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Research Article

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Some Remarks on the Fractal Structure of Irrigation Balls

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Abstract: The paper is related to a conjecture by Pegon, Santambrogio and Xia concerning the dimension of the boundary of some sets which we are calling "irrigation balls". We propose a notion of sub-balls and sub-spheres of prescribed radius and we prove that, generically, the only possible Minkowski dimension of sub-spheres is the one expected in the conjecture. At the same time, beside the scale transition properties and the dimension estimates on some significant sets, we propose a third approach to study the fractal regularity which relies on lower oscillation estimates on the landscape function, which turns out to behave as a Weierstrass-type function.

Keywords: Optimal Transport Problems, Branched Transport, Irrigation Models, Minkowski Dimension, Landscape Function, Weierstrass Function, Fractal Regularity

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1 Introduction

As nowadays is well known, the ramified structures which characterize river basins, blood vessels, leaves, trees and so on may be derived directly from a variational principle without relying on assumptions based on empirical observations (see [14, 22]). Indeed, if a cost functional is introduced which encourages joint transportation, the ramified structure comes out as the result of a compromise between the convenience of keeping together the fibers and the necessity of reaching a measure spread out on a large set. These recent developments, known as branched transport or irrigation models, can be included in the literature on the Monge transport problem, even if they are radically distinct from the approach proposed by Monge in [16]. Indeed, in the Monge–Kantorovitch model (see [10, 16]), the cost of the motion of a single particle is not influenced by interactions with the remaining part. In the context of irrigation models, one is not so much interested in knowing the final destination of a single particle (the so called "who goes where" problem) as in obtaining some information about the shape of the set of the trajectories, knowing, in particular, if particles move together giving rise to a river. The distinction is clearly reflected in the fact that V-shaped trajectories, optimal for the Monge–Kantorovitch functional, become Y-shaped ones, i.e. branching paths, for the irrigation functional. The branched transport has been introduced in [14, 22], where the existence of minima in an appropriate context is also proved. Further results are contained in [1, 3, 4, 12, 13, 20].

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From the point of view of the regularity problem, one has two completely different issues: the regularity of the curves which describe the branches out of the branching points, and the properties of the branching structure (fractal regularity). An answer to the first question is given in [17], where the authors prove that the derivative of the particles trajectories for an optimal irrigation pattern have a locally bounded variation (if the irrigation measure is the Lebesgue measure restricted to a sufficiently regular set). The first result about fractal regularity is the multiscale result given in [6, Theorem 6.17], in which the authors prove that, under suitable assumptions and modulo constants, the number of branches of length ε starting from a branch having length ℓ is $\frac{\ell}{\varepsilon}$. An alternative way to approach the fractal regularity is the proof that some significant sets connected to branched networks in the Euclidean space $\mathbb{R}^d$ have a fractal dimension.

This approach has led Pegon, Santambrogio and Xia to introduce in [18] the notion of unit (volume) irrigation ball and to investigate the dimension of its boundary (irrigation sphere in the present paper). They conjecture, see [18, Conjecture 4.3], that such a dimension should be $d - \beta$, where $\beta = d(\alpha - 1) + 1$ is a relevant constant in the irrigation problem, and they partially prove the assertion showing in [18, Theorem 4.2] that, if $1 - \frac{1}{d} < \alpha \leq 1$, the upper Minkowski dimension of the boundary of the unit volume ball is less than or equal to $d - \beta$. Unfortunately, this inequality still leaves open the possibility that the sphere is a smooth surface of dimension $d - 1$ and the conjecture is supported in [18] essentially by numerical computations.

The main aim of this paper is to supply more theoretical evidences by looking to some other significant sets which have a fractal dimension. We shall also propose a third approach to the fractal regularity by establishing *oscillation estimates* which are closely related to the scale transition approach in [6] (indeed the main result of this type is a consequence of [6, Theorem 6.17]) and to the approach in [18] because they are a fundamental tool for the dimension estimates. We shall recall the notion of what we call *irrigation ball*, we shall discuss some of its geometrical aspects and we shall introduce a natural notion of sub-balls (with the corresponding spheres). We shall prove some estimates on the lower and upper Minkowski dimensions of the spheres, which generically (both from a measure theoretic and from a topological point of view) hold true; see Theorem 2.5 and Corollary 2.7, respectively. Such results guarantee that the irrigation sub-spheres do not behave as regular sets, in the sense that, generically, the unique possible Minkowski dimension they may have is $d - \beta$. In addiction, we prove that the graph of the landscape function associated to an irrigation sphere (another significant set) has a Minkowski dimension equal to $d + 1 - \beta$.

The estimates on the landscape function oscillations have been a key tool to get the results and are, as already said, a further description of fractal regularity. We actually prove that the landscape function behaves as a Weierstrass-type function (see [9, 21]), well studied in the literature (see [2] and the references therein). In this regard it is worth remarking that the landscape function related to an optimal irrigation pattern, under the assumptions which will be introduced in Section 3, represents a significant example of a Weierstrass-type function which naturally arises in the study of branching structures and does not come from any artificial construction.

The paper is organized as follows: the main results will be formulated at the end of Section 2, after we recall the notion of irrigation ball introduced in [18], underline its geometrical aspects and propose a natural notion of concentric sub-ball of a given radius and related sphere. This preliminary part will give us the necessary notions needed for the statements. Section 3 is devoted to study the oscillations of the landscape function. In Section 4 we prove how the results obtained in the previous section allow to estimate the dimension of the level sets of a Hölder continuous function. Finally, in Section 5 we provide the proof of the main results.

2 Irrigation Balls and Concentric Sub-balls

The reader is supposed to be familiar with the literature on irrigation models or branched transport (see [1, 3, 4, 12–14, 20, 22]). Since the notation in the literature is not uniform, we shall list some of the symbols and basic notions we shall mainly use in the following. The references to the papers in which the reader can find a more detailed explanation are not necessarily the place in which each notion has been introduced for the first time:

- χ denotes any irrigation pattern; see [6, Definition 1.1].
- *Branch* of a pattern; see [7, Definition 6.2].
- D_χ is the flow zone and $\overline{D_\chi}$ is the domain of the pattern; see [6, Definition 1.8].
- $\preceq$ is the flow ordering; see [6, Definition 2.6].
- $\ell(x, y)$ is the branch distance; see [6, Definition 2.9].
- $\ell(x)$ is the residual distance; see [6, Definition 2.3].
- $I_\alpha(\chi)$ is the cost of the pattern; see [6, Definition 1.5] for $i = 0$.
- μ_χ is the irrigation measure; see [7, Definition 2.15].
- *Optimal pattern* is any irrigation pattern which minimizes I_α among the competitors with the same irrigation measure; see [7, Definition 4.1].
- *Simple pattern* is any irrigation pattern with no cycle; see [7, Definition 6.1].
- $\mu(x)$ is the mass flowing trough a point; see [6, Formula (1.1) (with $i = 0$) and Remark 1.12].
- Z is the (Santambrogio) landscape function introduced in [19]; see also [6, Definition 1.9].

For any Euclidean set $E \subset \mathbb{R}^d$, we shall denote by $|E|$ its (outer) Lebesgue measure (in the following, the reader can safely assume to work always with measurable sets and functions if he finds it more relaxing). Moreover, we shall denote by $\underline{\dim}_M(E)$ and $\overline{\dim}_M(E)$ the lower and upper, respectively, Minkowski dimension of the set E (see [15, Section 5.3]) and, when the two dimensions agree, we shall denote by $\dim_M(E)$ the Minkowski dimension of E. (The Minkowski dimension also has alternative names. It is sometimes called box-counting, metric, fractal or capacity dimension. Finally, the notion of Minkowski dimension has been extended to measures in [8, Definition 1.7]).

