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Trajectories of Affine Control Systems and Geodesics of a Spacetime with a Causal Killing Vector Field

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Abstract

We study the geodesic connectedness of a globally hyperbolic spacetime (M, g) admitting a complete smooth Cauchy hypersurface S and endowed with a complete causal Killing vector field K. The main assumptions are that the kernel distribution $\mathcal{D}$ of the one-form induced by K on S is non-integrable and that the gradient of $g(K, K)$ is orthogonal to $\mathcal{D}$. We approximate the metric g by metrics g_ε smoothly depending on a real parameter ε and admitting K as a timelike Killing vector field. A known existence result for geodesics of such type of metrics provides a sequence of approximating solutions, joining two given points, of the geodesic equations of (M, g) and whose Lorentzian energy turns out to be bounded thanks to an argument involving trajectories of some affine control systems related with $\mathcal{D}$.

Keywords Geodesics · Spacetime · Causal Killing field · Affine control systems

Mathematics Subject Classification (2010) 53C50 · 58E10 · 53C22 · 34H05

1 Introduction

The existence of a shortest path between any two given points on a Riemannian manifold or more generally on a locally compact length space is a basic result in metric geometry. It is a consequence of Ascoli-Arzelá theorem once metric completeness is assumed (see, e.g., [6, Section 2.5]). In contrast with Riemannian ones, Lorentzian metrics do not define a length metric structure due to their indefiniteness as bilinear symmetric tensors and their length functional is defined only for causal curves and not on the whole set of rectifiable curves between two points. On the other hand, taking the square root of the absolute value of $g(\dot\gamma, \dot\gamma)$ allows one to consider all the absolutely continuous curves between two points but

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produces a length functional which might have minimum value equal to zero between any couple of points due the possible presence of piecewise smooth (not future-pointing) null curves between them. It is then not surprising that establishing the existence of a geodesic of a Lorentzian metric between any two points is a highly non-trivial problem. A generalization of the notion of length structure in the setting of causality theory and Lorentzian geometry has been recently proposed in [15]. In particular, a classical existence result for future-pointing causal geodesics between two points admits an extension [15, Theorem 3.30]. The assumption that replaces completeness in this case is global hyperbolicity as in the classical causality theory (see, e.g., [4, Theorem 6.1]).

It is quite surprising that global hyperbolicity which, on a non-compact spacetime (M, g) is equivalent to the compactness of its causal diamonds (see [14]), also plays a fundamental role in the proof, obtained in [7], of the full geodesic connectedness of a spacetime (M, g) when g admits a complete Killing vector field K, which is timelike (i.e., $g(K, K) < 0$), and there exists a smooth, spacelike, *complete* Cauchy hypersurface in M. An analogous result has been obtained in [3] when the complete Killing vector field K is everywhere lightlike (i.e., $g(K, K) = 0$ and $K_p \neq 0$ for all $p \in M$).

In this work, we consider the case when K is causal, i.e., $g(K, K) \leq 0$, $K_p \neq 0$ for all $p \in M$. As far as we know, this case is open, apart from a recent result where a compact spacetime endowed with a causal Killing vector field satisfying the null generic condition and having globally hyperbolic universal covering is studied (see [2, Corollary 3.6]).

Let us give some further details on the geometric setting that we consider in connection with the ones in [3, 7].

Let (M, g) be a globally hyperbolic spacetime endowed with a complete causal Killing vector field K and a (smooth, spacelike) Cauchy hypersurface S[1]. Then there exists a diffeomorphism $\varphi : S \times \mathbb{R} \to M$ defined by the restriction of the flow K to $S \times \mathbb{R}$ and the induced metric on $S \times \mathbb{R}$ is

$$\varphi^* g = g_0 + \omega \otimes \mathrm{d}t + \mathrm{d}t \otimes \omega - \Lambda \mathrm{d}t^2, \tag{1}$$

where g_0 is the Riemannian metric induced by g on S that will be assumed to be complete, ω is the one-form metrically equivalent to the orthogonal projection of K on S, i.e., $\omega(v) = g(v, K)$ for all $v \in TS$ and $\Lambda : S \to \mathbb{R}$ is the non-negative function on S defined as $\Lambda = -g(K, K)|_S$ (see [7, Theorem 2.3], [3, Proposition 2.2]).

Henceforth, we will identify (M, g) with the spacetime $S \times \mathbb{R}$ endowed with the metric (1) that will be denoted with g as well. Moreover, we will assume that S (with the Riemannian metric g_0) is complete.

Notice that if K is timelike then $\Lambda(x) > 0$ for all $x \in S$ and the spacetime is called *standard stationary*; if K is lightlike then $\Lambda \equiv 0$, ω does not vanish at any point and the metric on M becomes equal to

$$g_0 + \omega \otimes \mathrm{d}t + \mathrm{d}t \otimes \omega.$$

A metric like (1) is a Lorentzian one if and only if

$$\Lambda(x) + |\omega_x|_0^2 > 0 \quad \text{for all } x \in S, \tag{2}$$

being $|\omega_x|_0$ the g_0-norm of ω_x in $T_x S$ (see [10, Proposition 3.3]).

[1]We refer to [4, Section 3.2] for causality notions as global hyperbolicity and the definition of a Cauchy hypersurface.

Before stating our main result, we observe that if $\omega_x \neq 0$ for all $x \in S$ and $\mathcal{D}_x := \ker(\omega_x)$, then

$$\mathcal{D} := \bigcup_{x \in S} \mathcal{D}_x \text{ is a distribution on } S \text{ and rank}\, \mathcal{D} = m - 1, \tag{3}$$

with $m = \dim(S)$. In the following we will denote by d_0 the distance on S induced by the complete Riemannian metric g_0 and by $\omega^\sharp$ the vector field g_0-metrically equivalent to ω.

Theorem 1 *Let (M, g) be a globally hyperbolic spacetime admitting a complete causal Killing vector field K and a smooth, spacelike, complete Cauchy hypersurface S. With the notations in (1), let us assume that*

(i) *there exists a constant $L \geq 0$ such that $\Lambda(x) \leq L$ for all $x \in S$;*
(ii) *$g_0(\nabla\Lambda, \omega^\sharp) = 0$;*
(iii) *$\mathcal{D}$ in (3) is non-integrable;*
(iv) *there exist $\nu > 0$, a point $\bar{x} \in S$, a constant $C = C_{\bar{x}} > 0$ and $\alpha \in [0, 1)$ such that*

$$\nu \leq |\omega_x|_0 \leq C\big(d_0(\bar{x}, x)^\alpha + 1\big), \quad \text{for all } x \in S. \tag{4}$$

Then (M, g) is geodesically connected.

As already recalled, the case when K is lightlike everywhere has been studied in [3, Theorem 1.2], where it is proved that any couple of points $p_0 = (x_0, t_0)$, $p_1 = (x_1, t_1)$ in $S \times \mathbb{R}$ can be connected by a geodesic provided that there exists a C^1 curve σ on S between x_0 and x_1 such that $\omega(\dot{\sigma})$ is constant. We notice here that the existence of a curve σ between any two points in S satisfying $\omega(\dot{\sigma}) = \text{const.}$ (in particular 0) follows by assuming the non-integrability of the distribution defined pointwise by the kernel of the one-form ω thanks to Chow-Rashevskii theorem (see, e.g., [1, Theorem 3.31]). On the other hand, [3, Example (c), p.22] shows that the integrability of ω is quite a natural obstruction to the existence of a geodesic between any couple of points of a spacetime endowed with a lightlike Killing vector field.

