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Weighted Subspace Designs from q-Polymatroids

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A B S T R A C T

The Assmus-Mattson Theorem gives a way to identify block designs arising from codes. This result was broadened to matroids and weighted designs by Britz et al. in 2009. In this work we present a further two-fold generalisation: first from matroids to polymatroids and also from sets to vector spaces. To achieve this, we study the characteristic polynomial of a q-polymatroid and outline several of its properties. We also derive a MacWilliams duality result and apply this to establish criteria on the weight enumerator of a q-polymatroid for which dependent spaces of the q-polymatroid form the blocks of a weighted subspace design.

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1. Introduction

The characteristic polynomial of a matroid is a well studied object. It was first introduced as a matroid generalisation of the chromatic polynomial of a graph. It arises in critical problems, analyses of the Tutte polynomial, and is the subject of numerous identities [21]. See also [31–33], for further reading.

In combinatorics, the concept of a q-analogue can be viewed as a generalisation from sets to vector spaces. Recently, the q-analogue of a matroid has been studied [19]. A generalisation of this is a q-polymatroid [17,18,28].

There are many interesting connections between q-(poly)matroids and rank-metric codes. In this paper we develop the theory of the characteristic polynomial of a q-polymatroid. We show a relation between the characteristic polynomial of a q-polymatroid and that of its dual, establishing a MacWilliams-like identity for q-polymatroids. In a similar line of research, Shiromoto [28] established a q-analogue of Greene's theorem.

Another motivation to study the characteristic polynomial is to establish a q-analogue of the Assmus-Mattson Theorem [1]. This theorem gives a criterion for identifying a t-design as a collection of supports of codewords of fixed weight in a linear code. Since its publication in 1969, it has seen a number of generalisations [10,23] and has been used widely to obtain new constructions of designs [16,27]. In one of these results [5], the authors define a weighted t-design as a generalisation of a classical t-design and give criteria for identifying such an object among the dependent sets of a matroid of a fixed cardinality. A weighted t-design is a collection of subsets $\mathcal{B}$ of a fixed cardinality k chosen from an n-set of points $\mathcal{P}$ together with a function f defined on $\mathcal{B}$ such that for any t-set $T \subset \mathcal{P}$ the sum $\sum_{B \in \mathcal{B}\,:\,T \subset B} f(B)$ is independent of T. In the case that $f(B) = 1$ for every block $B \in \mathcal{B}$, the weighted t-design is an ordinary design.

In this paper, we generalise the results of [5] to q-polymatroids, which is a two-fold generalisation: first from matroids to polymatroids and also from sets to vector spaces. Hence the results presented here give a q-analogue of their result. The q-analogue of a weighted t-design is a weighted subspace design; in the definition shown above we replace the collection of subsets $\mathcal{B}$ with a collection of subspaces of a fixed dimension k and T with a t-dimensional subspace.

In Section 2 we study q-polymatroids and the properties of q-polymatroids that are necessary for this work. In Section 3 we outline properties of the characteristic polynomial of a q-polymatroid that will be used later and in Section 4 we look at q-polymatroids arising from matrix codes. In Section 5 we give a version of the MacWilliams duality result for q-polymatroids. In Section 6 we give criteria for identifying when the dependent spaces of a q-polymatroid are the blocks of a weighted t-subspace design.

Notation 1. Throughout, we let n denote a fixed positive integer and we will let q denote a fixed prime power. We let E denote an n-dimensional vector space over the finite field $\mathbb{F}_q$ of order q. We let $\mathcal{L}(E)$ denote the lattice of all subspaces of E, ordered by inclusion, which we denote by $\leq$. We will write $U < V$ for $U, V \leq E$ if U is strictly contained in V. The join of a pair of subspaces is their vector space sum and the meet of a pair of subspaces is their intersection. For any positive integer ℓ, we write $[\ell] := \{1, \ldots, \ell\}$.

2. q-Polymatroids

q-Polymatroids and their connections to linear codes were introduced in [18] and [28]. Their properties have been further developed in [17]. In our presentation, we will not assume that q-polymatroids are *representable*, that is, we will not assume that the q-polymatroids under consideration here are constructed from rank-metric codes over $\mathbb{F}_q$. We use the following definition of a q-polymatroid from [28], since it suits our purposes to have an integer valued function in what follows.

Definition 2. A (q, r)-polymatroid is a pair $M = (E, \rho)$ for which $r \in \mathbb{N}_0$ and ρ is a function $\rho : \mathcal{L}(E) \longrightarrow \mathbb{N}_0$ satisfying the following axioms.

(R1) For all $A \leq E$, $0 \leq \rho(A) \leq r \dim(A)$.
(R2) For all $A, B \leq E$, if $A \leq B$, then $\rho(A) \leq \rho(B)$.
(R3) For all $A, B \leq E$, $\rho(A + B) + \rho(A \cap B) \leq \rho(A) + \rho(B)$.

Every (q, r)-polymatroid is also a (q, r')-polymatroid for any $r' \geq r$. Hence, all the definitions below involving r depend on the choice of r. If it is not necessary to specify r, we will simply refer to such an object as a q-polymatroid. If we need to specify the q-polymatroid M, we denote its rank function by ρ_M. Note that a $(q, 1)$-polymatroid is a q-matroid. Conversely, if we consider a q-matroid as a (q, r)-polymatroid, we will always take $r = 1$. In order to stress in a stronger way the distinction between matroids and their q-analogues, we may use the terminology "classical matroids" for matroids.

Recall that a lattice isomorphism between a pair of lattices $(\mathcal{L}_1, \vee_1, \wedge_1), (\mathcal{L}_2, \vee_2, \wedge_2)$ is a bijective function $\varphi : \mathcal{L}_1 \longrightarrow \mathcal{L}_2$ that preserves the meet and join, that is, for all $x, y \in \mathcal{L}_1$ we have that $\varphi(x \wedge_1 y) = \varphi(x) \wedge_2 \varphi(y)$ and $\varphi(x \vee_1 y) = \varphi(x) \vee_2 \varphi(y)$. It is well known that reversing the ordering of a lattice gives again a lattice, with the meet and join interchanged. Combining this operation with a lattice isomorphism gives a lattice anti-isomorphism. Formally, a lattice anti-isomorphism between a pair of lattices is a bijective function $\psi : \mathcal{L}_1 \longrightarrow \mathcal{L}_2$ that is order-reversing and interchanges the meet and join, that is, for all $x, y \in \mathcal{L}_1$ we have that $\psi(x \wedge_1 y) = \psi(x) \vee_2 \psi(y)$ and $\psi(x \vee_1 y) = \psi(x) \wedge_2 \psi(y)$. See [2, Pages 3–4]. We hence define a notion of equivalence and duality between q-polymatroids.

Definition 3. Let E_1, E_2 be $\mathbb{F}_q$-vector spaces. Let $M_1 = (E_1, \rho_1)$ and $M_2 = (E_2, \rho_2)$ be q-polymatroids. We say that M_1 and M_2 are **lattice-equivalent** if there exists a lattice isomorphism $\varphi : \mathcal{L}(E_1) \longrightarrow \mathcal{L}(E_2)$ such that $\rho_1(A) = \rho_2(\varphi(A))$ for all $A \leq E_1$. In this case we write $M_1 \cong M_2$.

Notation 4. Let F be an $\mathbb{F}_q$-vector space. We denote by $\perp(F)$ a fixed anti-isomorphism on $\mathcal{L}(F)$, which we require to be an involution. For any subspace $U \leq F$ we denote by $U^{\perp(F)}$ the image of U under $\perp(F)$, which we call the **dual** of U in F. Note that since an anti-isomorphism preserves the length of intervals, we have for any $U \leq F$ that $\dim(U^{\perp(F)}) = \dim(F) - \dim(U)$. In the case $F = E$, we simply write $\perp := \perp(E)$. For any subspace $U \leq E$, we write $U^{\perp} := U^{\perp(E)}$.

Remark 5. The notion of lattice-equivalence of polymatroids in Definition 3 is not the same as the definition of equivalence given in [17] and [18]. Indeed, in [17] and [18] two q-polymatroids (E_1, ρ_1) and (E_2, ρ_2) are said to be **equivalent** if there exists an $\mathbb{F}_q$-linear isomorphism $\tau : E_1 \longrightarrow E_2$ such that $\rho_1(A) = \rho_2(\tau(A))$ for all $A \leq E_1$. Since every vector space isomorphism induces a lattice isomorphism, equivalence implies lattice-equivalence for q-polymatroids. In particular, every non-degenerate symmetric bilinear form b_F on F induces a lattice anti-isomorphism, hence our definition of dual implies the usual definition of orthogonal complement for q-polymatroids.

The dual q-polymatroid was defined in [18,28].

Definition 6. Let $M = (E, \rho)$ be a (q, r)-polymatroid. For every subspace $A \leq E$, define $\rho^*(A) := r\dim(A) - \rho(E) + \rho(A^{\perp})$. Then $M^* := (E, \rho^*)$ is a (q, r)-polymatroid called the **lattice-dual** of M.

We call M^* the **dual** of M. As noted in [17], the definition of the dual of M depends on the choice of anti-isomorphism on E, but all such choices yield equivalent duals. We prove this for our more general notions of lattice-equivalence and lattice-duality. The following is a generalisation of [17, Theorem 2.8].

Lemma 7. *Let $M = (E, \rho)$ be a (q, r)-polymatroid and let $M' = (E, \rho')$ be a (q, r)-polymatroid that is lattice-equivalent to M. Let $\perp, \hat{\perp}$ be a pair of anti-isomorphisms on $\mathcal{L}(E)$. Let M^* and M'^* be the duals of M and M' with respect to $\perp$ and $M^{\hat{*}}$ the dual of M with respect to $\hat{\perp}$. Then $M^* \cong M'^*$ and $M^* \cong M^{\hat{*}}$.*

Proof. For the first part, notice that the proof of [18, Proposition 3.7] carries over directly from $\mathbb{F}_q$-isomorphisms to lattice-(anti-)isomorphisms. We include it here for completeness. Let $\varphi : \mathcal{L}(E) \longrightarrow \mathcal{L}(E)$ be the isomorphism such that $\rho(A) = \rho'(\varphi(A))$ for all $A \subseteq E$. Let $\psi : \mathcal{L}(E) \longrightarrow \mathcal{L}(E)$ be the isomorphism $\psi := \perp \circ\, \varphi \circ \perp$, so $\varphi \circ \perp = \perp \circ\, \psi$. Then

$$\begin{aligned}
\rho^*(A) &= r\dim(A) - \rho(E) + \rho(A^{\perp}) \\
&= r\dim(A) - \rho'(E) + \rho'(\varphi(A^{\perp})) \\
&= r\dim(\psi(A)) - \rho'(\psi(E)) + \rho'(\psi(A)^{\perp}) \\
&= \rho'^*(\psi(A)).
\end{aligned}$$

This shows that $M^* \cong M'^*$. For the second statement we proceed in a similar way. Let $\phi : \mathcal{L}(E) \longrightarrow \mathcal{L}(E)$ be the lattice isomorphism $\phi := \perp \circ \hat{\perp}$, so $\perp = \phi \circ \hat{\perp}$. Then

$$\begin{aligned}\rho^*(A) &= r\dim(A) - \rho(E) + \rho(A^\perp)\\ &= r\dim(\phi(A)) - \rho(\phi(E)) + \rho(\phi(A^{\hat{\perp}}))\\ &= \rho^{\hat{*}}(\phi(A)).\end{aligned}$$

This shows that $M^* \cong M^{\hat{*}}$. $\square$

Note that $M^{**} = M$ is an equality, because we assume that the anti-isomorphism $\perp$ is an involution.

It is easy to see that for a map $\rho : \mathcal{L}(E) \longrightarrow \mathbb{N}_0$ satisfying the axioms (R1)-(R3), the restriction of that map to $\mathcal{L}(T)$, for each subspace $T \leq E$, also yields a q-polymatroid.

Definition 8. Let $M = (E, \rho)$ be a (q, r)-polymatroid and let $T \leq E$. For every subspace $A \leq T$, define $\rho_{M|T}(A) := \rho(A)$. Then $M|T := (T, \rho_{M|T})$ is a (q, r)-polymatroid called the **restriction** of M to T.

Another way to construct a new q-polymatroid from an existing one is via contraction. It was proven in [17, Theorem 5.2] that this gives in fact a q-polymatroid.

Definition 9. Let $M = (E, \rho)$ be a (q, r)-polymatroid and let $T \leq E$. We define the map

$$\rho_{M/T} : \mathcal{L}(E/T) \longrightarrow \mathbb{Z}$$

via $\rho_{M/T}(A/T) = \rho(A) - \rho(T)$. Then $M/T := (E/T, \rho_{M/T})$ is a (q, r)-polymatroid called the **contraction** of M **by** T.

It will sometimes be more convenient for us to use the slightly less commonly used definition of contraction *to* a subspace.

Definition 10. Let $M = (E, \rho)$ be a (q, r)-polymatroid and let $X \leq E$. We denote by $M.X$ the q-polymatroid $M.X := (E/X^\perp, \rho_{M/X^\perp})$. We call $M.X$ the **contraction** of M **to** X.

In the language of classical matroids, the contraction of M to X is the contraction of M by $E - X$, that is $M.X = M/(E - X)$ (see [25, Chapter 3]). In the q-analogue we have $M.X := M/X^\perp$.

The following duality result is a straightforward extension of [19, Theorem 60]. It relates the contraction of a q-polymatroid by a subspace with a restriction of its dual q-polymatroid. We will make good use of this in Section 6, where we give a construction of weighted subspace designs from q-polymatroids.

Lemma 11. *Let $M = (E, \rho)$ be a (q, r)-polymatroid and let T be a subspace of E. Then,*

$$M^*/T \cong (M|T^\perp)^* \text{ and } (M/T)^* \cong M^*|T^\perp.$$

Proof. Let $\phi : \mathcal{L}(E/T) \longrightarrow \mathcal{L}(T^\perp)$ be defined by $\phi(X/T) = (X^\perp)^{\perp(T^\perp)}$, for each $X \leq E$ such that $T \leq X$ (in which case $X^\perp \leq T^\perp$). This map is the composition of two anti-isomorphisms: the anti-isomorphism between intervals $[T, E]$ and $[0, T^\perp]$ induced by $\perp(E)$, followed by the anti-isomorphism $\perp(T^\perp)$ on $\mathcal{L}(T^\perp)$. Hence ϕ is a lattice isomorphism.