Given $1 - \frac{1}{d} < \alpha \leq 1$, we define as *irrigation ball of unit volume and prescribed center* $x_0 \in \mathbb{R}^d$, a solution of the minimization problem consisting in finding an irrigation pattern χ of minimum cost $I_\alpha(\chi)$ among those which have source $S = x_0$ and irrigation measure μ_χ equal to the restriction of the Lebesgue measure to a measurable set $\mathcal{B}$ such that $|\mathcal{B}| = 1$. The existence of such a χ is proved in [18, Theorem 2.1], the uniqueness is out of question if $\alpha < 1$ because χ cannot be invariant under rotations around x_0. Sometimes we shall use the term irrigation ball in order to denote the set $\mathcal{B}$ (which is also not unique) when the context makes the use not ambiguous.

When an irrigation unit volume ball χ is given, it defines a landscape function Z (see [5, 18, 19]). In [18, Theorem 2.3] it is proved that Z takes a constant value Z^* on $\partial\mathcal{B}$, where

$$Z^* := \frac{e_\alpha}{\alpha}\Big(\alpha + \frac{1}{d}\Big), \quad \text{with } e_\alpha := I_\alpha(\chi).$$

We shall call Z^* the *radius of the unit volume irrigation ball* and we shall consider Z as the *radial distance function* from the center x_0. The scaling law in [18, Lemma 2.2] shows that for any given constant $V > 0$ the problem of minimizing I_α with the constraint $|\mathcal{B}| = V$, i.e. of finding an irrigation ball of volume V, can be solved by scaling χ by a factor $\lambda = V^{1/d}$, so obtaining a new pattern χ_V such that

$$I_\alpha(\chi_V) = \lambda^{\alpha d+1} e_\alpha = V^{\alpha + \frac{1}{d}} e_\alpha \tag{2.1}$$

and a new value of the corresponding landscape function on $\partial\mathcal{B}$ given by

$$R(V) = V^{\alpha + \frac{1}{d} - 1} Z^* = V^{\frac{\beta}{d}} Z^*. \tag{2.2}$$

The last equality gives the radius of the irrigation ball in terms of the volume. Inverting such a function, we find

$$V(R) = \Big(\frac{R}{Z^*}\Big)^{\frac{d}{\beta}}, \tag{2.3}$$

which gives the volume of an irrigation ball as a function of the radius R. In particular, we can compute the measure b_α of the unit ball (ball of radius 1) as

$$b_\alpha = \Big(\frac{1}{Z^*}\Big)^{\frac{d}{\beta}} = \Big(\frac{\alpha}{e_\alpha(\alpha + \frac{1}{d})}\Big)^{\frac{d}{\beta}} = \Big(\frac{\alpha d}{e_\alpha(\alpha d + 1)}\Big)^{\frac{d}{\beta}}, \tag{2.4}$$

which we prefer to use as a fundamental constant instead of e_α for its more geometric meaning. So (2.2) and (2.3) respectively become

$$R(V) = \left(\frac{V}{b_\alpha}\right)^{\frac{\beta}{d}}, \tag{2.5}$$
$$V(R) = b_\alpha R^{\frac{d}{\beta}}.$$

We finally observe that the least irrigation cost for an irrigation ball of radius R is given by $e_\alpha(R) = I_\alpha(\chi_{V(R)})$ and can be computed by (2.1), (2.4) and (2.5) as

$$\begin{aligned} e_\alpha(R) &= (V(R))^{\alpha+\frac{1}{d}} e_\alpha \\ &= (b_\alpha R^{\frac{d}{\beta}})^{\alpha+\frac{1}{d}} e_\alpha \\ &= \frac{\alpha}{b_\alpha^{\frac{\beta}{d}}(\alpha+\frac{1}{d})}(b_\alpha R^{\frac{d}{\beta}})^{\alpha+\frac{1}{d}} \\ &= \frac{\alpha}{\alpha+\frac{1}{d}} b_\alpha^{\alpha+\frac{1-\beta}{d}} R^{\frac{\alpha d+1}{\beta}} \\ &= \frac{\alpha d}{\alpha d+1} b_\alpha R^{\frac{d}{\beta}+1} \\ &= \frac{\alpha d}{\alpha d+1} R V(R), \end{aligned}$$

which shows that the mean value of the landscape function on the ball $\mathcal{B}$ is obtained by multiplying the radius R by $\frac{\alpha d}{\alpha d+1}$. One could be tempted to define the irrigation ball by fixing a radius, intended as an upper bound on the landscape function, and maximizing the volume, but this choice would not lead to an optimal pattern. Indeed, adding a useless length to the fibers which end at a small landscape level would help to keep the maximum level low. So the definition in [18] looks to be the most reasonable one.

The above geometrical setting widely motivates the following definition.

Definition 2.1. Let $x_0 \in \mathbb{R}^d$ and $R > 0$ and let $\mathcal{B}$ be an irrigation ball centered in x_0 and having radius R. For any $0 \le \rho \le R$ the sets

$$\begin{aligned} \mathcal{B}_\rho &:= \{x \in \mathcal{B} \mid Z(x) < \rho\}, \\ \overline{\mathcal{B}}_\rho &:= \{x \in \mathcal{B} \mid Z(x) \le \rho\}, \\ \mathcal{S}_\rho &:= \{x \in \mathcal{B} \mid Z(x) = \rho\} \end{aligned}$$

will be respectively called (concentric) open, closed sub-ball and sub-sphere of $\mathcal{B}$ of radius ρ.

It is worth explicitly remarking that, for $\alpha = 1$, the branched cost functional reduces to the usual Monge–Kantorovitch one, and so the notion of balls, concentric sub-balls and sub-spheres reduces to the usual one while the landscape function is nothing else than the Euclidean distance from the center (the source). Of course, in such a case, [18, Conjecture 4.3] is trivially true. On the contrary, when $\alpha < 1$, sub-balls are not irrigation balls of lower radius. We shall see soon that, for $\rho = R$, we have $\mathcal{B}_\rho = \mathcal{B}$ modulo a negligible set. We shall introduce functions defined on the interval $[0, R]$ by setting for all $\rho \in [0, R]$,

$$m(\rho) := |\mathcal{B}_\rho| \quad \text{and (temporarily)} \quad \overline{m}(\rho) := |\overline{\mathcal{B}}_\rho|. \tag{2.6}$$

The functions m and $\overline{m}$ are increasing, m is lower semicontinuous (left continuous) and $\overline{m}$ is upper semicontinuous (right continuous). In Section 5 we shall prove the following statements.

Proposition 2.2. *For any $\rho \in [0, R]$ we have $|\mathcal{S}_\rho| = 0$.*

Corollary 2.3. *For any $\rho \in [0, R]$ we have $m(\rho) = \overline{m}(\rho)$ (so we will not use $\overline{m}$ anymore) and m is a continuous function.*

Proposition 2.4. *For any $\rho \in [0, R]$ we have $\mathcal{S}_\rho = \partial\mathcal{B}_\rho$.*

We are now in a position to state the main results in the paper in which we shall implicitly assume that an irrigation ball $\mathcal{B}$ of radius R is given and we shall use the notation introduced above.

