.12), there exists $C > 0$ such that $|u_\theta|_p \leq C\|u_\theta\|_0 \leq C$, for $p \in [2^*, q^*]$ and any $\theta \in (0, \theta_1]$. So, using (4.7), we have that, for any $0 < r \leq R$ and $\theta \in (0, \theta_1]$,

$$|a_\theta(|u_\theta'(r)|^2)u_\theta'(r)| \leq \frac{1}{r^{N-1}}\int_0^r s^{N-1}|g(u_\theta(s))|\,ds \leq C.$$

If $r > R$, then

$$\begin{aligned}|a_\theta(|u_\theta'(r)|^2)u_\theta'(r)| &\leq \frac{1}{r^{N-1}}\int_0^r s^{N-1}|g(u_\theta(s))|\,ds \\ &\leq \frac{1}{r^{N-1}}\left(\int_0^R s^{N-1}|g(u_\theta(s))|\,ds + \int_R^r s^{N-1}|g(u_\theta(s))|\,ds\right)\end{aligned}$$

$$\leq \frac{C}{r^{N-1}} + \underbrace{\frac{c_1}{r^{N-1}} \int_1^r s^{N-1} |g(u_\theta(s))|\, ds}_{(A)} .$$

We have to estimate (A). First of all, by Lemma 2.1 and (4.12), for $r > 1$, we have that

$$|u_\theta(r)| \leq C r^{-\frac{N-2}{2}} |\nabla u_\theta|_2 \leq \bar{C} r^{-\frac{N-2}{2}} .$$

Hence, by (4.15) and (4.7), since $2^* < \gamma < q^*$,

$$\begin{aligned}(A) &\leq \frac{C}{r^{N-1}} \int_1^r s^{N-1} \left(|u_\theta(s)|^{\gamma-1} + |u_\theta(s)|^{q^*-1} \right) ds \\ &\leq \frac{C}{r^{N-1}} \int_1^r s^{N-1-\frac{N-2}{2}(\gamma-1)}\, ds \leq C \left(r^{1-\frac{N-2}{2}(\gamma-1)} + 1 \right) \leq C.\end{aligned}$$

Therefore the claim is proved.
Now we conclude as in the previous section. □

Data availability

No data was used for the research described in the article.

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