Let A be a subspace of E satisfying $T \leq A \leq E$. We claim that $\rho_{M^*/T}(A/T) = (\rho_{M|T^\perp})^*(\phi(A/T))$. Indeed, we have that:

$$\begin{aligned}
\rho_{M^*/T}(A/T) &= \rho^*(A) - \rho^*(T) \\
&= r\dim(A) - \rho(T^\perp) + \rho(A^\perp) - r\dim(T) \\
&= r\dim(A/T) - \rho_{M|T^\perp}(T^\perp) + \rho_{M|T^\perp}(A^\perp) \\
&= r\dim(\phi(A/T)) - \rho_{M|T^\perp}(T^\perp) + \rho_{M|T^\perp}(\phi(A/T)^{\perp(T^\perp)}) \\
&= (\rho_{M|T^\perp})^*(\phi(A/T)).
\end{aligned}$$

This shows that $M^*/T \cong (M|T^\perp)^*$. That $(M/T)^* \cong M^*|T^\perp$ holds can be seen by replacing M with M^* in the previous identity, taking duals and applying Lemma 7. $\square$

Remark 12. In fact, the above result holds even in terms of equivalence in the stronger sense [17, Definition 2.6 (b)], and not only lattice-equivalence, as it was shown in Theorem 5.3 of the same paper. Note that in establishing the equivalence of these q-polymatroids, the vector space isomorphism depends on the choice of the bilinear form arising in the construction of the lattice isomorphism.

Having established duality, restriction and contraction in terms of the rank function, we now introduce independent spaces.

Definition 13. Let $I \leq E$ and let $M = (E, \rho)$ be a (q, r)-polymatroid. We say that I is an **independent space** of M if $\rho(I) = r\dim(I)$. A subspace that is not independent is called a **dependent space** of M. We call $C \leq E$ a **circuit** of M if it is a minimal dependent space with respect to inclusion. We call $T \leq E$ a **cocircuit** of M if it is a circuit of M^*. A **loop** of M is a 1-dimensional space of rank zero.

For q-matroids, the following result is (I2) of the independence axioms (see [9, Definition 7]). We show that this holds for q-polymatroids.

Lemma 14. *Let $M = (E, \rho)$ be a (q, r)-polymatroid and let $I \leq E$ be an independent space of M. Then every subspace of I is independent.*

Proof. Since I is independent, we have $\rho(I) = r\dim(I)$. Let J, J' be subspaces of I such that I is a direct sum of J and J'. By (R1) and applying semimodularity (R3) to J and J' we get

$$\begin{aligned} r\dim(J) + r\dim(J') \geq \rho(J) + \rho(J') &\geq \rho(J + J') + \rho(J \cap J') \\ &= \rho(I) = r\dim(I) = r(\dim(J) + \dim(J')). \end{aligned}$$

Since $\rho(J) \leq r\dim(J)$ and $\rho(J') \leq r\dim(J')$ we must have that $\rho(J) = r\dim(J)$ and $\rho(J') = r\dim(J')$ and so the result follows. □

From the above lemma, it follows that a circuit cannot be contained in an independent space. The next lemma considers what happens to independent spaces and circuits under contraction of an independent space.

Lemma 15. *Let $M = (E, \rho)$ be a (q, r)-polymatroid and let $I \leq E$ be an independent space of M. Let $I \leq A \leq E$. Then A is independent in M if and only if A/I is independent in M/I. Moreover, if A is a circuit in M, then A/I is a circuit in M/I.*

Proof. Let A be independent in M. Then

$$r\dim(A/I) = r\dim(A) - r\dim(I) = \rho(A) - \rho(I) = \rho_{M/I}(A/I),$$

hence A/I is an independent space of M/I. Conversely, if A/I is independent in M/I, then

$$r\dim(A) - r\dim(I) = r\dim(A/I) = \rho_{M/I}(A/I) = \rho(A) - \rho(I) = \rho(A) - r\dim(I),$$

so $\rho(A) = r\dim(A)$.

Let A be a circuit in M. Any proper subspace of A/I has the form B/I for some unique $I \leq B < A$. Since A is a circuit, A/I is a dependent space in M/I, and B is an independent space of M. Therefore B/I is independent and so A/I is a circuit of M/I. □

We conclude this section with some examples given in [19, Example 4] and [17, Example 4.8(a)].

Example 16 *(The uniform q-matroid).* Let k be a positive integer, $k \leq n$. The *uniform q-matroid* is the q-matroid $M = (E, \rho)$ with rank function defined as follows:

$$\rho(U) := \begin{cases} \dim(U) & \text{if } \dim(U) \leq k, \\ k & \text{if } \dim(U) > k. \end{cases}$$

We denote this q-matroid by $U_{k,n}$.

Example 17 *(The Vámos q-matroid).* This q-matroid is constructed over $\mathcal{L}(\mathbb{F}_q^8)$. Choose the canonical basis for $\mathbb{F}_q^8$ denoted by $e_1, \dots, e_8$. Consider the following collection of subspaces.

$$\mathcal{C} := \{\langle e_1, e_2, e_3, e_4\rangle, \langle e_1, e_2, e_5, e_6\rangle, \langle e_3, e_4, e_5, e_6\rangle, \langle e_3, e_4, e_7, e_8\rangle, \langle e_5, e_6, e_7, e_8\rangle\}.$$

For each $A \leq \mathbb{F}_q^8$, we define $\rho(A)$ as follows:

$$\rho(A) := \begin{cases} \dim(A) & \text{if } \dim(A) \leq 3, \\ 3 & \text{if } A \in \mathcal{C}, \\ 4 & \text{if } \dim(A) \geq 4 \text{ and } A \notin \mathcal{C}. \end{cases}$$

It can be shown that ρ is the rank function of a q-matroid whose set of circuits of minimum dimension is the set $\mathcal{C}$ (see also [17, Prop. 4.6]).

3. Characteristic polynomial of a q-polymatroid

In this section, we introduce the characteristic polynomial of a q-polymatroid. This polynomial and its properties are well-studied in the case of a classical polymatroid [21, 32], in which case its coefficients are the Möbius values of the lattice of subsets of $[n]$. In the q-polymatroid case the underlying lattice is the subspace lattice of E. We will use the characteristic polynomial to obtain a version of the MacWilliams identities for q-polymatroids.

3.1. The Möbius function of a lattice

Throughout this paper we will use the Möbius function (see, e.g., [30, Chapter 25]), which is fundamental to the definition of a characteristic polynomial. We recall some basic results.

Let $(P, \leq)$ be a partially ordered set. The Möbius function for P is defined via the recursive formula

$$\mu(x, y) := \begin{cases} 1 & \text{if } x = y, \\ -\sum_{x \leq z < y} \mu(x, z) & \text{if } x < y, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 18 *(Möbius Inversion Formula). Let $(P, \leq)$ be a poset and let $f, g, h : P \longrightarrow \mathbb{Z}$ be any 3 functions on P. Then, we have:*

1. $f(x) = \sum_{x \leq y} g(y)$ *for all* $x \in P$ *if and only if* $g(x) = \sum_{x \leq y} \mu(x, y) f(y)$ *for all* $x \in P$,
2. $f(x) = \sum_{x \geq y} h(y)$ *for all* $x \in P$ *if and only if* $h(x) = \sum_{x \geq y} \mu(y, x) f(y)$ *for all* $x \in P$.

For the subspace lattice of E and for two subspaces U and V of dimensions u and v, we have that

$$\mu(U,V) = \begin{cases} (-1)^{v-u} q^{\binom{v-u}{2}} & \text{if } U \leq V \\ 0 & \text{otherwise.} \end{cases}$$

Definition 19. Given a pair of nonnegative integers a and b, the q-**binomial** or **Gaussian coefficient** counts the number of b-dimensional subspaces of an a-dimensional subspace over $\mathbb{F}_q$ and is given by:

$$\begin{bmatrix} a \\ b \end{bmatrix}_q := \prod_{i=0}^{b-1} \frac{q^a - q^i}{q^b - q^i},$$

if $a \geq b$ and is zero if $a < b$.

We will use the following identities

$$\begin{bmatrix} a \\ b \end{bmatrix}_q = \begin{bmatrix} a \\ a-b \end{bmatrix}_q \text{ and } \begin{bmatrix} a \\ b \end{bmatrix}_q \begin{bmatrix} b \\ c \end{bmatrix}_q = \begin{bmatrix} a \\ c \end{bmatrix}_q \begin{bmatrix} a-c \\ a-b \end{bmatrix}_q. \tag{1}$$

See also [13], for example, for a comprehensive account of the properties of Gaussian coefficients. In the following lemma, we note another identity.

Lemma 20. *Let I, J be subspaces of E of dimensions i and j, respectively, satisfying $I \cap J = \{0\}$ and $i + j \leq k$. Then, the number of k-dimensional subspaces of E that contain I and meet trivially with J is*

$$\sum_{s=0}^{j} (-1)^s q^{\binom{s}{2}} \begin{bmatrix} j \\ s \end{bmatrix}_q \begin{bmatrix} n-i-s \\ k-i-s \end{bmatrix}_q = q^{j(k-i)} \begin{bmatrix} n-i-j \\ k-i \end{bmatrix}_q, \tag{2}$$

where n is the dimension of E.

We omit the details of the proof of this lemma, since it is a straightforward computation. That the right-hand side of Equation (2) counts the number of k-dimensional subspaces of E that contain I and meet trivially with J was already observed, for example, in [15, Lemma 3], but is generally well-known. That this number is also given by the left-hand side formula can be established using Möbius inversion.

3.2. The characteristic polynomial

We now introduce the characteristic polynomial. First, we give another definition, which originates in weight enumeration of linear codes.

Definition 21. Let M be a (q,r)-polymatroid with ground-space E. For each $A \leq E$ we define

$$\ell_M(A) := \rho_M(E) - \rho_M(A).$$

By the definition of the rank function of a q-polymatroid, for each subspace A of E we see that $\ell_M(A)$ is non-negative integer in $\{0, \dots, \rho_M(E)\}$.

Notation 22. For the remainder, we fix r to be a positive integer and we let M denote a fixed (q,r)-polymatroid $M = (E, \rho)$. We write $\ell := \ell_M$ and $\rho := \rho_M$. For the dual q-polymatroid, we write $\ell^* := \ell_{M^*}$ and $\rho^* := \rho_{M^*}$.

Definition 23. The **characteristic polynomial** of M is the polynomial in $\mathbb{Z}[z]$ defined by

$$p(M; z) := \sum_{X\,:\,0 \leq X \leq E} \mu(0, X) z^{\ell(X)},$$

where μ is the Möbius function of the subspace lattice, $\mathcal{L}(E)$.

For the case $E = \{0\}$, we have $p(M; z) = 1$. If $E \neq \{0\}$, then $p(M; 1) = 0$ and so, unless $p(M; z)$ is identically zero, $z - 1$ is a factor of it in $\mathbb{Z}[z]$. For the (q,r)-polymatroid M, we have

$$p(M; z) := \sum_{j=0}^{n} (-1)^j q^{\binom{j}{2}} \sum_{X: X \leq E,\, \dim(X) = j} z^{\ell(X)}.$$

Example 24. We calculate the characteristic polynomial of the Vámos q-matroid of Example 17. From the definition of the rank function it follows that:

$$\ell(X) = \begin{cases} 4 - \dim(X) & \text{if } \dim(X) \leq 3, \\ 1 & \text{if } X \in \mathcal{C}, \\ 0 & \text{if } \dim(X) \geq 4 \text{ and } X \notin \mathcal{C}. \end{cases}$$

We treat the calculations of the coefficients by the powers of z. For the coefficient of z^4 we only have $X \leq E$ with $\dim(X) = 0$, i.e., the zero space. Then $\mu(0, X) = \mu(0, 0) = 1$ and we get the term z^4. For z^3 and z^2, we get:

$$\sum_{\dim(X)=1} \mu(0, X) z^{\ell(X)} = -\begin{bmatrix} 8 \\ 1 \end{bmatrix}_q z^3, \qquad \sum_{\dim(X)=2} \mu(0, X) z^{\ell(X)} = q \begin{bmatrix} 8 \\ 2 \end{bmatrix}_q z^2.$$

Consider the five circuits of dimension 4 and all spaces of dimension 3, from which we deduce that the coefficient of z is:

$$5q^6 - q^3 \begin{bmatrix} 8 \\ 3 \end{bmatrix}_q.$$

Finally, the constant term is determined by all spaces of dimension 4 that are not circuits, plus all spaces of higher dimension:

$$\begin{aligned} &q^6\left(\begin{bmatrix}8\\4\end{bmatrix}_q - 5\right) - q^{10}\begin{bmatrix}8\\5\end{bmatrix}_q + q^{15}\begin{bmatrix}8\\6\end{bmatrix}_q - q^{21}\begin{bmatrix}8\\7\end{bmatrix}_q + q^{28} \\ &= q^6\left(\begin{bmatrix}8\\4\end{bmatrix}_q - 5 - q^4\begin{bmatrix}8\\3\end{bmatrix}_q + q^9\begin{bmatrix}8\\2\end{bmatrix}_q - q^{15}\begin{bmatrix}8\\1\end{bmatrix}_q + q^{22}\right). \end{aligned}$$

The sum of all these terms gives the characteristic polynomial of the Vámos q-matroid. For example, for $q = 2$, we have $p(M; z) = z^4 - 255z^3 + 21590z^2 - 776920z + 755584 = (z - 1)(z^3 - 254z^2 + 21336z - 755584)$.

It is easily checked that the characteristic polynomial is an invariant of the lattice-equivalence class of a q-polymatroid.