Theorem 2.5. *For a.e. $\rho \in]0, R]$,*

$$\overline{\dim}_M(\mathcal{S}_\rho) \le d - \beta.$$

Theorem 2.6. *There exists a dense G_δ-set $G \subset [0, R]$ such that for all $\rho \in G$,*

$$\underline{\dim}_M(\mathcal{S}_\rho) \le d - \beta \le \overline{\dim}_M(\mathcal{S}_\rho).$$

Corollary 2.7. *If $\rho \in G$ as in Theorem 2.6, one of the two following alternatives holds:*

- $\underline{\dim}_M(\mathcal{S}_\rho) < \overline{\dim}_M(\mathcal{S}_\rho)$ *(i.e. the irrigation sphere $\mathcal{S}_\rho$ has not a Minkowski dimension).*
- $\dim_M(\mathcal{S}_\rho) = d - \beta$.

Theorem 2.8. *The graph of the landscape function has a Minkowski dimension equal to $d - \beta + 1$.*

3 Lower Oscillation Estimates on the Landscape Function

Definition 3.1. Let $A \subset \mathbb{R}^d$ and let $f : A \to \mathbb{R}$ be a function. We call the quantity

$$\omega_B(f) := \sup_{\substack{J \subset f(B) \\ J \text{ interval}}} |J|$$

the *connected oscillation* of f on $B \subset A$.

Note that $\omega_B(f) \le \operatorname{osc}_B f := \sup_B f - \inf_B f$ for any $B \subset A$ and the equality holds if f is continuous and B is a connected set. Note that in this section and in the next one β will denote a generic exponent in $[0, 1]$ (and we do not specify this every time), while in Proposition 3.9 (and related formulas) and in the last section, in which we shall go back to the irrigation problem, β will be taken as $1 + d(\alpha - 1)$ as specified in Section 1.

Definition 3.2. Let $A \subset \mathbb{R}^d$ be given. We say that a function $f : A \to \mathbb{R}$ is Hölder continuous with exponent β if there exists $C_H > 0$ such that

$$|f(x) - f(y)| \le C_H |x - y|^\beta \tag{HC}$$

for every $x, y \in A$. We say that f satisfies the *lower Hölder condition* if there exists a constant $C_L > 0$ such that for every $r > 0$ the connected oscillation of f on (the trace on A of) every ball B_r of radius r centered in a point of A satisfies

$$\omega_{B_r}(f) \ge C_L r^\beta. \tag{LHC}$$

Note that if A is a convex set and f is a continuous function, we get the usual definition of lower Hölder condition given in the literature on Weierstrass-type functions (defined on intervals). Note also that condition (LHC) with exponent $\beta < 1$ implies nondifferentiability (i.e. nonexistence of a finite derivative) of the function at every point. We give a local version of the previous definitions.

Definition 3.3. Let $A \subset \mathbb{R}^d$ be given. We say that a function $f : A \to \mathbb{R}$ is *Hölder continuous at a low scale* with exponent $\beta > 0$ if there exists a constant $\overline{R} > 0$ such that (HC) holds true for every $x, y \in A$ such that $|x - y| \le \overline{R}$. Analogously, we say that f satisfies the *lower Hölder condition at a low scale* with exponent $\beta > 0$ if there exists a constant $\overline{R} > 0$ such that (LHC) holds true for (the trace on A of) every ball B with radius $r \le \overline{R}$ centered in a point of A.

Sometimes we shall emphasize the constant $\overline{R}$, which can of course be changed by varying the constants C_H and C_L, by writing that (HC) or (LHC) is satisfied, with a given constant, at the scale $\overline{R}$.

Hölder continuous functions can easily be extended to larger sets. The result is probably better known for Lipschitz continuous functions, one just has to observe that Hölder continuity is the Lipschitz continuity with respect to the metric $d(x, y) = |x - y|^\beta$. A slightly less simple variant, which we shall use to avoid a useless loss of generality, allows to extend a Hölder continuous function at a low scale defined on a set A to a neighborhood $N_s(A)$ for a suitable $s > 0$ (depending on the scale $\overline{R}$), where, in correspondence to any $s > 0$ and $X \subset \mathbb{R}^d$, we denote by

$$N_s(X) := \{x \in \mathbb{R}^d \mid d(x, X) < s\}$$

the s-neighborhood of the set X.

Let $A \subset \mathbb{R}^d$ and let $f : A \to \mathbb{R}$ be a function which satisfies (HC) at a low scale $\overline{R}$. For all $x \in \mathbb{R}^d$ we set

$$\varphi(x) := \inf_{\substack{z\in A\\ |x-z|<\frac{2}{3}\overline{R}}} \Big(f(z) + \frac{3^\beta}{2^\beta - 1} C_H |x - z|^\beta\Big). \tag{3.1}$$

Lemma 3.4. *If $z_1, z_2 \in A$ and $x \in \mathbb{R}^d$ are such that*

$$|z_1 - x| \le \frac{1}{2}|z_2 - x| < \frac{1}{3}\overline{R}, \tag{3.2}$$

then

$$f(z_1) + \frac{3^\beta C_H}{2^\beta - 1}|z_1 - x| \le f(z_2) + \frac{3^\beta C_H}{2^\beta - 1}|z_2 - x|.$$

Proof. We can assume without any restriction that $x = 0$, and so from (3.2) we deduce $|z_1 - z_2| \le \frac{3}{2}|z_2| \le \overline{R}$. Moreover

$$|z_2|^\beta - |z_1|^\beta \ge (1 - 2^{-\beta})|z_2|^\beta \ge (1 - 2^{-\beta})\Big(\frac{2}{3}|z_1 - z_2|\Big)^\beta = \frac{2^\beta - 1}{3^\beta}|z_1 - z_2|^\beta.$$

So, by (HC), we get $f(z_1) - f(z_2) \le C_H|z_1 - z_2|^\beta \le \frac{3^\beta C_H}{2^\beta - 1}(|z_2|^\beta - |z_1|^\beta)$. □

Corollary 3.5. *For any $0 < s < \frac{\overline{R}}{3}$, if $x \in N_s(A)$, then*

$$\varphi(x) = \inf_{\substack{z\in A\\ |x-z|<2s}} \Big(f(z) + \frac{3^\beta C_H}{2^\beta - 1}|x - z|^\beta\Big). \tag{3.3}$$

Corollary 3.6. *The function φ is an extension of f, i.e. $\varphi_{|A} = f$.*

Proposition 3.7. *The restriction of the function φ to the set $N_{\overline{R}/6}(A)$ is Hölder continuous at scale $\frac{\overline{R}}{3}$ with a constant $\frac{3^\beta C_H}{2^\beta - 1}$.*

Proof. Set $s = \frac{\overline{R}}{6}$ and fix $x_1, x_2 \in N_s(A)$ such that $|x_1 - x_2| \le \frac{\overline{R}}{3}$. Then, for all $z \in A$ such that $|z - x_1| < 2s$, we have $|z - x_2| \le 2s + \frac{\overline{R}}{3} < \frac{2}{3}\overline{R}$, and so, by (3.1),

$$\varphi(x_2) - \Big(f(z) + \frac{3^\beta C_H}{2^\beta - 1}|x_1 - z|^\beta\Big) \le f(z) + \frac{3^\beta C_H}{2^\beta - 1}|x_2 - z|^\beta - \Big(f(z) + \frac{3^\beta C_H}{2^\beta - 1}|x_1 - z|^\beta\Big) \le \frac{3^\beta C_H}{2^\beta - 1}|x_1 - x_2|^\beta.$$

Then, by taking the supremum with respect to z and by applying (3.3), we deduce

$$\varphi(x_2) - \varphi(x_1) \le \frac{3^\beta C_H}{2^\beta - 1}|x_1 - x_2|^\beta.$$ □

We also recall some definitions concerning the regularity of a measure μ.

