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Some observations regarding the stationary Buckley–Leverett equation

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ABSTRACT

The basic hyperbolic–elliptic black-oil model describes oil–water displacement in a porous medium. Given its mathematical complexity, there is a need for particular simple solutions for validation of numerical methods. We present a class of stationary solutions, which are easy to compute, and in many cases are given by explicit formulae. These solutions are constructed by a nonlinear coupling of two linear equations, an elliptic pressure equation and a hyperbolic saturation equation.

1. Introduction

Flow through porous media, particularly when it involves multiple phases, is commonly modeled using a complex nonlinear system of partial differential equations (PDEs) [1–4]. These models are fundamental in various fields, especially in petroleum engineering for reservoir simulation and in hydrology for groundwater flow. The most basic models describing multi-phase flow in porous media are derived from mass conservation assuming Darcy's law for fluid flow.

The resulting model, henceforth referred to as the Buckley–Leverett system, is often used to model the displacement of oil by water (or vice versa) in a porous medium. This system consists of a fractional flow equation, describing the evolution of the saturation (volume fraction of water) s, coupled with a linear equation for the pressure p. To be more precise, let $\Omega \subset \mathbb{R}^d$ ($d = 2$ or $d = 3$) be a Lipschitz domain with boundary $\partial\Omega$ with unit normal ν. The unknowns s and p satisfy

$$\begin{cases} \partial_t s + \operatorname{div}(f(s)\mathbf{v}) = 0, & t > 0,\ x \in \Omega, \\ \operatorname{div}(\mathbf{v}) = 0, \quad \mathbf{v} = -\lambda(s)\nabla p, & t > 0,\ x \in \Omega, \\ s = s_b, \quad \partial_\nu p = p_b, & t > 0,\ x \in \partial\Omega, \\ s(0,x) = s_0(x), & x \in \Omega, \\ \int_\Omega p(t,x)\,dx = 0, & t > 0. \end{cases} \tag{1.1}$$

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Note that the evolution of $s = s(t,x)$ and $p = p(t,x)$ only depends on ∇p. The last condition $\int_\Omega p\,dx = 0$ is only needed to ensure solvability of the pressure equation $\operatorname{div}(\mathbf{v}) = 0$. Thus there is no need to worry about whether p is positive. The saturation s satisfies the natural bound $s \in [0,1]$, and the nonlinear functions λ and f satisfy $0 < \lambda_* \le \lambda(s) \le \lambda^*$, and f is increasing with $0 = f(0) \le f(s) \le f(1) = 1$ for $s \in [0,1]$.

Solutions to nonlinear conservation laws may evolve discontinuities, even for smooth data. This makes it necessary to interpret (1.1) in the weak sense, augmented with extra conditions, in the hope that these extra conditions make the system well-posed. The challenge of establishing the well-posedness of suitably defined entropy solutions for the Buckley–Leverett system (1.1) is a long-standing open problem. Partial results in this direction, contingent on strong assumptions regarding the velocity field $\mathbf{v}$ or the solution (s,p), have been established in [5–8]. The scenario improves notably when capillary effects are included in the model, see, e.g., [9–11]. However, our approach does not include terms modeling capillary pressure.

Analysis of the Buckley–Leverett system (1.1) is hampered by numerous challenges. In particular, s may be discontinuous, which in turn implies that ∇p has discontinuities, and consequently $\mathbf{v}$ becomes too irregular for the saturation equation to make sense. Consequently, numerical simulations play an indispensable role in understanding the behavior of multiphase flow in porous media.

Thus there is an interest in simplified and solvable models. Such models are useful in the development, validation, and benchmarking of numerical methods. In this note we consider such a simple model in the stationary case, when $s = s(x)$ and $p = p(x)$. Under this assumption the Buckley–Leverett system reads

$$\begin{cases} \operatorname{div}(f(s)\lambda(s)\nabla p) = 0, & x \in \Omega, \\ \operatorname{div}(\lambda(s)\nabla p) = 0, & x \in \Omega, \\ s = s_b, \quad \partial_\nu p = p_b, & x \in \partial\Omega, \\ \int_\Omega p(x)\,dx = 0. \end{cases} \tag{1.2}$$

Solutions of this time-independent system can be used to check numerical methods for (1.1). If the explicit solutions presented here are stable, one would like to use so called well-balanced schemes which preserve, or almost preserve, stable time-independent solutions. Currently, this stability remains an unsolved problem.

