

GLOBAL SOBOLEV REGULARITY FOR GENERAL ELLIPTIC EQUATIONS OF p -LAPLACIAN TYPE

SUN-SIG BYUN, DIAN K. PALAGACHEV, AND PILSOO SHIN

ABSTRACT. We derive global gradient estimates for $W_0^{1,p}(\Omega)$ -weak solutions to quasilinear elliptic equations of the form

$$\operatorname{div} \mathbf{a}(x, u, Du) = \operatorname{div} (|F|^{p-2} F)$$

over n -dimensional Reifenberg flat domains. The nonlinear term of the elliptic differential operator is supposed to be small-BMO with respect to x and merely continuous in u .

Our result highly improves the known regularity results available in the literature. Actually, we are able not only to weaken the Lipschitz continuity with respect to u of the nonlinearity to only uniform continuity, but we also find a very lower level of geometric assumption on the boundary of the domain to ensure a global character of the gradient estimates obtained.

1. INTRODUCTION

Solutions to important real world problems from science and technology turn out to realize minimal energy of suitable nonlinear functionals. Finding these solutions and examining closely their qualitative properties is a central problem of the Calculus of Variations, and the machinery of the nonlinear functional analysis is what serves to pursue that study. On the other hand, each minimizer of a variational functional solves weakly the corresponding Euler–Lagrange equation and this fact allows to rely on the powerful theory of PDEs as an additional tool in the Calculus of Variations. The Euler–Lagrange equations are divergence form PDEs of elliptic type, usually nonlinear, and their weak solutions (the minimizers) own some basic minimal smoothness. The *regularity theory* of general (non necessary variational) divergence form elliptic PDEs establishes how the smoothness of the data of a given problem influences the regularity of the solution, obtained under very general circumstances. Once having better smoothness, powerful tools of functional analysis apply to infer finer properties of the solution and the problem itself. The importance of these issues is even more evident in the settings of variational problems if dealing with discontinuous functionals over domains with non-smooth boundaries when many of the classical nonlinear analysis techniques fail.

Starting with the deep results of Caffarelli and Peral ([9]), a notable progress has been achieved in the last two decades in the regularity theory of nonlinear divergence form elliptic PDEs (see also [1–3, 7, 10, 13, 16] and the references therein). On the base of suitable L^p -estimates for the gradient Du of the weak solution a

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satisfactory Calderón–Zygmund type theory has been developed, firstly for equations with principal term depending only on Du , and later also dependence on the independent variables x has been allowed. Moreover, the minimal regularity requirements have been identified for the nonlinear terms of the equations and the boundary of the underlying domain in order the Calderón–Zygmund theory still holds true for large class of equations with generally x -discontinuous ingredients. In all that context, the possibility to deal with equations with general nonlinearity with respect to the solution u is a rather delicate matter, and the reason of this lies in the fact that such equations are not invariant under particular scaling and normalization, whereas these are crucial ingredients of the perturbation approach in [9].

We deal here with the Dirichlet problem

$$(1.1) \quad \begin{cases} \operatorname{div} \mathbf{a}(x, u, Du) = \operatorname{div} (|F|^{p-2} F) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a bounded and generally irregular domain, $\mathbf{a}: \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a Carathéodory map, $p > 1$ is an arbitrary exponent and $F \in L^p(\Omega, \mathbb{R}^n)$.

Our main goal is to obtain a Calderón–Zygmund type regularizing property of (1.1). Namely, assuming $F \in L^{p'}(\Omega, \mathbb{R}^n)$ for $p' > p$, under rather general structure and regularity hypotheses on $\mathbf{a}(x, z, \xi)$ and $\partial\Omega$, we derive global $L^{p'}(\Omega)$ -gradient estimate for any bounded $W_0^{1,p}(\Omega)$ weak solution of (1.1) in terms of $\|F\|_{L^{p'}(\Omega, \mathbb{R}^n)}$ showing this way that $F \in L^{p'}(\Omega, \mathbb{R}^n)$ implies $Du \in L^{p'}(\Omega, \mathbb{R}^n)$.

In the case when $\mathbf{a} = \mathbf{a}(x, Du)$, similar results have been obtained in [8] in the settings of classical Lebesgue spaces and in [6] for weighted Lebesgue spaces, assuming the standard ellipticity condition and allowing discontinuity of \mathbf{a} with respect to x , measured in terms of small-BMO seminorm. In the recent paper [17], the authors succeeded to obtain *interior* gradient estimates for (1.1) also in the case when \mathbf{a} depends on the solution u . The problems arising with the scaling and normalization in that situation are cleverly avoided by including the nonlinear differential operator into a *two parameter* class of elliptic operators, that turns out to be invariant with respect to dilations and rescaling of the domain. In order to run the approximation procedure of [9], a *uniform* control with respect to these two parameters is necessary, and the authors of [17] carry out it by means of a delicate compactness argument relying on the Minty trick. This approach, however, strongly requires *uniqueness* for the approximating equation and, that is why, $\mathbf{a}(x, z, \xi)$ is assumed to be *Lipschitz continuous with respect to z* in [17].

Here we suppose that $\mathbf{a}(x, z, \xi)$ is a small-BMO function with respect to x and it satisfies the standard uniform ellipticity condition in ξ but, in contrast to [17], \mathbf{a} is assumed to be *only locally uniformly continuous* with respect to the variable z . To get our main result, we combine the two-parameter approach from [17] with correct scaling arguments in the L^q -estimates for the maximal function of the gradient and Vitali type covering lemma. However, we rely here on the *higher gradient integrability* in the spirit of Gehring–Giaquinta rather than on the uniqueness of the approximating equation, and this allows us to weaken the z -Lipschitz continuity of \mathbf{a} to only continuity.

We start with considering two appropriate reference problems with *only gradient nonlinear terms*, given by the z -compositions of $\mathbf{a}(x, z, \xi)$ first with the weak solution $u(x)$ and then with its local average \bar{u} . Thanks to the uniform ellipticity of the associated nonlinearities, the reference solutions support higher integrability results and Hölder continuity properties. We then combine these properties with the z -continuity of \mathbf{a} and the comparison estimates of [6], regarding nonlinear terms like $\mathbf{a}(x, Du)$, in order to obtain the desired comparison estimates. Once having these, the main result follows by maximal function techniques and a Vitali type covering lemma.

Another advantage of the approach here adopted is that it works also near the boundary of Ω and this allows to obtain *global* gradient estimates for the solutions of (1.1). Indeed, this requires some “good” geometric properties of $\partial\Omega$ and these are ensured when Ω belongs to the class of the Reifenberg flat domains.

The paper is organized as follows. In Section 2 we list the hypotheses imposed on the data and state the main result, Theorem 2.2. Some comments about the structure and regularity assumptions required are given as well. Section 3 provides an analysis of how the equation in (1.1) and the hypotheses on the nonlinear term behave under the two-parameter scaling and normalization. Section 4 forms the analytic heart of the paper. We derive there good gradient estimates for solutions to appropriate limiting problems to which (1.1) compares. With these estimates at hand, we employ in Section 5 a Vitali type covering lemma and scaling arguments in order to prove Theorem 2.2 by obtaining suitable decay estimates for the upper-level sets of the Hardy–Littlewood maximal function of the gradient.

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2. HYPOTHESES AND MAIN RESULTS

Throughout the paper, we will use standard notations and will assume that the functions and sets considered are measurable.

We denote by $B_\rho(x)$ (or simply B_ρ if there is no ambiguity) the n -dimensional open ball with center $x \in \mathbb{R}^n$ and radius ρ , and $\Omega_\rho(x) := \Omega \cap B_\rho(x)$ for an open set $\Omega \subset \mathbb{R}^n$. The Lebesgue measure of a measurable set $A \subset \mathbb{R}^n$ will be denoted by $|A|$ while, for any integrable function u defined on A ,

$$\bar{u}_A := \fint_A u(x) \, dx = \frac{1}{|A|} \int_A u(x) \, dx$$

stands for its integral average. If $u \in L^1_{\text{loc}}(\mathbb{R}^n)$, then the Hardy–Littlewood maximal function of u is given by

$$\mathcal{M}u(x) := \sup_{\rho>0} \fint_{B_\rho(x)} |u(y)| \, dy,$$

while $\mathcal{M}_A u := \mathcal{M}(\chi_A u)$ when u is defined on a bounded set A , with the characteristic function χ_A of the set A .

We will denote by $C_0^\infty(\Omega)$ the space of infinitely differentiable functions over a bounded domain $\Omega \subset \mathbb{R}^n$ with compact support contained in Ω , and $L^p(\Omega)$ stands for the standard Lebesgue space with a given $p \in [1, \infty]$. The Sobolev space $W_0^{1,p}(\Omega)$ is defined, as usual, by the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{W^{1,p}(\Omega)} := \|u\|_{L^p(\Omega)} + \|Du\|_{L^p(\Omega)}$$

for $p \in [1, \infty)$.