Lemma 25. *Let E_1, E_2 be $\mathbb{F}_q$-vector spaces. Let $M_1 = (E_1, \rho_1)$ and $M_2 = (E_2, \rho_2)$ be a pair of lattice-equivalent q-polymatroids. Then $p(M_1; z) = p(M_2; z)$.*

Proof. Let $\phi : \mathcal{L}(E_1) \longrightarrow \mathcal{L}(E_2)$ be a lattice isomorphism such that $\rho_2(\phi(X)) = \rho_1(X)$ for all $X \in \mathcal{L}(E_1)$. Since $\mathcal{L}(E_1)$ and $\mathcal{L}(E_2)$ are equivalent lattices, we have that $\dim(X) = \dim(\phi(X))$ for all $X \in \mathcal{L}(E_1)$ and in particular $\mu_1(0, X) = \mu_2(0, \phi(X))$, where μ_i denotes the Möbius function on $\mathcal{L}(E_i)$. Moreover, $X \leq Y$ in $\mathcal{L}(E_1)$ if and only if $\phi(X) \leq \phi(Y)$ in $\mathcal{L}(E_2)$. By assumption, $\ell_{M_1}(X) = \ell_{M_2}(\phi(X))$ for each $X \in \mathcal{L}(E_1)$ and so the result follows. $\square$

We have the following results on the characteristic polynomial of the contraction of M to a subspace T. These will be important later when we define the q-polymatroid version of the rank weight enumerator.

Lemma 26. *Let $T \leq E$ and $M = (E, \rho)$ a q-polymatroid. The following hold.*

1. $\ell_{M.T}(X/T^\perp) = \ell_{M/T^\perp}(X/T^\perp) = \ell(X)$.
2. $p(M.T; z) = \displaystyle\sum_{X : T^\perp \leq X \leq E} \mu(T^\perp, X) z^{\ell(X)}$.
3. $p(M/T; z) = \displaystyle\sum_{X : T \leq X \leq E} \mu(T, X) z^{\ell(X)}$.

Proof. The first part follows from a direct computation:

$$\begin{aligned} \ell_{M.T}(X/T^\perp) = \ell_{M/T^\perp}(X/T^\perp) &= \rho_{M/T^\perp}(E/T^\perp) - \rho_{M/T^\perp}(X/T^\perp) \\ &= \rho(E) - \rho(T^\perp) - \rho(X) + \rho(T^\perp) \\ &= \rho(E) - \rho(X) = \ell(X). \end{aligned}$$

Let $\bar{\mu}$ denote the Möbius function on the subspace lattice of $E/T^{\perp}$. Then, using the second equality obtained at item 1 we have:

$$p(M.T;z) = p(M/T^{\perp};z) = \sum_{X:T^{\perp}\leq X\leq E} \bar{\mu}(0, X/T^{\perp}) z^{\ell_{M/T^{\perp}}(X/T^{\perp})}$$
$$= \sum_{X:T^{\perp}\leq X\leq E} \mu(T^{\perp}, X) z^{\ell(X)},$$

which proves item 2. The last item follows by replacing T by $T^{\perp}$ in the equation at item 2 and the fact that $M.T = M/T^{\perp}$ (Definition 10). □

Clearly, if T has dimension t, then

$$p(M.T;z) = \sum_{j=0}^{t} (-1)^j q^{\binom{j}{2}} \sum_{\substack{Y:\, T^{\perp}\leq Y, \\ \dim(Y)=n-t+j}} z^{\ell(Y)}.$$

Example 27. Let T be a subspace of $E = \mathbb{F}_q^8$. We calculate $p(M.T;z)$ where M is the Vámos q-matroid (Example 17). If T has dimension 5, then $\dim(T^{\perp}) = 3$. We only need to consider two cases, depending on whether or not $T^{\perp}$ is contained in a circuit (a member of $\mathcal{C}$). Note that the circuits intersect pairwise in dimension 2 or 0, so $T^{\perp}$ cannot be in more than one circuit.

Suppose $T^{\perp}$ is in none of the circuits. Then for all X such that $T^{\perp} < X \leq E$ we have that $\ell(X) = 0$. For $X = T^{\perp}$, we have $\ell(X) = 1$. So the q-matroid $M.T$ is lattice-equivalent to the uniform q-matroid $U_{1,5}$. Its characteristic polynomial is $p(M.T;z) = z - 1$.

Suppose now that $T^{\perp}$ is contained in a circuit $C \in \mathcal{C}$. Among all X such that $T^{\perp} \leq X \leq E$ we have that $\ell(X) = 1$ for $X = T^{\perp}$ and $X = C$. Otherwise, $\ell(X) = 0$. The q-matroid $M.T$ has rank 1 and all 1-dimensional spaces are independent, except for the loop $C/T^{\perp}$. For the characteristic polynomial we get the following:

$$p(M.T;z) = \mu(T^{\perp}, T^{\perp})z + \mu(T^{\perp}, C)z + \sum_{X\,:\,T^{\perp}\leq X\leq E} \mu(T^{\perp}, X) - \sum_{X\,:\,T^{\perp}\leq X\leq C} \mu(T^{\perp}, X)$$
$$= 0,$$

since $\mu(T^{\perp}, C) = -1$.

We continue to develop technical properties of the characteristic polynomial of the contraction $M.T$. In Section 6, we will use the fact that the characteristic polynomial of $M.T$ is identically zero when T is an independent space of the dual q-polymatroid.

Lemma 28. *A subspace T of E is an independent space of M^* if and only if $\ell(T^{\perp}) = 0$.*

Proof. We have

$$\ell(T^\perp) = \rho(E) - \rho(T^\perp) = \rho(E) - (\rho^*(T) + \rho(E) - r\dim(T)) = r\dim(T) - \rho^*(T).$$

Hence T is an independent space of M^* if and only if $\ell(T^\perp) = 0$. □

Lemma 29. *Let T be a subspace of E such that $\dim(T) > 0$. If T is an independent space of M^*, then $p(M.T; z) = 0$.*

Proof. By Lemma 28, $\ell(T^\perp) = 0$. Since all subspaces of an independent space are independent, we have that $\ell(X) = 0$ for all X such that $T^\perp \leq X$. We use this to compute the characteristic polynomial. Since $T^\perp \neq E$, we get

$$p(M.T; z) = \sum_{T^\perp \leq X \leq E} \mu(T^\perp, X) z^{\ell(X)} = \sum_{T^\perp \leq X \leq E} \mu(T^\perp, X) = 0. \quad \square$$

Lemma 30. *Let $T \leq E$ be a circuit of $M^* = (E, \rho^*)$. Then $p(M.T; z) = z^{\ell(T^\perp)} - 1$.*

Proof. Let $X \leq E$. If $T^\perp$ is strictly contained in X, then $X^\perp$ is strictly contained in T, and so $X^\perp$ is independent in M^*. Therefore, Lemma 28 gives that $\ell(X) = 0$. Hence

$$\begin{aligned} p(M.T; z) &= \sum_{T^\perp \leq X \leq E} \mu(T^\perp, X) z^{\ell(X)} \\ &= z^{\ell(T^\perp)} + \sum_{T^\perp < X \leq E} \mu(T^\perp, X) \\ &= z^{\ell(T^\perp)} - \mu(T^\perp, T^\perp) = z^{\ell(T^\perp)} - 1. \quad \square \end{aligned}$$

Remark 31. Note that if M is a q-matroid, a cocircuit T of M has $\ell(T^\perp) = \dim(T) - \rho^*(T) = \dim(T) - (\dim(T) - 1) = 1$ hence $p(M.T; z) = z - 1$.

Lemma 32. *Let $M = (E, \rho)$ be a q-polymatroid and let $T \leq E$ be an independent space of M^*. The following hold.*

1. $\rho(E) = \rho(T^\perp)$.
2. *For any subspace $U \leq T^\perp$, we have $\ell_{M|T^\perp}(U) = \ell(U)$.*

Proof. By definition of the dual q-polymatroid, we have $\rho(T^\perp) = \rho^*(T) - r\dim(T) + \rho(E)$. Since T is independent in M^*, $\rho^*(T) = r\dim(T)$ and so we get $\rho(T^\perp) = \rho(E)$, which establishes 1. Therefore, $\ell_{M|T^\perp}(U) = \rho_{M|T^\perp}(T^\perp) - \rho_{M|T^\perp}(U) = \rho(T^\perp) - \rho(U) = \rho(E) - \rho(U) = \ell(U)$, which proves item 2. □

Corollary 33. *Let T and U be subspaces of E such that $T \leq U$ and T is independent* *in M^*. If U/T is a circuit in M^*/T, then*

$$p((M^*/T)^*.(U/T); z) = p(M|T^\perp.(U^\perp)^{\perp(T^\perp)}; z) = z^{\ell(U^\perp)} - 1.$$

Proof. Recall from Lemma 11 that $M^*/T \cong (M|T^\perp)^*$ (and hence $(M^*/T)^* \cong M|T^\perp$) under the map $\phi : A/T \mapsto (A^\perp)^{\perp(T^\perp)}$ for any $A \leq E$ with $T \leq A$. In particular, if U/T is a circuit in M^*/T, then $\phi(U/T)$ is a circuit in $(M|T^\perp)^*$. Moreover $\phi(U/T)^{\perp(T^\perp)} = U^\perp$. From Lemmas 32 and 30 we have

$$p((M^*/T)^*.(U/T); z) = p(M|T^\perp.\phi(U/T); z) = z^{\ell_{M|T^\perp}(U^\perp)} - 1 = z^{\ell(U^\perp)} - 1. \quad \square$$

The following result will be used in the proof of Corollary 68.

Lemma 34. *Let $W \leq E$ and let $T \leq W$ be an independent space of M^*. Then*

$$p(M|T^\perp/W^\perp; z) = \sum_{A\,:\,A+T=W} p(M.A; z).$$

Proof. By Lemmas 26 and 32, we have $\ell_{M|T^\perp/W^\perp}(U/W^\perp) = \ell(U)$ for any subspace U satisfying $T \leq U^\perp \leq W$. Since $p(M/U; z) = \sum_{A\,:\,U\leq A\leq E} \mu(U, A)z^{\ell(A)}$, by applying the Möbius inversion formula we have $z^{\ell(U)} = \sum_{A\,:\,U\leq A\leq E} p(M/A; z)$. Therefore, we have

$$\begin{aligned}
p(M|T^\perp/W^\perp; z) &= \sum_{U\,:\,W^\perp\leq U\leq T^\perp} \mu(W^\perp, U)z^{\ell(U)} \\
&= \sum_{U\,:\,W^\perp\leq U\leq T^\perp} \mu(W^\perp, U) \sum_{A\,:\,U\leq A\leq E} p(M/A; z) \\
&= \sum_{U\,:\,W^\perp\leq U\leq T^\perp} \mu(W^\perp, U) \sum_{A\,:\,U\leq A\leq E} p(M.A^\perp; z) \\
&= \sum_{V\,:\,W^\perp\leq V^\perp\leq T^\perp} \mu(W^\perp, V^\perp) \sum_{A\,:\,V^\perp\leq A\leq E} p(M.A^\perp; z) \\
&= \sum_{V\,:\,T\leq V\leq W} \mu(W^\perp, V^\perp) \sum_{A\,:\,0\leq A\leq V} p(M.A; z) \\
&= \sum_{A\,:\,0\leq A\leq W} p(M.A; z) \sum_{V\,:\,A+T\leq V\leq W} \mu(W^\perp, V^\perp) \\
&= \sum_{A\,:\,A+T=W} p(M.A; z),
\end{aligned}$$

where the last equality follows from the fact that

$$\sum_{V\,:\,A+T\leq V\leq W}\mu(W^\perp,V^\perp)=\sum_{V\,:\,W^\perp\leq V^\perp\leq A^\perp\cap T^\perp}\mu(W^\perp,V^\perp)=\begin{cases}1 & \text{if } A^\perp\cap T^\perp=W^\perp,\\ 0 & \text{otherwise.}\end{cases}$$

□

We now present some further results on the characteristic polynomial.

Lemma 35. *Let e be a one-dimensional subspace of E. The following are equivalent:*

1. $p(M.e;z)=0$,
2. $\rho(e^\perp)=\rho(E)$,
3. *e is independent in M^*.*

Proof. We have $p(M.e;z)=z^{\ell(e^\perp)}-z^{\ell(E)}=z^{\ell(e^\perp)}-1$, which is zero if and only if $\ell(e^\perp)=\rho(E)-\rho(e^\perp)=0$. This shows that the equalities at items 1 and 2 are equivalent. The one-dimensional space e is independent in M^* if and only if $\rho^*(e)=r$. Since $\rho^*(e)=r\dim(e)-\rho(E)+\rho(e^\perp)=r-\rho(E)+\rho(e^\perp)$, this occurs if and only if $\rho(e^\perp)=\rho(E)$, which shows that statements 2 and 3 are equivalent. □

Remark 36. For a polymatroid M, parts 1 and 2 of the above lemma (with $e^\perp$ replaced by the set theoretic complement $e^c:=E-\{e\}$) are implied if e is a loop in M. Indeed, if e is a loop, then by semimodularity we get that $\rho(e^c)=\rho(E)$. For a q-polymatroid, however, it may occur that $e\leq e^\perp$, in which case if e is a loop, semimodularity does not imply that $\rho(e^\perp)=\rho(E)$. Note that in the polymatroid case, e being a loop in M implies that $\rho^*(e)=r\cdot|e|-\rho(E)+\rho(e^c)=r$, i.e., that e is independent in M^*.

Definition 37. For each $A\in\mathcal{L}(E)$, define $c(A):=\{X\leq E : A\leq X, \rho(A)=\rho(X)\}$. The **closure** of A in the (q,r)-polymatroid M is denoted by $\mathrm{cl}(A)$ and is defined to be the vector space sum of the members of $c(A)$; that is, $\mathrm{cl}(A):=\sum_{X\in c(A)}X$.

Remark 38. If M is a q-matroid, $\mathrm{cl}(\{0\})$ is the space spanned by its loops.

