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Uniform L^∞-Estimates for Quasilinear Elliptic Systems

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Abstract. The aim of this work is to provide uniform L^∞-estimates for the solutions of a quite general class of (p, q)-quasilinear elliptic systems depending on two parameters α and δ.

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1. Introduction

Let us consider the following autonomous quasilinear system

$$\begin{cases} -\mathrm{div}\left((\alpha + |\nabla u|^2)^{\frac{p-2}{2}}\nabla u\right) = H_s(\delta, u, v) \text{ in } \Omega \\ -\mathrm{div}\left((\alpha + |\nabla v|^2)^{\frac{q-2}{2}}\nabla v\right) = H_t(\delta, u, v) \text{ in } \Omega \\ u = v = 0 \qquad\qquad\qquad\qquad\qquad \text{on } \partial\Omega \end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain of $\mathbb{R}^N$, $N \geq 3$, $p, q \in [2, N)$, $\alpha \geq 0$ and $H : I \times \mathbb{R}^2 \to \mathbb{R}$ is a function, where $I \subset \mathbb{R}$ is an interval and $H(\delta, \cdot, \cdot) \in C^1(\mathbb{R}^2, \mathbb{R})$ for any $\delta \in I$.

Moreover, we assume that

$(*)$ there are $p' \in (p, p^*)$, $q' \in (q, q^*)$ and $C_0 > 0$ such that

$$H_s(\delta, s, t) \leq C_0 \left(|s|^{p'-1} + |t|^{q'\frac{p'-1}{p'}} + 1\right)$$

$$H_t(\delta, s, t) \leq C_0 \left(|s|^{p'\frac{q'-1}{q'}} + |t|^{q'-1} + 1\right)$$

for any $(\delta, s, t) \in I \times \mathbb{R}^2$.

Let X be the product space $W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ endowed with the norm

$$\|z\| = \|u\|_{1,p} + \|v\|_{1,q}$$

where $z = (u, v) \in X$. In what follows we shall denote respectively by $\|\cdot\|_s$ and $\|\cdot\|_{1,s}$ the usual norms in $L^s(\Omega)$ and $W_0^{1,s}(\Omega)$.

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Weak solutions of problem (1.1) correspond to critical points of the Euler functional $I_{\alpha,\delta} : X \to \mathbb{R}$ defined as

$$I_{\alpha,\delta}(z) = I_{\alpha,\delta}(u,v) = \frac{1}{p}\int_\Omega \left(\alpha + |\nabla u(x)|^2\right)^{\frac{p}{2}}\, \mathrm{d}x + \frac{1}{q}\int_\Omega \left(\alpha + |\nabla v(x)|^2\right)^{\frac{q}{2}}\, \mathrm{d}x$$
$$- \int_\Omega H(\delta, u(x), v(x))\, \mathrm{d}x \qquad \text{for any } z = (u,v) \in X.$$

By $(*)$, the functional $I_{\alpha,\delta}$ is C^1 on X and, for any $z_0 = (u_0, v_0)$ and $z = (u,v)$ in X, it results

$$\langle I'_{\alpha,\delta}(z_0), z\rangle = \int_\Omega (\alpha + |\nabla u_0|^2)^{\frac{p-2}{2}} \nabla u_0 \nabla u + \int_\Omega (\alpha + |\nabla v_0|^2)^{\frac{q-2}{2}} \nabla v_0 \nabla v$$
$$- \int_\Omega H_s(\delta, u_0, v_0) u + H_t(\delta, u_0, v_0) v.$$

Systems involving this kind of quasilinear operators model some phenomena in non-Newtonian mechanics, nonlinear elasticity and glaciology, combustion theory, population biology; see [7,9,11,12]. Existence, nonexistence and regularity results for such quasilinear elliptic systems are obtained by various authors, see for instance [1,3,6,8,14].

More recently we proved that any weak solution of the following system, not depending on δ,

$$\begin{cases} -\mathrm{div}\left((\alpha + |\nabla u|^2)^{\frac{p-2}{2}} \nabla u\right) = H_s(u,v) \text{ in } \Omega \\ -\mathrm{div}\left((\alpha + |\nabla v|^2)^{\frac{q-2}{2}} \nabla v\right) = H_t(u,v) \text{ in } \Omega \\ u = v = 0 \qquad\qquad \text{on } \partial\Omega \end{cases}$$

is in $(L^\infty(\Omega))^2$ (see [4, Theorem 1.1]).