Stationary solutions can also be used as a building block in the formulation of an appropriate solution concept. For an example of how this has been used for conservation laws with heterogeneous coefficients, see [12] (and the references therein).

Let us mention that the distinction between our problem and the one addressed in [7] lies in the boundary conditions for p: we consider Neumann boundary conditions, whereas the authors in [7] deal with Dirichlet boundary conditions. While the arguments in [7] are highly technical and rely on sophisticated and delicate estimates, our approach is straightforward and uses only basic identities.

Our primary insight of this note is that the solvability of the nonlinear stationary system (1.2) can be reduced to solving two linear PDEs. This simplification makes the stationary Buckley–Leverett system (1.2) almost analytically solvable, contrasting with the complexities of the time-dependent system (1.1). Our (stationary) solutions to (1.2) are obtained as follows. Consider a function $p = p(x)$, in the Sobolev space $H^2(\Omega)$, that satisfies the linear boundary value problem:

$$\begin{cases} \Delta p = 0, & \text{in } \Omega, \\ \partial_\nu p = p_b, & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

under the conditions $\int_\Omega p(x)\,dx = 0$ and $\int_{\partial\Omega} p_b(x)\,d\sigma = 0$, where Δ denotes the Laplacian and ∂_ν the derivative of p in the direction of the normal ν. Observe the this boundary value problem has a unique solution. Next, let $s \in L^\infty(\Omega)$ be a solution to the linear boundary value problem

$$\operatorname{div}(s\nabla p) = 0 \text{ in } \Omega, \quad s = s_b \text{ on } \partial\Omega \cap \{p_b < 0\}, \tag{1.4}$$

where div and ∇ denote the divergence and gradient operators respectively, and s_b is the specified boundary value for s. Elementary manipulations shows that s solves the linear transport equation

$$\mathbf{v}\cdot\nabla s = 0 \text{ in } \Omega, \quad s = s_b \text{ on } \partial\Omega \cap \{p_b < 0\}, \tag{1.5}$$

with $\mathbf{v} = \nabla p \in H^1(\Omega;\mathbb{R}^d)$. Solutions to the boundary problems (1.3) and (1.4) or (1.5) are easily found. The pressure p can be any function which is harmonic in Ω (this fixes p_b) and s can be any function which is constant along the characteristics $\mathbf{c}_y$ defined by

$$\mathbf{c}_y'(\xi) = \mathbf{v}(\mathbf{c}_y(\xi)), \ \mathbf{c}_y(0) = y.$$

Formally, if p and s satisfy (1.3) and (1.4), we derive the following:

$$\begin{aligned} \operatorname{div}(f(s)\lambda(s)\nabla p) &= (f\lambda)'(s)\nabla s\cdot\nabla p + (f\lambda)(s)\Delta p \\ &= (f\lambda)'(s)\left[\operatorname{div}(s\nabla p) - s\,\Delta p\right] + (f\lambda)(s)\,\Delta p \\ &= (f\lambda)'(s)\underbrace{\operatorname{div}(s\nabla p)}_{=0} + \left[(f\lambda)(s) - s\,(f\lambda)'(s)\right]\underbrace{\Delta p}_{=0} = 0 \end{aligned}$$

and

$$\operatorname{div}(\lambda(s)\nabla p) = \lambda'(s)\nabla s\cdot\nabla p + \lambda(s)\Delta p$$

$$= \lambda'(s)\underbrace{\operatorname{div}(s\nabla p)}_{=0} + \big[\lambda(s) - s\lambda'(s)\big]\underbrace{\Delta p}_{=0} = 0.$$

Thus, the pair (s, p) constitutes a solution to the time-independent Buckley–Leverett system (1.2).

In the remainder of this note we formalize and outline a proof of these observations.