In what follows, we will consider a bounded domain $\Omega \subset \mathbb{R}^n$ with $n \geq 2$, the boundary $\partial\Omega$ of which is *Reifenberg flat* in the sense of the following definition.

Definition 2.1. *The domain Ω is said to be (δ, R) -Reifenberg flat if there exist positive constants δ and R with the property that for each $x_0 \in \partial\Omega$ and each $\rho \in (0, R)$ there is a local coordinate system $\{x_1, \dots, x_n\}$ with origin at the point x_0 , and such that*

$$B_\rho(x_0) \cap \{x : x_n > \rho\delta\} \subset B_\rho(x_0) \cap \Omega \subset B_\rho(x_0) \cap \{x : x_n > -\rho\delta\}.$$

Turning back to problem (1.1), the nonlinear term is given by the Carathéodory map $\mathbf{a} : \Omega \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ where $\mathbf{a}(x, z, \xi) = (a^1(x, z, \xi), \dots, a^n(x, z, \xi))$. We suppose moreover that $\mathbf{a}(x, z, \xi)$ is differentiable with respect to $\xi \neq 0$, and $D_\xi \mathbf{a}$ is a Carathéodory map.

Throughout the paper the following structure and regularity conditions on the data will be assumed:

- *Uniform ellipticity and growth conditions:* There exists a constant $\gamma > 0$ such that

$$(2.1) \quad \begin{cases} \gamma|\xi|^{p-2}|\eta|^2 \leq \langle D_\xi \mathbf{a}(x, z, \xi)\eta, \eta \rangle, \\ |\mathbf{a}(x, z, \xi)| + |\xi| |D_\xi \mathbf{a}(x, z, \xi)| \leq \gamma^{-1}|\xi|^{p-1} \end{cases}$$

for a.a. $x \in \Omega$, $\forall (z, \xi) \in \mathbb{R} \times \mathbb{R}^n \setminus \{0\}$ and $\forall \eta \in \mathbb{R}^n$.

It is worth noting that the uniform ellipticity condition (2.1) implies easily the following monotonicity property:

$$(2.2) \quad \langle \mathbf{a}(x, z, \xi_1) - \mathbf{a}(x, z, \xi_2), \xi_1 - \xi_2 \rangle \geq \begin{cases} \tilde{\gamma}|\xi_1 - \xi_2|^p & \text{if } p \geq 2, \\ \tilde{\gamma}|\xi_1 - \xi_2|^2(|\xi_1| + |\xi_2|)^{p-2} & \text{if } 1 < p < 2, \end{cases}$$

where $\tilde{\gamma}$ depends only on γ , n and p .

- *Local uniform continuity:* For each $M > 0$ there is a non-decreasing function $\omega_M : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ with $\lim_{\rho \rightarrow 0^+} \omega_M(\rho) = 0$ such that

$$(2.3) \quad |\mathbf{a}(x, z_1, \xi) - \mathbf{a}(x, z_2, \xi)| \leq \omega_M(|z_1 - z_2|)|\xi|^{p-1}$$

for a.a. $x \in \Omega$, $\forall z_1, z_2 \in [-M, M]$ and $\forall \xi \in \mathbb{R}^n$.

- *(δ, R) -vanishing property:* For each constant $M > 0$ there exist $R > 0$ and $\delta > 0$, depending on M , such that

$$(2.4) \quad \sup_{z \in [-M, M]} \sup_{0 < \rho \leq R} \sup_{y \in \mathbb{R}^n} \int_{B_\rho(y)} \Theta(\mathbf{a}; B_\rho(y))(x, z) dx \leq \delta,$$

where the function Θ is defined by

$$\Theta(\mathbf{a}; B_\rho(y))(x, z) := \sup_{\xi \in \mathbb{R}^n \setminus \{0\}} \frac{|\mathbf{a}(x, z, \xi) - \bar{\mathbf{a}}_{B_\rho(y)}(z, \xi)|}{|\xi|^{p-1}},$$

and $\bar{\mathbf{a}}_{B_\rho(y)}(z, \xi)$ is the integral average of $\mathbf{a}(x, z, \xi)$ in the variables x for a fixed couple $(z, \xi) \in \mathbb{R} \times \mathbb{R}^n$, that is,

$$\bar{\mathbf{a}}_{B_\rho(y)}(z, \xi) = \fint_{B_\rho(y)} \mathbf{a}(x, z, \xi) \, dx.$$

To make clear the meaning of the above assumptions, we should note that, thanks to the scaling invariance property of $\partial\Omega$, R could be any number greater than or equal to 1 in Definition 2.1, while R could be taken equal to $\text{diam}\Omega$ in (2.4). For what concerns δ instead, the definitions of (δ, R) -Reifenberg flatness and (δ, R) -vanishing property are significant only for small values, say $\delta \in (0, 1/8]$. Roughly speaking, the Reifenberg flatness of Ω means that $\partial\Omega$ is well approximated by hyperplanes at every point and at every scale. In particular, domains with C^1 -smooth boundary or with boundary that is locally given as graph of a Lipschitz continuous function with small Lipschitz constant are Reifenberg flat. Actually, the class of the Reifenberg flat domains is much wider and contains sets with rough fractal boundaries such as the von Koch snowflake that is a Reifenberg flat when the angle of the spike with respect to the horizontal is small enough. As for the (δ, R) -vanishing property (2.4), it exhibits a sort of smallness in terms of BMO for what concerns the behaviour of $\mathbf{a}(x, z, \xi)$ with respect to the x -variables. For instance, (2.4) is satisfied when $\mathbf{a} \in C_x^0$ or even VMO_x . This way, (2.4) allows x -discontinuity of the nonlinearity which is controlled in terms of small-BMO.

Turning back to the Dirichlet problem (1.1), recall that a function $u \in W_0^{1,p}(\Omega)$ is said to be a *weak solution* if

$$\int_{\Omega} \langle \mathbf{a}(x, u(x), Du(x)), D\phi(x) \rangle \, dx = \int_{\Omega} \langle |F(x)|^{p-2} F(x), D\phi(x) \rangle \, dx$$

for each test function $\phi \in W_0^{1,p}(\Omega)$.

Throughout the paper, we will denote by C a universal constant, depending on $\gamma, \omega_M(\cdot), n, p, q, M$ and Ω , which may vary within the same formula.

Our main result is as follows.

Theorem 2.2. *Suppose (2.1) and (2.3), and let $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ be a bounded weak solution of (1.1). Assume that $\|u\|_{L^\infty(\Omega)} \leq M$ and $|F|^p \in L^q(\Omega)$ for some $q \in (1, \infty)$. Then there is a small constant $\delta = \delta(\gamma, n, p, q, \omega_M(\cdot), M)$ such that if \mathbf{a} is (δ, R) -vanishing and Ω is (δ, R) -Reifenberg flat, then $|Du|^p \in L^q(\Omega)$ with the estimate*

$$\int_{\Omega} |Du|^{pq} \, dx \leq C \int_{\Omega} |F|^{pq} \, dx,$$

where $C > 0$ depends only on $\gamma, n, p, q, M, \omega_M(\cdot)$ and $|\Omega|$.

As already mentioned before, Theorem 2.2 provides *global (up to the boundary) Calderón–Zygmund* property for all bounded weak solutions to the Dirichlet problem (1.1) and it generalizes [17, Theorem 1.1] where *interior* gradient estimates have been derived. Moreover, our main result requires merely continuity of $\mathbf{a}(x, u, Du)$ with respect to u , while \mathbf{a} is u -Lipschitz continuous in [17]. For what concerns the assumption $\|u\|_{L^\infty(\Omega)} \leq M$ in Theorem 2.2, sufficient conditions, ensuring *essential boundedness* of any weak solution to (1.1) can be found in [14, Chapter IV] and these require suitable growths of $\mathbf{a}(x, u, Du)$ in u and Du , together with appropriate control with respect to x in terms of Lebesgue spaces. We refer the reader also

to our recent results [4, 5] where boundedness and Hölder continuity of the weak solutions are obtained when the x -behaviour of $\mathbf{a}(x, u, Du)$ is controlled in Morrey spaces and Ω is a Reifenberg flat domain.

3. SCALING AND NORMALIZATION PROPERTIES

In this section, we will show how the scaling and normalization reflect on the structure conditions and regularity assumptions imposed on the data.