In this work we want to extend the previous result to the class of systems (1.1) depending also on δ. Moreover here we show carefully that, for any arbitrary $z_0 \in X$ and $r > 0$, the $(L^\infty(\Omega))^2$-norm of the weak solutions to (1.1) belonging to $B_r(z_0)$ depends just on r and z_0, but is independent on α and δ.

The main result of this work is the following:

Theorem 1.1. *If (u,v) is a solution of (1.1) and $(*)$ holds, then $(u,v) \in (L^\infty(\Omega))^2$.*
Moreover, for any fixed $(u_0, v_0) \in X$, $r > 0$, $\alpha \geq 0$ and $\delta \in I$, denoting by

$$D_{r,\alpha,\delta}(u_0,v_0) = \{(u,v) \in X \;\; \|(u,v) - (u_0,v_0)\| \leq r,\; I'_{\alpha,\delta}(u,v) = 0\},$$

there exists $C > 0$, depending on r and (u_0, v_0) but independent of α and δ, such that

$$\|u\|_\infty,\, \|v\|_\infty \leq C \qquad \forall\, (u,v) \in D_{r,\alpha,\delta}(u_0,v_0).$$

This uniform L^∞-estimate will be used in the forthcoming paper [2] in which we derive some crucial existence results about system (1.1), studying the interaction of the spectrum of the quasilinear operators with the nonlinearity H which grows (p, q)-linearly at infinity, in continuity with the Amann–Zehnder type results obtained in [5] for a class of quasilinear elliptic equations.

2. Proof of Theorem 1.1

We first introduce the following result.

Lemma 2.1. *Let $s \in (1, N)$ and denote by s^* the conjugate Sobolev exponent of s, namely $s^* = sN/(N-s)$. If $r, \varepsilon > 0$, $u_0 \in W_0^{1,s}(\Omega)$ and $s' \in [1, s^*)$, there is $\sigma > 0$ such that*

$$\int_{\{|u(x)|\, \geq \sigma\}} |u(x)|^{s'}\, dx < \varepsilon$$

for any $u \in B_r(u_0) = \{u \in W_0^{1,s}(\Omega) \;\; \|u - u_0\|_{1,s} \leq r\}$.

Proof. By contradiction, assume that there are $r, \varepsilon > 0$, $u_0 \in W_0^{1,s}(\Omega)$, $s' < s^*$, $h_n \geq n$ and $u_n \in B_r(u_0)$ such that

$$\int_{\{|u_n(x)|\, \geq h_n\}} |u_n(x)|^{s'}\, \mathrm{d}x \geq \varepsilon \tag{2.1}$$

for any $n \in \mathbb{N}$.

Up to subsequences, u_n strongly converges to some $\bar{u}$ in $L^{s'}(\Omega)$.

Moreover, denoting by $E_n = \{x \in \Omega \;\; |u_n(x)| \, \geq h_n\}$, we claim that

$$|E_n| \to 0. \tag{2.2}$$

Otherwise, if not, we should have, up to subsequences, $|E_n| \geq \alpha > 0$ for any n, hence

$$\int_\Omega |u_n(x)|^{s'}\, \mathrm{d}x \geq \int_{E_n} |u_n(x)|^{s'}\, \mathrm{d}x \geq \alpha\, n^{s'} \quad \Rightarrow \quad \int_\Omega |u_n(x)|^{s'}\, \mathrm{d}x \to \infty$$

while $\int_\Omega |u_n(x)|^{s'}\, \mathrm{d}x \to \int_\Omega |\bar{u}(x)|^{s'}\, \mathrm{d}x$. This proves (2.2), hence the Vitali convergence theorem gives that

$$\int_{E_n} |u_n(x)|^{s'}\, \mathrm{d}x \to 0$$

which contradicts (2.1). □

Now, inspired by [4] and [10], we prove the main result.