A class of examples satisfying the assumptions in Theorem 1 is the following. Let us consider a product manifold $S \times \mathbb{R}$ where $S = \mathbb{R}^3$ is endowed with spherical coordinates (r, θ, ϕ). Let g be the Lorentzian metric on $S \times \mathbb{R}$ defined as

$$\mathrm{d}r^2 + 2a(r, \theta)(\mathrm{d}\theta + \mathrm{d}\phi)\mathrm{d}t + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2) - \Lambda(r)dt^2,$$

where $\Lambda = \Lambda(r)$ is a smooth, non-negative, bounded function which is 0 on the interval $[0, R]$ and a is a bounded function such that $a(r, \theta) \geq \nu_1 > 0$, having nowhere vanishing partial derivative a_r. The vector field ∂_t is a causal Killing vector field which is lightlike on $[0, R] \times S^2 \times \mathbb{R}$ and timelike otherwise. The one-form ω is given by $a(\mathrm{d}\theta + \mathrm{d}\phi)$ and, being $\omega \wedge \mathrm{d}\omega = -a_r a \mathrm{d}r \wedge \mathrm{d}\theta \wedge \mathrm{d}\phi$, it is a contact form on S and then its kernel distribution is non-integrable. Notice that $\nabla\Lambda = \Lambda'(r)\partial_r$, thus it is contained in the kernel of ω. We notice also that $(S \times \mathbb{R}, g)$ is globally hyperbolic with complete Cauchy hypersurfaces $S \times \{t\}$, $t \in \mathbb{R}$, see Remark 7.

We emphasize that Theorem 1 extends [3, Theorem 1.2] and its proof is independent of it. On the other hand, the idea of approximating the metric g with metrics g_ε, $\varepsilon > 0$, such that the vector field K is Killing and timelike for each metric g_ε, is the same as in [3] and in [8]. A novelty of the present work is the use of some affine control systems associated with $\mathcal{D}$ and drifts depending on ε, on the t-components $t_0, t_1 \in \mathbb{R}$ of the fixed points and collinear with the vector field g_0-metrically equivalent to ω. Thanks to appropriate curves constructed by concatenating solutions of these control systems, we get a bound from above of the critical values of some *special* (in a sense that will be explained in Section 3) connecting geodesics of the approximating metrics g_ε. We mention that similar affine control systems (but with a fixed drift) have been recently used in [9] to study multiplicity of geodesics between two points on some singular Finsler spaces.

The paper is organized as follows: in Section 2 we introduce some preliminary remarks involving the distribution $\mathcal{D}$ and the causality of M. Then, in Section 3 we adapt control systems to stationary perturbations (M, g_ε), $\varepsilon > 0$, of (M, g). Exploiting the result in [7], we consider a special family $\gamma_\varepsilon = (\rho_\varepsilon, t_\varepsilon)$ of geodesics of g_ε joining two points (x_0, t_0), $(x_1, t_1) \in M$ and we construct, by means of a control system having the drift smoothly depending on ε, a family σ_ε of curves connecting x_0 to x_1 and having bounded g_0-energy. In Section 4 we show that the family $\{\gamma_\varepsilon\}$ is bounded in the C^1-topology, so that, up to pass to a subsequence, for any sequence $\varepsilon_n \to 0$, $\{\gamma_{\varepsilon_n}\}$ converges (in the C^∞-topology) to a geodesic of (M, g) joining (x_0, t_0) and (x_1, t_1).

Let us finally specify some notations. We do not explicitly write the point where a vector field or a tensor is applied, except for some cases where the point might appear as an index (as, e.g., ω_x). If we look at a vector field X on a manifold S as a vector field along a curve σ, then we write $X(\sigma)$. An exception is when it is clear from the context that a vector field must be restricted to a given curve (as, e.g., in the expression $g_0(\nabla_{\dot\rho}\dot\rho, \omega^\sharp)$, where ρ is a curve). On the other hand we always write the evaluation of a function on S at a point x (as, e.g., $\Lambda(x)$) and of a one-form or a $(1,1)$-tensor field at a vector $v \in TS$ (as, e.g., in $\Omega^\sharp(v)$). Analogously, for a function defined on TS, as the Finsler type functions $F^\pm$ in (6), we write $F^\pm(v)$ without specifying the point $x \in S$ where v is applied.

2 Non-Integrability of ω and Causality

In [5, 11] the homotopy properties of the trajectories of an affine control system on a manifold S are studied. A *trajectory* $\sigma : [0, b] \to S$ is an absolutely continuous curve solving the system

$$\dot\sigma = V(\sigma) + \sum_{i=1}^{d} u_i X_i(\sigma), \quad \sigma(0) = x_0 \in S$$

for some functions $u = (u_1, \dots, u_d)$ called *controls*, where $V, X_1, \dots, X_d$ are vector fields, with V playing the role of a *drift* (which in some cases — as in the sub-Riemannian one — is the null vector field) and $X_1, \dots, X_d$ satisfy the *bracket generating* condition (see, e.g., [1, Definition 3.1]). The regularity assumption on the controls determines the topology on the space Ω of the trajectories. Henceforth, we will consider L^2 controls and, called u the d-tuple $(u_1, \dots, u_d) \in L^2([0, b], \mathbb{R}^d)$, for some $b > 0$, we will denote by $\|u\|_2$ the L^2 norm of u, i.e., $\|u\|_2 := \left(\sum_{i=1}^d \int_0^b u_i^2(s)\, \mathrm{d}s\right)^{1/2}$.

The *end-point map* is the differentiable (see [1, Proposition 8.5]) map

$$\mathcal{F} : \Omega \to S, \quad \sigma \mapsto \sigma(b) \in S,$$

i.e., $\mathcal{F}$ associates to each trajectory its endpoint. The set

$$\Omega(x) = \mathcal{F}^{-1}(x), \quad x \in S$$

is the set of trajectories joining x_0 to x.

Now let $\{X_1, \dots X_d\}$ be a set of globally defined smooth vector fields on S, with $d \geq \operatorname{rank} \mathcal{D}$, which generate $\mathcal{D}$ as in (3) (see [1, Corollary 3.27]) and W a smooth vector field on S. Let us consider the affine control system

$$\dot{\sigma} = -W(\sigma) + \sum_{i=1}^{d} u_i X_i(\sigma), \quad \sigma(0) = x_0 \in S. \tag{5}$$

Remark 2 Being $\mathcal{D}$ non-integrable and of rank $m-1$ by (iii), for all $x_1 \in S$ there exist controls $u_1, \dots, u_d \in L^2([0,1], \mathbb{R})$ and a solution of (5) parametrized on [0, 1] which is a curve in

$$\Omega_{x_0 x_1}(S) := \{\sigma : [0,1] \to S : \sigma \text{ is absolutely continuous,} \\ \int_0^1 g_0(\dot{\sigma}, \dot{\sigma}) \mathrm{d}s < +\infty,\ \sigma(0) = x_0, \sigma(1) = x_1\}.$$

This is a consequence of a far more general result [5, Theorem 5]; we notice that by (iii) in Theorem 1 and since the rank of $\mathcal{D}$ is $m-1$, the exponent p_c in [5, Theorem 5] is equal to $+\infty$ and then $p = 2$ is allowed in our setting (see last remark at the end of the proof of Proposition 2 in [5]).

We recall that on a Lorentzian manifold (M, g) a tangent vector $w \in TM$ is *timelike* (resp. *lightlike*; *spacelike*; *causal*) if $g(w, w) < 0$ (resp. $g(w, w) = 0$ and $w \neq 0$; $g(w, w) > 0$ or $w = 0$; w is either timelike or lightlike). It is well known that the set of causal vectors at each tangent space has a structure of double cone called *causal cones*. In the spacetime $(S \times \mathbb{R}, g)$ the function $(x, t) \in S \times \mathbb{R} \mapsto t \in \mathbb{R}$ is a temporal function, i.e., it is smooth and strictly increasing when composed with any future-pointing causal curve in $(S \times \mathbb{R}, g)$. The notion of being future-pointing for a vector or a curve is related to the opposite of the gradient of the function $(x, t) \in S \times \mathbb{R} \mapsto t \in \mathbb{R}$. In fact, it can be proved that $-\nabla t$ is timelike and then it gives a time-orientation to $(S \times \mathbb{R}, g)$ in the sense that it allows us to choose, continuously and globally, one of the two causal cones at $T_p(S \times \mathbb{R})$, $p \in S \times \mathbb{R}$. The selected ones (containing $-\nabla t$) constitute the set of future-pointing causal vectors in $T(S \times \mathbb{R})$; with our convention on the signature of the metric g, they are non-zero vectors $w \in T(S \times \mathbb{R})$, such that $g(w, w) \leq 0$ and $\mathrm{d}t(w) > 0$, so that ∂_t is future-pointing as well. Thus, a causal vector $w \in T(S \times \mathbb{R})$ is future-pointing if and only if $g(w, \partial_t) \leq 0$.