Lemma 39. *Let $L=\mathrm{cl}(\{0\})$. Let X be a subspace of E such that $X^\perp\leq L$. Then*

$$p(M.X;z)=\begin{cases} z^{\rho(E)}+\displaystyle\sum_{A\,:\,X^\perp\lneq A\leq E}\mu(X^\perp,A)z^{\ell(A)} & \text{if } X=L^\perp,\\ \displaystyle\sum_{A\,:\,X^\perp\leq A\leq E, A\not\leq L}\mu(X^\perp,A)z^{\ell(A)} & \text{otherwise.}\end{cases}$$

If $X^\perp=L$, then $p(M.X;z)$ is a monic polynomial of degree $\rho(E)$ in z. In particular, if M has no loops, then $p(M;z)$ is a monic polynomial of degree $\rho(E)$.

Proof. From Lemma 26 we have that

$$p(M.X;z)=\sum_{A\,:\,X^\perp\leq A\leq E}\mu(X^\perp,A)z^{\ell(A)}$$

$$= z^{\rho(E)} \sum_{A\,:\,X^\perp \leq A \leq L} \mu(X^\perp, A) + \sum_{A\,:\,X^\perp \leq A \leq E, A \nleq L} \mu(X^\perp, A) z^{\ell(A)}.$$

By the definition of the Möbius function, $\sum_{A\,:\,X^\perp \leq A \leq L} \mu(X^\perp, A) = 0$ unless $X^\perp = L$. If $A \nleq L$, then $\ell(A) = \rho(E) - \rho(A) < \rho(E)$, so if $L = X^\perp$, then $p(M.X; z)$ is a monic polynomial with leading term $z^{\rho(E)}$. Furthermore, setting $X = E$, we obtain that if M is a q-matroid with no loops, then its characteristic polynomial is monic of degree $\rho(E)$. □

In the q-matroid case, cryptomorphisms between axiom systems such as those relating to independent spaces, the closure function, flats, hyperplanes etc., were established in [9]. We therefore have the following facts, as in the case for classical matroids. The reader is referred to [9] and the references therein for further details. A subspace F is called a **flat** of a q-matroid if $\mathrm{cl}(F) = F$. For each $B \leq E$, there is a unique flat F such that $\mathrm{cl}(B) = F$, in which case $\rho(B) = \rho(F)$. Moreover, if M is a q-matroid, its collection of flats forms a semi-modular lattice [8]. A **hyperplane** $H < E$ is a flat that is maximal with respect to containment, that is, if $H \leq F$ for some flat F, then either $F = E$ or $F = H$. Every flat of M is an intersection of hyperplanes and for every hyperplane H, we have that $H^\perp$ is a cocircuit of M. Therefore, for every flat F of M, $F^\perp$ is the vector space sum of a collection of cocircuits.

The following result will be used in Lemma 73.

Theorem 40. *Let M be a q-matroid. Let X be a subspace of E and suppose that X contains a unique cocircuit C. Then*

$$p(M.X; z) = \begin{cases} z - 1 & \text{if } X = C, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. If $X = C$, then by Remark 31 we have that $p(M.X; z) = z - 1$. Assume now that $C \lneq X$. Then X is not a sum of cocircuits of M and hence $X^\perp$ is not a flat. Clearly X is a dependent space of M^* and by the uniqueness of C, any subspace of X that is dependent in M^* contains C. Therefore, by Lemma 28, $\ell(A^\perp) = 0$ for every $A \leq X$ such that $C \nleq A$.

Let F be a flat of M. For any $A \leq E$ such that $\mathrm{cl}(A^\perp) = F$, we have $\rho(A^\perp) = \rho(F)$ and hence $\ell(A^\perp) = \ell(F)$. Furthermore, $\mathrm{cl}(C^\perp) = C^\perp$ since C is a cocircuit of M. This will be used in the following computation of $p(M.X; z)$:

$$\begin{aligned} p(M.X; z) &= \sum_{A\,:\,X^\perp \leq A \leq E} \mu(X^\perp, A) z^{\ell(A)} = \sum_{A\,:\,A \leq X} \mu(A, X) z^{\ell(A^\perp)} \\ &= \sum_{A\,:\,C \leq A \leq X} \mu(A, X) z^{\ell(A^\perp)} + \sum_{A\,:\,A \leq X, C \nleq A} \mu(A, X) \end{aligned}$$

$$= \sum_{\substack{F\,:\,\mathrm{cl}(F)=F\\ X^\perp \le F \le C^\perp}} \sum_{\substack{A\,:\,X^\perp \le A^\perp \le C^\perp\\ \mathrm{cl}(A^\perp)=F}} \mu(X^\perp, A^\perp) z^{\ell(F)} + \sum_{A:A\le X} \mu(A,X)$$

$$- \sum_{A:C\le A\le X} \mu(A,X)$$

$$= \sum_{\substack{F\,:\,\mathrm{cl}(F)=F\\ X^\perp \le F \le C^\perp}} z^{\ell(F)} \sum_{\substack{A\,:\,X^\perp \le A^\perp \le C^\perp\\ \mathrm{cl}(A^\perp)=F}} \mu(X^\perp, A^\perp).$$

Since $X^\perp$ is not a flat, by [32, Proposition 3.3], we have $\sum_{\substack{A\,:\,X^\perp \le A^\perp \le C^\perp\\ \mathrm{cl}(A^\perp)=F}} \mu(X^\perp, A^\perp) = 0$, and so the result follows. □

Remark 41. In fact, by a similar argument (also essentially the same as for classical matroids), for a q-matroid M, we have $p(M.X; z) = 0$ unless $X^\perp$ is a flat in M. Equivalently, we have that $p(M.X; z) = 0$ unless X is a sum of cocircuits of M.

3.3. The weight enumerator of a q-polymatroid

We next define the *weight enumerator* of a q-polymatroid. In Section 5, we will show that its values satisfy a duality property and in Section 6, we will apply this duality result to establish a criterion for identifying a weighted subspace design determined by a q-polymatroid.

Definition 42. We define the **weight enumerator** of the (q,r)-polymatroid M to be the list $[A_M(i; z) : 0 \le i \le n]$, where for each i we define

$$A_M(i; z) := \sum_{X:X\le E, \dim(X)=i} p(M.X; z) = \sum_{X:X\le E, \dim(X)=i} p(M/X^\perp; z).$$

Lemma 43. *Let T be a subspace of E. The following hold.*

1. *If $T \le Z \le E$, then $p((M/T)/(Z/T); z) = p(M.Z^\perp; z)$.*
2. *$A_{M/T}(j; z) = \sum_{X \le T^\perp : \dim(X)=j} p(M.X; z)$.*

Proof. Let $T \le Z \le Y \le E$. Then $(Y/T)/(Z/T)$ and Y/Z are isomorphic. Let $V = (E/T)/(Z/T)$ and write $M_V = (M/T)/(Z/T)$. We have a lattice isomorphism between $\mathcal{L}(E/Z)$ and $\mathcal{L}(V)$. Moreover, it is easy to check that $\rho_{M_V}((Y/T)/(Z/T)) = \rho_{M/Z}(Y/Z)$. Therefore, M_V and M/Z are lattice-equivalent q-polymatroids. We thus have

$$p(M_V; z) = p(M/Z; z) = p(M.Z^\perp; z).$$

Let $X \le T^\perp$. It is straightforward to check that $\dim((X^\perp/T)^{\perp(E/T)}) = \dim(X)$. Therefore,

$$A_{M/T}(j;z) = \sum_{\substack{X:X^\perp/T \le E/T,\\ \dim((X^\perp/T)^{\perp(E/T)})=j}} p((M/T)/(X^\perp/T);z) = \sum_{\substack{X:X\le T^\perp\\ \dim(X)=j}} p(M.X;z). \quad \square$$

4. Matrix codes and q-polymatroids

We consider properties of a q-polymatroid arising from an $\mathbb{F}_q$-linear rank-metric code. There are several papers outlining properties of rank-metric codes. The q-polymatroids associated with these structures have been studied in [17,18,28].

Notation 44. Throughout this section, we let m be a positive integer and $E = \mathbb{F}_q^n$. We write $U^\perp$ to denote the orthogonal complement of $U \le E$ with respect to a non-degenerate symmetric bilinear form b_E on E. By abuse of notation, we also write $U^\perp$ to denote the orthogonal complement of

- $U \le \mathbb{F}_q^{n\times m}$ with respect to the inner product $b_{\mathbb{F}_q^{n\times m}}$ defined by $b_{\mathbb{F}_q^{n\times m}}(X,Y) = \mathrm{Tr}(XY^T)$ for all $X, Y \in \mathbb{F}_q^{n\times m}$ and
- $U \le \mathbb{F}_{q^m}^n$ with respect to the dot product defined by $x \cdot y = \sum_{i=1}^n x_i y_i$ for all $x = (x_1, \dots, x_n), y = (y_1, \dots y_n) \in \mathbb{F}_{q^m}^n$.

Definition 45. We say that $C \subseteq \mathbb{F}_q^{n\times m}$ is a **linear rank-metric code**, or a **matrix code** if C is a subspace of $\mathbb{F}_q^{n\times m}$. The **minimum distance** of C is the minimum rank of any nonzero member of C. We say that C is an $\mathbb{F}_q$-$[n\times m, k, d]$ rank-metric code if it has $\mathbb{F}_q$-dimension k and minimum distance d. The **dual code** of C is given by $C^\perp := \{Y \in \mathbb{F}_q^{n\times m} : \mathrm{Tr}(XY^T) = 0 \, \forall X \in C\}$. Finally, for each $i \in \{0, \dots, n\}$, we define $W_i(C) := |\{A \in C : \mathrm{rank}(A) = i\}|$. The list $[W_i(C) : 0 \le i \le n]$ is called the **weight distribution** of C.

For $X \le E$ we write $\mathrm{colsp}(X)$ to denote the column space of X over $\mathbb{F}_q$.

Definition 46. Let $X \in \mathbb{F}_q^{n\times m}$ and let $U \le E$. We say that U is the support of X if $\mathrm{colsp}(X) = U$. Let C be an $\mathbb{F}_q$-$[n\times m, k, d]$ rank-metric code. We say that U is a **support** of C if there exists some $X \in C$ with support U.

Definition 47. Let m be a positive integer and let C be an $\mathbb{F}_q$-$[n\times m, k, d]$ rank-metric code. For each subspace $U \le E$, we define

$$C_U := \{A \in C : \mathrm{colsp}(A) \le U^\perp\} \text{ and } C_{=U} := \{A \in C : \mathrm{colsp}(A) = U^\perp\}.$$

Let $\rho : \mathcal{L}(E) \longrightarrow \mathbb{N}_{\ge 0}$ be defined by $\rho(U) := k - \dim(C_U)$. Then (E, ρ) is a (q, m)-polymatroid [18, Theorem 5.3] and we denote it by M_C.

Clearly, we have $\ell(U) = \dim(C_U)$ for every $U \leq E$.

Lemma 48. *Let C be an $\mathbb{F}_q$-$[n \times m, k, d]$ rank-metric code. The following hold.*

1. $M_{C^\perp} = (M_C)^*$.
2. $p(M_C/U; q) = |C_{=U}|$.
3. $W_i(C) = A_{M_C}(i; q)$ *for each* $i \in [n]$.
4. $A_{M_C}(i; q) = 0$ *if and only if* $p(M_C.U; q) = 0$ *for every i-dimensional subspace* $U \leq E$.
5. *If* $A_{M_C}(i; q) = 0$, *then* $A_{M_C/T}(i; q) = 0$ *for every subspace* $T \leq E$.

Proof. Item 1 has been established in [18, Theorem 7.1]. Let $M = M_C$. Since $|C_U| = \sum_{V:U\leq V} |C_{=V}|$, by Möbius inversion we have

$$|C_{=U}| = \sum_{V:U\leq V} \mu(U,V)|C_V| = \sum_{V:U\leq V} \mu(U,V)q^{\ell(V)} = p(M/U; q) = p(M.U^\perp; q).$$

Therefore 2 holds. The number of codewords of C that have rank i over $\mathbb{F}_q$ is

$$W_i(C) = \sum_{U\,:\,\dim(U)=n-i} |C_{=U}| = \sum_{U\,:\,\dim(U)=n-i} p(M.U^\perp; q) = \sum_{U\,:\,\dim(U)=i} p(M.U; q)$$
$$= A_M(i; q),$$

which gives 3. Clearly, $A_M(i; q) = 0$ if and only if $p(M.U; q) = 0$ for each $U \leq E$ of dimension i, which gives the statement at item 4. Let T be a subspace of E. By Lemma 43 we have

$$A_{M/T}(i; q) = \sum_{X\leq T^\perp\,:\,\dim(X)=i} p(M.X; q).$$

If $A_M(i; q) = 0$, then from item 4 we have $p(M.X; q) = 0$ for each i-dimensional subspace X, and so we get $A_{M/T}(i; q) = 0$, which proves the statement in item 5. $\square$

Remark 49. Note that Part 2 of Lemma 48 is an instance of the Critical Theorem [14] for q-polymatroids and matrix codes.

Remark 50. In [18], the authors define a pair of q-polymatroids associated with a matrix code. The one given above is the q-polymatroid whose rank function is determined by the column-spaces of the codewords. A second q-polymatroid is the one whose rank function is determined by the row spaces of the codewords.

One way to construct an $\mathbb{F}_q$-$[n \times m, k, d]$ rank-metric code is by taking a subspace of $\mathbb{F}_{q^m}^n$, and expanding its elements with respect to a basis of $\mathbb{F}_{q^m}$ over $\mathbb{F}_q$. Such rank-metric codes are referred to as vector rank-metric codes.

Definition 51. Let Γ be a basis of $\mathbb{F}_{q^m}$ over $\mathbb{F}_q$. For each $x \in \mathbb{F}_{q^m}^n$, we write $\Gamma(x)$ to denote the $n \times m$ matrix over $\mathbb{F}_q$ whose ith row is the coordinate vector of the ith coefficient of x with respect to the basis Γ. The **rank** of x is the rank of the matrix $\Gamma(x)$. Note that the rank of x is well-defined, being independent of the choice of basis Γ.