Recall that Ω is assumed to be a (δ, R) -Reifenberg flat domain and the nonlinearity \mathbf{a} satisfies the conditions (2.1), (2.3) and the (δ, R) -vanishing property (2.4). Let σ be a large enough positive constant which is to be determined later in a universal way so that it will depend only on the given data such as n, p, q, γ and M . Then for each fixed $\lambda > 0$ and $0 < r \leq \frac{R}{\sigma}$, we define a bounded domain

$$\tilde{\Omega} = \left\{ \frac{1}{r}x : x \in \Omega \right\},$$

a Carathéodory map $\tilde{\mathbf{a}}: \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ by

$$\tilde{\mathbf{a}}(x, z, \xi) = \frac{\mathbf{a}(rx, \lambda rz, \lambda \xi)}{\lambda^{p-1}},$$

a function $\tilde{u} \in W_0^{1,p}(\tilde{\Omega})$ and a measurable function $\tilde{F} \in L^p(\tilde{\Omega})$ by

$$\tilde{u}(x) = \frac{u(rx)}{\lambda r} \quad \text{and} \quad \tilde{F}(x) = \frac{F(rx)}{\lambda}.$$

Straightforward calculations yield the following properties:

- $\tilde{\mathbf{a}}$ satisfies the uniform ellipticity and growth condition (2.3) with the same constant γ . That is,

$$(3.1) \quad \begin{cases} \gamma |\xi|^{p-2} |\eta|^2 \leq \langle D_\xi \tilde{\mathbf{a}}(x, z, \xi) \eta, \eta \rangle, \\ |\tilde{\mathbf{a}}(x, z, \xi)| + |\xi| |D_\xi \tilde{\mathbf{a}}(x, z, \xi)| \leq \gamma^{-1} |\xi|^{p-1} \end{cases}$$

for a.a. $x \in \mathbb{R}^n$, $\forall (z, \xi) \in \mathbb{R} \times \mathbb{R}^n \setminus \{0\}$ and $\forall \eta \in \mathbb{R}^n$.
Moreover, the monotonicity

$$(3.2) \quad \begin{aligned} & \langle \tilde{\mathbf{a}}(x, z, \xi_1) - \tilde{\mathbf{a}}(x, z, \xi_2), \xi_1 - \xi_2 \rangle \\ & \geq \begin{cases} \tilde{\gamma} |\xi_1 - \xi_2|^p & \text{if } p \geq 2, \\ \tilde{\gamma} |\xi_1 - \xi_2|^2 (|\xi_1| + |\xi_2|)^{p-2} & \text{if } 1 < p < 2, \end{cases} \end{aligned}$$

does follow with the same constant $\tilde{\gamma}$.

- $\tilde{\mathbf{a}}$ satisfies

$$(3.3) \quad |\tilde{\mathbf{a}}(x, z_1, \xi) - \tilde{\mathbf{a}}(x, z_2, \xi)| \leq \omega_M(\lambda r |z_1 - z_2|) |\xi|^{p-1}$$

for a.a. $x \in \tilde{\Omega}$, $\forall z_1, z_2 \in [-\frac{M}{\lambda r}, \frac{M}{\lambda r}]$ and $\forall \xi \in \mathbb{R}^n$.

- $\tilde{\mathbf{a}}$ is $(\delta, \frac{R}{r})$ -vanishing. Namely,

$$\sup_{z \in [-\frac{M}{\lambda r}, \frac{M}{\lambda r}]} \sup_{0 < \rho \leq \frac{R}{r}} \sup_{y \in \mathbb{R}^n} \int_{B_\rho(y)} \Theta(\tilde{\mathbf{a}}; B_\rho(y))(x, z) dx \leq \delta.$$

- $\tilde{\Omega}$ is $(\delta, \frac{R}{r})$ -Reifenberg flat.

- If $u \in W_0^{1,p}(\Omega)$ is a weak solution of (1.1) with $\|u\|_{L^\infty(\Omega)} \leq M$, then $\tilde{u} \in W_0^{1,p}(\tilde{\Omega})$ is a weak solution of the problem

$$(3.4) \quad \begin{cases} \operatorname{div} \tilde{\mathbf{a}}(x, \tilde{u}(x), D\tilde{u}(x)) = \operatorname{div} (|\tilde{F}|^{p-2} \tilde{F}) & \text{in } \tilde{\Omega}, \\ \tilde{u} = 0 & \text{on } \partial\tilde{\Omega} \end{cases}$$

$$\text{with } \|\tilde{u}\|_{L^\infty(\tilde{\Omega})} \leq \frac{M}{\lambda r}.$$

4. COMPARISON ESTIMATES

A crucial step in the proof of the main result is ensured by an appropriate comparison of a given weak solution to (3.4) with those of the associated reference problems (4.5), (4.6) and (4.9) below. Throughout the section, for the sake of simplicity, we will use the notations u , F , \mathbf{a} and Ω , instead of \tilde{u} , \tilde{F} , $\tilde{\mathbf{a}}$ and $\tilde{\Omega}$, respectively.

We start with the following useful lemma.

Lemma 4.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Assume that $\mathbf{a} : \Omega \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ satisfies (3.2) for a.a. $x \in \Omega$ and for some $p \in (1, 2)$. Then, for any $\zeta_1, \zeta_2 \in W^{1,p}(\Omega_\rho)$, any non-negative function $\eta \in C^0(\overline{B_\rho})$, any bounded function ϕ defined on Ω_ρ and any constant $\tau > 0$, we have*

$$\begin{aligned} \int_{\Omega_\rho} |D\zeta_1 - D\zeta_2|^p \eta \, dx &\leq \tau \int_{\Omega_\rho} |D\zeta_1|^p \eta \, dx \\ &\quad + C \int_{\Omega_\rho} \langle \mathbf{a}(x, \phi, D\zeta_1) - \mathbf{a}(x, \phi, D\zeta_2), D\zeta_1 - D\zeta_2 \rangle \eta \, dx \end{aligned}$$

with $C > 0$ depending only on γ , p and τ .

Proof. See the proofs of [6, Lemma 3.7] and [17, Lemma 3.1]. \square

Let $\sigma \geq 6$ be a universal constant which will be chosen later in Lemma 4.6, and consider a localized solution u in Ω_σ of the problem

$$(4.1) \quad \begin{cases} \operatorname{div} \mathbf{a}(x, u(x), Du(x)) = \operatorname{div} (|F|^{p-2} F) & \text{in } \Omega_\sigma, \\ u = 0 & \text{on } \partial\Omega \cap B_\sigma. \end{cases}$$

Suppose further that for any $\rho \geq 6$,

$$(4.2) \quad \|u\|_{L^\infty(\Omega_\rho)} \leq \frac{M}{\lambda r},$$

$$(4.3) \quad \frac{1}{|B_\rho|} \int_{\Omega_\rho} |Du|^p \, dx \leq 1,$$

and

$$(4.4) \quad \frac{1}{|B_\rho|} \int_{\Omega_\rho} |F|^p \, dx \leq \delta^p.$$

We let next $h \in W^{1,p}(\Omega_\sigma)$ to be the weak solution of

$$(4.5) \quad \begin{cases} \operatorname{div} \mathbf{a}(x, u(x), Dh(x)) = 0 & \text{in } \Omega_\sigma, \\ h = u & \text{on } \partial\Omega_\sigma, \end{cases}$$

and $f \in W^{1,p}(\Omega_5)$ the weak solution of

$$(4.6) \quad \begin{cases} \operatorname{div} \mathbf{a}(x, \bar{u}_{\Omega_5}, Df(x)) = 0 & \text{in } \Omega_5, \\ f = h & \text{on } \partial\Omega_5. \end{cases}$$

Moreover, we assume that

$$(4.7) \quad \frac{1}{|B_5|} \int_{\Omega_5} \Theta(\mathbf{a}; \Omega_5)(x, \bar{u}_{\Omega_5}) dx \leq \delta$$

for both interior and boundary case, and

$$(4.8) \quad B_5^+ \subset \Omega_5 \subset B_5 \cap \{x : x_n > -10\delta\}$$

for the boundary case.

We consider finally the limiting problem

$$(4.9) \quad \operatorname{div} \mathbf{A}(Dv(x)) = 0 \quad \text{in } B_4,$$

for the interior case and

$$(4.10) \quad \begin{cases} \operatorname{div} A(Dv(x)) = 0 & \text{in } B_4^+, \\ v = 0 & \text{on } B_4 \cap \{x_n = 0\}. \end{cases}$$

for the boundary case. Here the map $\mathbf{A} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is given by

$$\mathbf{A}(\xi) := \frac{1}{|B_4|} \int_{B_4} \mathbf{a}(x, \bar{u}_{\Omega_5}, \xi) dx$$

for the interior case and

$$\mathbf{A}(\xi) := \frac{1}{|B_4|} \int_{B_4^+} \mathbf{a}(x, \bar{u}_{\Omega_5}, \xi) dx$$

for the boundary case.

The following is the main result of this section.

Lemma 4.2. *For any small constant $\varepsilon \in (0, 1)$, there exist two constants $\sigma = \sigma(\gamma, \omega_M(\cdot), n, p, M, \varepsilon) \geq 6$ and $\delta = \delta(\gamma, \omega_M(\cdot), n, p, M, \varepsilon) \in (0, \frac{1}{8})$ such that if $u \in W^{1,p}(\Omega_\sigma)$ is a weak solution of (4.1) with (4.2)–(4.4) and (4.7), and (4.8) holds for the boundary case, then there exists a weak solution $v \in W^{1,p}(B_4)$ of (4.9) for the interior case or a weak solution $v \in W^{1,p}(B_4^+)$ of (4.10) for the boundary case such that*

$$\|D\bar{v}\|_{L^\infty(\Omega_3)} \leq N_0 \quad \text{and} \quad \frac{1}{|B_4|} \int_{\Omega_4} |Du - D\bar{v}|^p dx \leq \varepsilon^p$$

for some constant $N_0 = N_0(\gamma, n, p) > 1$. Here, the function $\bar{v} \in W^{1,p}(\Omega_4)$ is equal to v for the interior case and \bar{v} is the zero extension of v from B_4^+ to B_4 for the boundary case.