Remark 3 Let $M = S \times \mathbb{R}$ be endowed with a metric g as in (1). Taking W equal to one of the two vector fields

$$W^{\pm} := \pm \frac{\omega^{\sharp}}{|\omega^{\sharp}|_0^2},$$

then $W_x^{\pm} \notin \mathcal{D}_x$ for all $x \in S$. By Remark 2, there exists a solution $\sigma^{\pm}$ of (5) with $W = W^{\pm}$. Thus, $\omega(\dot{\sigma}^{\pm}) = -\omega\big(W^{\pm}\big) = \mp 1$. By [10, Proposition 3.12, Corollary 3.16] any trajectory $\sigma^{\pm}$ of (5) with $W = W^{\pm}$ can be lifted to a future-pointing (resp. past-pointing) lightlike curve $\gamma^{\pm}$ starting from a $p_0 = (x_0, t_0) \in S \times \mathbb{R}$ and given by

$$\gamma^{\pm}(s) = \left(\sigma^{\pm}(s), t_0 \pm \int_0^s F^{\pm}\big(\dot{\sigma}^{\pm}(r)\big) \mathrm{d}r\right),$$

where

$$F^{\pm}(v) := \frac{g_0(v, v)}{\mp\omega(v) + \sqrt{\Lambda g_0(v, v) + \omega^2(v)}}. \tag{6}$$

Notice that $F^{-}(v) = F^{+}(-v)$, hence F^{+}, F^{-} are defined on T_xS if $\Lambda(x) > 0$, while if $\Lambda(x) = 0$, F^{+} and F^{-} are respectively defined on those vectors $v \in T_xS$ such that $\omega_x(v) < 0$ and $\omega_x(v) > 0$. This implies that any point p_0 and any integral line of ∂_t can be joined by at least one causal curve.

Recall that, given $p, q \in M$, we say that p is in the causal past of q, and we write $p < q$, if there exists a future-directed causal curve from p to q. Moreover, we denote by $p \leq q$ either $p < q$ or $p = q$. For each $p \in M$, the *causal past* $J^{-}(p)$ and the *causal future* $J^{+}(p)$ are defined as

$$J^{-}(p) = \{q \in M : q \leq p\} \quad \text{and} \quad J^{+}(p) = \{q \in M : p \leq q\}.$$

Moreover, fixed $p_0 = (x_0, t_0) \in M = S \times \mathbb{R}$ and $x_1 \in S$, let us define

$$\begin{aligned} A &:= \{t \in \mathbb{R} : (x_1, t) \in J^{+}(x_0, t_0)\} \\ B &:= \{t \in \mathbb{R} : (x_1, t) \in J^{-}(x_0, t_0)\}, \end{aligned}$$

which are non-empty by Remarks 2 and 3. Let then set

$$\begin{aligned} \Delta^{+}(x_0, x_1) &:= \inf A \\ \Delta^{-}(x_0, x_1) &:= \sup B. \end{aligned}$$

Remark 4 Notice that being the line $t \in \mathbb{R} \mapsto (x_1, t) \in S \times \mathbb{R}$ causal and future-pointing, $(x_1, t) \in J^{+}(x_0, t_0)$ for all $t > \Delta^{+}(x_0, x_1)$ and $(x_1, t) \in J^{-}(x_0, t_0)$ for all $t < \Delta^{-}(x_0, x_1)$. Moreover, since a globally hyperbolic spacetime is causally simple (see, e.g., [4, Proposition 3.16]), if (M, g) is globally hyperbolic then $J^{\pm}(x_0, t_0)$ are closed and $\big(x_1, \Delta^{\pm}(x_0, x_1)\big) \in J^{\pm}(x_0, t_0)$.

The following proposition holds.

Proposition 5 *Let* $M = S \times \mathbb{R}$ *be endowed with a metric* g *as in* (1). *Assume that* (M, g) *is globally hyperbolic and (iii) in Theorem 1 holds. Then for all* $p_0 = (x_0, t_0) \in M$ *and* $x_1 \in S$, $\Delta^{+}(x_0, x_1) \in [t_0, +\infty)$ *and* $\Delta^{-}(x_0, x_1) \in (-\infty, t_0]$. *Moreover, if* $x_1 \neq x_0$ *then* $\Delta^{+}(x_0, x_1) > t_0$ *and* $\Delta^{-}(x_0, x_1) < t_0$.

Proof We notice that by the first part of Remark 4, if $x_1 = x_0$ then $\Delta^{+}(x_0, x_1) \leq t_0$ and $\Delta^{-}(x_0, x_1) \geq t_0$; since the function $t : S \times \mathbb{R} \to \mathbb{R}$ is strictly increasing (resp. decreasing) along all the future-pointing (resp. past-pointing) causal curves we then get $\Delta^{\pm}(x_0, x_1) = t_0$. By Remark 3, $\Delta^{+}(x_0, x_1) \in [t_0, +\infty)$ when $x_0 \neq x_1$. Now we notice that $\Delta^{+}(x_0, x_1)$ cannot be equal to t_0, otherwise by the second part of Remark 4 $\big(x_1, \Delta^{+}(x_0, x_1)\big) \in J^{+}(x_0, t_0)$ and there would exist a future-pointing causal curve between (x_0, t_0) and (x_1, t_0), in contradiction with the strict monotonicity of the function t along future-pointing causal curves. A similar reasoning holds also for $\Delta^{-}(x_0, x_1)$. □

3 Control Systems Adapted to Stationary Approximations

The function Λ in (1) is non-negative, thus $\Lambda(x) + \varepsilon > 0$ for all $\varepsilon > 0, x \in S$ and the corresponding metric g_ε on $S \times \mathbb{R}$

$$g_\varepsilon := g_0 + \omega \otimes \mathrm{d}t + \mathrm{d}t \otimes \omega - (\Lambda + \varepsilon)\mathrm{d}t^2 \tag{7}$$

has larger future-causal cones than g ($g \prec g_\varepsilon$, for all $\varepsilon > 0$); moreover, $g_\varepsilon \prec g_{\varepsilon'}$ for all $0 < \varepsilon < \varepsilon'$. In particular, the vector field ∂_t becomes timelike for g_ε, remaining a Killing vector field, thus $(S \times \mathbb{R}, g_\varepsilon)$ is a standard stationary spacetime for each $\varepsilon > 0$ (see Section 1).

By computing the Euler-Lagrange equation of the energy functional

$$\gamma \mapsto \frac{1}{2}\int_0^1 g_\varepsilon(\dot\gamma, \dot\gamma)\,\mathrm{d}s$$

(defined on the space of piecewise smooth curves parametrized on [0, 1] and connecting two given points $p_0, p_1 \in S \times \mathbb{R}$), it follows that a curve $\gamma_\varepsilon = (\rho_\varepsilon, t_\varepsilon)$ is a geodesic of the metric g_ε if and only if it is smooth and satisfies the following system of differential equations:

$$\begin{cases} \nabla_{\dot\rho_\varepsilon}\dot\rho_\varepsilon - \dot t_\varepsilon \Omega^\sharp(\dot\rho_\varepsilon) + \omega^\sharp(\rho_\varepsilon)\ddot t_\varepsilon + \frac{1}{2}\dot t_\varepsilon^2 \nabla\Lambda(\rho_\varepsilon) = 0 \\ \omega(\dot\rho_\varepsilon) - (\Lambda(\rho_\varepsilon) + \varepsilon)\dot t_\varepsilon = C_{\gamma_\varepsilon,\varepsilon}, \end{cases} \tag{8}$$

for some constant $C_{\gamma_\varepsilon,\varepsilon}$, where ∇ is the covariant derivative associated to the Levi-Civita connection of metric g_0 and $\Omega^\sharp$ is the (1, 1)-tensor field g_0-metrically equivalent to $\Omega := \mathrm{d}\omega$. We point out that the second equation in (8) is equivalent to the conservation law $h(\dot\gamma, K) = \text{const.}$ that any geodesic of a pseudo-Riemannian metric h endowed with a Killing vector field K must satisfy.