For the remainder, we fix Γ to be a basis of $\mathbb{F}_{q^m}$ over $\mathbb{F}_q$.

Definition 52. A (linear rank-metric) **vector code** C is an $\mathbb{F}_{q^m}$-subspace of $\mathbb{F}_{q^m}^n$. The **minimum distance** of C is the minimum rank of any non-zero element of C. We say that C is an $\mathbb{F}_{q^m}$-$[n,k,d]$ code if it has $\mathbb{F}_{q^m}$-dimension k and $\Gamma(C)$ has minimum rank distance d. The code $C^\perp$ denotes the **dual code** of C with respect to the standard dot product on $\mathbb{F}_{q^m}^n$.

Each vector rank-metric code determines a q-matroid, as follows.

Definition 53. Let C be an $\mathbb{F}_{q^m}$-$[n,k,d]$ rank-metric code. Let $U \leq E$ and let $x \in C$. We say that U is a **support** of x if U is the column space of $\Gamma(x)$ and we write $\sigma(x) = U$. For each subspace $U \leq E$, we define

$$C_U := \{x \in C : \sigma(x) \leq U^\perp\} \text{ and } C_{=U} := \{x \in C : \sigma(x) = U^\perp\}.$$

Let $\rho : \mathcal{L}(E) \longrightarrow \mathbb{N}_{\geq 0}$ be defined by $\rho(U) := k - \dim_{\mathbb{F}_{q^m}}(C_U)$. Then (E,ρ) is a q-matroid [19, Theorem 24] and we denote it by M_C.

Remark 54. Note that in the definition given above, the rank function for the q-matroid of C is the rank function of the associated (q,m)-polymatroid as defined in Definition 47, divided by m. Since C is $\mathbb{F}_{q^m}$-linear, C_U is an $\mathbb{F}_{q^m}$-vector space for each subspace U and so has $\mathbb{F}_q$-dimension a multiple of m. Therefore the results of Lemma 48 hold with q^m in place of q. For example, with respect to the characteristic polynomial of the q-matroid, we have $p(M/U; q^m) = |C_{=U}|$ for an $\mathbb{F}_{q^m}$-$[n,k,d]$ code C and subspace U.

Let C be an $\mathbb{F}_{q^m}$-$[n,k,d]$ code. Recall that for any $U \leq \mathbb{F}_q^n$ we have

$$\ell_{M_C}(U) = \dim_{\mathbb{F}_{q^m}}(C_U) = \dim(\{x \in C : \sigma(x) \leq U^\perp\}).$$

Now $U^\perp$ is independent in $M_{C^\perp}$ if and only if $\ell_{M_C}(U) = 0$, which occurs if and only if no subspace of $U^\perp$ is a support of C. Therefore every support of C corresponds to a dependent space of $M_{C^\perp}$.

In the following example we illustrate the notions discussed in Sections 3 and 4. We calculate the characteristic polynomial of M_C by carefully studying the structure of the q-matroid and its dual.

Example 55. Let α be a root of $x^6+x^4+x^3+x+1 \in \mathbb{F}_{2^6}[x]$. Then α is a primitive element of $\mathbb{F}_{2^6}$. Let C be the $\mathbb{F}_{2^6}$-$[6,3,3]$ vector rank-metric code generated by the matrix:

$\ell(U)$ \ $\dim(U)$	0	1	2	3	4	5	6
0	0	0	0	$\begin{bmatrix}6\\3\end{bmatrix}_2 - 9 = 1386$	$\begin{bmatrix}6\\4\end{bmatrix}_2 = 651$	$\begin{bmatrix}6\\5\end{bmatrix}_2 = 63$	1
1	0	0	$\begin{bmatrix}6\\2\end{bmatrix}_2 = 651$	9	0	0	0
2	0	$\begin{bmatrix}6\\1\end{bmatrix}_2 = 63$	0	0	0	0	0
3	1	0	0	0	0	0	0

Fig. 1. Number of subspaces for each possible $\ell(U)$ value.

$$G = \begin{bmatrix} 1 & 0 & 0 & \alpha^{13} & \alpha^{47} & \alpha^{35} \\ 0 & 1 & 0 & \alpha^{44} & \alpha^{62} & \alpha^{32} \\ 0 & 0 & 1 & \alpha^{34} & \alpha^{22} & \alpha^{19} \end{bmatrix}.$$

With respect to the basis $\Gamma = \{1, \ldots, \alpha^5\}$, the rows of G are expanded to the following binary matrices:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \end{bmatrix}.$$

A basis of $\Gamma(C)$ over $\mathbb{F}_2$, which has 18 elements, is found by multiplying each row of G by successive powers of α and expanding with respect to Γ. We have that $\Gamma(C)$ is an $\mathbb{F}_2$-$[6 \times 6, 18, 3]$ rank-metric code with rank-metric weight distribution $[1, 0, 0, 567, 37044, 142884, 81648]$. Moreover, C is formally self-dual, that is, its dual code has the same weight distribution as C. Now consider the q-matroid $M := M_C$ arising from C, with rank function satisfying $\rho(U) = 3 - \dim_{\mathbb{F}_{2^6}}(C_U)$ for each $U \leq \mathbb{F}_2^6$. In Fig. 1 we write down the number of subspaces of $\mathbb{F}_2^6$ for each possible value of $\ell(U) = \dim_{\mathbb{F}_{2^6}}(C_U)$.

Using the entries of the table of Fig. 1, we write down the characteristic polynomial of M:

$$\begin{aligned} p(M; z) &= \sum_{U : 0 \leq U \leq E} \mu(0, U) z^{\ell(U)} \\ &= z^3 + \sum_{U : 0 \lneq U \leq E, \ell(U)=2} \mu(0, U) z^2 + \sum_{U : 0 \lneq U \leq E, \ell(U)=1} \mu(0, U) z \\ &\quad + \sum_{U : 0 \lneq U \leq E, \ell(U)=0} \mu(0, U) \\ &= z^3 - 63z^2 + 1230z - 1168 = (z-1)(z^2 - 62z + 1168). \end{aligned}$$

We will explain the values in this table column by column from right to left: recall that to say something about $\ell(U)$, we have to consider how the supports of C relate to $U^\perp$. Along the way, we will compute the different possible values of $p(M/U;q^m) = |C_{=U}|$, which, by Lemma 48, counts the number of codewords of C with support equal to $U^\perp$.

Since the rank distance of C is 3, C has no supports of dimension less than 3 apart from $\{0\}$. Hence $\ell(U) = \dim_{\mathbb{F}_2^6}(C_U) = 0$ for each of the 651 subspaces $U \leq \mathbb{F}_2^6$ of dimension 4 and each of the 63 spaces of dimension 5. So by Lemma 28, the respective 1 and 2-dimensional orthogonal complements of these spaces are independent in M^*. We remark that by Lemma 48, we have $M^* = M_{C^\perp}$.

We now consider the 3-dimensional subspaces. Since any proper subspace of a 3-dimensional subspace $U \leq \mathbb{F}_2^6$ is independent in M^*, it must be the case that if U is dependent in M^*, it is a cocircuit of M. Then by Remark 31, we have $\ell(U^\perp) = 1$ (indeed $p(M.U;z) = z-1$) and $p(M.U;2^6) = |C_{=U^\perp}| = 2^6 - 1 = 63$.

Therefore, by inspection of the weight enumerator, we see that there are $9 = 567/63$, different 3-dimensional spaces that are supports of C. We list the 3-dimensional cocircuits of M below:

$$\begin{aligned}
&\langle(010011),(001010),(000100)\rangle, \langle(101100),(010000),(000001)\rangle,\\
&\langle(100001),(011000),(000010)\rangle, \langle(100111),(010010),(001101)\rangle,\\
&\langle(100110),(010101),(001001)\rangle, \langle(100010),(001011),(000111)\rangle,\\
&\langle(110001),(000101),(000011)\rangle, \langle(100100),(010100),(001111)\rangle,\\
&\langle(100000),(010110),(001000)\rangle.
\end{aligned}$$

Every other 3-dimensional subspace U is a non-support of C, as are all its nontrivial subspaces, hence $\ell(U^\perp) = \dim_{\mathbb{F}_{2^6}}(C_{U^\perp}) = 0$. We remark that Lemma 28 says for such U that $U^\perp$ is independent in M^*, and Lemma 29 gives that $p(M.U;z) = 0$.

By computation, we obtain that there are 588 4-dimensional supports of C and that none of these spaces contains a cocircuit of dimension 3. Therefore, each such subspace U is itself a cocircuit and so we have that $\ell(U^\perp) = 1$, $p(M.U;z) = z-1$, and $p(M.U;2^6) = |C_{=U^\perp}| = 2^6 - 1$.

There remain a further $\begin{bmatrix}6\\4\end{bmatrix}_2 - 588 = 63$ 4-dimensional subspaces that are not supports of C. Every 3-dimensional cocircuit is contained in $\begin{bmatrix}6-3\\4-3\end{bmatrix}_2 = \begin{bmatrix}3\\1\end{bmatrix}_2 = 7$ different 4-dimensional spaces and every pair of 3-dimensional cocircuits span $\mathbb{F}_2^6$. Therefore, every 4-dimensional non-support of C contains at most one 3-dimensional cocircuit and since there are $9 \cdot 7 = 63$ such 4-dimensional non-supports altogether, each of them contains a unique 3-dimensional cocircuit. It follows that $\ell(U^\perp) = 1$ for every 4-dimensional subspace U.

By direct computation it can be checked that there are 63 5-dimensional supports of C and of course the only 6-dimensional support is the entire space $\mathbb{F}_2^6$. Each 5-dimensional support U is the support of exactly 2268 different codewords, so $p(M.U; 2^6) = |C_{=U^\perp}| = 2268$. Therefore, $\ell(U^\perp) = \dim_{\mathbb{F}_{2^6}}(C_{U^\perp}) \geq 2$. If $\ell(U^\perp) = 3$ then the support of every codeword is contained in U, which is impossible as C has words of rank 6. It follows that $\ell(U^\perp) = 2$.

All computations carried out in this example were done using Magma [3].

5. MacWilliams identities for q-polymatroids

We establish a version of the MacWilliams identities for the (q, r)-polymatroids, which we shall use in establishing criteria for the existence of a weighted t-design over $\mathbb{F}_q$. Duality via the rank polynomial of a q-polymatroid was considered in [28]. We start with a result that relates the characteristic polynomial of a q-polymatroid to that of its dual. The statements of Theorem 56 and Corollary 57 may be regarded as q-analogues of [6, Corollary 12]. However, unlike the proofs given here, which rely only on Möbius inversion, the proof of [6, Corollary 12] relies on an existing version of the MacWilliams duality theorem for matroids, which shows that the weight enumerator polynomial of the dual of a matroid can be retrieved from the weight enumerator polynomial of the original matroid by a substitution of variables. Recall that M denotes an arbitrary but fixed (q, r)-polymatroid (E, ρ).

Theorem 56. *Let $U \leq E$. Then*

$$\sum_{A:A\leq U} p(M^*.A; z) = z^{r\dim(U)-\rho(E)} \sum_{A:A\leq U^\perp} p(M.A; z).$$

Proof. We have by Lemma 26 and then replacing X with $A^\perp$ that

$$p(M^*.U; z) = \sum_{X\,:\,U^\perp\leq X\leq E} \mu(U^\perp, X) z^{\ell^*(X)} = \sum_{A\,:\,A\leq U} \mu(A, U) z^{\ell^*(A^\perp)}.$$

To this we apply Möbius inversion (Lemma 18 part (2)), duality (Definitions 6 and 21) and Möbius inversion again to get

$$\begin{aligned}\sum_{A\,:\,U\leq A\leq E} p(M^*.A; z) &= z^{\ell^*(U)} \\ &= z^{\ell(U^\perp)-\rho(E)+r\dim(U^\perp)} \\ &= z^{r\dim(U^\perp)-\rho(E)} \sum_{A\,:\,U^\perp\leq A\leq E} p(M.A; z). \quad \square\end{aligned}$$

We now show that for any subspace $U \leq E$, the characteristic polynomial of $M^*.U$ is completely determined by the set $\{(p(M.V; z), V) : V \leq E\}$.

Corollary 57. *Let $U \leq E$. We have the identity:*

$$z^{\rho(E)}p(M^*.U;z) = \sum_{V \leq E} p(M.V;z) \sum_{j=0}^{\dim(U\cap V^\perp)} \begin{bmatrix} \dim(U \cap V^\perp) \\ j \end{bmatrix}_q (-1)^{\dim(U)-j} q^{\binom{\dim(U)-j}{2}} z^{jr}.$$

Proof. From Lemma 56, we have:

$$\sum_{A\,:\,A\leq U} p(M^*.A;z) = z^{r\dim(U)-\rho(E)} \sum_{V\,:\,V\leq U^\perp} p(M.V;z). \tag{3}$$

Apply the Möbius inversion formula to Equation (3) to get the identity

$$p(M^*.U;z) = \sum_{A\,:\,A\leq U} \mu(A,U) z^{r\dim(A)-\rho(E)} \sum_{V\,:V\leq A^\perp} p(M.V;z).$$

We thus have that:

$$\begin{aligned}
&z^{\rho(E)}p(M^*.U;z) \\
&\quad= \sum_{A\,:\,A\leq U} \mu(A,U) z^{r\dim(A)} \sum_{V:V\leq A^\perp} p(M.V;z) \\
&\quad= \sum_{V:V\leq E} p(M.V;z) \sum_{A\,:\,A\leq U\cap V^\perp} \mu(A,U) z^{r\dim(A)} \\
&\quad= \sum_{V:V\leq E} p(M.V;z) \sum_{j=0}^{\dim(U\cap V^\perp)} \begin{bmatrix} \dim(U\cap V^\perp) \\ j \end{bmatrix}_q (-1)^{\dim(U)-j} q^{\binom{\dim(U)-j}{2}} z^{jr}. \qquad \square
\end{aligned}$$

We now have the following MacWilliams identity, relating the weight enumerators of M and M^*. This version of the identity, or rather its corollary, will be used to prove the main theorem of Section 6.