Definition 3.8. A measure μ is Ahlfors regular from below in dimension d if there exists a constant $c_A > 0$ such that

$$\mu(B(x, r)) \ge c_A r^d \quad \text{for all } r \in [0, 1],\ x \in \operatorname{supp}(\mu), \tag{LAR}$$

while μ is Ahlfors regular from above in dimension d if there exists a constant $C_A > 0$ such that

$$\mu(B(x, r)) \le C_A r^d \quad \text{for all } r > 0. \tag{UAR}$$

We shall say that μ is Ahlfors regular (in dimension d) if it is Ahlfors regular both from above and below. Finally, we shall say that μ is inner lower (resp. upper) Ahlfors regular on a set $A \subset \mathbb{R}^d$ if (LAR) (resp. (UAR)) holds on balls contained in A.

In what follows we shall take a set $\emptyset \neq B \subset \overline{D_\chi}$ which satisfies the following conditions:

$$\text{for all } x, y \in D_\chi \text{ such that } y \preceq x: \text{ if } x \in B, \text{ then } y \in B, \tag{A1}$$

which states that the set B is backward-stable with respect to the flow induced by the pattern χ,

$$\mu_\chi \text{ is (UAR) and inner (LAR) on } B, \tag{A2}$$

$$\text{for all } x, y \in D_\chi \cap B \text{ such that } y \preceq x: \text{ if } \operatorname{dist}(x, \partial B) \leq \ell(y, x), \text{ then } Z(y) + C_B(\ell(y, x))^\beta \leq Z(x), \tag{A3}$$

$$\inf_{\partial B} Z \geq C_Z(\mu_\chi(\mathbb{R}^d))^{\frac{\beta}{d}}, \tag{A4}$$

where C_B and C_Z are given positive constants and $\mu_\chi(\mathbb{R}^d) = \mu(x_0)$ represents the total mass of the irrigation measure, i.e. the mass in the source. The main aim of this section is the proof of the following statement.

Proposition 3.9. *Let Z satisfy* (HC) *with $\beta = 1 + d(\alpha - 1)$ on a set $\emptyset \neq B \subset \overline{D_\chi}$ which satisfies conditions* (A1), (A2), (A3) *and* (A4). *Then Z satisfies* (LHC) *with the same value of β at a scale $\overline{R}$ and with a constant C_L which depends on α, the dimension d, the global mass $\mu_\chi(\mathbb{R}^d)$ and the constants C_H, c_A, C_A, C_B and C_Z respectively appearing in* (HC), (LAR), (UAR), (A3) *and* (A4).

In the following, we shall assume that such values are given, so that the expression *universal constant* will be intended as a constant which only depends on the above quantities.

We recall that the variation of the landscape function between two points x and $y \in D_\chi$ which are comparable by the flow order $\preceq$ is obtained by the following relation:

$$Z(x) - Z(y) = \int_y^x (\mu(z))^{\alpha-1} \, d\mathcal{H}^1(z) \quad \text{for all } y \preceq x, \tag{3.4}$$

where $\mu(z)$ denotes the mass flowing in the point z. Proposition 3.9 is a consequence of the fractal regularity result [6, Theorem 6.17], whose application is not straightforward just because we are only assuming the inner version of (LAR) instead of the global one. Note that the constant ε_0 appearing in that statement is quantified in the proof and it can be written as

$$\varepsilon_0 := \frac{c_\alpha}{2}\Big(\frac{C_A}{c_A} 2^d\Big)^{\alpha-2} \ell = C_T \ell,$$

where c_A and C_A are as above and $c_\alpha = \frac{\alpha(1-\alpha)}{2}$. So C_T is a scale transition universal constant and ℓ represents the length of the main branch. The proof shows that the role of condition (LAR) consists in an estimate from below on the measure of a tubular neighborhood of a branch Γ. Of course the inner version of (LAR) works in the same way when we assume, as we shall do, that such a neighborhood is contained in B. In [6], (LAR) is assumed globally, but, of course, all the estimates which are quantified in terms of constants which do not involve c_A but only depend on the other constants (in particular C_H and C_A) can be used without any restriction in this setting. Finally, we warn the reader that some estimates in [6] seem to explicitly depend on c_A while they actually depend on C_H. Indeed, the Hölder continuity of Z is usually derived from (LAR) (see [5, Theorem 6.2]), while we are assuming it directly in the statement of Proposition 3.9.

Proof of Proposition 3.9. Set $C_1 := C_T^{-1} C_A^{1/d} > 0$, $\overline{R} = C_1^{-1}(\mu_\chi(\mathbb{R}^d))^{1/d}$ and fix $0 < r \leq \overline{R}$ and $\overline{x} \in B$. We shall initially assume that

$$\mu(y) \leq (C_1 r)^d \quad \text{for all } y \preceq \overline{x} \text{ such that } \ell(y, \overline{x}) < r. \tag{3.5}$$

In particular, by our choice of $\overline{R}$, we have $\ell(x_0, \overline{x}) > r$. Then, starting from $\overline{x}$, we can proceed backward along the flow, toward the source x_0, and find $y \preceq \overline{x}$ such that $\ell(y, \overline{x}) = r$. Then, by applying (3.4) and (3.5), we get that $Z(\overline{x}) - Z(y) \geq C_1^{\beta-1} r^\beta$ and we can conclude the proof easily. On the other hand, (3.5) does not hold when

$$\text{there exists } \overline{y} \preceq \overline{x} \text{ and } \ell(\overline{y}, \overline{x}) < r \text{ such that } \mu(\overline{y}) > (C_1 r)^d. \tag{3.6}$$

Then set $r_T := C_T^{-1} r$. As a consequence, $\ell(\overline{y}) > r_T$ (indeed, otherwise condition (A2) would imply

$$\mu(\overline{y}) \leq \mu_\chi(B_{\ell(\overline{y})}(\overline{y})) \leq C_A(\ell(\overline{y}))^d \leq C_A(r_T)^d = (C_1 r)^d,$$

in contradiction to (3.6)). Then, by applying [6, Lemma 2.17], we deduce the existence of a branch $\overline{\Gamma}$ starting from $\overline{y}$ having length r_T. Let

$$N_r(\overline{\Gamma}) := \{z \in \mathbb{R}^d \mid \operatorname{dist}(z, \overline{\Gamma}) < r\}$$

be the tubular neighborhood of $\overline{\Gamma}$ of radius r. We shall assume that $N_r(\overline{\Gamma}) \subset B$, in such a way to only use the inner version of (LAR) in order to estimate $\mu_\chi(N_r(\overline{\Gamma}))$. Then we can argue as in [6, Theorem 6.17], getting the existence of a branch $\tilde{\Gamma}$, starting from a point of Γ and with a length $\tilde{\ell}$ such that $r \le \tilde{\ell} \le Wr$, where W is the "scale window" universal constant introduced in [6]. So we can deduce, by applying as usual (UAR), that $\mu(z) \le C_A(Wr)^d$ for all $z \in \tilde{\Gamma}$. Then, by taking two points $x_1 \preceq x_2 \in \tilde{\Gamma}$ such that $\ell(x_1, x_2) \ge r$, by applying (3.4), we obtain that $\omega_{\tilde{\Gamma}}(Z) \ge C_A^{\alpha-1} W^{\beta-1} r^\beta$. For the arbitrariness of r we can conclude that (LHC) holds.