Proof of Theorem 1.1. For every γ, t, $k > 1$ we define

$$h_{k,\gamma}(s) = \begin{cases} s|s|^{\gamma-1} & |s| \le k, \\ \gamma k^{\gamma-1}s + \operatorname{sign}(s)(1-\gamma)k^{\gamma} & |s| > k, \end{cases}$$

$$\Phi_{k,t,\gamma}(s) = \int_0^s \left|h'_{k,\gamma}(r)\right|^{\frac{t}{\gamma}} \mathrm{d}r.$$

Observe that $h_{k,\gamma}$ and $\Phi_{k,t,\gamma}$ are C^1-functions with bounded derivative, depending on γ, t and k. Thus if $(u,v) \in X = W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$, then $\Phi_{k,t,\gamma}(u) \in W_0^{1,p}(\Omega)$ and $\Phi_{k,t,\gamma}(v) \in W_0^{1,q}(\Omega)$. Moreover, for every $t \ge \gamma$, there exists a positive constant C, depending on γ and t but independent of k, such that

$$|s|^{\frac{t}{\gamma}-1}|\Phi_{k,t,\gamma}(s)| \le C|h_{k,\gamma}(s)|^{\frac{t}{\gamma}} \tag{2.3}$$

$$|\Phi_{k,t,\gamma}(s)| \le C|h_{k,\gamma}(s)|^{\frac{1}{\gamma}(1+t\frac{\gamma-1}{\gamma})} \tag{2.4}$$

and

$$\left|h_{k,\gamma}\left(|s|^{\frac{q'}{p'}}\right)\right|^{p'} \le C\left|h_{k^{\frac{p'}{q'}},\gamma}(s)\right|^{q'}. \tag{2.5}$$

Let us fix $r > 0$, $\alpha \ge 0$, $\delta \in I$ and consider an arbitrary $\bar{z} = (\bar{u}, \bar{v}) \in D_{r,\alpha,\delta}(u_0, v_0)$.

In particular,

$$\langle I'_{\alpha,\delta}(\bar{z}), (\Phi_{k,\gamma p,\gamma}(\bar{u}), 0)\rangle = 0$$

for any $k, \gamma > 1$.

So, as $W_0^{1,p}(\Omega) \hookrightarrow L^{p'}(\Omega)$, there is $c > 0$ such that

$$\begin{aligned} \left(\int_\Omega |h_{k,\gamma}(\bar{u})|^{p'}\right)^{\frac{p}{p'}} &\le c\int_\Omega |\nabla h_{k,\gamma}(\bar{u})|^p = c\int_\Omega |\nabla\bar{u}|^p|h'_{k,\gamma}(\bar{u})|^p \\ &\le c\int_\Omega (\alpha + |\nabla\bar{u}|^2)^{\frac{p-2}{2}}|\nabla\bar{u}|^2|h'_{k,\gamma}(\bar{u})|^p = c\int_\Omega (\alpha + |\nabla\bar{u}|^2)^{\frac{p-2}{2}}\nabla\bar{u}\cdot\nabla\Phi_{k,\gamma p,\gamma}(\bar{u}) \\ &= c\int_\Omega H_s(\delta, \bar{u}, \bar{v})\Phi_{k,\gamma p,\gamma}(\bar{u}). \end{aligned}$$

By $(*)$, we get

$$\begin{aligned} &\left(\int_\Omega |h_{k,\gamma}(\bar{u})|^{p'}\right)^{\frac{p}{p'}} \\ &\quad\le c\,C_0\left(\int_\Omega (|\bar{u}|^{p'-1}+1)|\Phi_{k,\gamma p,\gamma}(\bar{u})| + \int_\Omega |\bar{v}|^{q'\frac{p'-1}{p'}}|\Phi_{k,\gamma p,\gamma}(\bar{u})|\right). \end{aligned} \tag{2.6}$$