Remark 6 In particular, for $\varepsilon = 0$ the above equations give the geodesic ones for the metric g.

Remark 7 We emphasize that assumptions (i) and (iv) in Theorem 1 imply that the standard stationary spacetimes $(S \times \mathbb{R}, g_\varepsilon)$ are globally hyperbolic for all $\varepsilon > 0$ and the hypersurfaces $S \times \{t_1\}$ are Cauchy hypersurfaces for each t_1 in $\mathbb{R}$ (see [17, Proposition 3.1 and Corollary 3.4]). Moreover, given a family of metrics g_ε on $S \times \mathbb{R}$ as in (7), $\varepsilon \geq 0$, if one of them is globally hyperbolic with Cauchy hypersurfaces $S \times \{t_1\}$, say $g_{\bar\varepsilon}$, $\bar\varepsilon > 0$, then all of them with $\varepsilon \in [0, \bar\varepsilon)$ are globally hyperbolic with the same Cauchy hypersurfaces. This follows simply observing that a future-pointing (resp. past-pointing) causal curve in $(S \times \mathbb{R}, g_\varepsilon)$, $\varepsilon \in [0, \bar\varepsilon)$, is future-pointing (resp. past-pointing) and timelike in $(S \times \mathbb{R}, g_{\bar\varepsilon})$.

From [7, Theorem 1.1] we know that for all $\varepsilon > 0$ and for all $(x_0, t_0), (x_1, t_1) \in M$, there exists a geodesic $\gamma_\varepsilon = (\rho_\varepsilon, t_\varepsilon)$ of $(S \times \mathbb{R}, g_\varepsilon)$ connecting (x_0, t_0) to (x_1, t_1). In particular, these geodesics γ_ε have the S-components ρ_ε which are minimizers of the functional

$$\begin{aligned} &\mathcal{J}_\varepsilon : \Omega_{x_0x_1}(S) \to \mathbb{R}, \\ &\mathcal{J}_\varepsilon(\rho) = \frac{1}{2}\int_0^1 \big(g_0(\dot\rho, \dot\rho) + (\Lambda(\rho) + \varepsilon)\dot t^2\big)\mathrm{d}s + C_{\gamma,\varepsilon}(t_1 - t_0), \end{aligned} \tag{9}$$

where $\Omega_{x_0x_1}(S)$ has been introduced in Remark 2. Let us observe that curves $\gamma(s) = \big(\rho(s), t(s)\big)$ and constants $C_{\gamma,\varepsilon}$ in (8) and (9) are linked by the equation

$$C_{\gamma,\varepsilon} = g_\varepsilon(\dot\gamma, K) = \omega(\dot\rho) - (\Lambda(\rho) + \varepsilon)\dot t \tag{10}$$

so that $t \in H^1([0, 1], \mathbb{R})$ is the function such that $t(0) = t_0$, $t(1) = t_1$ and

$$\dot t = \frac{\omega(\dot\rho) - C_{\gamma,\varepsilon}}{\Lambda(\rho) + \varepsilon} \tag{11}$$

with

$$C_{\gamma,\varepsilon} = \left(\int_0^1 \frac{\omega(\dot\rho)}{\Lambda(\rho)+\varepsilon}\,\mathrm{d}s - (t_1 - t_0)\right)\left(\int_0^1 \frac{1}{\Lambda(\rho)+\varepsilon}\,\mathrm{d}s\right)^{-1} \tag{12}$$

(see [12, pp. 347–351], [7, p. 526]). From (11) and (12) we infer that actually $\mathcal{J}_\varepsilon$ is a functional depending only on the S-component of a curve. As shown in [13, Theorem 3.3], once a splitting $S\times\mathbb{R}$ of M is chosen, $\mathcal{J}_\varepsilon$ coincides with the restriction of the energy functional of the stationary metric g_ε, defined on the Sobolev manifold of the H^1-curves between (x_0, t_0) and (x_1, t_1) (parametrized on the interval [0, 1]) to its submanifold constituted by the curves γ satisfying the conservation law (10) a.e. on [0, 1].

Our aim is to prove that a subsequence γ_{ε_n}, $\varepsilon_n \to 0$, of these connecting geodesics converges in C^∞-topology to a geodesic of the metric g between (x_0, t_0) and (x_1, t_1) (see Section 4). In order to do this, here we seek for a family of curves σ_ε connecting x_0 to x_1 and having bounded g_0-energy: these curves will be used to control from above the minimum values of the functionals $\mathcal{J}_\varepsilon$. To this end, we modify the control system (5) introducing a family of drifts smoothly depending on the parameter ε. For $\varepsilon \geq 0$, let W_ε be the vector field on S defined as

$$W_\varepsilon := 2(\Lambda+\varepsilon)(t_1 - t_0)\frac{\omega^\sharp}{|\omega^\sharp|_0^2}. \tag{13}$$

We notice that if $t_1 - t_0 \neq 0$ then, for all $\varepsilon > 0$, $(W_\varepsilon)_x \notin \mathcal{D}_x$, while for $\varepsilon = 0$, $(W_0)_x \in \mathcal{D}_x$ at those x where $\Lambda(x) = 0$ and, if $t_1 = t_0$, then $W_\varepsilon \equiv 0$ for all $\varepsilon \geq 0$.

Let us then consider for $\varepsilon \geq 0$ the control systems

$$\dot\tau = W_\varepsilon(\tau) + \sum_{i=1}^d u_i X_i(\tau), \qquad \tau(0) = x_0 \in S, \tag{14}$$

with control functions $u = (u_1, \ldots, u_d) \in L^2([0, 1/2], \mathbb{R}^d)$, so that the trajectories belong to the Sobolev manifold of absolutely continuous curves $\tau : [0, 1/2] \to S$ between x_0 and $\tau(1/2)$ with $\int_0^{1/2} g_0(\dot\tau, \dot\tau)\,\mathrm{d}s < +\infty$. For any $\varepsilon \geq 0$ we denote by Ω_ε the set of trajectories of (14) endowed with the H^1-topology and by $\mathcal{F}_\varepsilon$ the associated end-point map, hence $\mathcal{F}_\varepsilon(\tau_\varepsilon) = \tau_\varepsilon(1/2)$ for all $\tau_\varepsilon \in \Omega_\varepsilon$.

For the following result we use some ideas contained in the proof of [18, Proposition 3.1].

Lemma 8 *Let $t_1 \neq t_0$ and $x_1 \in S$. Denote by $\tau_0 : [0, 1/2] \to S$ a trajectory of (14) for $\varepsilon = 0$ and some control functions $u_0 = (u_{01}, \ldots, u_{0d}) \in L^2\left([0, 1/2], \mathbb{R}^d\right)$, such that $\tau_0(1/2) = x_1$. Then, there exists $\bar\varepsilon \in (0, 1]$ such that, for each $\varepsilon \in (0, \bar\varepsilon)$, (14) with fixed control functions u_0 admits a solution τ_ε defined on $[0, 1/2]$ and such that*

$$\lim_{\varepsilon\to 0} \mathcal{F}_\varepsilon(\tau_\varepsilon) = \mathcal{F}_0(\tau_0) = x_1. \tag{15}$$

Proof Taken $x_1 \in S$, the existence of u_0 and τ_0 follows from [5, Theorem 5], see Remark 2.