If our extra assumption in this proof is false, we can deduce that the ball B_0 centered in $\overline{x}$ with radius $(2 + C_T^{-1})r$ is not contained in B. So, for all $\varepsilon > 0$ there exists a point $x_\varepsilon \in D_\chi \cap B_0$ such that $\operatorname{dist}(x_\varepsilon, \partial B) < \varepsilon$. Then, starting from x_ε, if it is possible, we proceed backward along the flow of a distance r up to reach a point y_ε belonging to the ball B_1 centered in $\overline{x}$ with radius $(3 + C_T^{-1})r$. Since by construction $\ell(x_\varepsilon, y_\varepsilon) = r$, by (A3) we get that $\omega_{B_1}(Z) \ge Z(x_\varepsilon) - Z(y_\varepsilon) \ge C_B r^\beta$, so (LHC) holds (under a suitable choice of C_L). Finally, note that we cannot reach y_ε only when, during the backward motion, we find the source x_0, and so, since $\operatorname{dist}(x_\varepsilon, \partial B) < \varepsilon$ and $Z(x_0) = 0$, we get, by (HC) and (A4) and by our choice of $\overline{R}$, that

$$\omega_{B_1}(Z) \ge Z(x_\varepsilon) \ge \inf_{\partial B} Z - C_H \varepsilon^\beta \ge C_Z C_1^\beta \overline{R}^\beta - C_H \varepsilon^\beta.$$

So, for ε small, we see that (LHC) holds in every case (under a suitable choice of C_L). □

Proposition 3.10. *Condition* (A3) *follows from the following variant:*

$$\text{there exists } C_B' > 0 \text{ such that for all } x \in B \cap D_\chi \text{ we have } \ell(x) \le C_B' \operatorname{dist}(x, \partial B). \tag{A3'}$$

Proof. Fix $y \preceq x \in B$ such that $\operatorname{dist}(x, \partial B) \le \ell(y, x) =: \ell$. Then for all $y \preceq z \preceq x$ we have that $\operatorname{dist}(z, \partial B) \le 2\ell$, so by combining this last inequality with (A3′), we deduce $\ell(z) \le 2C_B'\ell$. Since μ_χ is (UAR), we deduce that

$$\mu(z) \le \mu_\chi(B_{2C_B'\ell}(z)) \le C_A(2C_B'\ell)^d.$$

Therefore, by (3.4), since $d(\alpha - 1) = \beta - 1$, condition (A3) follows with $C_B = C_A^{\alpha-1}(2C_B')^{\beta-1}$. □

We conclude this section by giving an estimate on the margin of growth of the landscape function with respect to the residual distance.

Lemma 3.11. *If Z satisfies* (HC) *and μ_χ satisfies* (UAR) *globally, then, for a suitable constant $C_M > 0$,*

$$\text{for all } x \in D_\chi \text{ there exists } z \succeq x \text{ such that } Z(z) - Z(x) \ge C_M \ell(x)^\beta.$$

Proof. Fix $x \in D_\chi$. Then, by applying [6, Lemma 2.17] (which, as already pointed out, can be applied thanks to the Hölder continuity of Z), for every $\varepsilon > 0$ there exists $y_\varepsilon \succeq x$ such that $\ell(x, y_\varepsilon) = \ell(x) - \varepsilon$. By (3.4) and (UAR) we get

$$Z(y_\varepsilon) - Z(x) = \int_x^{y_\varepsilon} (\mu(z))^{\alpha-1} \, d\mathcal{H}^1(z) \ge C_A^{\alpha-1} \ell(x)^{\beta-1} \ell(x, y_\varepsilon).$$

Then the thesis follows by taking $C_M < C_A^{\alpha-1}$ and ε small enough. □

4 Dimension Estimates on Level Sets

The main results of this paper will follow from two estimates respectively derived from properties (HC) and (LHC) at a low scale.

Fixing a function $f : A \to \mathbb{R}$, for any $a \le b \in \mathbb{R}$ we set

$$L_a^b := \{x \in A \mid a \le f(x) \le b\}$$

and $L_c = L_c^c$ when $a = b = c \in \mathbb{R}$ to denote a level set of the function f.

Lemma 4.1. *Let $A \subset \mathbb{R}^d$ and let $f : A \to \mathbb{R}$ be a function which satisfies* (HC) *at scale $\overline{R}$ with exponent β. Then, for any $a < b \in \mathbb{R}$ and for any*

$$0 < \delta < \frac{C_H}{2^{\beta-1}(2^\beta - 1)}\overline{R}^\beta,$$

setting

$$h = \Big(\frac{(2^\beta - 1)\delta}{3^\beta 2 C_H}\Big)^{\frac{1}{\beta}},$$

we have that, for any large $k \in \mathbb{N}$, if $(c_i)_{1 \le i \le k} \subset [a + \frac{\delta}{2}, b - \frac{\delta}{2}]$ is any family of levels such that

$$|c_i - c_j| \ge \delta \quad \text{for all } i \ne j, \tag{4.1}$$

then

$$\sum_{i=1}^{k} |N_h(L_{c_i})| \le |N_h(L_a^b)|. \tag{4.2}$$

Proof. Set $N_i := N_h(L_{c_i})$ for all i. We shall prove the thesis by showing that

$$N_i \cap A \subset L_a^b \quad \text{for all } i \in \{1, \dots, k\}, \tag{4.3}$$

and

$$N_i \cap N_j = \emptyset \quad \text{for all } i, j \in \{1, \dots, k\},\ i \ne j. \tag{4.4}$$

By fixing i and $x \in N_i \cap A$, there exists some $\overline{x}_i \in L_{c_i}$ such that $|x - \overline{x}_i| < h < \overline{R}$. Then, by (HC), we deduce that

$$|f(x) - c_i| \le C_H |x - \overline{x}_i|^\beta < \frac{\delta}{2}\Big(\frac{2^\beta - 1}{3^\beta}\Big) < \frac{\delta}{2}.$$

So, since $c_i \in [a + \frac{\delta}{2}, b - \frac{\delta}{2}]$, we deduce that $x \in L_a^b$ and, by the arbitrariness of x, we deduce (4.3). As proved in Proposition 3.7, the restriction of f to the set L_a^b can be extended to $N_h(L_a^b)$ if h is small enough (i.e. k is large enough) in such a way to remain Hölder continuous at scale $\frac{\overline{R}}{3}$ with a constant $\frac{3^\beta}{2^\beta - 1} C_H$. So, by identifying f with its extension, we can prove (4.4) by showing that $f(N_i) \cap f(N_j) = \emptyset$ for $i \ne j$. If, by contradiction, $f(N_i) \cap f(N_j) \ne \emptyset$ for some $i \ne j$, we get the existence of $x_i \in N_i$ and $x_j \in N_j$ such that $f(x_i) = f(x_j)$ and the existence of $\overline{x}_i \in L_{c_i}$ and $\overline{x}_j \in L_{c_j}$ such that

$$\max(|x_i - \overline{x}_i|, |x_j - \overline{x}_j|) < h < \frac{\overline{R}}{6}.$$

Then, by Hölder continuity, $\max(|f(x_i) - c_i|, |f(x_j) - c_j|) < C_H h^\beta \le \frac{\delta}{2}$. Then, since $f(x_i) = f(x_j)$, by the triangular inequality, we get a contradiction to (4.1). □

A corresponding statement which gives the opposite estimate is the following lemma.