The proof is based on the following Lemmas 4.3, 4.6 and 4.7.

Lemma 4.3. *Let $\sigma \geq 6$ and $\varepsilon_1 \in (0, 1)$. Then there exists a constant $\delta \in (0, \frac{1}{8})$ depending on n, p, γ, σ and ε_1 such that if $u \in W^{1,p}(\Omega_\sigma)$ is a weak solution of (4.1) with (4.2)–(4.4) and $h \in W^{1,p}(\Omega_\sigma)$ is the weak solution of (4.5), then*

$$(4.11) \quad \frac{1}{|B_6|} \int_{\Omega_6} |Du - Dh|^p dx \leq \varepsilon_1^p.$$

Proof. The proof will be divided into two cases.

Case 1 : $1 < p < 2$. Taking $u - h$ as a test function for equations (4.1) and (4.5), it follows from the Young inequality with $\tau_1 > 0$ that

$$(4.12) \quad \int_{\Omega_\sigma} \langle \mathbf{a}(x, u, Du) - \mathbf{a}(x, u, Dh), Du - Dh \rangle dx = \int_{\Omega_\sigma} \langle |F|^{p-2} F, Du - Dh \rangle dx \\ \leq \tau_1 \int_{\Omega_\sigma} |Du - Dh|^p dx + C(\tau_1) \int_{\Omega_\sigma} |F|^p dx.$$

Then Lemma 4.1 implies

$$\int_{\Omega_\sigma} |Du - Dh|^p dx \\ \leq \tau \int_{\Omega_\sigma} |Du|^p dx + C_0(\tau) \int_{\Omega_\sigma} \langle \mathbf{a}(x, u, Du) - \mathbf{a}(x, u, Dh), Du - Dh \rangle dx \\ \leq \tau \int_{\Omega_\sigma} |Du|^p dx + C_0 \tau_1 \int_{\Omega_\sigma} |Du - Dh|^p dx + C(\tau, \tau_1) \int_{\Omega_\sigma} |F|^p dx.$$

Setting $\tau_1 = \frac{1}{2C_0}$ in the above inequality, we obtain

$$\int_{\Omega_\sigma} |Du - Dh|^p dx \leq 2\tau \int_{\Omega_\sigma} |Du|^p dx + C(\tau) \int_{\Omega_\sigma} |F|^p dx,$$

and so (4.4) yields

$$\frac{1}{|B_6|} \int_{\Omega_6} |Du - Dh|^p dx \leq \left(\frac{\sigma}{6}\right)^n \frac{1}{|B_\sigma|} \int_{\Omega_\sigma} |Du - Dh|^p dx \\ \leq \left(\frac{\sigma}{6}\right)^n \frac{2\tau}{|B_\sigma|} \int_{\Omega_\sigma} |Du|^p dx + C(\tau, \sigma) \frac{1}{|B_\sigma|} \int_{\Omega_\sigma} |F|^p dx \\ \leq 2\tau \left(\frac{\sigma}{6}\right)^n + C(\tau, \sigma) \delta^p.$$

Now, taking the constants τ and δ sufficiently small so that

$$2\tau \left(\frac{\sigma}{6}\right)^n \leq \frac{\varepsilon_1^p}{2} \quad \text{and} \quad C(\tau, \sigma) \delta^p \leq \frac{\varepsilon_1^p}{2},$$

we obtain the conclusion (4.11) when $1 < p < 2$.

Case 2 : $p \geq 2$. Having in mind (3.2) and (4.12), we get

$$\int_{\Omega_\sigma} |Du - Dh|^p dx \leq \tilde{\gamma}^{-1} \int_{\Omega_\sigma} \langle \mathbf{a}(x, u, Du) - \mathbf{a}(x, u, Dh), Du - Dh \rangle dx \\ \leq \tilde{\gamma}^{-1} \tau_1 \int_{\Omega_\sigma} |Du - Dh|^p dx + C(\tau_1) \int_{\Omega_\sigma} |F|^p dx$$

for any $\tau_1 > 0$. Taking $\tau_1 = \frac{\tilde{\gamma}}{2}$ in the above inequality, it follows from (4.4) that

$$\frac{1}{|B_6|} \int_{\Omega_6} |Du - Dh|^p dx \leq \left(\frac{\sigma}{6}\right)^n \frac{1}{|B_\sigma|} \int_{\Omega_\sigma} |Du - Dh|^p dx \\ \leq \frac{C_1(\gamma, p) \sigma^n}{|B_\sigma|} \int_{\Omega_\sigma} |F|^p dx \leq C_1 \sigma^n \delta^p.$$

We choose now the constant δ small enough to have $C_1 \sigma^n \delta^p \leq \varepsilon_1^p$, and this gives the claim of Lemma 4.3. \square

We need the following higher integrability result for the equation (4.5).

Proposition 4.4. ([12, Theorem 1.1], [8, Theorem 2.2] [6, Lemma 3.2]) *Let $h \in W^{1,p}(\Omega_\sigma)$ be a weak solution of (4.5). Then there is a positive constants $p_0 > p$ depending only on γ , n and p such that for any $p_1 \in (p, p_0)$,*

$$\left(\int_{\Omega_5} |Dh|^{p_1} dx \right)^{\frac{1}{p_1}} \leq C \left(\int_{\Omega_6} |Dh|^p dx \right)^{\frac{1}{p}}$$

holds, where $C > 0$ depends only on γ , n , p and p_0 .

We also need the following oscillation theorem for the equation (4.5).

Proposition 4.5. ([18, Theorem 4.2], [11, Theorem 7.7]) *Let $h \in W^{1,p}(\Omega_\sigma)$ be a weak solution of (4.5). Then there is a positive constant $\beta \in (0, 1)$, depending only on γ , n and p , such that*

$$\text{osc}_{\Omega_5} h \leq C \left(\frac{5}{\sigma} \right)^\beta \|h\|_{L^\infty(\Omega_\sigma)}$$

holds, where $C > 0$ depends only on γ , n and p .

Now, we compare the weak solution $h \in W^{1,p}(\Omega_\sigma)$ of (4.5) with the weak solution $f \in W^{1,p}(\Omega_5)$ of (4.6) to have the following result.

Lemma 4.6. *For any $\varepsilon > 0$, there are two constants $\delta \in (0, \frac{1}{8})$ and $\sigma \geq 6$ depending only on γ , n , p , $\omega_M(\cdot)$, M and ε , such that if $h \in W^{1,p}(\Omega_\sigma)$ is a weak solution of (4.5) and $f \in W^{1,p}(\Omega_5)$ is the weak solution of (4.6) with (4.2)–(4.4) and (4.8) for the boundary case, then*

$$\frac{1}{|B_5|} \int_{\Omega_5} |Dh - Df|^p dx \leq \varepsilon^p.$$

Proof. The proof will be divided into two cases.

Case 1 : $1 < p < 2$. We first prove the following inequality:

$$(4.13) \quad \int_{\Omega_5} |Dh - Df|^p dx \leq C_0 \int_{\Omega_6} |h|^p dx$$

where C_0 depends only on γ , n and p .

Let $\eta \in C_0^\infty(B_6)$ be a cut-off function with the properties $0 \leq \eta \leq 1$, $\eta \equiv 1$ on B_5 and $|D\eta| \leq 2$. Taking $\eta^p h$ as a test function for the equation (4.5), we have

$$\begin{aligned} \int_{\Omega_6} \langle \mathbf{a}(x, u, Dh) - \mathbf{a}(x, u, 0), \eta^p Dh \rangle dx &= \int_{\Omega_6} \langle \mathbf{a}(x, u, Dh), \eta^p Dh \rangle dx \\ &= -p \int_{\Omega_6} \langle \mathbf{a}(x, u, Dh), \eta^{p-1} h D\eta \rangle dx \\ &\leq \gamma^{-1} p \int_{\Omega_6} \eta^{p-1} |h| |Dh|^{p-1} |D\eta| dx \\ &\leq \tau \int_{\Omega_6} \eta^p |Dh|^p dx + C(\tau) \int_{\Omega_6} |h|^p |D\eta|^p dx \end{aligned}$$

as consequence of the Young inequality with $\tau > 0$. By Lemma 4.1, we have

$$\begin{aligned} \int_{\Omega_6} \eta^p |Dh|^p dx &\leq \frac{1}{4} \int_{\Omega_6} \eta^p |Dh|^p dx + C \int_{\Omega_6} \langle \mathbf{a}(x, u, Dh) - \mathbf{a}(x, u, 0), \eta^p Dh \rangle dx \\ &\leq \frac{1}{2} \int_{\Omega_6} \eta^p |Dh|^p dx + C \int_{\Omega_6} |h|^p |D\eta|^p dx, \end{aligned}$$

and so

$$(4.14) \quad \int_{\Omega_5} |Dh|^p dx \leq C \int_{\Omega_6} |h|^p dx.$$

Further on, taking $h - f$ as a test function for (4.5) and (4.6), we obtain

$$(4.15) \quad \int_{\Omega_6} \langle \mathbf{a}(x, u, Dh), Dh - Df \rangle dx = \int_{\Omega_6} \langle \mathbf{a}(x, \bar{u}_{\Omega_5}, Df), Dh - Df \rangle dx.$$