From (13), (i) in Theorem 1 and the first inequality in (4), we have that the vector fields W_ε are uniformly bounded on S:

$$|W_\varepsilon|_0 \leq 2(L+1)|t_1 - t_0|\frac{1}{\nu}, \qquad \text{for all } \varepsilon \in [0, 1]. \tag{16}$$

Since the image of τ_0 is compact, we can cover it by a finite number of coordinate charts $\{(U_l, \varphi_l)\}$, $l \in \{1, \ldots, h\}$, and look at the system (14) on the open subsets $\varphi_l(U_l) \subset \mathbb{R}^m$.

Let $\{s_l\}$, $l \in \{0, \dots, h\}$, be a partition of the interval $[0, 1/2]$ such that $\tau_0([s_{l-1}, s_l]) \subset U_l$ for each $l \in \{1, \dots h\}$. Identifying then the vector fields W_ε, X_i with their images by $\mathrm{d}\varphi_l$ on $\varphi_l(U_l)$ we have that they all are bounded and Lipschitz on $\varphi_l(U_l)$. Let $\bar{L}$ be a common Lipschitz constant for the above vector fields on all the subsets $\varphi_l(U_l)$. Thus, following the proof of [16, Lemma D.3], thanks to the uniform bound (16) and the equi-Lipschitz property, we see that, for each $\varepsilon \in (0, 1]$, the trajectories τ_ε of (14) are defined on the same interval $[0, b)$, $b < 1/2$, and contained in U_1. Being such trajectories uniformly $\frac{1}{2}$-Hölder continuous on $[0, b)$ (as it can be easily seen by using the integral representation of the solutions of (14)), they can be extended at b.

We will actually prove that

$$\lim_{\varepsilon \to 0} \tau_\varepsilon(r) = \tau_0(r), \qquad \text{uniformly on } [0, b]. \tag{17}$$

Using that $\{W_\varepsilon\}_{\varepsilon \in (0,1]}$ are equi-Lipschitz,

$$\begin{aligned} &|W_\varepsilon(\tau_\varepsilon(s)) - W_0(\tau_0(s))| \\ &= |W_\varepsilon(\tau_\varepsilon(s)) - W_\varepsilon(\tau_0(s)) + W_\varepsilon(\tau_0(s)) - W_0(\tau_0(s))| \\ &\le \bar{L}|\tau_\varepsilon(s) - \tau_0(s)| + \frac{2\varepsilon|t_1 - t_0|}{\nu}. \end{aligned}$$

Then for all $r \in [0, b]$ and for $C := \frac{2|t_1 - t_0|}{\nu}$,

$$\begin{aligned} |\tau_\varepsilon(r) - \tau_0(r)| &= \\ &\left| \int_0^r [W_\varepsilon(\tau_\varepsilon) - W_0(\tau_0)]\,\mathrm{d}s + \int_0^r \sum_{i=1}^d u_{0i}[X_i(\tau_\varepsilon) - X_i(\tau_0)]\,\mathrm{d}s \right| \\ &\le \bar{L} \int_0^r \left(1 + \sum_{i=1}^d |u_{0i}|\right) |\tau_\varepsilon(s) - \tau_0(s)|\,\mathrm{d}s + C\varepsilon r. \end{aligned}$$

Hence, by the Gronwall inequality

$$|\tau_\varepsilon(r) - \tau_0(r)| \le C\varepsilon r e^{\int_0^r h(s)\,\mathrm{d}s} \le C\varepsilon b e^{\bar{L}\sqrt{b}(\sqrt{b} + \|u_0\|_2)}, \qquad \text{for all } r \in [0, b],$$

where $h(s) := \bar{L}(1 + \sum_{i=1}^d |u_{0i}(s)|)$. Since $\tau_\varepsilon(b) \to \tau_0(b)$, as $\varepsilon \to 0$, we can repeat the above reasoning, starting at $s = b$ and, in a finite number of steps, we obtain that the trajectories τ_ε uniformly converge to τ_0 on $[0, s_1]$ and do not leave U_1. Then we can repeat the same argument on the interval $[s_1, s_2]$ and so on, covering the whole interval $[0, 1/2]$. □

In order to concatenate the curve τ_ε in Lemma 8 with trajectories of (14) with $W_\varepsilon \equiv 0$ and starting from x_1, we consider the system

$$\dot{\nu} = \sum_{i=1}^d u_i X_i(\nu), \qquad \nu(0) = x_1 \in S. \tag{18}$$

For any $\varepsilon > 0$, we denote by $\nu_\varepsilon : [0, \frac{1}{2}] \to S$ a trajectory of (18) such that $\nu_\varepsilon(1/2) = x_\varepsilon := \mathcal{F}_\varepsilon(\tau_\varepsilon)$ (recall [5, Theorem 5]). Since the end-point map is continuous in the H^1-topology (see, e.g., [1, Proposition 8.5]) we get the following result:

Lemma 9 *For all $\delta > 0$ there exists $\bar{\varepsilon} > 0$ such that for all $\varepsilon \in (0, \bar{\varepsilon})$:*

$$\int_0^{1/2} |\dot{\nu}_\varepsilon|_0^2\,\mathrm{d}s < \delta. \tag{19}$$

Proof We can modify the fields X_i outside a given compact subset of S containing x_1 and the points x_ε, for ε small enough, in order to get bounded vector fields $\tilde{X}_i$. Then we consider the system

$$\dot{\nu} = \sum_{i=1}^{d} u_i \tilde{X}_i(\nu), \quad \nu(0) = x_1 \in S. \tag{20}$$

By the continuity of the end-point map, there exists a neighborhood in $L^2([0,1/2],\mathbb{R}^d)$ of the zero control and $\bar{\varepsilon} > 0$ such that, for any $\varepsilon \in (0,\bar{\varepsilon})$, a control u_ε in such a neighborhood and an associated trajectory ν_ε of (20) connecting x_1 to x_ε do exist. We then have

$$\int_0^{1/2} |\dot{\nu}_\varepsilon|_0^2 \, \mathrm{d}s = \int_0^{1/2} \left| \sum_{i=1}^{d} u_i \tilde{X}_i(\nu) \right|_0^2 \mathrm{d}s \le d\, M^2 \|u_\varepsilon\|_2^2,$$

where $M := \max_{i\in\{1,\dots,d\}} \left(\max_{x\in S} |\tilde{X}_i(x)|_0 \right)$. This implies that (19) holds and the curves ν_ε are in a small compact set containing x_1, hence they are also trajectories of (18). □

Using, for each $\varepsilon \in (0,\bar{\varepsilon})$, a curve τ_ε as in Lemma 8 and one ν_ε as in Lemma 9, we define the curves $\sigma_\varepsilon : [0,1] \to S$

$$\sigma_\varepsilon(s) = \begin{cases} \tau_\varepsilon(s) & \text{for all } s \in [0,1/2] \\ \nu_\varepsilon(1-s) & \text{for all } s \in (1/2,1]. \end{cases} \tag{21}$$

which connect x_0 to x_1.