For any $\sigma > 1$ and w in $W_0^{1,p}(\Omega)$ or w in $W_0^{1,q}(\Omega)$, we denote by

$$\Omega_{\sigma,w} = \{x \in \Omega \ \ |w(x)| > \sigma\}.$$

Therefore, using (2.3), (2.4) and redefining from now on, when necessary, the positive constant C, depending on γ but independent of k and σ, we have

$$\begin{aligned}
&\int_\Omega (|\bar u|^{p'-1}+1)|\Phi_{k,\gamma p,\gamma}(\bar u)| \\
&\quad \le (\sigma^{p'-1}+1)\int_\Omega |\Phi_{k,\gamma p,\gamma}(\bar u)| + \int_{\Omega_{\sigma,\bar u}} |\bar u|^{p'-p}|\bar u|^{p-1}|\Phi_{k,\gamma p,\gamma}(\bar u)| \\
&\quad \le 2\sigma^{p'-1}\int_\Omega |\Phi_{k,\gamma p,\gamma}(\bar u)| + C\int_{\Omega_{\sigma,\bar u}} |\bar u|^{p'-p}|h_{k,\gamma}(\bar u)|^p \\
&\quad \le C\sigma^{p'-1}\int_\Omega |h_{k,\gamma}(\bar u)|^{\frac{p\gamma+1-p}{\gamma}} + C\int_{\Omega_{\sigma,\bar u}} |\bar u|^{p'-p}|h_{k,\gamma}(\bar u)|^p.
\end{aligned}$$

Using Hölder inequality we deduce

$$\begin{aligned}
\int_\Omega (|\bar u|^{p'-1}+1)|\Phi_{k,\gamma p,\gamma}(\bar u)| &\le C\sigma^{p'-1}\left(\int_\Omega |h_{k,\gamma}(\bar u)|^{p'}\right)^{\frac{p}{p'}\frac{\gamma p+1-p}{\gamma p}} \\
&\quad + C\|\bar u\|^{p'-p}_{L^{p'}(\Omega_{\sigma,\bar u})}\left(\int_\Omega |h_{k,\gamma}(\bar u)|^{p'}\right)^{\frac{p}{p'}}. \quad (2.7)
\end{aligned}$$

We deal with the second integral in (2.6) and similarly, using (2.3), (2.4), (2.5) and the fact that $\Phi_{k,\gamma p,\gamma}$ is non decreasing, we obtain

$$\begin{aligned}
&\int_\Omega |\bar v|^{q'\frac{p'-1}{p'}}|\Phi_{k,\gamma p,\gamma}(\bar u)| \\
&\le \sigma^{q'\frac{p'-1}{p'}}\int_\Omega |\Phi_{k,\gamma p,\gamma}(\bar u)| + \int_{\Omega_{\sigma,\bar v}\cap\{|\bar v|^{\frac{q'}{p'}}\le|\bar u|\}} |\bar v|^{\frac{q'}{p'}(p'-p)}|\bar u|^{p-1}|\Phi_{k,\gamma p,\gamma}(\bar u)| \\
&\quad + \int_{\Omega_{\sigma,\bar v}\cap\{|\bar v|^{\frac{q'}{p'}}\ge|\bar u|\}} |\bar v|^{\frac{q'}{p'}(p'-p)}\big(|\bar v|^{\frac{q'}{p'}}\big)^{p-1}|\Phi_{k,\gamma p,\gamma}(|\bar v|^{\frac{q'}{p'}})| \\
&\le C\sigma^{q'\frac{p'-1}{p'}}\int_\Omega |h_{k,\gamma}(\bar u)|^{\frac{p\gamma+1-p}{\gamma}} + C\int_{\Omega_{\sigma,\bar v}\cap\{|\bar v|^{\frac{q'}{p'}}\le|\bar u|\}} |\bar v|^{\frac{q'}{p'}(p'-p)}|h_{k,\gamma}(\bar u)|^p \\
&\quad + C\int_{\Omega_{\sigma,\bar v}\cap\{|\bar v|^{\frac{q'}{p'}}\ge|\bar u|\}} |\bar v|^{\frac{q'}{p'}(p'-p)}|h_{k,\gamma}\big(|\bar v|^{\frac{q'}{p'}}\big)|^p \\
&\le C\sigma^{q'\frac{p'-1}{p'}}\left(\int_\Omega |h_{k,\gamma}(\bar u)|^{p'}\right)^{\frac{p}{p'}\frac{\gamma p+1-p}{\gamma p}}
\end{aligned}$$

$$+ C\|\bar v\|_{L^{q'}(\Omega_{\sigma,\bar v})}^{\frac{q'}{p'}(p'-p)} \left(\left(\int_\Omega |h_{k,\gamma}(\bar u)|^{p'} \right)^{\frac{p}{p'}} + \left(\int_\Omega |h_{k^{\frac{p'}{q'}},\gamma}(\bar v)|^{q'} \right)^{\frac{p}{p'}} \right).$$