In view of Lemma 4.1 with $\eta \equiv 1$, (3.1) and the Young inequality, we obtain that

$$\begin{aligned} &\int_{\Omega_5} |Dh - Df|^p dx \\ &\leq C \int_{\Omega_5} |Dh|^p dx + C \int_{\Omega_5} \langle \mathbf{a}(x, \bar{u}_{\Omega_5}, Dh) - \mathbf{a}(x, \bar{u}_{\Omega_5}, Df), Dh - Df \rangle dx \\ &= C \int_{\Omega_5} |Dh|^p dx + C \int_{\Omega_5} \langle \mathbf{a}(x, \bar{u}_{\Omega_5}, Dh) - \mathbf{a}(x, u, Dh), Dh - Df \rangle dx \\ &\leq C \int_{\Omega_5} |Dh|^p dx + C \int_{\Omega_5} |Dh|^{p-1} |Dh - Df| dx \\ &\leq \frac{1}{2} \int_{\Omega_5} |Dh - Df|^p dx + C \int_{\Omega_5} |Dh|^p dx. \end{aligned}$$

Thus, the claim (4.13) follows by (4.14).

Recalling $\|u\|_{L^\infty(\Omega_6)} \leq \frac{M}{\lambda r}$, the maximum principle implies $\|h\|_{L^\infty(\Omega_6)} \leq \frac{M}{\lambda r}$. Therefore, (4.13) yields

$$\int_{\Omega_5} |Dh - Df|^p dx \leq C_0 \int_{\Omega_6} |h|^p dx \leq C_0 |B_6| \left(\frac{M}{\lambda r} \right)^p.$$

If $C_0 |B_6| \left(\frac{M}{\lambda r} \right)^p \leq \varepsilon^p$, then we get the conclusion.

So, assume alternatively that $C_0 |B_6| \left(\frac{M}{\lambda r} \right)^p > \varepsilon^p$. In view of (3.3), (4.15) and Lemma 4.1, we have

$$\begin{aligned} &\int_{\Omega_5} |Dh - Df|^p dx \\ &\leq \frac{\tau}{2} \int_{\Omega_5} |Dh|^p dx + C(\tau) \int_{\Omega_5} \langle \mathbf{a}(x, \bar{u}_{\Omega_5}, Dh) - \mathbf{a}(x, \bar{u}_{\Omega_5}, Df), Dh - Df \rangle dx \\ &= \frac{\tau}{2} \int_{\Omega_5} |Dh|^p dx + C(\tau) \int_{\Omega_5} \langle \mathbf{a}(x, \bar{u}_{\Omega_5}, Dh) - \mathbf{a}(x, u, Dh), Dh - Df \rangle dx \\ &\leq \frac{\tau}{2} \int_{\Omega_5} |Dh|^p dx + C(\tau) \int_{\Omega_5} \omega_M(\lambda r |u - \bar{u}_{\Omega_5}|) |Dh|^{p-1} |Dh - Df| dx. \end{aligned}$$

The Young inequality gives

$$\begin{aligned} & \int_{\Omega_5} |Dh - Df|^p dx \\ & \leq \frac{\tau}{2} \int_{\Omega_5} |Dh|^p dx + C(\tau) \int_{\Omega_5} (\omega_M(\lambda r|u - \bar{u}_{\Omega_5}|))^{\frac{p}{p-1}} |Dh|^p dx + \frac{1}{2} \int_{\Omega_5} |Dh - Df|^p dx, \end{aligned}$$

and this implies

$$\int_{\Omega_5} |Dh - Df|^p dx \leq \tau \int_{\Omega_5} |Dh|^p dx + \widehat{C}(\tau) \int_{\Omega_5} (\omega_M(\lambda r|u - \bar{u}_{\Omega_5}|))^{\frac{p}{p-1}} |Dh|^p dx.$$

Here $\widehat{C}(\tau)$ behaves as $\tau^{\frac{p-1}{p-1}} \left(\frac{1+\tau}{\tau}\right)^{\frac{2}{p-1}}$ (see [17, Lemma 3.1]) whence $\frac{\tau}{\widehat{C}(\tau)} \sim \tau^{\frac{1}{p-1}}$ for $\tau \in (0, 1)$.

Since $\omega_M(\cdot)$ is a modulus of continuity, there exists a constant $\delta_0 > 0$ such that $0 < \omega_M(\rho) < \left(\frac{\tau}{\widehat{C}(\tau)}\right)^{\frac{p-1}{p}}$ if $0 < \rho < \delta_0$. On the other hand, for any $\alpha > 0$ satisfying $\frac{1}{2} \leq \delta_0^\alpha$, we have

$$\omega_M(\rho) \leq \omega_M(2M) \leq 2\omega_M(2M)\delta_0^\alpha \leq 2\omega_M(2M)\rho^\alpha$$

for all $\rho \in [\delta_0, 2M]$. Therefore it follows that

$$\omega_M(\rho)^{\frac{p}{p-1}} \leq \frac{\tau}{\widehat{C}(\tau)} + (2\omega_M(2M))^{\frac{p}{p-1}} \rho^{\frac{\alpha p}{p-1}}$$

for all $\rho \in (0, 2M]$. Remembering (4.2), we get

$$\begin{aligned} & \int_{\Omega_5} (\omega_M(\lambda r|u - \bar{u}_{\Omega_5}|))^{\frac{p}{p-1}} |Dh|^p dx \\ & \leq \int_{\Omega_5} \left(\frac{\tau}{\widehat{C}(\tau)} + (2\omega_M(2M))^{\frac{p}{p-1}} (\lambda r|u - \bar{u}_{\Omega_5}|)^{\frac{\alpha p}{p-1}} \right) |Dh|^p dx \end{aligned}$$

which implies

$$(4.16) \quad \int_{\Omega_5} |Dh - Df|^p dx \leq 2\tau \int_{\Omega_5} |Dh|^p dx + \widehat{C}(\tau) \int_{\Omega_5} (\lambda r|u - \bar{u}_{\Omega_5}|)^{\frac{\alpha p}{p-1}} |Dh|^p dx.$$

To estimate the second term in the above inequality, we first take constants α and p_1 such that

$$\frac{1}{2} \leq \delta_0^\alpha, \quad p < p_1 < p_0, \quad \text{and} \quad p_1 = \frac{p(p-1)}{p-\alpha-1},$$

where p_0 is as in Proposition 4.4. We use then the Hölder inequality and Proposition 4.4 to find that

$$\begin{aligned} & \int_{\Omega_5} (\lambda r|u - \bar{u}_{\Omega_5}|)^{\frac{\alpha p}{p-1}} |Dh|^p dx \\ & \leq \left(\int_{\Omega_5} (\lambda r|u - \bar{u}_{\Omega_5}|)^{\frac{\alpha p p_1}{(p-1)(p_1-p)}} dx \right)^{\frac{p_1-p}{p_1}} \left(\int_{\Omega_5} |Dh|^{p_1} dx \right)^{\frac{p}{p_1}} \\ & \leq C \left(\int_{B_5} (\lambda r|u - \bar{u}_{\Omega_5}|)^{\frac{\alpha p p_1}{(p-1)(p_1-p)}} dx \right)^{\frac{p_1-p}{p_1}} \int_{\Omega_6} |Dh|^p dx \\ & = C \left(\int_{\Omega_5} (\lambda r|u - \bar{u}_{\Omega_5}|)^p dx \right)^{\frac{p_1-p}{p_1}} \int_{\Omega_6} |Dh|^p dx, \end{aligned}$$

and thus (4.16) becomes

$$(4.17) \quad \int_{\Omega_5} |Dh - Df|^p dx \leq \left[2\tau + C(\tau) \left(\underbrace{\int_{\Omega_5} (\lambda r |u - \bar{u}_{\Omega_5}|)^p dx}_{I_1} \right)^{\frac{p_1-p}{p_1}} \right] \underbrace{\int_{\Omega_6} |Dh|^p dx}_{I_2}.$$