Lemma 4.2. *Let $A \subset \mathbb{R}^d$ and let $f : A \to \mathbb{R}$ be a function which satisfies* (LHC) *at scale $\overline{R}$ with exponent β. Then, for any $a < b \in \mathbb{R}$ and for any $0 < \delta < C_L \overline{R}^\beta$, setting $h = (\frac{\delta}{2C_L})^{1/\beta}$, we have that for any $k \in \mathbb{N}$, if $(c_i)_{1 \le i \le k}$ is a $\frac{\delta}{2}$-net of $[a, b]$, then*

$$\sum_{i=1}^{k} |N_h(L_{c_i})| \ge |L_a^b|. \tag{4.5}$$

Proof. Set $N_i := N_h(L_{c_i})$ for all i. The thesis follows since, by construction, $(N_i)_{1 \le i \le k}$ is a covering of L_a^b. Indeed, fixing $x \in L_a^b$, since by assumption $h < \overline{R}$, and setting $B = B(x, h)$, by (LHC) we have that $\omega_B(f) \ge C_L h^\beta = \frac{\delta}{2}$, and so $f(B(x, h) \cap A)$ must contain an interval whose length is at least $\frac{\delta}{2}$. Then, since $(c_i)_{1 \le i \le k}$ is a $\frac{\delta}{2}$-net of $[a, b]$, there exists $y \in B$ such that $f(y) = c_i$ for some $i \in \{1, \dots, k\}$. Then, since $|x - y| < h$, we deduce that $x \in N_i$. □

The two following propositions are easy consequences of the above lemmas.

Proposition 4.3. *Let $A \subset \mathbb{R}^d$ and let $f : A \to \mathbb{R}$ be a function which satisfies* (HC) *at scale $\overline{R}$ with exponent β. Let $\overline{h} > 0$, $\gamma < \beta$ and $a < b \in \mathbb{R}$ such that $|N_{\overline{h}}(L_a^b)| < +\infty$. Then there exist an open set $V \subset [a, b]$ and a positive constant $h < \overline{h}$ such that*

$$\frac{|N_h(L_c)|}{h^\gamma} \le 1 \quad \text{for all } c \in V. \tag{4.6}$$

Proof. Fix $k \in \mathbb{N}$, $k \geq 1$ and set $\delta = \frac{b-a}{2k}$. We can split the interval $[a, b]$ into $2k + 1$ sub-intervals $V_i \subset [a, b]$ such that the first and last interval V_1 and V_{2k+1} have length $|V_1| = |V_{2k+1}| = \frac{\delta}{2}$ and the remaining $2k - 1$ have length δ. Setting

$$h = \Big(\frac{(2^\beta - 1)\delta}{3^\beta 2C_H}\Big)^{\frac{1}{\beta}},$$

we claim that for every k large enough there exists $\bar{\imath} \in \{1, \dots, k\}$ such that

$$|N_h(L_c)| \leq \frac{|N_h(L_a^b)|}{k} \quad \text{for all } c \in V_{2\bar{\imath}}. \tag{4.7}$$

Indeed, on the contrary, for all $i \in \{1, \dots, k\}$ would exist $c_i \in V_{2i}$ such that $|N_h(L_{c_i})| > |N_h(L_a^b)|/k$, i.e. there would exist a finite family of levels $(c_i)_{1\leq i\leq k}$ which, by construction, is contained in $[a + \frac{\delta}{2}, b - \frac{\delta}{2}]$ and satisfies (4.1). Therefore, $\sum_{i=1}^k |N_h(L_{c_i})| > |N_h(L_a^b)|$, but since, for k large enough,

$$\delta = \frac{b-a}{2k} < \frac{C_H}{2^{\beta-1}(2^\beta - 1)}\overline{R}^\beta$$

and Lemma 4.1 applies, we get a contradiction to (4.2).

So we take $V = V_{2\bar{\imath}}$ as in (4.7). For all $c \in V$,

$$|N_h(L_c)| \leq \frac{|N_h(L_a^b)|}{k} \leq \frac{4C_H}{b-a}\frac{3^\beta}{2^\beta - 1}h^\beta|N_h(L_a^b)| = \frac{4C_H}{b-a}\frac{3^\beta}{2^\beta - 1}h^{\beta-\gamma}h^\gamma|N_h(L_a^b)|,$$

and so (4.6) follows since the constant $\frac{4C_H}{b-a}\frac{3^\beta}{2^\beta-1}h^{\beta-\gamma}|N_h(L_a^b)|$ is less than 1 (since $\beta > \gamma$ and h is small) when k is large enough. □

Proposition 4.4. *Let $A \subset \mathbb{R}^d$ and let $f : A \to \mathbb{R}$ be a function which satisfies* (LHC) *at scale $\overline{R}$ with exponent $\beta \in]0, 1]$. Let $\overline{h} > 0$, $\gamma < \beta$ and $a < b \in \mathbb{R}$ such that $|L_a^b| > 0$. Then there exist an open set $V \subset [a, b]$ and a positive constant $h < \overline{h}$ such that*

$$\frac{|N_h(L_c)|}{h^\gamma} \geq 1 \quad \textit{for all } c \in V. \tag{4.8}$$

Proof. Fix $k \in \mathbb{N}$, $k \geq 1$ and set $\delta = \frac{3(b-a)}{k}$. We can split the interval $[a, b]$ into $2k + 1$ sub-intervals $V_i \subset [a, b]$ such that the first and last interval V_1 and V_{2k+1} have length $|V_1| = |V_{2k+1}| = \frac{\delta}{12}$ and the remaining $2k - 1$ have length $\frac{\delta}{6}$. Setting

$$h = \Big(\frac{\delta}{2C_L}\Big)^{\frac{1}{\beta}},$$

we claim that for every k large enough there exists $\bar{\imath} \in \{1, \dots, k\}$ such that

$$|N_h(L_c)| \geq \frac{|L_a^b|}{k} \quad \text{for all } c \in V_{2\bar{\imath}}. \tag{4.9}$$

Indeed, on the contrary, for all $i \in \{1, \dots, k\}$ would exist $c_i \in V_{2i}$ such that $|N_h(L_{c_i})| \leq |L_a^b|/k$, i.e. there would exist a finite family of levels $(c_i)_{1\leq i\leq k}$ which, by construction, is a $\frac{\delta}{2}$-net of $[a, b]$ and satisfies $\sum_{i=1}^k |N_h(L_{c_i})| \leq |L_a^b|$. Since, for k large enough,

$$\delta = \frac{3(b-a)}{k} < C_L\overline{R}^\beta,$$

and Lemma 4.2 applies, we get a contradiction to (4.5).