Combining with (2.6) and (2.7), we get

$$\begin{aligned}
&\left(\int_\Omega |h_{k,\gamma}(\bar u)|^{p'} \right)^{\frac{p}{p'}} \\
&\quad \le C\sigma^{p'-1} \left(\int_\Omega |h_{k,\gamma}(\bar u)|^{p'} \right)^{\frac{p}{p'}\frac{\gamma p+1-p}{\gamma p}} + C\|\bar u\|_{L^{p'}(\Omega_{\sigma,\bar u})}^{p'-p} \left(\int_\Omega |h_{k,\gamma}(\bar u)|^{p'} \right)^{\frac{p}{p'}} \\
&\quad + C\sigma^{\frac{q'}{p'}(p'-1)} \left(\int_\Omega |h_{k,\gamma}(\bar u)|^{p'} \right)^{\frac{p}{p'}\frac{\gamma p+1-p}{\gamma p}} \\
&\quad + C\|\bar v\|_{L^{q'}(\Omega_{\sigma,\bar v})}^{\frac{q'}{p'}(p'-p)} \left(\left(\int_\Omega |h_{k,\gamma}(\bar u)|^{p'} \right)^{\frac{p}{p'}} + \left(\int_\Omega |h_{k^{\frac{p'}{q'}},\gamma}(\bar v)|^{q'} \right)^{\frac{p}{p'}} \right).
\end{aligned}$$

Through Lemma 2.1, there is $\sigma_1 > 1$ such that, for any $\sigma \ge \sigma_1$ and for any $k, \gamma > 1$:

$$\begin{aligned}
\frac{1}{2} \left(\int_\Omega |h_{k,\gamma}(\bar u)|^{p'} \right)^{\frac{p}{p'}} &\le C\Big(\sigma^{p'-1} + \sigma^{\frac{q'}{p'}(p'-1)}\Big) \left(\int_\Omega |h_{k,\gamma}(\bar u)|^{p'} \right)^{\frac{p}{p'}\frac{\gamma p+1-p}{\gamma p}} \\
&\quad + C\,\|\bar v\|_{L^{q'}(\Omega_{\sigma,\bar v})}^{\frac{q'}{p'}(p'-p)} \left(\int_\Omega |h_{k^{\frac{p'}{q'}},\gamma}(\bar v)|^{q'} \right)^{\frac{p}{p'}}. \qquad (2.8)
\end{aligned}$$

If $\eta \in (0,1)$, using Young inequality we obtain that

$$a x^\eta \le \frac{x}{4} + (4a)^{1/(1-\eta)} \qquad \forall\, a, x \ge 0.$$

In particular, as $\frac{\gamma p+1-p}{\gamma p} < 1$,

$$\begin{aligned}
&C\Big(\sigma^{p'-1} + \sigma^{\frac{q'}{p'}(p'-1)}\Big) \left(\int_\Omega |h_{k,\gamma}(\bar u)|^{p'} \right)^{\frac{p}{p'}\frac{\gamma p+1-p}{\gamma p}} \\
&\quad \le \frac{1}{4} \left(\int_\Omega |h_{k,\gamma}(\bar u)|^{p'} \right)^{\frac{p}{p'}} + C\left(\sigma^{p'-1} + \sigma^{\frac{q'}{p'}(p'-1)}\right)^{\frac{\gamma p}{p-1}}
\end{aligned}$$

so that (2.8) becomes

$$\frac{1}{4}\left(\int_\Omega |h_{k,\gamma}(\bar u)|^{p'}\right)^{\frac{p}{p'}}$$
$$\leq C\Big(\sigma^{p'-1}+\sigma^{\frac{q'}{p'}(p'-1)}\Big)^{\frac{\gamma p}{p-1}} + C\,\|\bar v\|^{\frac{q'}{p'}(p'-p)}_{L^{q'}(\Omega_{\sigma,\bar v})}\left(\int_\Omega |h_{k^{\frac{p'}{q'}},\gamma}(\bar v)|^{q'}\right)^{\frac{p}{p'}}.$$