2. Statement of the result and a sketch of its proof

We denote by $C^1_b(\mathbb{R})$ the collection of all bounded C^1 functions with bounded derivatives, and assume that the data of the Buckley–Leverett system (1.2) satisfy:

(H.1) Ω is a smooth bounded open domain in $\mathbb{R}^d$.
(H.2) $f \in C^1([0,1])$, $f(0) = 0$, $f'(s) \geq 0$.
(H.3) $\lambda \in C^2([0,1])$ and $\lambda(\cdot) \geq \lambda_* > 0$.
(H.4) $p_b \in H^{k+1/2}(\partial\Omega) \cap L^\infty(\partial\Omega)$ for some $k > \max\{0, \frac{d}{2} - 2\}$, $\int_{\partial\Omega} p_b \, d\sigma = 0$, $s_b \in L^\infty(\partial\Omega)$.

Definition 2.1. A pair (s, p) is called a renormalized solution of (1.2) if

(A.1) $s \in L^\infty(\Omega)$ admits a trace γs on $\{p_b > 0\}$, $p \in H^2(\Omega)$ and $\int_\Omega p(x)\,dx = 0$;
(A.2) for every test function $\varphi \in C^\infty(\mathbb{R}^N)$ and every $\beta \in C^1(\mathbb{R})$,

$$\int_\Omega b(s)\lambda(s)\nabla p \cdot \nabla\varphi \, dx - \int_{\{p_b<0\}} b(s_b)\lambda(s_b)p_b\varphi \, d\sigma - \int_{\{p_b>0\}} b(\gamma s)\lambda(\gamma s)p_b\varphi \, d\sigma = 0, \tag{2.1}$$

$$\int_\Omega \lambda(s)\nabla p \cdot \nabla\varphi \, dx - \int_{\{p_b<0\}} \lambda(s_b)p_b\varphi \, d\sigma - \int_{\{p_b>0\}} \lambda(\gamma s)p_b\varphi \, d\sigma = 0, \tag{2.2}$$

where $b' = \beta' f'$.

A pair (s, p) qualifies as a weak solution to (1.2) if it meets the conditions specified in **(A.1)** and **(A.2)**, where $\beta(u) = u$, and thereby $b(u) = f(u)$.

The main result of this note is the following theorem:

Theorem 2.2. *Assume that* **(H.1)**, **(H.2)**, **(H.3)** *and* **(H.4)** *hold. Then the boundary value problem* (1.2) *has a weak solution* (s, p) *in the sense of Definition* 2.1. *Moreover, this solution* (s, p) *is found by solving the Laplace boundary value problem* (1.3) *and the linear hyperbolic boundary value problem* (1.4).

We note that the uniqueness of (weak or renormalized) solutions to the stationary Buckley–Leverett system (1.2) remains an open problem. In our analysis, we employ renormalized solutions as a technical tool to ensure the uniqueness of weak solutions and the existence of strong traces for the weak solutions of (1.5), for a given, possibly irregular velocity field $\mathbf{v}$.

2.1. A sketch of the proof

The existence and uniqueness of traces for weak and renormalized solutions to time-dependent transport equations were proved in [13]. The subsequent theorem is a time-independent version of [13, Thm. 3.1].

Theorem 2.3. *Fix* $q \in [1, \infty]$ *and set* $q' = \frac{q}{q-1}$. *Suppose* $f \in L^q(\Omega)$ *and that the vector field* $\mathbf{v}$ *satisfies*

$$\begin{cases} \mathbf{v} \in W^{1,q'}(\Omega, \mathbb{R}^d), \quad \operatorname{div}(\mathbf{v}) = 0, \\ (\mathbf{v}\cdot\nu) \in L^\alpha(\partial\Omega), \qquad \text{for a given } \alpha > 1. \end{cases}$$

For any such $\mathbf{v}$, *introduce the following measures on* $\partial\Omega$:

$$d\mu_{\mathbf{v}} = (\mathbf{v}\cdot\nu)\,d\sigma, \quad d\mu^\alpha_{\mathbf{v}} = |\mathbf{v}\cdot\nu|^\alpha \, d\sigma.$$

Moreover, let $d\mu^+_{\mathbf{v}} := (\mathbf{v}\cdot\nu)^+ \, d\sigma$ *and* $d\mu^-_{\mathbf{v}} := (\mathbf{v}\cdot\nu)^- \, d\sigma$ *be the positive and negative parts of* $d\mu_{\mathbf{v}}$, *so that*

$$d\mu_{\mathbf{v}} = d\mu^+_{\mathbf{v}} - d\mu^-_{\mathbf{v}}, \quad d\,|\mu_{\mathbf{v}}| = d\mu^+_{\mathbf{v}} + d\mu^-_{\mathbf{v}}.$$