So we take $V = V_{2\bar{\imath}}$ as in (4.9). For all $c \in V$,

$$|N_h(L_c)| \geq \frac{2C_L}{3(b-a)}h^\beta|L_a^b| = \frac{2C_L}{3(b-a)}h^{\beta-\gamma}h^\gamma|L_a^b|,$$

and so (4.8) follows since the constant $\frac{2C_L}{3(b-a)}h^{\beta-\gamma}|L_a^b|$ is greater than 1 (since $\beta < \gamma$ and h is small) when k is large enough. □

Given a measurable set $A \subset \mathbb{R}^d$ and a real-valued function $f : A \to \mathbb{R}$, we define $F : A \times \mathbb{R} \to \mathbb{R}$ by setting $F(x, y) := f(x) - y$ for all $x \in A$ and $y \in \mathbb{R}$. Of course we have that the graph of f is the zero level set of the function F. We point out that if f satisfies (LHC) (at a low scale), so does F with the same constant C_L and the same scale $\overline{R}$. On the other hand, if f is (HC), then F is only (HC) (with the same exponent β) at a low scale for any $\overline{R} > 0$ with a different constant $\widetilde{C}_H$. Finally, note that, if f is measurable, the level sets of the function F can easily be estimated as

$$|L_a^b| = (b-a)|A| \quad \text{for all } a < b \in \mathbb{R}. \tag{4.10}$$

For a general f just one of the two inequalities holds and only the lower bound to $|L_a^b|$ given by (4.10) can be applied.

Theorem 4.5. *Let $A \subset \mathbb{R}^d$ be a measurable set such that $|N_s(A)| < +\infty$ for some $s > 0$. Let $f : A \to \mathbb{R}$ be Hölder continuous with exponent β. Then*

$$\overline{\dim}_M(\operatorname{graph} f) \leq d + 1 - \beta. \tag{4.11}$$

Proof. Fix $k \in \mathbb{N}$ suitably large and let $\{-1 < c_1 < \cdots < c_{2k+1} < 1\}$ be a partition of the interval $[-1, 1]$ such that $c_1 + 1 = 1 - c_{2k+1} = \frac{1}{2k+1}$ and $c_{i+1} - c_i = \frac{2}{2k+1}$ for all $i \in \{1, \dots, 2k\}$. Then fix a real number $\overline{R} > 0$ such that the function F is (HC) at scale $\overline{R}$ for some constant $\widetilde{C}_H$. We can apply Lemma 4.1 to the function F (with d replaced by $d + 1$) by taking $a = -1$, $b = 1$ and k large enough to get that $\delta := \frac{1}{2k+1} < \widetilde{C}_H \overline{R}^{\beta}$. Then set

$$h = \Big(\frac{\delta}{2\widetilde{C}_H}\Big)^{\frac{1}{\beta}}.$$

By a translation in the y-variable of F we easily see that the value of $|N_h(L_{c_i})|$ is the same for all indexes i, so (4.2) and (4.10) give $|N_h(L_{c_i})| \leq \frac{1}{2k+1}|N_h(L_{-1}^1)| = \frac{2}{2k+1}|N_h(A)|$ for all $i \in \{1, \dots, 2k+1\}$. Since, by construction $c_{k+1} = 0$, we get that

$$|N_h(\operatorname{graph} f)| \leq \frac{1}{2k+1}|N_h(L_{-1}^1)| = \frac{2}{2k+1}|N_h(A)| \leq 4\widetilde{C}_H|N_h(A)|h^{\beta}$$

since

$$h = \Big(\frac{\delta}{2\widetilde{C}_H}\Big)^{\frac{1}{\beta}} = \Big(\frac{1}{2(2k+1)\widetilde{C}_H}\Big)^{\frac{1}{\beta}},$$

and so $\frac{1}{2k+1} = 2\widetilde{C}_H h^{\beta}$. The values assumed by h for k varying in $\mathbb{N}$ are dense enough to estimate the limit for $h \to 0$. So we get an upper bound on the Minkowski content (see [15, Section 5.5])

$$M_c^s(\operatorname{graph} f) := \lim_{h \to 0^+} \frac{|N_h(\operatorname{graph} f)|}{h^{d+1-s}} \leq 2\widetilde{C}_H|N_h(L_{-1}^1)| = 4\widetilde{C}_H|N_h(A)|$$

when the dimension s equals $d + 1 - \beta$. □

Theorem 4.6. *Let $A \subset \mathbb{R}^d$ be a measurable set such that $|A| > 0$. Let $f : A \to \mathbb{R}$ satisfy the lower Hölder condition with exponent β. Then*

$$\underline{\dim}_M(\operatorname{graph} f) \geq d + 1 - \beta. \tag{4.12}$$

Proof. The proof follows the same argument already used to prove Theorem 4.5 by applying Lemma 4.2 instead of Lemma 4.1 and using (4.10) as a lower bound. □

By combining (4.12) with (4.11), we get the following result.

Corollary 4.7. *The graph of any function defined on a measurable set $A \subset \mathbb{R}^d$ such that $|A| > 0$ and $|N_s(A)| < +\infty$ for some $s > 0$, which satisfies* (HC) *and* (LHC) *with the same exponent β, has a Minkowski dimension equal to $d + 1 - \beta$.*

We introduce some notation. Fixing $\overline{h} > 0$ and $\gamma \in \mathbb{R}$, we set

$$S^-(\gamma, \overline{h}) := \{c \in f(A) \mid \text{there exists } h < \overline{h} \text{ such that } |N_h(L_c)| \leq h^{\gamma}\},$$
$$S^+(\gamma, \overline{h}) := \{c \in f(A) \mid \text{there exists } h < \overline{h} \text{ such that } |N_h(L_c)| \geq h^{\gamma}\},$$

and state the following conditions on the measure of the sets $|L_a^b|$ related to f:

$$\text{for all } a, b \in f(A),\ a < b, \text{ there exists } s > 0 \text{ such that } |N_s(L_a^b)| < +\infty, \tag{M$^-$}$$
$$\text{for all } a, b \in f(A),\ a < b, \text{ we have } |L_a^b| > 0. \tag{M$^+$}$$

Remark 4.8. Note that conditions (M^-) and (M^+) are trivially satisfied when A is bounded or, respectively, A is an open connected set and f is a continuous function.

Propositions 4.3 and 4.4 respectively guarantee that, if condition ($M^\pm$) is satisfied and $\pm\gamma > \pm\beta$, the inner part of the set $S^\pm(\gamma, \bar{h})$ is dense in $f(A)$. So, setting $S^\pm(\gamma) := \bigcap_{\bar{h}>0} S^\pm(\gamma, \bar{h})$, we get, by the Baire Theorem, that both $S^\pm(\gamma)$ are dense G_δ-sets if conditions ($M^\pm$) are both satisfied. Obviously, by construction, if $c \in S^-(\gamma)$, then $\underline{\dim}_M(L_c) \leq d - \gamma$, while if $c \in S^+(\gamma)$, then $\overline{\dim}_M(L_c) \geq d - \gamma$.

Finally, the sets

$$\mathcal{S}^\pm := \bigcap_{\substack{\pm\gamma > \pm\beta \\ \gamma \in \mathbb{Q}}} S^\pm(\gamma)$$

are dense G_δ-sets if conditions ($M^\pm$) are true.