Lemma 10 *Let* $\{\sigma_\varepsilon\}_{\varepsilon\in(0,\bar{\varepsilon})}$ *be the family of curves in* (21). *Then there exists* $\bar{C} > 0$ *such that for all* $\varepsilon \in (0,\bar{\varepsilon})$*:*

$$\int_0^1 |\dot{\sigma}_\varepsilon|_0^2 \, \mathrm{d}s \le \bar{C}.$$

Proof From (19), it is enough to prove that there exist $C > 0$ and $\bar{\epsilon} > 0$ such that

$$\int_0^1 |\dot{\tau}_\varepsilon|_0^2 \, \mathrm{d}s \le \bar{C}, \quad \text{for all } \varepsilon \in (0,\bar{\varepsilon}).$$

By Lemma 8, the trajectories τ_ε are definitively contained in a compact subset K of S thus, recalling that W_ε is g_0-orthogonal to $\mathcal{D}$, we obtain

$$\begin{aligned} \int_0^{1/2} |\dot{\tau}_\varepsilon|_0^2 \, \mathrm{d}s &= \int_0^{1/2} \left| W_\varepsilon(\tau_\varepsilon) \right|_0^2 \mathrm{d}s + \int_0^{1/2} \left| \sum_{i=1}^{d} u_{0i} X_i(\tau_\varepsilon) \right|_0^2 \mathrm{d}s \\ &\le \frac{2(L+1)^2(t_1-t_0)^2}{\nu^2} + \int_0^{1/2} \left(\sum_{i=1}^{d} |u_{0i}| \left| X_i(\tau_\varepsilon) \right|_0 \right)^2 \mathrm{d}s \\ &\le \frac{2(L+1)^2(t_1-t_0)^2}{\nu^2} + d\, M_X^2 \|u_0\|_2^2 \end{aligned} \tag{22}$$

where $M_X := \max_{i\in\{1,\dots,d\}} \left(\max_{x\in K} |X_i(x)|_0 \right)$. □

For all $\varepsilon \in (0,\bar{\varepsilon})$ we pair the family σ_ε in (21) with a function $t : [0,1] \to \mathbb{R}$ assuming values t_0 and t_1, $t_1 \neq t_0$, at the endpoints and so that (10) and (11) are satisfied for a zero constant:

$$t(s) = \begin{cases} t_0 + 2s(t_1 - t_0) & \text{if } s \in [0,1/2] \\ t_1 & \text{if } s \in (1/2,1]. \end{cases} \tag{23}$$

Let also $\bar{\sigma}$ be a trajectory of

$$\dot{\sigma} = \sum_{i=1}^{d} u_i X_i(\sigma),$$

for some control functions $\bar{u}_1, \ldots, \bar{u}_d$ parametrized on $[0, 1]$ and connecting x_0 to x_1. Finally we set $\eta_\varepsilon(s) := \big(\sigma_\varepsilon(s), t(s)\big)$, such that

$$\begin{aligned} &\sigma_\varepsilon \text{ is given in (21) and } t \text{ in (23)}, && \text{if } t_1 \neq t_0, \\ &\sigma_\varepsilon \equiv \bar{\sigma} \text{ and } t \equiv t_0, && \text{otherwise.} \end{aligned} \tag{24}$$

Proposition 11 *For each* $\varepsilon \in (0, \bar{\varepsilon})$*, let* $\eta_\varepsilon = \big(\sigma_\varepsilon, t\big)$ *be defined as in* (24)*. Then* η_ε *satisfies* (11) *with* $C_{\eta_\varepsilon,\varepsilon} = 0$.

Proof Let us consider first the case $t_1 \neq t_0$. Being τ_ε a trajectory of (14) for $s \in [0, 1/2]$, we get

$$\omega(\dot{\tau}_\varepsilon) = \omega\big(W_\varepsilon(\tau_\varepsilon)\big) = 2(\Lambda(\tau_\varepsilon) + \varepsilon)(t_1 - t_0).$$

Then $t(s)$ satisfies (11) on $[0, 1/2]$ since by (10) $C_{\eta_\varepsilon,\varepsilon} = 0$. On the other hand, both $\omega(\dot{\sigma}_\varepsilon)$ and $\dot{t}$ vanish on the interval $(1/2, 1]$ and then $C_{\eta_\varepsilon,\varepsilon}$ is 0 also there. If $t_1 = t_0$, as $\omega(\dot{\bar{\sigma}}) = 0$, both $C_{\eta_\varepsilon,\varepsilon}$ and $\dot{t}$ vanish, thus (11) holds. □

Remark 12 For each $\varepsilon \in (0, \bar{\varepsilon})$ let $\gamma_\varepsilon = (\rho_\varepsilon, t_\varepsilon)$ be a geodesic between (x_0, t_0) and (x_1, t_1), being ρ_ε a minimizer of $\mathcal{J}_\varepsilon$ in (9). Then, taking $\eta_\varepsilon = \big(\sigma_\varepsilon, t\big)$ as in Proposition 11, if $t_1 \neq t_0$, from Lemma 10 and assumption (i) in Theorem 1, we obtain

$$\begin{aligned} \mathcal{J}_\varepsilon(\rho_\varepsilon) \leq \mathcal{J}_\varepsilon(\sigma_\varepsilon) &= \frac{1}{2}\int_0^1 |\dot{\sigma}_\varepsilon|_0^2 \, ds + 2(t_1 - t_0)^2 \int_0^{1/2} (\Lambda(\sigma_\varepsilon) + \varepsilon) \, ds \\ &\leq \frac{\bar{C}}{2} + 2(t_1 - t_0)^2 (L + 1) \end{aligned}$$

for all $\varepsilon \in (0, \bar{\varepsilon})$. Recall that, in the case $t_1 = t_0$, the family of curves η_ε is a constant (w.r.t. ε) equal to $(\bar{\sigma}, t_0)$, hence $\mathcal{J}_\varepsilon(\rho_\varepsilon) \leq \frac{1}{2}\int_0^1 |\dot{\bar{\sigma}}|_0^2 \, ds$ for all $\varepsilon \in (0, \bar{\varepsilon})$.

4 Geodesic Connectedness

In this section we prove Theorem 1. Let us start by showing that thanks to Remark 12 the minimizers ρ_ε constitute a family of bounded curves in $\Omega_{x_0 x_1}(S)$.

Lemma 13 *Under assumptions (i)–(iv) in Theorem 1, for each* $\varepsilon > 0$*, let* $\gamma_\varepsilon : [0, 1] \to S \times \mathbb{R}$*,* $\gamma_\varepsilon = (\rho_\varepsilon, t_\varepsilon)$ *be a geodesic of* $(S \times \mathbb{R}, g_\varepsilon)$*, with* g_ε *as in* (7)*, between* (x_0, t_0) *and* (x_1, t_1)*, such that* ρ_ε *is a minimizer of* $\mathcal{J}_\varepsilon$ *in* (9)*. Then there exists* $\bar{\varepsilon} > 0$ *such that the family of curves* $\{\rho_\varepsilon\}_{\varepsilon \in (0,\bar{\varepsilon})} \subset \Omega_{x_0 x_1}(S)$ *is bounded in the* H^1*-topology and therefore, up to pass to a subsequence, for any sequence* $\varepsilon_n \to 0$*,* $\{\rho_{\varepsilon_n}\}$ *uniformly converges to some continuous curve* ρ *connecting* x_0 *to* x_1.

Proof From (10) and (iv) in Theorem 1 we get

$$
\begin{aligned}
|C_{\gamma_\varepsilon,\varepsilon}| &\le C\left(\int_0^1 |\dot\rho_\varepsilon|_0\,\mathrm{d}s\right)^{\alpha+1} + C\int_0^1 |\dot\rho_\varepsilon|_0\,\mathrm{d}s + \int_0^1 \big(\Lambda(\rho_\varepsilon)+\varepsilon\big)|\dot t_\varepsilon|\,\mathrm{d}s\\
&\le 2C\left(1+\left(\int_0^1 |\dot\rho_\varepsilon|_0\,\mathrm{d}s\right)^{\alpha+1}\right)\\
&\quad + \left(\int_0^1 \big(\Lambda(\rho_\varepsilon)+\varepsilon\big)\right)^{\frac12}\left(\int_0^1 \big(\Lambda(\rho_\varepsilon)+\varepsilon\big)\dot t_\varepsilon^2\,\mathrm{d}s\right)^{\frac12}\\
&\le 2C\left(1+\left(\int_0^1 |\dot\rho_\varepsilon|_0\,\mathrm{d}s\right)^{\alpha+1}\right) + \sqrt{L+1}\left(\int_0^1 \big(\Lambda(\rho_\varepsilon)+\varepsilon\big)\dot t_\varepsilon^2\,\mathrm{d}s\right)^{\frac12}
\end{aligned}
$$

and from (9) and Remark 12 we get that $\int_0^1 |\dot\rho_\varepsilon|_0\,\mathrm{d}s$ is bounded w.r.t. $\varepsilon\in(0,\bar\varepsilon)$. Taking into account that the curves ρ_ε connect the fixed points x_0 and x_1, by Ascoli-Arzelà theorem we have that any sequence $\varepsilon_n\to 0$ admits a subsequence ε_{n_k} such that $\{\rho_{\varepsilon_{n_k}}\}$ uniformly converges to a continuous curve ρ connecting x_0 to x_1. □

Remark 14 We notice that the proof of Lemma 13 implies that the family $\left\{\int_0^1\big(\Lambda(\rho_\varepsilon)+\varepsilon\big)\dot t_\varepsilon^2\,\mathrm{d}s\right\}_{\varepsilon\in(0,\bar\varepsilon)}$ is bounded as well.