Let $\varrho \in L^q(\Omega)$ *be a distributional (weak) solution to the stationary transport equation*

$$\mathbf{v}\cdot\nabla\varrho = f \quad \text{in } \Omega. \tag{2.3}$$

There exists a function (trace) $\gamma\varrho$ *that is measurable on* $\partial\Omega$ *such that for any* $\beta \in C^1_b(\mathbb{R})$, *for any test function* $\varphi \in C^1(\overline{\Omega})$, *we have*

$$\int_\Omega \beta(\varrho)\mathbf{v}\cdot\nabla\varphi \, dx - \int_{\partial\Omega} \beta(\gamma\varrho)\,\varphi\big(\mathbf{v}\cdot\nu\big)^+ \, d\sigma + \int_\Omega \beta'(p) f\varphi \, d\mathbf{x} = 0. \tag{2.4}$$

The trace function $\gamma\varrho$ is uniquely determined a.e. with respect to $d\left|\mu_{\mathbf{v}}\right|$ on Ω.

If $q=\infty$, the function $\gamma\varrho$ belongs to $L^\infty(\partial\Omega, d\left|\mu_{\mathbf{v}}\right|)$, and the renormalization formula (2.4) holds for any $\beta\in C^1(\mathbb{R})$ without the requirement of boundedness. Specifically, by selecting $\beta(\xi)=\xi$, it follows from (2.4) that ϱ satisfies the weak formulation of the transport Eq. (2.3) taking into account the trace $\gamma\varrho$ of the solution ϱ on $\partial\Omega$.

As previously mentioned, we construct a solution of (1.2) by splitting it into two equations, the first of which is Laplace's equation in Ω with Neumann boundary conditions:

$$\begin{cases} \Delta p=0, & x\in\Omega,\\ \partial_\nu p=p_b, & x\in\partial\Omega,\\ \int_\Omega p(x)\,dx=0. \end{cases} \tag{2.5}$$

Note that by [14, Prop. 7.7], the unique solution p of (2.5) is in $H^{k+2}(\Omega)\cap L^\infty(\Omega)$, since $k+2>d/2$. Since p is continuous, all points are Lebesgue points, and by the unique continuation principle [15, Thm. 1.2], p is not constant on any set with positive measure in Ω, and consequently ∇p is not zero on any such set. The remaining part of (1.2) is the hyperbolic transport problem:

$$\begin{cases} \operatorname{div}(s\nabla p)=0, & x\in\Omega,\\ s=s_b, & x\in\partial\Omega\cap\left\{p_b<0\right\}, \end{cases} \tag{2.6}$$

In the context of a Sobolev regular velocity field $\mathbf{v}=\nabla p\in H^1$ and irregular solutions $s\in L^\infty$, we use the following definition of a weak solution of (2.6):

Definition 2.4. The function s is a weak solution of (2.6) if

(C.1) $s\in L^\infty(\Omega)$ and s admits a trace $\gamma s\in L^\infty(\partial\Omega)$ on $\{p_b>0\}$;
(C.2) for every test function $\varphi\in C^\infty(\mathbb{R}^d)$,

$$\int_\Omega s\nabla p\cdot\nabla\varphi\,dx-\int_{\{p_b<0\}}s_bp_b\varphi\,d\sigma-\int_{\{p_b>0\}}(\gamma s)p_b\varphi\,d\sigma=0.$$

The existence of the trace appearing in 2.4 is provided by Theorem 2.3. Observe that Definition 2.4 says that s satisfies (2.6) in the distributional sense. Consequently, Theorem 2.3 implies that s is a renormalized solution defined as follows:

Definition 2.5. A function s is a renormalized solution of (2.6) if

(D.1) $s\in L^\infty(\Omega)$ and s admits a trace $\gamma s\in L^\infty(\partial\Omega)$ on $\{p_b>0\}$;
(D.2) for every $\varphi\in C^\infty(\mathbb{R}^d)$ and every $\beta\in C^1(\mathbb{R})$,

$$\int_\Omega \beta(s)\nabla p\cdot\nabla\varphi\,dx-\int_{\{p_b<0\}}\beta(s_b)p_b\varphi\,d\sigma-\int_{\{p_b>0\}}\beta(\gamma s)p_b\varphi\,d\sigma=0. \tag{2.7}$$

We shall need the following stability estimate.