Theorem 4.9. *Let $A \subset \mathbb{R}^d$ and let $f : A \to \mathbb{R}$ be a real-valued function defined on A which satisfies* (HC) *with exponent β. Then, if condition* (M^-) *holds true, there exists a dense G_δ-set $\mathcal{S} \subset f(A)$ such that*

$$\underline{\dim}_M(L_c) \leq d - \beta \quad \text{for all } c \in \mathcal{S}. \tag{4.13}$$

Proof. Just take as $\mathcal{S}$ in (4.13) the set $\mathcal{S}^-$. □

Theorem 4.10. *Let $A \subset \mathbb{R}^d$ and let $f : A \to \mathbb{R}$ be a real-valued function defined on A which satisfies* (LHC) *(at a low scale) with exponent β. Then, if condition* (M^+) *holds true, there exists a dense G_δ-set $\mathcal{S} \subset f(A)$ such that*

$$\overline{\dim}_M(L_c) \geq d - \beta \quad \text{for all } c \in \mathcal{S}. \tag{4.14}$$

Proof. Just take as $\mathcal{S}$ in (4.14) the set $\mathcal{S}^+$. □

Finally, by combining Theorem 4.9 with Theorem 4.10, we get the following corollary which states a topological generic estimate on the Minkowski dimension (when it exists) of the level sets of a function which satisfies (HC) and (LHC) with the same exponent β.

Corollary 4.11. *Let $A \subset \mathbb{R}^d$ be an open set and let $f : A \to \mathbb{R}$ be a real-valued function defined on A which satisfies* (HC) *and* (LHC) *(at a low scale) with exponent β. Then, if both* ($M^\pm$) *are true, there exists a dense G_δ-set $\mathcal{S} \subset f(A)$ such that*

$$\underline{\dim}_M(L_c) \leq d - \beta \leq \overline{\dim}_M(L_c) \quad \text{for all } c \in \mathcal{S}.$$

As a consequence, if $c \in \mathcal{S}$ and there exists $\dim_M(L_c)$, *then* $\dim_M(L_c) = d - \beta$.

Proof. Just take $\mathcal{S} = \mathcal{S}^- \cap \mathcal{S}^+$, which is still a dense G_δ-set, and combine (4.13) with (4.14). □

5 Conclusions

To the aim to apply to the irrigation ball $\mathcal{B}$ and to its landscape function Z the results in the previous section, we need to preliminarily establish property (LHC), and therefore to prove the assumptions needed in Section 3.

By [18, Theorem 3.7], the landscape function Z satisfies (HC) with exponent $\beta = 1 + d(\alpha - 1)$. Moreover, since μ_χ is the restriction of the Lebesgue measure to the irrigation ball $\mathcal{B}$, we have that Z satisfies (UAR) globally and the inner (LAR) on $\mathcal{B}$ (with the same constants C_A and c_A). So, $\mathcal{B}$ satisfies (A2) and, trivially, (A1). Condition (A4) follows from (2.5) with $C_Z = b_\alpha^{-\beta/d}$.

Lemma 5.1. *The irrigation ball $\mathcal{B}$ satisfies condition* (A3′).

Proof. Fix $x \in \mathcal{B}$. Since $Z = R$ on $\partial\mathcal{B}$, we deduce by (HC) that $R - Z(x) \leq C_H \operatorname{dist}(x, \partial B)^\beta$. On the other hand, by Lemma 3.11, there exists $z \geq x$ such that $Z(x) \leq Z(z) - C_M \ell(x)^\beta \leq R - C_M \ell(x)^\beta$. By combining the two inequalities above, we get $C_M \ell(x)^\beta \leq C_H \operatorname{dist}(x, \partial\mathcal{B})^\beta$, and so the thesis follows with $C'_B = (C_H C_M^{-1})^{1/\beta}$. □

By Proposition 3.10, Z also satisfies (A3). So, by Proposition 3.9, Z satisfies (LHC) on $\mathcal{B}$. Since the irrigation sub-spheres are level sets of Z, the main results of this paper (stated in Section 2) follow from Section 4.

Proof of Proposition 2.2. We prove that any $\mathcal{S}_\rho$ does not contain Lebesgue points. Let $\overline{x} \in \mathcal{S}_\rho$ and $B = B(\overline{x}, r)$ with $r < \overline{R}$. By Proposition 3.9 we deduce the existence of a point $x \in B$ such that $|Z(x) - Z(\overline{x})| \geq C_L r^\beta$. Then, setting $r' := (\frac{C_L}{2C_H})^{1/\beta} r$ (which is of the same order as r), we have $B(x, r') \cap \mathcal{S}\rho = \emptyset$. □

Proof of Proposition 2.4. The inclusion $\partial\mathcal{B}_\rho \subset \mathcal{S}_\rho$ is trivial. On the other hand, fixing $\overline{x} \in \mathcal{S}_\rho$, we shall prove that for all (small) $\delta > 0$ such that $|\overline{x} - x_0| > \delta$ in the Euclidean ball $B_\delta(\overline{x})$ there exists a point y such that $Z(y) < \rho$, i.e. $y \in \mathcal{B}_\rho$. Indeed, fixing $0 < \varepsilon < \delta$, we get a point $x \in D_\chi \cap B_\varepsilon(\overline{x})$, and since $x_0 \notin B_\delta(x)$, we can fix a point y which precedes x in the flow order such that $y \in \partial B_\delta(\overline{x})$. Obviously, we have that $\ell(y, x) \geq \delta - \varepsilon$, and so we get, by (3.4), $Z(x) - Z(y) \geq |\mathcal{B}|^{\alpha-1}\ell(y, x) \geq |\mathcal{B}|^{\alpha-1}(\varepsilon - \delta)$. Therefore, since Z satisfies (HC),

$$Z(y) \leq Z(y) - Z(x) + \operatorname{osc}_{B(x_0,\varepsilon)} Z + |Z(x_0)| \leq \rho + C_H \varepsilon^\beta - |\mathcal{B}|^{\alpha-1}(\delta - \varepsilon).$$

So, by letting ε go to zero, we have $Z(y) < \rho$. □

Proof of Theorem 2.5. Given $\rho \in [0, R]$ and $\delta > 0$, setting $h = (\frac{\delta}{C_H})^{1/\beta}$, we deduce that

$$|N_h(L_c)| \leq |L_{c-\delta}^{c+\delta}|.$$

On the other hand, by Definition 2.6, we have that $|L_{c-\delta}^{c+\delta}| = m(c + \delta) - m(c - \delta)$. Then, since $\frac{1}{h^\beta} = \frac{C_H}{\delta}$, we deduce that

$$\frac{|N_h(L_c)|}{h^\beta} \leq C_H\Big[\frac{m(c + \delta) - m(c)}{\delta} + \frac{m(c) - m(c - \delta)}{\delta}\Big]. \tag{5.1}$$

Since the function m is monotone, by the Lebesgue theorem on differentiability of monotone functions (see [11]), for a.e. c the following upper Dini derivative is finite:

$$\overline{D}m(c) := \limsup_{\sigma\to0^+} \frac{m(c + \sigma) - m(c)}{\sigma}.$$

So, we deduce by (5.1) that for a.e. c,

$$\limsup_{h\to0} \frac{|N_h(L_c)|}{h^\beta} \leq 2C_H\overline{D}m(c) < +\infty,$$

and therefore that $\overline{\dim}_M(\mathcal{S}_\rho) = \overline{\dim}_M(L_c) \leq d - \beta$. □

Finally, since Z is continuous on $\mathcal{B}$, which is a bounded open connected set, both conditions ($M^\pm$) are satisfied (see Remark 4.8). Then Theorem 2.6 and Theorem 2.8 respectively follow from Corollary 4.11 and Corollary 4.7 applied to the landscape function Z.

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