Theorem 58. *Let $s \in \{0,\dots,n\}$. Then*

$$\sum_{i=0}^{n-s} \begin{bmatrix} n-i \\ s \end{bmatrix}_q A_M(i;z) = z^{\rho(E)-rs} \sum_{i=0}^{s} \begin{bmatrix} n-i \\ s-i \end{bmatrix}_q A_{M^*}(i;z).$$

Proof. We start with the left-hand-side of the equation and rewrite it, noting that $\begin{bmatrix} n-i \\ s \end{bmatrix}_q = \begin{bmatrix} n-i \\ n-s-i \end{bmatrix}_q$ counts the number of $(n-s)$-dimensional subspaces of E that contain a fixed space of dimension i. This yields:

$$\sum_{i=0}^{n-s} \begin{bmatrix} n-i \\ s \end{bmatrix}_q A_M(i;z) = \sum_{i=0}^{n-s} \begin{bmatrix} n-i \\ n-s-i \end{bmatrix}_q \sum_{X:\dim(X)=i} p(M.X;z)$$

$$= \sum_{U\,:\,\dim(U)=n-s} \sum_{X \le U} p(M.X;z).$$

From Lemma 56, this gives:

$$\begin{aligned}
\sum_{i=0}^{n-s} \begin{bmatrix} n-i \\ s \end{bmatrix}_q A_M(i;z) &= \sum_{U\,:\,\dim(U)=n-s} z^{\rho(E)-r\dim(U^\perp)} \sum_{X \le U^\perp} p(M^*.X;z) \\
&= \sum_{V\,:\,\dim(V)=s} z^{\rho(E)-rs} \sum_{X : X \le V} p(M^*.X;z) \\
&= z^{\rho(E)-rs} \sum_{i=0}^{s} \begin{bmatrix} n-i \\ s-i \end{bmatrix}_q \sum_{X:\,\dim(X)=i} p(M^*.X;z) \\
&= z^{\rho(E)-rs} \sum_{i=0}^{s} \begin{bmatrix} n-i \\ s-i \end{bmatrix}_q A_{M^*}(i;z). \qquad \square
\end{aligned}$$

Theorem 58 shows that the weight enumerator of a q-polymatroid and that of its dual are related by invertible q-Pascal matrices. The minors of such matrices have been studied as q-analogues of the classical Pascal matrices. We will use the following result from [22, Theorem 2.2].

Lemma 59. *Let $r_1, \ldots, r_n$ be a sequence of non-negative integers. We have*

$$\det\left(\begin{bmatrix} r_i \\ j-1 \end{bmatrix}_q\right)_{1\le i,j\le n} = q^{\binom{n}{2}} \prod_{1\le i<j\le n} \frac{q^{r_j}-q^{r_i}}{q^j-q^i}.$$

The next corollary (cf. [5, Corollary 3.2] for matroids) is the main device used to prove Theorem 66, which identifies sufficiency criteria for the existence of weighted subspace designs arising from the dependent spaces of a q-polymatroid (cf. [5, Theorem 3.3]). We remark that the reasoning used here is similar to that of the original Assmus-Mattson Theorem and its generalizations.

Corollary 60. *Let $S \subseteq \{1, \ldots, n\}$. The pair of lists*

$$[A_{M^*}(i;z) : |S| \le i \le n] \text{ and } [A_M(j;z) : j \in S],$$

is determined uniquely by the pair of lists

$$[A_{M^*}(i;z) : 1 \le i \le |S|-1] \text{ and } [A_M(j;z) : j \in [n]-S].$$

Proof. Let $A_M(z) := (A_M(i;z))_{0\le i\le n}$ and let $A_{M^*}(z) := (A_{M^*}(i;z))_{0\le i\le n}$. Note that $A_M(0;z) = A_{M^*}(0;z) = 1$, and in particular are known. From Theorem 58, we have the matrix equation

$$\left(\begin{bmatrix} n-i \\ s \end{bmatrix}_q\right)_{0\le i,s\le n} A_M(z) = \operatorname{diag}(z^{\rho(E)-rs})_{0\le s\le n} \left(\begin{bmatrix} n-i \\ n-s \end{bmatrix}_q\right)_{0\le i,s\le n} A_{M^*}(z).$$

Let $t = |S|$ and write $S = \{\ell_1, \dots, \ell_t\}$. By Lemma 59, we have

$$\det\left(\begin{bmatrix} n-\ell_i \\ s-1 \end{bmatrix}_q\right)_{1\le i,s\le t} = q^{\binom{t}{2}} \prod_{1\le i<s\le t} \frac{q^{n-\ell_s} - q^{n-\ell_i}}{q^s - q^i},$$

which is non-zero, as the ℓ_i are distinct. Now suppose that the coefficients $A_M(j;z)$ are known for $j \notin S$ and that the $A_{M^*}(j;z)$ are known for $0 \le j \le t-1$. Then we can solve for the unknown $A_M(j;z)$ via

$$\begin{aligned}(A_M(\ell_i;z))_{1\le i\le t} &= \left(\begin{bmatrix} n-\ell_j \\ s-1 \end{bmatrix}_q\right)^{-1}_{1\le j,s\le t} \\ &\times \left(\operatorname{diag}(z^{\rho(E)-rs})_{0\le s\le t-1} \left(\begin{bmatrix} n-j \\ n-s \end{bmatrix}_q\right)_{0\le s,j\le t-1} (A_{M^*}(j;z))_{0\le j\le t-1}\right. \\ &\left. - \left(\begin{bmatrix} n-j \\ s-1 \end{bmatrix}_q\right)_{\substack{1\le s\le t \\ j\in\{0,\dots,n\}-S}} (A_M(j;z))_{j\in\{0,\dots,n\}-S}\right).\end{aligned}$$

Once the list $[A_M(j;z) : j \in S]$ is determined, since $A_M(z)$ is now known, Theorem 58 can be applied to retrieve $[A_{M^*}(i;z) : t \le i \le n]$. $\square$

6. Weighted subspace designs from q-polymatroids

6.1. *Weighted subspace designs*

In [5], the authors define a *weighted design*, which generalizes a t-design. A t-(n,k,λ) design, with t, k, λ positive integers, is a collection of k-subsets of an n-set (called blocks) with the property that every t-subset of the n-set is contained in exactly λ blocks. A q-analogue of this notion is that of a t-design over $\mathbb{F}_q$, which is a collection of k-dimensional subspaces of E called blocks, with the property that every t-dimensional subspace of E is contained in the same number of blocks. Similarly, there is a q-analogue of a weighted t-design.

Definition 61. Let $\mathbb{G}$ be an additive group, let t, k be positive integers, and let $\lambda \in \mathbb{G}$. A weighted t-$(n,k,\lambda;q)$ design $\mathcal{D}$ is a triple $(E, \mathcal{B}, f)$ for which $\mathcal{B}$ is a collection of k-dimensional subspaces of E (called blocks) and $f : \mathcal{B} \mapsto \mathbb{G}$ is a weight function such that for every t-dimensional spaces T of E, $\sum_{B:T\le B} f(B) = \lambda$. We say that $\mathcal{D}$ is a **weighted subspace design** or is a **weighted design over** $\mathbb{F}_q$.

A subspace design (a design over $\mathbb{F}_q$) can be interpreted as a weighted subspace design with the weight function $f(B) := 1$ for all $B \in \mathcal{B}$, and $\mathbb{G} = (\mathbb{Z}, +)$. For an excellent survey on subspace designs, see [4]. In general, obtaining new subspace designs is a difficult problem, often highly dependent on computer search, which is exacerbated by the number of subspaces involved (which is exponential in comparison to classical designs for the same parameters). For example, it is not yet known if a 3-$(8, 4, 1; 2)$ subspace design exists; such a design would have 6477 blocks, chosen from an ambient space having 200,787 4-dimensional subspaces. Its classical analogue, the extended Fano plane, has 14 blocks, chosen from a collection of 70 4-sets. In [8], a construction of a q-analogue of a perfect matroid design (q-PMD) was given, which is a q-matroid for which all flats of the same dimension have the same rank. This q-PMD yields a construction of a subspace design from a q-Steiner system. In the following sections we will show another way that subspace designs and weighted subspace designs can arise from q-polymatroids satisfying certain rigidity properties.

The **intersection numbers** of a weighted subspace design are important invariants and can be used to establish non-existence results. Their values are the same as for subspace designs; see, for example [20, Fact 1.5] or [29].

Theorem 62. *Let $(E, \mathcal{B}, f)$ be a t-$(n, k, \lambda; q)$ weighted subspace design and let I, J be two subspaces of E of dimension i and j, respectively, such that $I \cap J = \{0\}$. If $i + j \leq t$, then*

$$\sum_{B \in \mathcal{B}\,:\, I \leq B, B \cap J = \{0\}} f(B) = q^{(k-i)j} \begin{bmatrix} n-i-j \\ k-i \end{bmatrix}_q \begin{bmatrix} n-t \\ k-t \end{bmatrix}_q^{-1} \lambda.$$

In particular, this number is independent of the choice of I of dimension i and J of dimension j. We denote it by $\lambda_{i,j}$.

Proof. If X is a subspace of E of dimension $x \leq t$, then since $(E, \mathcal{B}, f)$ is a weighted subspace design, we have

$$\begin{aligned} \begin{bmatrix} k-x \\ t-x \end{bmatrix}_q \sum_{B \in \mathcal{B}\,:\, X \leq B} f(B) &= \sum_{B \in \mathcal{B}\,:\, X \leq B} \quad \sum_{T\,:\, X \leq T \leq B, \dim(T)=t} f(B), \qquad (4) \\ &= \sum_{T\,:\, X \leq T, \dim(T)=t} \quad \sum_{B \in \mathcal{B}\,:\, T \leq B} f(B) = \begin{bmatrix} n-x \\ t-x \end{bmatrix}_q \lambda. \end{aligned}$$

Now restrict to a subspace X of the form $X = I + L$ for some $L \leq J$ of dimension s. Then $I \cap L = \{0\}$ and $\dim(I + L) = i + s$ and so Equation (4) becomes:

$$g(L) := \sum_{B \in \mathcal{B}\,:\, I+L \leq B} f(B) = \begin{bmatrix} n-(i+s) \\ t-(i+s) \end{bmatrix}_q \begin{bmatrix} k-(i+s) \\ t-(i+s) \end{bmatrix}_q^{-1} \lambda.$$

Define $h(K) = \sum_{B\in\mathcal{B}\,:\,I\le B, B\cap J=K} f(B)$, for each $K \le J$. Then we have that

$$g(L) = \sum_{K\,:\,L\le K\le J} h(K)$$

and so, by Möbius inversion on the lattice $\mathcal{L}(J)$,

$$h(L) = \sum_{K\,:\,L\le K\le J} \mu(L,K)g(K).$$

Substituting $L = \{0\}$ now gives

$$\begin{aligned}
\sum_{B\in\mathcal{B}\,:\,I\le B, B\cap J=\{0\}} f(B) = h(\{0\}) &= \sum_{K\le J} \mu(0,K)g(K) \\
&= \sum_{s=0}^{j} \begin{bmatrix} j \\ s \end{bmatrix}_q (-1)^s q^{\binom{s}{2}} \begin{bmatrix} n-i-s \\ t-i-s \end{bmatrix}_q \begin{bmatrix} k-i-s \\ t-i-s \end{bmatrix}_q^{-1} \lambda \\
&= \lambda \begin{bmatrix} n-t \\ k-t \end{bmatrix}_q^{-1} \sum_{s=0}^{j} (-1)^s q^{\binom{s}{2}} \begin{bmatrix} j \\ s \end{bmatrix}_q \begin{bmatrix} n-i-s \\ k-i-s \end{bmatrix}_q \\
&= \lambda \begin{bmatrix} n-t \\ k-t \end{bmatrix}_q^{-1} q^{j(k-i)} \begin{bmatrix} n-i-j \\ k-i \end{bmatrix}_q
\end{aligned}$$

The third line follows from applying Equation (1) with $a = n-i-s$, $b = k-i-s$, $c = t-i-s$, and the last equality follows from Lemma 20. $\square$

The proof outlined above is a direct q-analogue of [5, Theorem 2.6]. The intersection numbers for subspace designs were given in [12,29], for which the authors proposed an inductive argument.

We have the following constructions of weighted subspace designs from a given one (cf. [20,29]).

Corollary 63. *Let $\mathcal{D} := (E, \mathcal{B}, f)$ be a weighted t-$(n, k, \lambda; q)$ design.*

1. *For $0 \le i \le t$, $\mathcal{D}$ is an i-$(n, k, \lambda_i; q)$ weighted subspace design with*

$$\lambda_i = \begin{bmatrix} n-i \\ k-i \end{bmatrix}_q \begin{bmatrix} n-t \\ k-t \end{bmatrix}_q^{-1} \lambda = \begin{bmatrix} n-i \\ t-i \end{bmatrix}_q \begin{bmatrix} k-i \\ t-i \end{bmatrix}_q^{-1} \lambda.$$

2. *Define $\mathcal{B}^\perp := \{B^\perp : B \in \mathcal{B}\}$. If $k \le n-t$ then $\mathcal{D}^\perp = (E, \mathcal{B}^\perp, f^\perp)$ is a t-$(n, n-k, \lambda^\perp; q)$ weighted subspace design with $f^\perp(B^\perp) := f(B)$ for all $B \in \mathcal{B}$ and $\lambda^\perp := \begin{bmatrix} n-k \\ t \end{bmatrix}_q \begin{bmatrix} k \\ t \end{bmatrix}_q^{-1} \lambda$.*

Proof. To see that item 1 holds, apply Theorem 62 with $\lambda_i := \lambda_{i,0}$. Let I be an i-dimensional subspace of E. We have $\lambda_{i,0} = \sum_{B\in\mathcal{B}\,:\,I\leq B} f(B) = \begin{bmatrix} n-i \\ k-i \end{bmatrix}_q \begin{bmatrix} n-t \\ k-t \end{bmatrix}_q^{-1}$. The rest follows from Equation (1).