It follows from the triangle and the Jensen inequalities that

$$\begin{aligned} I_1 &= \int_{\Omega_5} (\lambda r |u - \bar{u}_{\Omega_5}|)^p dx \leq C \int_{\Omega_5} (\lambda r |u - h|)^p dx + C \int_{\Omega_5} (\lambda r |h - \bar{h}_{\Omega_5}|)^p dx \\ &\quad + C \int_{\Omega_5} (\lambda r |\bar{h}_{\Omega_5} - \bar{u}_{\Omega_5}|)^p dx \\ &\leq C \underbrace{\int_{\Omega_5} (\lambda r |u - h|)^p dx}_{I_3} + C \underbrace{\int_{\Omega_5} (\lambda r |h - \bar{h}_{\Omega_5}|)^p dx}_{I_4}. \end{aligned}$$

Remembering that $C_0 |B_6| \left(\frac{M}{\varepsilon}\right)^p > (\lambda r)^p$ and using the Poincaré inequality and Lemma 4.3, we have

$$\begin{aligned} I_3 &= \int_{\Omega_5} (\lambda r |u - h|)^p dx \leq \int_{\Omega_6} (\lambda r |u - h|)^p dx \leq C \int_{\Omega_6} (\lambda r |Du - Dh|)^p dx \\ &\leq C \left(\frac{1}{\varepsilon}\right)^p \int_{\Omega_6} |Du - Dh|^p dx \leq C \left(\frac{\varepsilon_1}{\varepsilon}\right)^p. \end{aligned}$$

On the other hand, Proposition 4.5 yields

$$I_4 = \int_{\Omega_5} (\lambda r |h - \bar{h}_{\Omega_5}|)^p dx \leq C (\lambda r)^p \left(\frac{5}{\sigma}\right)^{p\beta} \|h\|_{L^\infty(B_\sigma)}^p \leq \frac{C}{\sigma^{p\beta}}$$

because of $\|h\|_{L^\infty(B_\sigma)} \leq \frac{M}{\lambda r}$. Consequently,

$$I_1 \leq C \left(\left(\frac{\varepsilon_1}{\varepsilon}\right)^p + \frac{1}{\sigma^{p\beta}} \right).$$

Further on, using Lemma 4.3, the fact that $\varepsilon_1 < 1$ and (4.3), we find

$$I_2 = \int_{\Omega_6} |Dh|^p dx \leq C \left(\int_{\Omega_6} |Du - Dh|^p dx + \int_{\Omega_6} |Du|^p dx \right) \leq C(\varepsilon_1^p + 1) \leq C_1,$$

and thus (4.17) takes on the form

$$\begin{aligned} \int_{\Omega_5} |Dh - Df|^p dx &\leq C_1 \left(\tau + C_2(\tau) \left(\left(\frac{\varepsilon_1}{\varepsilon}\right)^p + \frac{1}{\sigma^{p\beta}} \right)^{\frac{p_1-p}{p_1}} \right) \\ &\leq C_1 \tau + C_1 C_2(\tau) \left(\frac{\varepsilon_1}{\varepsilon}\right)^{\frac{p(p_1-p)}{p_1}} + C_1 C_2(\tau) \left(\frac{1}{\sigma}\right)^{\frac{p\beta(p_1-p)}{p_1}}. \end{aligned}$$

Choosing τ and ε_1 sufficiently small, and σ sufficiently large so that

$$C_1 \tau \leq \frac{\varepsilon^p}{3}, \quad C_1 C_2(\tau) \left(\frac{\varepsilon_1}{\varepsilon}\right)^{\frac{p(p_1-p)}{p_1}} \leq \frac{\varepsilon^p}{3} \quad \text{and} \quad C_1 C_2(\tau) \left(\frac{1}{\sigma}\right)^{\frac{p\beta(p_1-p)}{p_1}} \leq \frac{\varepsilon^p}{3},$$

we get the claim.

Case 2 : $p \geq 2$. By using (3.2) instead of Lemma 4.1 in the above proof, we can obtain the conclusion in a similar manner. \square

Lemma 4.7. *Under the hypotheses of Lemma 4.6, we further assume (4.7), and (4.8) for the boundary case. Then there exists a weak solution $v \in W^{1,p}(B_4)$ of (4.9) for the interior case or a weak solution $v \in W^{1,p}(B_4^+)$ of (4.10) for the boundary case such that*

$$\|D\bar{v}\|_{L^\infty(\Omega_3)} \leq N_0 \quad \text{and} \quad \frac{1}{|B_4|} \int_{\Omega_4} |Df - D\bar{v}|^p dx \leq \varepsilon^p$$

for some constant $N_0 = N_0(\gamma, n, p) > 1$. Here, the function $\bar{v} \in W^{1,p}(\Omega_4)$ is equal to v for the interior case and \bar{v} is the zero extension of v from B_4^+ to B_4 for the boundary case.

Proof. According to Lemma 4.3, Lemma 4.6 and (4.3), it follows from the triangle inequality that

$$\int_{\Omega_5} |Df|^p dx \leq C \int_{\Omega_5} |Du|^p + |Du - Dh|^p + |Dh - Df|^p dx \leq C,$$

where C depends only on n and p . Then we proceed in doing comparison estimates from standard perturbation argument, as in Lemma 3.1 and Lemma 3.7 of [6], in order to obtain the desired conclusion. \square

Proof of Lemma 4.2. The proof follows directly from the triangle inequality and Lemmas 4.3, 4.6 and 4.7. \square

5. GLOBAL GRADIENT ESTIMATES

This section is devoted to the proof of the main result, Theorem 2.2. We start with a modified Vitali covering lemma for the problem (1.1).

Proposition 5.1. (see [7, 17]) *Let \mathcal{C} and \mathcal{D} be measurable sets with $\mathcal{C} \subset \mathcal{D} \subset \Omega$. Assume that Ω is (δ, R) -Reifenberg flat. Suppose that there exist $\varepsilon \in (0, 1)$ and $\sigma \geq 6$ for which*

- (1) $|\mathcal{C}| < \varepsilon |B_{R/\sigma}|$;
- (2) for all $x \in \Omega$ and $r \in (0, \frac{R}{\sigma}]$ with $|\mathcal{C} \cap B_r(x)| \geq \varepsilon |B_r(x)|$, there holds $\Omega \cap B_r(x) \subset \mathcal{D}$.

Then we have

$$|\mathcal{C}| \leq \left(\frac{10}{1-\delta} \right)^n \varepsilon |\mathcal{D}|.$$

We now return to the scaled and normalized problem (3.4).

Lemma 5.2. *Assume that $\tilde{\mathbf{a}}$ satisfies (3.1) and (3.3). Let $\tilde{u} \in W^{1,p}(\tilde{\Omega})$ be a bounded weak solution of (3.4) with $\|\tilde{u}\|_{L^\infty(\tilde{\Omega})} \leq \frac{M}{\lambda r}$. Then there exists a constant $N_1 = N_1(\gamma, n, p) > 1$ so that for any $\varepsilon \in (0, 1)$, there exist constants $\delta = \delta(\gamma, \omega_M, n, p, M, \varepsilon) \in (0, \frac{1}{8})$ and $\sigma = \sigma(\gamma, \omega_M, n, p, M, \varepsilon) \geq 6$ such that if $\tilde{\mathbf{a}}(x, z, \xi)$ is (δ, σ) -vanishing and $\tilde{\Omega}$ is (δ, σ) -Reifenberg flat, and if*

$$(5.1) \quad \left\{ x \in \tilde{\Omega}_1 : \mathcal{M}(|D\tilde{u}|^p) \leq \left(\frac{6}{7} \right)^n \right\} \cap \left\{ x \in \tilde{\Omega}_1 : \mathcal{M}(|\tilde{F}|^p) \leq \left(\frac{6}{7} \right)^n \delta^p \right\} \neq \emptyset,$$

then

$$\left| \left\{ x \in \tilde{\Omega}_1 : \mathcal{M}(|D\tilde{u}|^p)(x) > \left(\frac{6}{7} \right)^n N_1^p \right\} \right| < \varepsilon |B_1|.$$

Here, \mathcal{M} stands for the Hardy–Littlewood maximal operator.

Proof. Remembering (5.1), there is a point $\tilde{x} \in \tilde{\Omega}_1$ such that for all $\rho > 0$,

$$(5.2) \quad \frac{1}{|B_\rho|} \int_{\tilde{\Omega}_\rho(\tilde{x})} |D\tilde{u}|^p dx \leq \left(\frac{6}{7}\right)^n$$

and

$$(5.3) \quad \frac{1}{|B_\rho|} \int_{\tilde{\Omega}_\rho(\tilde{x})} |\tilde{F}|^p dx \leq \left(\frac{6}{7}\right)^n \delta^p.$$

We note that $\tilde{\Omega}_\rho \subset \tilde{\Omega}_{\rho+1}(\tilde{x})$ for any $\rho \geq 6$. Then it follows from (5.3) that

$$\frac{1}{|B_\rho|} \int_{\tilde{\Omega}_\rho} |\tilde{F}|^p dx \leq \left(\frac{7}{6}\right)^n \frac{1}{|B_{\rho+1}|} \int_{\tilde{\Omega}_{\rho+1}(\tilde{x})} |\tilde{F}|^p dx \leq \delta^p$$

for any $\rho \geq 6$. Similarly, (5.2) yields

$$\frac{1}{|B_\rho|} \int_{\tilde{\Omega}_\rho} |D\tilde{u}|^p dx \leq 1$$

for any $\rho \geq 6$. Thus, we are under the hypotheses of Lemma 4.2, which implies that there exist two constants $\sigma = \sigma(\gamma, \omega_M, n, p, M, \varepsilon) \geq 6$ and $\delta = \delta(\gamma, \omega_M, n, p, M, \varepsilon) \in (0, \frac{1}{8})$ such that the conclusion of Lemma 4.2 holds for such \bar{v} and N_0 .