Thus there are $C>0$ and $\sigma_1>1$ such that

$$\int_\Omega |h_{k,\gamma}(\bar u)|^{p'}$$
$$\leq C\Big(\sigma^{p'-1}+\sigma^{\frac{q'}{p'}(p'-1)}\Big)^{\frac{\gamma p'}{p-1}} + C\,\|\bar v\|^{\frac{q'}{p}(p'-p)}_{L^{q'}(\Omega_{\sigma,\bar v})}\int_\Omega |h_{k^{\frac{p'}{q'}},\gamma}(\bar v)|^{q'} \quad (2.9)$$

for any $(\bar u,\bar v)\in D_{r,\alpha,\delta}(u_0,v_0)$, any $k,\gamma>1$ and any $\sigma\geq\sigma_1$.

Reasoning in a similar way and exploiting that $\langle I'_{\alpha,\delta}(\bar z),(0,\Phi_{k,\gamma p,\gamma}(\bar v))\rangle = 0$, we find $C>0$ and $\sigma_2\geq\sigma_1$ such that

$$\int_\Omega |h_{\tilde k,\gamma}(\bar v)|^{q'}$$
$$\leq C\Big(\sigma^{q'-1}+\sigma^{\frac{p'}{q'}(q'-1)}\Big)^{\frac{\gamma q'}{q-1}} + C\,\|\bar u\|^{\frac{p'}{q}(q'-q)}_{L^{p'}(\Omega_{\sigma,\bar u})}\int_\Omega |h_{\tilde k^{\frac{q'}{p'}},\gamma}(\bar u)|^{p'} \quad (2.10)$$

for any $(\bar u,\bar v)\in D_{r,\alpha,\delta}(u_0,v_0)$, any $\tilde k,\gamma>1$ and any $\sigma\geq\sigma_2$.

Setting $\tilde k = k^{\frac{p'}{q'}}$ in (2.10) and substituting in (2.9) we obtain

$$\int_\Omega |h_{k,\gamma}(\bar u)|^{p'} \leq C\Big(\sigma^{p'-1}+\sigma^{\frac{q'}{p'}(p'-1)}\Big)^{\frac{\gamma p'}{p-1}}$$
$$+ C\,\|\bar v\|^{\frac{q'}{p}(p'-p)}_{L^{q'}(\Omega_{\sigma,\bar v})}\left(\Big(\sigma^{q'-1}+\sigma^{\frac{p'}{q'}(q'-1)}\Big)^{\frac{\gamma q'}{q-1}} + \|\bar u\|^{\frac{p'}{q}(q'-q)}_{L^{p'}(\Omega_{\sigma,\bar u})}\int_\Omega |h_{k,\gamma}(\bar u)|^{p'}\right).$$

Using again Lemma 2.1 and choosing a suitable $\sigma\geq\sigma_2$, we find $C>0$ such that, for any $k,\gamma>1$

$$\int_\Omega |h_{k,\gamma}(\bar u)|^{p'}\leq C$$

where C depends on r and γ but is independent of k.

Analogously, we can prove that there is $C>0$, independent of k, such that

$$\int_\Omega |h_{k,\gamma}(\bar v)|^{p'}\leq C$$

for any $k,\gamma>1$.

Thus we can use Fatou Lemma and, passing to the limit for $k \to +\infty$, we get

$$\int_\Omega |\bar{u}|^{\gamma p'}, \int_\Omega |\bar{v}|^{\gamma q'} \le C \qquad \forall (\bar{u}, \bar{v}) \in D_{r,\alpha,\delta}(u_0, v_0) \tag{2.11}$$

where C depends on r and γ.

Since $\gamma > 1$ is an arbitrary number, we have that $\bar{u}, \bar{v} \in L^t(\Omega)$ for any $t > 1$.

In particular, by $(*)$, we derive that there is $m > \max\{N/p, N/q\}$ such that $H_s(\delta, \bar{u}, \bar{v})$, $H_t(\delta, \bar{u}, \bar{v}) \in L^m(\Omega)$, for any $(\bar{u}, \bar{v}) \in D_{r,\alpha,\delta}(u_0, v_0)$, and

$$\|H_s(\delta, \bar{u}, \bar{v})\|_m, \ \|H_t(\delta, \bar{u}, \bar{v})\|_m \le C_1 \tag{2.12}$$

where the constant $C_1 > 0$ depends on r but is independent of α and δ.