We will compute the value $\lambda^\perp$. A t-dimensional subspace T is contained in $B^\perp \in \mathcal{B}^\perp$ if and only if $B \leq T^\perp$. Now consider the set $S := \{(B,X) : B \in \mathcal{B}, \dim(X) = n-t, B \leq X\}$. We will compute the sum of the $f(B)$ over all pairs (B,X) in S in two ways. On the one hand, we have:

$$\begin{aligned}\sum_{(B,X)\in S} f(B) &= \sum_{B\in\mathcal{B}} \sum_{\substack{X:B\leq X\\ \dim(X)=n-t}} f(B) = \begin{bmatrix} n-k \\ n-t-k \end{bmatrix}_q \sum_{B\in\mathcal{B}} f(B) = \begin{bmatrix} n-k \\ t \end{bmatrix}_q \lambda_{0,0} \\ &= \begin{bmatrix} n-k \\ t \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n-t \\ k-t \end{bmatrix}_q^{-1} \lambda = \begin{bmatrix} n-k \\ t \end{bmatrix}_q \begin{bmatrix} n \\ t \end{bmatrix}_q \begin{bmatrix} k \\ t \end{bmatrix}_q^{-1} \lambda.\end{aligned}$$

The last equality follows from applying Equation (1) with $a = n$, $b = k$, $c = t$. On the other hand,

$$\sum_{(B,X)\in S} f(B) = \sum_{\substack{X\leq E:\\ \dim(X)=n-t}} \sum_{B\in\mathcal{B},B\leq X} f(B) = \begin{bmatrix} n \\ t \end{bmatrix}_q \sum_{B\in\mathcal{B},B\leq X} f(B).$$

It follows, by comparing the two right-hand sides, that

$$\lambda^\perp := \sum_{B^\perp\in\mathcal{B}^\perp,T\leq B^\perp} f(B) = \sum_{B\in\mathcal{B},B\leq T^\perp} f(B) = \begin{bmatrix} n-k \\ t \end{bmatrix}_q \begin{bmatrix} k \\ t \end{bmatrix}_q^{-1} \lambda. \quad \square$$

6.2. Subspace designs from q-polymatroids

We now present criteria for the existence of a weighted subspace design arising from the dependent spaces of a q-polymatroid. The approach is in essence a generalization of the original argument given by Assmus and Mattson [1]. To do this, we obtain a q-analogue of [5, Theorem 3.3]. Throughout this section we let $\mathbb{F}$ denote an arbitrary field (which need not bear any relation to $\mathbb{F}_q$). We remind the reader that M denotes a (q,r)-polymatroid (E,ρ). Since $p(M;z) \in \mathbb{Z}[z]$, it gives a well-defined function on any field, viewed as a $\mathbb{Z}$-module. We define the following (cf. [5]).

Definition 64. Let $\theta \in \mathbb{F}$. We define:

- $D_M(i;\theta) := \{X \leq E : \dim(X) = i, p(M.X;\theta) \neq 0\}$,
- $R_M(t;\theta) := \{j \in [n-t] : A_{M^*}(j;\theta) \neq 0\}$,
- $S_M(t;\theta) := \{j \in [n-t] : A_{M^*}(j;\theta) = 0\}$,
- $d_M := \min\{\dim(X) : X \leq E, X \text{ is a cocircuit of } M\}$.

The sets $D_M(i;\theta)$ will, in certain circumstances, form the blocks of weighted subspace designs.

Proposition 65. *Let $\theta \in \mathbb{F}$ such that $\theta^s \neq 1$ for any $s \in [r]$. For each $i \in [n]$ every member of $D_M(i;\theta)$ is a dependent space of M^*. Moreover $D_M(d_M;\theta)$ is precisely the set of circuits of M^* of dimension d_M.*

Proof. If $A \in D_M(i;\theta)$, then $p(M.A;\theta) \neq 0$, which by Lemma 29 means that A is a dependent space of M^*. We show that for circuits of M^* (i.e., minimal dependent spaces of M^*) the converse also holds. By Lemma 30, for any circuit X of M^* we have $p(M.X;z) = z^{\ell(X^\perp)}-1$. Since X is not independent in M^*, $\ell(X^\perp) = r\dim(X)-\rho^*(X) = r\dim(X) - r\dim(X) + s = s$ for some $s \in [r]$. Therefore, by our choice of θ, we have that $p(M.X;\theta) = \theta^{\ell(X^\perp)} - 1 \neq 0$ and so $X \in D_M(\dim(X);\theta)$. In particular, $D_M(d_M;\theta)$ is precisely the set of all circuits of M^* of dimension d_M. $\square$

We will now present the main results of this section: Theorem 66 and its two corollaries. Together they form a q-analogue of [5, Theorem 3.3].

Theorem 66. *Let $\theta \in \mathbb{F}$ such that $\theta^s \neq 1$ for any $s \in [r]$. Let $t < d_M$ be a positive integer and suppose that:*

(1) $\sigma^* := |R_M(t;\theta)| \leq d_M - t$ and
(2) for each t-dimensional subspace T, we have that $A_{M^*/T}(j;\theta) = 0$ for all $j \in S_M(t;\theta)$.

Then $(E, D_M(d_M;\theta), f)$ is a weighted t-design over $\mathbb{F}_q$ with $f(X) := p(M.X;\theta)$ for all $X \leq E$.

Proof. Let T be a t-dimensional subspace of E. Since $t < d_M$, T is independent in M^*. By Lemma 15, any dependent space A of M^*/T has the form $A = B/T$ for a dependent space B of M^*. Therefore, for any such A and B we have

$$\sigma^* \leq d_M - t \leq \dim(B) - t = \dim(A). \tag{5}$$

In other words, no dependent space of M^*/T has dimension less than σ^*. By Lemma 29, if X is non-trivial and independent in M^*/T, then $p((M^*/T)^*.X;\theta) = 0$. Therefore,

$$A_{(M^*/T)^*}(i;\theta) = \sum_{X \leq E/T\,:\,\dim(X)=i} p((M^*/T)^*.X;\theta) = 0, \text{ for all } 1 \leq i \leq \sigma^* - 1.$$

By hypothesis, $A_{M^*/T}(j;\theta) = 0$ for all $j \in S_M(t;\theta)$ and so the coefficients,

$$[A_{M^*/T}(j;\theta) : j \in S_M(t;\theta)] \text{ and } [A_{(M^*/T)^*}(i;\theta) : 1 \leq i \leq \sigma^* - 1],$$

are known. Now apply Corollary 60 to the set $R_M(t;\theta)$ to see that the coefficients

$$[A_{M^*/T}(j;\theta) : j \in R_M(t;\theta)] \text{ and } [A_{(M^*/T)^*}(i;\theta) : \sigma^* \leq i \leq n-t]$$

are uniquely determined and independent of our choice of T of dimension t. It follows that the $A_{(M^*/T)^*}(i;\theta)$ are uniquely determined for $0 \leq i \leq n-t$. We will now show that

$$\sum_{X \in D_M(d_M;\theta)\,:\,T \leq X} p(M.X;\theta) = A_{(M^*/T)^*}(d_M - t;\theta),$$

which will establish that $(E, D_M(d_M;\theta), f)$ is a weighted t-design over $\mathbb{F}_q$ with $f(X) := p(M.X;\theta)$.

We claim there is a one-to-one correspondence between the members of $D_M(d_M;\theta)$ that contain T and the members of $D_{(M^*/T)^*}(d_M - t;\theta)$. Let B be a circuit of M^* that contains T such that $\dim(B) = d_M$. From Lemma 15, B/T is a circuit of M^*/T and $\dim(B/T) = \dim(B) - t = d_M - t$. Conversely, if A is a circuit of M^*/T satisfying $\dim(A) = d_M - t$, then $A = B/T$ for a dependent space B of M^* of dimension $\dim(B) = d_M$, which is therefore a circuit of M^*, as it has minimal dimension. By Proposition 65, $D_M(d_M;\theta)$ is the set of all cocircuits of M of dimension d_M and hence there is a one-to-one correspondence between the members of $D_M(d_M;\theta)$ that contain T and the circuits of M^*/T of dimension $d_M - t$. By Equation (5), any dependent space of M^*/T of dimension $d_M - t$ is a circuit of M^*/T and hence is a member of $D_{(M^*/T)^*}(d_M - t;\theta)$. This establishes the claim.

From Corollary 33, for any circuit X/T of M^*/T we have

$$p((M^*/T)^*.(X/T);\theta) = \theta^{\ell(X^\perp)} - 1.$$

Therefore,

$$\begin{aligned}
\sum_{X \in D_M(d_M;\theta) : T \leq X} p(M.X;\theta) &= \sum_{X \in D_M(d_M;\theta)\,:\,T \leq X} (\theta^{\ell(X^\perp)} - 1), \\
&= \sum_{X/T \in D_{(M^*/T)^*}(d_M - t;\theta)} (\theta^{\ell(X^\perp)} - 1), \\
&= \sum_{X/T \leq E/T\,:\,\dim(X/T) = d_M - t} p((M^*/T)^*.(X/T);\theta), \\
&= A_{(M^*/T)^*}(d_M - t;\theta),
\end{aligned}$$

which is independent of our choice of T of dimension t. It follows that $(E, D_M(d_M;\theta), f)$ is a weighted t-design over $\mathbb{F}_q$ with $f(X) := p(M.X;\theta)$. $\square$

Remark 67. In the proof of Theorem 66, we saw that with the hypothesis of the theorem, that the $A_{(M^*/T)^*}(i;\theta)$ (and therefore the $A_{M^*/T}(i;\theta)$) are uniquely determined for

$0 \le i \le n-t$. By Lemma 11, it follows that the $A_{M|T^\perp}(i;\theta)$ are uniquely determined for $0 \le i \le n-t$.

Corollary 68. *Let $\theta \in \mathbb{F}$ such that $\theta^s \neq 1$ for any $s \in [r]$. Let $t < d_M$ be a positive integer and suppose that:*

(1) $\sigma^* := |R_M(t;\theta)| \le d_M - t$ and
(2) for each t-dimensional subspace T, we have that $A_{M^*/T}(j;\theta) = 0$ for all $j \in S_M(t;\theta)$.

Then for each $j \in \{d_M, \dots, n-t\}$, $(E, D_M(j;\theta), f)$ is a weighted t-design over $\mathbb{F}_q$ with $f(X) := p(M.X;\theta)$ for each $X \le E$.

Proof. We will prove by induction on $w \in \{d_M, \dots, n-t\}$ that $(E, D_M(w;\theta), f)$ is a weighted t-design. The first step was proved in Theorem 66. Suppose now that $(E, D_M(j;\theta), f)$ is a weighted t-design for each $j \in \{d_M, \dots, w-1\}$. We will show that $(E, D_M(w;\theta), f)$ is also a weighted t-design.

Let $T \le E$ have dimension t. We will show that the following sum depends only on t:

$$\sum_{W \in D_M(w;\theta),\, T \le W} p(M.W;\theta) = \sum_{W\,:\, T \le W,\, \dim(W)=w} p(M.W;\theta).$$

Note that since $D_M(w;\theta)$ is the set of w-dimensional subspaces of E for which $p(M.W;\theta) \neq 0$, the above equality holds. From Lemma 34, for any $T \le W \le E$ we have that

$$p(M|T^\perp/W^\perp;\theta) = \sum_{A\,:\, A+T=W} p(M.A;\theta).$$

Let $\phi : \mathcal{L}(E/T) \longrightarrow \mathcal{L}(T^\perp)$ be defined by $\phi(A/T) = (A^\perp)^{\perp(T^\perp)}$, for each $A \le E$ such that $T \le A$ (as in Lemma 11). For any subspace W containing T and subspace $X = \phi(W/T)$, we have that $(W^\perp)^{\perp(T^\perp)} = X$, so $X^{\perp(T^\perp)} = W^\perp$ and hence $M|T^\perp.X = (M|T^\perp)/W^\perp$. Then clearly, $\dim(W) = \dim(X) + \dim(T)$. It follows that if T is a t-dimensional space, then for any $j \in \{d_M, \dots, n-t\}$, we have:

$$A_{M|T^\perp}(j;z) = \sum_{X \le T^\perp\,:\, \dim(X)=j} p(M|T^\perp.X;z) = \sum_{W:\, T \le W,\, \dim(W)=j+t} p(M|T^\perp/W^\perp;z).$$

Therefore, we have:

$$A_{M|T^\perp}(w-t;\theta) = \sum_{\substack{W\,:\,T \le W,\\ \dim(W)=w}} p(M|T^\perp/W^\perp;\theta) = \sum_{\substack{W\,:\,T \le W,\\ \dim(W)=w}} \sum_{A\,:\,A+T=W} p(M.A;\theta).$$

For any $I \leq T$, we write I_T to denote an arbitrary fixed subspace of T satisfying $I \oplus I_T = T$. Clearly, if $I = A \cap T$ we have $I \leq A$ and $A \cap I_T = \{0\}$. Conversely, if $I \leq A$ and $I_T \cap A = \{0\}$ then $A \cap T = A \cap (I + I_T) = I$. Moreover, if W is a w-dimensional subspace for which $A + T = W$ and $A \cap T = I$, then $\dim(A) = w - t + \dim(I)$. Therefore, we can rewrite the double summation as follows:

$$\begin{aligned} A_{M|T^\perp}(w-t;\theta) &= \sum_{i=0}^{t} \sum_{\substack{I\,:\,I\leq T,\\ \dim(I)=i}} \sum_{\substack{A\,:\,I\leq A,\, I_T\cap A=\{0\},\\ \dim(A)=w-t+i}} p(M.A;\theta) \\ &= \sum_{\substack{A\,:\,T\leq A,\\ \dim(A)=w}} p(M.A;\theta) + \sum_{i=0}^{t-1} \sum_{\substack{I\,:\,I\leq T\\ \dim(I)=i}} \sum_{\substack{A\,:\,I\leq A,\, I_T\cap A=\{0\},\\ \dim(A)=w-t+i}} p(M.A;\theta). \end{aligned}$$

Let $I \leq T$ such that $\dim(I) = i < t$ and so $d_M - t \leq w - t \leq w - t + i \leq w - 1$. By hypothesis, for each $1 \leq j \leq w - 1$, $(E, D_M(j;\theta), f)$ is a weighted t-design with $f(X) := p(M.X;\theta)$, and so by Theorem 62,

$$\sum_{\substack{A\,:\,I\leq A,\, I_T\cap A=\{0\},\\ \dim(A)=w-t+i}} p(M.A;\theta) = \Lambda^w_{i,t-i}(M;\theta),$$

for $\Lambda^w_{i,t-i}(M;\theta)$ that depend only on t, w, i. It follows that

$$\sum_{\substack{W\,:\,T\leq W,\\ \dim(W)=w}} p(M.W;\theta) = A_{M|T^\perp}(w-t;\theta) - \sum_{i=0}^{t-1} \begin{bmatrix} t \\ i \end{bmatrix}_q \Lambda^w_{i,t-i}(M;\theta).$$

By Remark 67, $A_{M|T^\perp}(w-t;\theta)$ is independent of our choice of T of dimension t and so the result follows. $\square$

Corollary 69. *Let $\theta \in \mathbb{F}$ such that $\theta^s \neq 1$ for any $s \in [r]$. Let $t < d_M$ be a positive integer and suppose that:*

(1) $\sigma^* := |R_M(t;\theta)| \leq d_M - t$ and
(2) for each t-dimensional subspace T, we have that $A_{M^*/T}(j;\theta) = 0$ for all $j \in S_M(t;\theta)$.