Let us now rewrite the geodesic (8) for the metrics g_ε, $\varepsilon>0$, and g (recall Remark 6) as a system of second-order differential equations in normal form.

Proposition 15 *Let $\varepsilon\ge 0$; a curve $\gamma\colon[0,1]\to M$, $\gamma(s)=\big(\rho(s),t(s)\big)$, is a geodesic of the metric g_ε if $\varepsilon>0$, or of the metric g if $\varepsilon=0$, if and only if t satisfies the following equation*

$$
\begin{aligned}
\ddot t =& \frac{g_0(\dot\rho,\nabla_{\dot\rho}\,\omega^\sharp)}{\Lambda(\rho)+\varepsilon+|\omega^\sharp(\rho)|_0^2} + \frac{g_0(\Omega^\sharp(\dot\rho),\omega^\sharp)-g_0(\nabla\Lambda,\dot\rho)}{\Lambda(\rho)+\varepsilon+|\omega^\sharp(\rho)|_0^2}\,\dot t\\
&-\frac{g_0\big(\nabla\Lambda(\rho),\omega^\sharp(\rho)\big)}{\Lambda(\rho)+\varepsilon+|\omega^\sharp(\rho)|_0^2}\,\frac{\dot t^2}{2},
\end{aligned}
\tag{25}
$$

and ρ satisfies the first equation in (8) with $\ddot t$ replaced by the expression in (25).

Proof Let us assume that γ is a geodesic of g_ε or g. Taking the product of the first equation in (8) by the vector field $\omega^\sharp$ along ρ (recall also Remark 6), we obtain:

$$
\begin{aligned}
&g_0(\nabla_{\dot\rho}\dot\rho,\omega^\sharp)-\dot t g_0(\Omega^\sharp(\dot\rho),\omega^\sharp)+g_0\big(\omega^\sharp(\rho),\omega^\sharp(\rho)\big)\ddot t\\
&+\frac12 g_0\big(\nabla\Lambda(\rho),\omega^\sharp(\rho)\big)\dot t^2=0.
\end{aligned}
\tag{26}
$$

By the second equation in (8), we get

$$
\frac{\mathrm{d}}{\mathrm{d}s}\omega(\dot\rho)=g_0(\nabla\Lambda,\dot\rho)\dot t+(\Lambda(\rho)+\varepsilon)\ddot t,
$$

whenever $\varepsilon > 0$ or $\varepsilon = 0$ and $\Lambda \circ \rho$ is not a constant equal to 0, otherwise we get $\frac{\mathrm{d}}{\mathrm{ds}}\omega(\dot\rho) = 0$ on [0, 1]. Using

$$g_0(\nabla_{\dot\rho}\dot\rho, \omega^\sharp) = -g_0(\dot\rho, \nabla_{\dot\rho}\,\omega^\sharp) + \frac{\mathrm{d}}{\mathrm{ds}}\omega(\dot\rho) \tag{27}$$

and plugging in (26), we get (25) on [0, 1].
For the other implication, we notice that by assumption ρ satisfies the first equation in (8). For the second one, by (27) and (26) we get

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}s}\big(\omega(\dot\rho) - (\Lambda(\rho) + \varepsilon)\dot t\big)\\
&\quad = g_0(\nabla_{\dot\rho}\dot\rho, \omega^\sharp) + g_0(\dot\rho, \nabla_{\dot\rho}\,\omega^\sharp) - g_0(\nabla\Lambda, \dot\rho)\dot t - (\Lambda(\rho) + \varepsilon)\ddot t\\
&\quad = \dot t g_0(\Omega^\sharp(\dot\rho), \omega^\sharp) - g_0\big(\omega^\sharp(\rho), \omega^\sharp(\rho)\big)\ddot t - \frac{1}{2}g_0\big(\nabla\Lambda(\rho), \omega^\sharp(\rho)\big)\dot t^2\\
&\qquad + g_0(\dot\rho, \nabla_{\dot\rho}\,\omega^\sharp) - g_0(\nabla\Lambda, \dot\rho)\dot t - (\Lambda(\rho) + \varepsilon)\ddot t,
\end{aligned}$$

and then using (25) for replacing $\ddot t$, we get

$$\frac{\mathrm{d}}{\mathrm{d}s}\big(\omega(\dot\rho) - (\Lambda(\rho) + \varepsilon)\dot t\big) = 0,$$

i.e., the second equation in (8) is satisfied too. □

Remark 16 Notice that (25) is invariant by affine reparametrization (and then the first equation in (8) also remains invariant by affine reparametrization when $\ddot t$ is replaced with its value in (25)).

Remark 17 Equation (25) and the first equation in (8) with $\ddot t$ replaced by (25) make clear that the geodesic equations of the metrics g_ε and g smoothly depend on ε on the interval $[0, +\infty)$.

We are now ready to prove Theorem 1.

Proof of Theorem 1 We at first notice that by condition (ii) (25) reduces to

$$\ddot t = \frac{g_0(\dot\rho, \nabla_{\dot\rho}\,\omega^\sharp)}{\Lambda(\rho) + \varepsilon + |\omega^\sharp(\rho)|_0^2} + \frac{g_0(\Omega^\sharp(\dot\rho), \omega^\sharp) - g_0(\nabla\Lambda, \dot\rho)}{\Lambda(\rho) + \varepsilon + |\omega^\sharp(\rho)|_0^2}\,\dot t. \tag{28}$$

Let now $\gamma_\varepsilon = (\rho_\varepsilon, t_\varepsilon)$, ε_n, ρ_{ε_n} and ρ be as in Lemma 13, and let $\Delta_t := t_1 - t_0$. For all $n \in \mathbb{N}$, let also $s^{(1)}_{\varepsilon_n}$ be the smallest instant in [0, 1) such that

$$\Delta_t = \int_0^1 \dot t_{\varepsilon_n}\,\mathrm{d}s = \dot t_{\varepsilon_n}(s^{(1)}_{\varepsilon_n}).$$

For $s \ge s^{(1)}_{\varepsilon_n}$, integrating $\ddot t_{\varepsilon_n}$ on $[s^{(1)}_{\varepsilon_n}, s]$ and recalling that by the first inequality in (4), $\Lambda(\rho_{\varepsilon_n}) + \varepsilon_n + |\omega^\sharp_{\rho_{\varepsilon_n}}|_0^2 \ge \nu^2$, we get from (28):

$$\begin{aligned}
|\dot t_{\varepsilon_n}(s)| \le |\Delta_t| + \frac{1}{\nu^2}\int_{s^{(1)}_{\varepsilon_n}}^{s} \big|g_0(\dot\rho_{\varepsilon_n}, \omega^\sharp)\big|\,\mathrm{d}r\\
+\frac{1}{\nu^2}\int_{s^{(1)}_{\varepsilon_n}}^{s} \big|g_0(\Omega^\sharp(\dot\rho_{\varepsilon_n}), \omega^\sharp) - g_0(\nabla\Lambda(\rho_{\varepsilon_n}), \dot\rho_{\varepsilon_n})\big||\dot t_{\varepsilon_n}|\,\mathrm{d}r.
\end{aligned} \tag{29}$$