Further on, we will show that there exists a constant $N_1 = N_1(\gamma, n, p) > 1$ such that

$$(5.4) \quad \left\{ y \in \tilde{\Omega}_1 : \mathcal{M}(|D\tilde{u}|^p) > \left(\frac{6}{7}\right)^n N_1^p \right\} \subset \left\{ y \in \tilde{\Omega}_1 : \mathcal{M}_{\tilde{\Omega}_4}(|D\tilde{u} - D\bar{v}|^p) > N_0^p \right\}.$$

To do this, let $\tilde{y} \in \left\{ y \in \tilde{\Omega}_1 : \mathcal{M}_{\tilde{\Omega}_4}(|D\tilde{u} - D\bar{v}|^p) \leq N_0^p \right\}$. Then

$$\frac{1}{|B_\rho|} \int_{B_\rho(\tilde{y})} \chi_{\tilde{\Omega}_4} |D\tilde{u} - D\bar{v}|^p dx \leq N_0^p$$

for any $\rho > 0$. If $\rho > 2$, then $\tilde{\Omega}_\rho(\tilde{y}) \subset \tilde{\Omega}_{2\rho}(\tilde{x})$ and it follows from (5.2) that

$$\frac{1}{|B_\rho|} \int_{\tilde{\Omega}_\rho(\tilde{y})} |D\tilde{u}|^p dx \leq \frac{1}{|B_\rho|} \int_{\tilde{\Omega}_{2\rho}(\tilde{x})} |D\tilde{u}|^p dx \leq 2^n.$$

On the other hand, if $\rho \in (0, 2]$, then $\tilde{\Omega}_\rho(\tilde{y}) \subset \tilde{\Omega}_3$ and so we have

$$\begin{aligned} \frac{1}{|B_\rho|} \int_{\tilde{\Omega}_\rho(\tilde{y})} |D\tilde{u}|^p dx &\leq 2^{p-1} \frac{1}{|B_\rho|} \int_{\tilde{\Omega}_\rho(\tilde{y})} |D\tilde{u} - D\bar{v}|^p + |D\bar{v}|^p dx \\ &\leq 2^{p-1} N_0^p + 2^{p-1} \frac{1}{|B_\rho|} \int_{\tilde{\Omega}_\rho(\tilde{y})} |D\bar{v}|^p dx \leq (2N_0)^p. \end{aligned}$$

Taking $N_1^p = \left(\frac{7}{6}\right)^n \max\{2^n, (2N_0)^p\}$, the claim (5.4) follows.

We now use (5.4), the weak (1, 1)-estimate for the Hardy–Littlewood maximal function and Lemma 4.2, to observe that

$$\begin{aligned} \left| \left\{ y \in \tilde{\Omega}_1 : \mathcal{M}(|D\tilde{u}|^p) > \left(\frac{6}{7}\right)^n N_1^p \right\} \right| &\leq \left| \left\{ y \in \tilde{\Omega}_1 : \mathcal{M}_{\tilde{\Omega}_4}(|D\tilde{u} - D\bar{v}|^p) > N_0^p \right\} \right| \\ &\leq \frac{C(n, p)}{N_0^p} \int_{\tilde{\Omega}_4} |D\tilde{u} - D\bar{v}|^2 dx \leq C\varepsilon^p |B_1|. \end{aligned}$$

Thus, the claim follows in view of the arbitrariness of $\varepsilon > 0$. \square

Turning back to the problem (1.1), scaling and normalization give

Corollary 5.3. *Assume that \mathbf{a} satisfies (2.1) and (2.3). Let $u \in W^{1,p}(\Omega)$ be a bounded weak solution of (1.1) with $\|u\|_{L^\infty(\Omega)} \leq M$. Then for any $\varepsilon \in (0, 1)$, there exist two constant $\delta = \delta(\gamma, \omega_M, n, p, M, \varepsilon) \in (0, \frac{1}{8})$ and $\sigma = \sigma(\gamma, \omega_M, n, p, M, \varepsilon) \geq 6$ such that if $\mathbf{a}(x, z, \xi)$ is (δ, R) -vanishing and Ω is (δ, R) -Reifenberg flat, and if*

$$B_r(y) \cap \left\{ x \in \Omega : \mathcal{M}(|Du|^p) \leq \left(\frac{6}{7}\right)^n \lambda^p \right\} \\ \cap \left\{ x \in \Omega : \mathcal{M}(|F|^p) \leq \left(\frac{6}{7}\right)^n \lambda^p \delta^p \right\} \neq \emptyset,$$

then

$$\left| \left\{ x \in \Omega_r(y) : \mathcal{M}(|Du|^p)(x) > \left(\frac{6}{7}\right)^n \lambda^p N_1^p \right\} \right| < \varepsilon |B_r|$$

whenever $r \in (0, \frac{R}{\sigma}]$ and $y \in \Omega$.

We now take N_1 , ε and the corresponding δ and σ from Corollary 5.3.

Lemma 5.4. *Assume that Ω is (δ, R) -Reifenberg flat, and \mathbf{a} satisfies (2.1), (2.3) and (2.4). Let $F \in L^p(\Omega, \mathbb{R}^n)$ and $u \in W_0^{1,p}(\Omega)$ be a bounded weak solution of (1.1) with $\|u\|_{L^\infty(\Omega)} \leq M$. Then*

$$\left| \left\{ x \in \Omega : \mathcal{M}(|Du|^p)(x) > \left(\frac{6}{7}\right)^n \lambda^p N_1^p \right\} \right| \\ \leq 20^n \varepsilon \left(\left| \left\{ x \in \Omega : \mathcal{M}(|Du|^p)(x) > \left(\frac{6}{7}\right)^n \lambda^p \right\} \right| \right. \\ \left. + \left| \left\{ x \in \Omega : \mathcal{M}(|F|^p)(x) > \left(\frac{6}{7}\right)^n \lambda^p \delta^p \right\} \right| \right)$$

for all $\lambda \geq \lambda_0$, where λ_0 is a constant satisfying

$$(5.5) \quad \frac{C_4}{\lambda_0^p N_1^p} \int_{\Omega} |F|^p dx < \varepsilon |B_{R/\sigma}| \leq \frac{C_4}{\lambda_0^p} \int_{\Omega} |F|^p dx$$

with $C_4 = C_4(n, p, \gamma)$.

Proof. Taking u as a test function for (1.1), we have

$$\int_{\Omega} \langle \mathbf{a}(x, u, Du) - \mathbf{a}(x, u, 0), Du \rangle dx = \int_{\Omega} \langle \mathbf{a}(x, u, Du), Du \rangle dx \\ = \int_{\Omega} \langle |F|^{p-2} F, Du \rangle dx \\ \leq \int_{\Omega} |F|^{p-1} |Du| dx \\ \leq \tau \int_{\Omega} |Du|^p dx + C(\tau) \int_{\Omega} |F|^p dx.$$

Lemma 4.1 and (3.2) give

$$\int_{\Omega} |Du|^p dx \leq C_3 \int_{\Omega} \langle \mathbf{a}(x, u, Du) - \mathbf{a}(x, u, 0), Du \rangle dx \\ \leq C_3 \tau \int_{\Omega} |Du|^p dx + C(\tau) \int_{\Omega} |F|^p dx,$$

and selecting $\tau = \frac{1}{2C_3}$, we obtain

$$\int_{\Omega} |Du|^p dx \leq C \int_{\Omega} |F|^p dx.$$

This estimate and the weak type $(1, 1)$ -estimate for the maximal function yield

$$\begin{aligned} \left| \left\{ x \in \Omega : \mathcal{M}(|Du|^p)(x) > \left(\frac{6}{7}\right)^n \lambda^p N_1^p \right\} \right| &\leq \frac{C}{\lambda^p} \int_{\Omega} |Du|^p dx \\ &\leq \frac{C_4}{\lambda^p} \int_{\Omega} |F|^p dx, \end{aligned}$$

for some positive constant $C_4 = C_4(n, p, \gamma)$. Selecting a number λ_0 for which (5.5) holds, we find that for all $\lambda \geq \lambda_0$,

$$\begin{aligned} \left| \left\{ x \in \Omega : \mathcal{M}(|Du|^p)(x) > \left(\frac{6}{7}\right)^n \lambda^p N_1^p \right\} \right| &\leq \frac{C_4}{\lambda^p} \int_{\Omega} |F|^p dx \\ &\leq \frac{C_4}{\lambda_0^p} \int_{\Omega} |F|^p dx < \varepsilon |B_{R/\sigma}|. \end{aligned}$$

We write now

$$\mathcal{C} = \left\{ x \in \Omega : \mathcal{M}(|Du|^p)(x) > \left(\frac{6}{7}\right)^n \lambda^p N_1^p \right\}$$

and

$$\begin{aligned} \mathcal{D} &= \left\{ x \in \Omega : \mathcal{M}(|Du|^p)(x) > \left(\frac{6}{7}\right)^n \lambda^p \right\} \\ &\quad \cup \left\{ x \in \Omega : \mathcal{M}(|F|^p)(x) > \left(\frac{6}{7}\right)^n \lambda^p \delta^p \right\}, \end{aligned}$$

in order to apply Proposition 5.1. Then the first assumption of Proposition 5.1 follows directly, while the second one comes from Corollary 5.3. Consequently, we obtain the conclusion of Lemma 5.4. \square

With all these tools in hand, we are in a position now to prove Theorem 2.2.