We want to prove that for any $(\bar{u}, \bar{v}) \in D_{r,\alpha,\delta}(u_0, v_0)$, $\bar{u}$ and $\bar{v}$ are in $L^\infty(\Omega)$ and

$$\|\bar{u}\|_\infty, \ \|\bar{v}\|_\infty \le C_2$$

where the constant $C_2 > 0$ still depends on r but is independent of α and δ.

Denoting by $p^{*\prime} = \frac{p^*}{p^*-1}$, from $m > N/p$ we see that

$$m > p^{*\prime} \quad \text{and} \quad \frac{p^*}{p-1}\left(\frac{1}{p^{*\prime}} - \frac{1}{m}\right) > 1.$$

Once fixed $(\bar{u}, \bar{v}) \in D_{r,\alpha,\delta}(u_0, v_0)$, for any $k \in \mathbb{N}$, let us denote by $A_k = \{x \in \Omega \ \ |\bar{u}(x)| \ge k\}$ and by

$$G_k(r) = \begin{cases} 0 & \text{if } |r| \le k, \\ r - k & \text{if } r \ge k, \\ r + k & \text{if } r \le -k. \end{cases}$$

As $G_k(\bar{u}) \in W_0^{1,p}(\Omega)$ and $\langle I'_{\alpha,\delta}(\bar{u}, \bar{v}), (G_k(\bar{u}), 0)\rangle = 0$, denoting by $\bar{f} = H_s(\delta, \bar{u}, \bar{v})$ and redefining, when necessary, a positive constant C independent on k, we have

$$\begin{aligned}
\left(\int_\Omega |G_k(\bar{u})|^{p^*}\right)^{p/p^*} &\le C \int_\Omega |\nabla G_k(\bar{u})|^p = C \int_\Omega |\nabla \bar{u}|^p G'_k(\bar{u}) \\
&\le C \int_\Omega (\alpha + |\nabla \bar{u}|^2)^{\frac{p-2}{2}} \left(\nabla \bar{u} \cdot \nabla G_k(\bar{u})\right) \\
&= C \int_\Omega \bar{f} G_k(\bar{u}) = C \int_{A_k} \bar{f} G_k(\bar{u}) \le C \left(\int_{A_k} |\bar{f}|^{p^{*\prime}}\right)^{1/p^{*\prime}} \left(\int_\Omega |G_k(\bar{u})|^{p^*}\right)^{1/p^*}
\end{aligned}$$

hence

$$\left(\int_\Omega |G_k(\bar u)|^{p^*}\right)^{(p-1)/p^*} \leq C\left(\int_{A_k}|\bar f|^{p^{*\prime}}\right)^{1/p^{*\prime}} \leq C\left(\int_{A_k}|\bar f|^{m}\right)^{1/m}|A_k|^{\frac{1}{p^{*\prime}}-\frac{1}{m}}. \tag{2.13}$$

Moreover, for any $h > k$

$$\left(\int_\Omega |G_k(\bar u)|^{p^*}\right)^{(p-1)/p^*} \geq \left(\int_{A_h}|G_k(\bar u)|^{p^*}\right)^{(p-1)/p^*} \geq \left(\int_{A_h}(h-k)^{p^*}\right)^{(p-1)/p^*} = (h-k)^{p-1}|A_h|^{\frac{p-1}{p^*}},$$

so that, combining with (2.12) and (2.13),

$$|A_h| \leq \frac{C}{(h-k)^{p^*}}\ |A_k|^{\frac{p^*}{p-1}(\frac{1}{p^{*\prime}}-\frac{1}{m})}$$

where $\frac{p^*}{p-1}(\frac{1}{p^{*\prime}}-\frac{1}{m}) > 1$.

Thereby, applying Lemma 4.1 in [13], there is C_2, depending just on r, such that

$$|A_h| = 0 \qquad \forall h \geq C_2$$

which means that $\bar u \in L^\infty(\Omega)$ and $\|\bar u\|_\infty \leq C_2$. As $m > N/q$, reasoning in the same way, we find that also $\bar v \in L^\infty(\Omega)$ and $\|\bar v\| \leq C_2$, choosing a suitable $C_2 > 0$. □

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Declarations

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