By the smoothness of $\omega^\sharp$, $\Omega^\sharp$, Λ and the fact that the curves ρ_{ε_n} are contained in a compact subset $\mathcal{K}$ of S, there exists a non-negative constant C_1, depending on $\mathcal{K}$ but independent of n, such that

$$\begin{aligned} &\left|g_0(\dot\rho_{\varepsilon_n}, \nabla_{\dot\rho_{\varepsilon_n}}\omega^\sharp)\right| \le C_1|\dot\rho_{\varepsilon_n}|_0^2, \qquad \text{for all } s\in[0,1].\\ &\left|g_0(\Omega^\sharp(\dot\rho_{\varepsilon_n}),\omega^\sharp) - g_0(\nabla\Lambda(\rho_{\varepsilon_n}),\dot\rho_{\varepsilon_n})\right| \le C_1|\dot\rho_{\varepsilon_n}|_0, \end{aligned} \tag{30}$$

From (29) and (30) we obtain

$$|\dot t_{\varepsilon_n}(s)| \le |\Delta_t| + \frac{C_1}{\nu^2}\int_{s_{\varepsilon_n}^{(1)}}^s |\dot\rho_{\varepsilon_n}|_0^2\,\mathrm{d}r + \frac{C_1}{\nu^2}\int_{s_{\varepsilon_n}^{(1)}}^s |\dot\rho_{\varepsilon_n}|_0|\dot t_{\varepsilon_n}|\,\mathrm{d}r,$$

for all $s\in[s_{\varepsilon_n^{(1)}},1]$ and $n\in\mathbb{N}$. By the Gronwall inequality we then get

$$|\dot t_{\varepsilon_n}(s)| \le \left(|\Delta_t| + \frac{C_1}{\nu^2}\int_{s_{\varepsilon_n^{(1)}}}^s |\dot\rho_{\varepsilon_n}|_0^2\,\mathrm{d}r\right)\exp\left(\frac{C_1}{\nu^2}\int_{s_{\varepsilon_n^{(1)}}}^s |\dot\rho_{\varepsilon_n}|_0\,\mathrm{d}r\right), \tag{31}$$

for all $s\in[s_{\varepsilon_n}^{(1)},1]$ and $n\in\mathbb{N}$. From Lemma 13, $\{\int_0^1|\dot\rho_{\varepsilon_n}|_0^2\,\mathrm{d}s\}$ is bounded, thus there exists a non-negative constant $C_2\ge 0$ such that

$$|\dot t_{\varepsilon_n}(s)| \le (|\Delta_t| + C_2)e^{C_2}, \quad \text{for all } s\in[s_{\varepsilon_n}^{(1)},1] \text{ and } n\in\mathbb{N}.$$

For each $n\in\mathbb{N}$, let now $s_{\varepsilon_n}^{(2)}$ be the smallest instant in $\big[0, s_{\varepsilon_n}^{(1)}\big)$ such that

$$\begin{aligned} &|\dot t_{\varepsilon_n}(s_{\varepsilon_n}^{(2)})| = |\Delta_t| && \text{if } |t_{\varepsilon_n}(s_{\varepsilon_n}^{(1)}) - t_0| > |\Delta_t|,\\ &|\dot t_{\varepsilon_n}(s_{\varepsilon_n}^{(2)})| = \left|\int_0^{s_{\varepsilon_n}^{12)}} \dot t_{\varepsilon_n}\,\mathrm{d}r\right| && \text{otherwise.} \end{aligned}$$

Repeating the reasoning as in the step above, we get (31) for all $s\in[s_{\varepsilon_n}^{(2)}, s_{\varepsilon_n}^{(1)}]$ and $n\in\mathbb{N}$ and then

$$|\dot t_{\varepsilon_n}(s)| \le (|\Delta_t| + C_2)e^{C_2}, \qquad \text{for all } s\in[s_{\varepsilon_n}^{(2)}, s_{\varepsilon_n}^{(1)}] \text{ and } n\in\mathbb{N}. \tag{32}$$

In this way we construct, for each $n\in\mathbb{N}$, a sequence $\{s_{\varepsilon_n}^{(k)}\}_{k\ge1}\subset[0,1)$ such that $s_{\varepsilon_n}^{(k)}\to 0$ as $k\to+\infty$ and such that (32) holds on each interval $[s_{\varepsilon_n}^{(k+1)}, s_{\varepsilon_n}^{(k)}]$ and each $n\in\mathbb{N}$. As the functions $\dot t_{\varepsilon_n}$ are continuous at 0, (32) is valid for $\dot t_{\varepsilon_n}(0)$ and all $n\in\mathbb{N}$, as well. Summing up, we have obtained

$$|\dot t_{\varepsilon_n}(s)| \le (|\Delta_t| + C_2)e^{C_2}, \qquad \text{for all } s\in[0,1] \text{ and } n\in\mathbb{N}. \tag{33}$$

Being $t_{\varepsilon_n}(0) = t_0$ for all n, we infer from (33) that $\{t_{\varepsilon_n}\}$ is bounded in $H^1([0,1],\mathbb{R})$ and then, up to a subsequence, $\{t_{\varepsilon_n}\}$ uniformly converges on $[0,1]$ to a function $\tilde t\in H^1([0,1],\mathbb{R})$ such that $\tilde t(0)=t_0$, $\tilde t(1)=t_1$.

Let us finally show that the curve $\gamma(s) := (\rho(s),\tilde t(s))$, $s\in[0,1]$, which connects (x_0,t_0) to (x_1,t_1), is a geodesic of the metric g. Recalling that the values of $\mathcal{J}_{\varepsilon_n}$ on its critical points ρ_{ε_n} coincide with $2E_{\gamma_{\varepsilon_n}}$, where $E_{\gamma_{\varepsilon_n}}$ is the constant of motion equal to $g_{\varepsilon_n}(\dot\gamma_{\varepsilon_n},\dot\gamma_{\varepsilon_n})$, we get from (7):

$$|\dot\rho_{\varepsilon_n}|_0^2 = 2\mathcal{J}_{\varepsilon_n}(\rho_{\varepsilon_n}) - 2\omega(\dot\rho_{\varepsilon_n})\dot t_{\varepsilon_n} + (\Lambda(\rho_{\varepsilon_n}) + \varepsilon_n)\dot t_{\varepsilon_n}^2. \tag{34}$$

Recalling that the curves ρ_{ε_n} are contained in the compact set $\mathcal{K}$, by Remark 12, (i) in Theorem 1, (33) and (34), we get

$$|\dot\rho_{\varepsilon_n}|_0^2 \le C_3(1 + |\dot\rho_{\varepsilon_n}|_0),$$

for some constant $C_3 > 0$. Hence, $\{|\dot{\rho}_{\varepsilon_n}(s)|_0\}$ is bounded on $[0, 1]$. Thus, recalling (33), up to pass to a subsequence, the initial vectors $\{(\dot{\rho}_{\varepsilon_n}(0), \dot{t}_{\varepsilon_n}(0))\}$ converge to a vector $v_0 \in T_{(x_0,t_0)}(S \times \mathbb{R})$. By Proposition 15 and Remark 17 we conclude, by the smooth dependence of solutions of (25) and the first equation in (8) (with $\ddot{t}_\varepsilon$ replaced by the expression in (25)) from initial conditions and the parameter ε, that the sequence $\{(\rho_{\varepsilon_n}, t_{\varepsilon_n})\}$ converges in the C^∞-topology to a geodesic $\bar{\gamma} : [0, a) \to M$ of the metric g. Since $\{(\rho_{\varepsilon_n}, t_{\varepsilon_n})\}$ uniformly converges to γ on $[0, 1]$, by the C^1-bounds on ρ_{ε_n} and t_{ε_n} obtained above, we conclude that $a > 1$ and $\bar{\gamma} = \gamma$ on $[0, 1]$. □

As a final remark we notice that from Remark 4, if

$$\Delta^-(x_0, x_1) < t_1 < \Delta^+(x_0, x_1),$$

then the geodesic γ between (x_0, t_0) and (x_1, t_1) is necessarily spacelike.

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Declarations

Competing interests The authors declare no competing interests.

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