Proof of Theorem 2.2. In view of a truncation argument (see step 4 in the proof of [15, Proposition 5.4]), we may assume that $\mathcal{M}(|Du|^p) \in L^q(\Omega)$.

Straightforward calculations yield

$$\begin{aligned} \int_{\Omega} \mathcal{M}(|Du|^p)^q dx &= C \int_0^{\infty} \lambda^{pq-1} \left| \left\{ x \in \Omega : \mathcal{M}(|Du|^p)(x) > \left(\frac{6}{7}\right)^n \lambda^p N_1^p \right\} \right| d\lambda \\ &= C \int_0^{\lambda_0} \lambda^{pq-1} \left| \left\{ x \in \Omega : \mathcal{M}(|Du|^p)(x) > \left(\frac{6}{7}\right)^n \lambda^p N_1^p \right\} \right| d\lambda \\ &\quad + C \int_{\lambda_0}^{\infty} \lambda^{pq-1} \left| \left\{ x \in \Omega : \mathcal{M}(|Du|^p)(x) > \left(\frac{6}{7}\right)^n \lambda^p N_1^p \right\} \right| d\lambda \\ &\leq C \lambda_0^{pq} \\ &\quad + C \int_{\lambda_0}^{\infty} \lambda^{pq-1} \left| \left\{ x \in \Omega : \mathcal{M}(|Du|^p)(x) > \left(\frac{6}{7}\right)^n \lambda^p N_1^p \right\} \right| d\lambda, \end{aligned}$$

where N_1 is given in Lemma 5.2 and λ_0 is given in Lemma 5.4.

Keeping in mind (5.5), we have

$$\lambda_0^{pq} \leq \left(\frac{C_4}{\varepsilon |B_{R/\sigma}|} \int_{\Omega} |F|^p dx \right)^q \leq C \int_{\Omega} |F|^{pq} dx.$$

Further on, Lemma 5.4 yields

$$\begin{aligned} & \int_{\lambda_0}^{\infty} \lambda^{pq-1} \left| \left\{ x \in \Omega : \mathcal{M}(|Du|^p)(x) > \left(\frac{6}{7}\right)^n \lambda^p N_1^p \right\} \right| d\lambda \\ & \leq 20^n \varepsilon \left(\int_{\lambda_0}^{\infty} \lambda^{pq-1} \left| \left\{ x \in \Omega : \mathcal{M}(|Du|^p)(x) > \left(\frac{6}{7}\right)^n \lambda^p \right\} \right| d\lambda \right. \\ & \quad \left. + \int_{\lambda_0}^{\infty} \lambda^{pq-1} \left| \left\{ x \in \Omega : \mathcal{M}(|F|^p)(x) > \left(\frac{6}{7}\right)^n \lambda^p \delta^p \right\} \right| d\lambda \right) \\ & \leq C_5 \varepsilon \left(\int_{\Omega} \mathcal{M}(|Du|^p)^q dx + \frac{1}{\delta^{pq}} \int_{\Omega} \mathcal{M}(|F|^p)^q dx \right). \end{aligned}$$

We take now $\varepsilon > 0$ small enough to have $C_5 \varepsilon = \frac{1}{2}$, in order to conclude that

$$\int_{\Omega} \mathcal{M}(|Du|^p)^q dx \leq C \int_{\Omega} |F|^{pq} dx + C \int_{\Omega} \mathcal{M}(|F|^p)^q dx.$$

At this point, applying the strong type (q, q) -estimate for the maximal function, we complete the proof of Theorem 2.2. \square

REFERENCES

- [1] E. ACERBI, G. MINGIONE, Gradient estimates for a class of parabolic systems, *Duke Math. J.* **136** (2007), no. 2, 282–320.
- [2] V. BÖGELEIN, F. DUZAAR, G. MINGIONE, Degenerate problems with irregular obstacles, *J. Reine Angew. Math.* **650** (2011), 107–160.
- [3] S.-S. BYUN, Y. KIM, Elliptic equations with measurable nonlinearities in nonsmooth domains, *Adv. Math.* **288** (2016), 152–200.
- [4] S.-S. BYUN, D.K. PALAGACHEV, P. SHIN, Global continuity of solutions to quasilinear equations with Morrey data, *Compt. Rend. Math. Paris* **353** (2015), no. 8, 717–721.
- [5] S.-S. BYUN, D.K. PALAGACHEV, P. SHIN, Global Hölder continuity of solutions to quasilinear equations with Morrey data, *arXiv:1501.06192* (2015).
- [6] S.-S. BYUN, S. RYU, Global weighted estimates for the gradient of solutions to nonlinear elliptic equations, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **30** (2013), no. 2, 291–313.
- [7] S.-S. BYUN, L. WANG, Elliptic equations with BMO coefficients in Reifenberg domains, *Comm. Pure Appl. Math.* **57** (2004), no. 10, 1283–1310.
- [8] S.-S. BYUN, L. WANG, Nonlinear gradient estimates for elliptic equations of general type, *Calc. Var. Partial Differential Equations* **45** (2012), no. 3–4, 403–419.
- [9] L.A. CAFFARELLI, I. PERAL, On $W^{1,p}$ estimates for elliptic equations in divergence form, *Comm. Pure Appl. Math.* **51** (1998), no. 1, 1–21.
- [10] M. COLOMBO, G. MINGIONE, Calderón-Zygmund estimates and non-uniformly elliptic operators, *J. Funct. Anal.* **136** (2016), no. 4, 1416–1478.
- [11] E. GIUSTI, *Direct Methods in the Calculus of Variations*, World Scientific Publishing Co., Inc., River Edge, NJ, 2003.
- [12] T. KILPELÄINEN, P. KOSKELA, Global integrability of the gradients of solutions to partial differential equations, *Nonlinear Anal.* **23** (1994), no. 7, 899–909.
- [13] T. KUUSI, G. MINGIONE, Universal potential estimates, *J. Funct. Anal.* **262** (2012), no. 10, 4205–4269.
- [14] O.A. LADYZHENSKAYA, N.N. URAL'TSEVA, *Linear and Quasilinear Equations of Elliptic Type*, 2nd Edition revised, Nauka, Moscow, 1973 (in Russian).

- [15] C. SCHEVEN, Existence of localizable solutions to nonlinear parabolic problems with irregular obstacles, *Manuscripta Math.* **146** (2015), no. 1-2, 7–63.
- [16] T. MENGESHA, N. PHUC, Global estimates for quasilinear elliptic equations on Reifenberg flat domains, *Arch. Ration. Mech. Anal.* **203** (2012), no. 1, 189–216.
- [17] T. NGUYEN, T. PHAN, Interior gradient estimates for quasilinear elliptic equations, *Calc. Var. Partial Differential Equations* **55** (2016), no. 3, Art. 59, 33 pp.
- [18] N.S. TRUDINGER, On Harnack type inequalities and their application to quasilinear elliptic equations, *Comm. Pure Appl. Math.* **20** (1967), 721–747.

SUN-SIG BYUN: SEOUL NATIONAL UNIVERSITY, DEPARTMENT OF MATHEMATICAL SCIENCES AND RESEARCH INSTITUTE OF MATHEMATICS, SEOUL 08826, KOREA

E-mail address: `byun@snu.ac.kr`

DIAN K. PALAGACHEV: POLITECNICO DI BARI, DIPARTIMENTO DI MECCANICA, MATEMATICA E MANAGEMENT, VIA EDOARDO ORABONA 4, 70125 BARI, ITALY

E-mail address: `dian.palagachev@poliba.it`

PILSOO SHIN: SEOUL NATIONAL UNIVERSITY, DEPARTMENT OF MATHEMATICAL SCIENCES, SEOUL 08826, KOREA

E-mail address: `shinpilsoo@snu.ac.kr`