Lemma 2.6. *Let s_1 and s_2 be two renormalized solutions of the boundary value problem (2.6) in the sense of Definition 2.4, corresponding to the boundary data $s_{b,1}$ and $s_{b,2}$. Then we have that*

$$\left\|(s_1-s_2)\nabla p\right\|_{L^2(\Omega)}\le\sqrt{3\left\|p_b\right\|_{L^\infty(\partial\Omega)}\|p\|_{L^\infty(\Omega)}}\left\|s_{b,1}-s_{b,2}\right\|_{L^2(\{p_b<0\})}. \tag{2.8}$$

In particular, there exists at most one renormalized solution of (2.6).

Proof. Observe that s_1-s_2 is a weak solution of

$$\begin{cases} \operatorname{div}\left((s_1-s_2)\nabla p\right)=0, & x\in\Omega,\\ s_1-s_2=s_{b,1}-s_{b,2}, & x\in\{p_b<0\}, \end{cases}$$

and due to (2.7), for every $\varphi\in C^\infty(\mathbb{R}^N)$ and $\beta\in C^1(\mathbb{R})$,

$$\int_\Omega\beta(s_1-s_2)\nabla p\cdot\nabla\varphi\,dx-\int_{\{p_b<0\}}\beta(s_{b,1}-s_{b,2})p_b\varphi\,d\sigma-\int_{\{p_b>0\}}\beta(\gamma s_1-\gamma s_2)p_b\varphi\,d\sigma=0.$$

Using

$$\beta(\xi)=\xi^2,\qquad\varphi=p-2\|p\|_{L^\infty(\Omega)}<0,$$

we have

$$\int_\Omega(s_1-s_2)^2|\nabla p|^2\,dx$$

$$= \int_{\{p_b<0\}} (s_{b,1}-s_{b,2})^2 p_b\big(p-2\|p\|_{L^\infty(\Omega)}\big)\,d\sigma + \underbrace{\int_{\{p_b>0\}} (\gamma s_1-\gamma s_2)^2 p_b\big(p-2\|p\|_{L^\infty(\Omega)}\big)\,d\sigma}_{\leq 0}$$

$$\leq 3\|p_b\|_{L^\infty(\partial\Omega)}\|p\|_{L^\infty(\Omega)}\|s_{b,1}-s_{b,2}\|^2_{L^2(\{p_b<0\})},$$

which proves (2.8). □

The next theorem establishes the existence and uniqueness of a renormalized solution for (2.6).

Theorem 2.7 (*Well-Posedness of* (2.6))**.** *The linear boundary value problem* (2.6) *has a unique renormalized solution in the sense of Definition* 2.5 *such that*

$$\|s\|_{L^\infty(\Omega)} \leq \|s_b\|_{L^\infty(\partial\Omega)}.$$

Proof. By the regularity of the coefficient ∇p, the proof follows using classical arguments based on characteristics [16, Thm. 1] when s_b is smooth. For non-smooth s_b as required in **(H.4)**, we construct a sequence $\{s_{b,n}\}$ of smooth functions converging to s_b in L^2. For each $s_{b,n}$, a smooth solution s_n exists. Using the stability estimate with respect to the boundary data (Lemma 2.6) and that $\nabla p \neq 0$ a.e., the sequence $\{s_n\}$ is shown to be Cauchy in L^2, and its limit is a weak solution. By Theorem 2.3, this weak solution is also a renormalized solution. □

Proof of Theorem 2.2. We have established that p is sufficiently regular to be used as a first component in a weak solution. Choosing $\beta(s) = f(s)\lambda(s)$ in (2.7) gives (2.1) with $b = f$ and choosing $\beta(s) = \lambda(s)$ gives (2.2). In other words, (s, p) is a weak solution to (1.2). □

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Data availability

No data was used for the research described in the article.

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