Then for each $j \in \{d_{M^*}, \ldots, n-t\}$, $(E, D_{M^*}(j;\theta), f)$ is a weighted t-design over $\mathbb{F}_q$ with $f(X) := p(M^*.X;\theta)$ for all subspaces $X \leq E$.

Proof. For each j such that $d_{M^*} \leq j \leq n-t$, define the set $\mathcal{D}_j := \{X^\perp : X \in D_{M^*}(j;\theta)\}$. Let T be a t-dimensional subspace of E. Now for each $X \leq T^\perp$ we have $(E/T)/(X^\perp/T) \cong E/X^\perp$ and it is easy to see that the corresponding q-polymatroids are lattice-equivalent.

Let $\phi : \mathcal{L}(E/T) \longrightarrow \mathcal{L}(T^\perp)$ be defined by $\phi(X/T) = (X^\perp)^{\perp(T^\perp)}$, for all $X \leq E$ such that $T \leq X$. We get that

$$\begin{aligned} M^*.X &\cong (M^*/T)/(X^\perp/T) \cong (M|T^\perp)^*/\phi(X^\perp/T) = (M|T^\perp)^*.\phi(X^\perp/T)^{\perp(T^\perp)} \\ &\cong (M|T^\perp)^*.X. \end{aligned}$$

Therefore, for each $j \in \{d_{M^*}, \dots, n-t\}$, we have:

$$\begin{aligned} \sum_{X \in \mathcal{D}_j : T \leq X} p(M^*.X^\perp; \theta) = \sum_{\substack{X : X \leq T^\perp, \\ \dim(X) = j}} p(M^*.X; \theta) &= \sum_{\substack{X : X \leq T^\perp, \\ \dim(X) = j}} p((M|T^\perp)^*.X; \theta) \\ &= A_{(M|T^\perp)^*}(j; \theta). \end{aligned}$$

From Remark 67, for each $j \leq n-t$, $A_{(M|T^\perp)^*}(j;\theta)$ is independent of the choice of T of dimension t. It follows that $(E, \mathcal{D}_j, f^*)$ is a weighted subspace design with f^* defined by $f^*(X) = p(M^*.X^\perp; \theta)$ for each $X \leq E$. The result now follows by Corollary 63: the required subspace design is the dual of $(E, \mathcal{D}_j, f^*)$. □

Remark 70. The results of Proposition 65, Theorem 66 and Corollaries 68 and 69 all hold with indeterminate z in place of a specific choice of θ in $\mathbb{F}$. In particular, $p(M.X; z)$ is a non-zero polynomial in $\mathbb{Z}[z]$ for any cocircuit X of M.

In general, a (q,r)-polymatroid M may satisfy the hypothesis of Corollary 68 for one choice of θ, but fail for another choice. However, if the hypothesis holds for indeterminate z, then a weighted t-design over $\mathbb{F}_q$ can be constructed for any choice of θ that doesn't vanish on $p(M.X; z)$ for a cocircuit X of M.

Example 71. Let $M = (E, \rho)$ be the uniform q-matroid $U_{k,n}$, as described in Example 16. We will show that this q-matroid satisfies the hypothesis of Corollary 68 with indeterminate z in place of a specific choice of θ in some field $\mathbb{F}$.

The dual q-matroid $M^* = (E, \rho^*)$ is the uniform q-matroid $U_{n-k,n}$, whose independent spaces are exactly those of dimension $n-k$ or less, and for which all other spaces are dependent and have rank $n-k$. Therefore, every cocircuit of M has dimension $d_M = n-k+1$. Now $p(M^*.X; z) = 0$ for all subspaces X such that $\dim(X) \in [k]$, as these are the independent spaces of M (see Lemma 29), and so $A_{M^*}(i; z) = 0$ for all $i \in [k]$. Therefore for any $t \leq d_M - 1 = n-k$, we have $R_M(t; z) \subseteq \{k+1, \dots, n-t\}$ and so $|R_M(t; z)| \leq n-t-k \leq d_M - t$. Therefore, for any $t < d_M$, hypothesis (1) of Corollary 68 holds for indeterminate z.

We now show that hypothesis (2) of Corollary 68 holds for indeterminate z; that is, for all $j \leq n-t$, if $A_{M^*}(j; z) = 0$, then $A_{M^*/T}(j; z) = 0$. Let T be a t-dimensional subspace of E for some $t \in [n-k]$. By Lemma 43 we have

$$A_{M^*/T}(j;z) = \sum_{X \leq T^\perp \,:\, \dim(X)=j} p(M^*.X;z).$$

Since $p(M^*.X;z) = 0$ for all subspaces X such that $\dim(X) \in [k]$, we have that $A_{M^*/T}(j;z) = 0$ for all $j \in [k]$.

Next, we consider the case $k+1 \leq j \leq n-t$. Let $X \leq E$ be a subspace of dimension at least $k+1$. We claim that the q-matroid $M^*.X$ has no loops, in which case by Lemma 39, $p(M^*.X;z)$ will be a monic polynomial of degree $n-k-\dim(X^\perp)$ and hence $A_{M^*}(j;z) \neq 0$ for $k+1 \leq j \leq n-1$, i.e. the condition holds vacuously. Consider a subspace U that strictly contains $X^\perp$. Since $M^* = U_{n-k,n}$, we have $\rho^*(X^\perp) = \dim(X^\perp)$ and so

$$\begin{aligned}
\rho_{M^*/X^\perp}(U/X^\perp) &= \rho^*(U) - \rho^*(X^\perp) \\
&= \rho^*(U) - \dim(X^\perp) \\
&= \dim(U) - \rho(E) + \rho(U^\perp) - \dim(X^\perp) \\
&= \dim(U) - \dim(X^\perp) + \rho(U^\perp) - k
\end{aligned}$$

We have that $\rho(U^\perp) = \min\{\dim(U^\perp), k\}$. Substituting both cases in the equation above and using that $\dim(X) - k \geq 1$ and $\dim(U) - \dim(X^\perp) \geq 1$, respectively, we find that $\rho_{M^*/X^\perp}(U/X^\perp) \geq 1$. This implies that the q-matroid $M^*.X$ has no loops.

We conclude that $M = U_{k,n}$ satisfies the hypothesis of Corollary 68 for indeterminate z. Therefore, $(E, D_{U_{k,n}}(i;z), f)$ is a weighted t-design for $n-k+1 \leq i \leq n-t$, where $f : D_{U_{k,n}} \longrightarrow \mathbb{Z}[z]$ is defined by $f(X) = p(U_{k,n}.X;z)$ for all $X \in D_{U_{k,n}}(i;z)$. However, for any j-dimensional subspace X such that $n-k+1 \leq j \leq n-t$ we have that $M.X$ has no loops and so $p(M.X;z)$ is non-zero. Hence, for each such j, $D_M(j;z)$ comprises all the j-dimensional subspaces of E, which means the corresponding weighted subspace designs are trivial.

On the other hand, from what we have just shown, for any $\theta \in \mathbb{F}$ such that $\theta^s \neq 1$, $s \in [r]$, we have that $(E, D_{U_{k,n}}(i;\theta), f)$ is a weighted subspace design for $n-k+1 \leq i \leq n-t$, and $f(X) = p(U_{k,n}.X;\theta)$ for all $X \in D_{U_{k,n}}(i;\theta)$. One may ask if there exists θ such that $(E, D_{U_{k,n}}(i;\theta), f)$ is non-trivial for some i. In fact there does not. If X has dimension i and there exists θ such that $p(U_{k,n}.X;\theta) = 0$, then $p(U_{k,n}.Y;\theta) = 0$ for any subspace Y of the same dimension i as X since the polynomial $p(U_{k,n}.X;z)$ depends only on the dimension of X. This means that for each θ, either $D_{U_{k,n}}(i;\theta)$ is empty, or comprises all the subspaces of dimension i.

6.3. Further implications

We now obtain a weaker form of the Assmus-Mattson Theorem for matrix codes as a direct consequence of Theorem 66. Note that the result for subspace designs (those weighted designs for which $f(B) = 1$ for every block B) obtained from rank-metric codes was shown in [10] with the further assumption that the number of codewords with a given support was dependent only on the dimension i of that space for some range of i.

Corollary 72. *Let C be an $\mathbb{F}_q$-$[n\times m, k, d]$ rank-metric code. Let $t < d$ be a positive integer and let $C^\perp$ have no more than $d-t$ distinct rank weights in the set $[n-t]$. For each $i \in \{d, \dots, n-t\}$, let*

$$B(i) = \{U \leq E : \dim(U) = i, |C_{=U^\perp}| \neq 0\}.$$

Then for each $i \in \{d, \dots, n-t\}$, $(E, B(i), f)$ is a weighted t-design over $\mathbb{F}_q$ with $f(X) := |C_{=X^\perp}|$ for all $X \leq E$.

Proof. Let $M := M_C$. By Lemma 48, we have that $M^* = M_{C^\perp}$ and for any $i \in [n]$, $W_i(C^\perp) = A_{M^*}(i;q)$. Also, $p(M.X;q) = |C_{=X^\perp}|$ for any subspace $X \leq E$. Now $d_M = \min\{\dim(X) : X$ is a cocircuit of $M\}$, which by Proposition 65, is the minimum dimension of any subspace X such that $p(M.X;q) \neq 0$.

Since C has minimum distance d, by Lemma 48 (4) there exists a d-dimensional subspace $X \leq E$ such that $|C_{=X^\perp}| = p(M.X;q) \neq 0$, while $p(M.U;q) = 0$ for every subspace $U \leq E$ with $\dim(U) < d$. Therefore, $d = d_M$. By hypothesis, at most $d - t = d_M - t$ of the integers $W_i(C^\perp)$ are non-zero for $i \in \{1, \dots, n-t\}$. By Lemma 48, if $A_{M^*}(i;q) = 0$, then $A_{M^*/T}(i;q) = 0$, for any t-dimensional subspace $T \leq E$. Therefore M satisfies the hypothesis of Corollary 68 and so the result follows. $\square$

In the case of a q-matroid M satisfying the hypothesis of Theorem 66, with an extra assumption on the cocircuits of M, our results imply the existence of a subspace design. These results form a direct q-analogue of the classical case (cf. [5, Section 3]).

Lemma 73. *Let M be a q-matroid, $\theta \in \mathbb{F}, \theta \neq 1$ and let p be the greatest integer such that any subspace $X \leq E$ of dimension at most p contains at most one cocircuit of M. Then for each $i \in \{d_M, \dots, p\}$, we have $D_M(i;\theta) = \{C \leq E : C$ a cocircuit of $M, \dim(C) = i\}$.*

Proof. If C is a cocircuit of M, then $p(M.C;\theta) = \theta - 1 \neq 0$ and so $C \in D_M(\dim(C);\theta)$. Now let $X \in D_M(i;\theta)$ for some $i \leq p$. Then $p(M.X;\theta) \neq 0$ and X is a dependent space of M^* of dimension at most p, so X contains a unique circuit of M^*. By Theorem 40, we have $X = C$ and the result follows. $\square$

Corollary 74. *Let M be a q-matroid that has at least one circuit and one cocircuit. Let $t < d_M$ be a positive integer such that the hypothesis of Theorem 66 holds for some $\theta \in \mathbb{F}, \theta \neq 1$. Let p be the greatest integer such that any subspace $X \leq E$ of dimension at most p contains at most one cocircuit (respectively, at most one circuit) of M. Then for each $i \in \{d_M, \dots, p\}$ (respectively, $\{d_{M^*}, \dots, p\}$) the set of cocircuits (respectively, the set of circuits) of M of dimension $\min\{i, n-t\}$ forms the blocks of a t-subspace design. Consequently, for each $i \in \{d_M, \dots, p\}$ (respectively, $\{d_{M^*}, \dots, p\}$), the set of* *hyperplanes of M (respectively, of M^*) of dimension $n-i$ is the set of blocks of a t-subspace design.*

Proof. From Lemma 73, for each $i \in \{d_M, \dots, p\}$ we have that $C_i := D_M(i;\theta)$ is the set of cocircuits of M of dimension i. Then by Corollary 68, for each $i \in \{d_M, \dots, p\}$, C_i is the set of blocks of a weighted t-subspace design with $f(X) = p(M.X;\theta) = \theta - 1$. Define a function $\hat{f} : D_M(i;\theta) \longrightarrow \mathbb{F}$ by $\hat{f}(X) = (\theta-1)^{-1} f(X)$. This yields a t-subspace design $\mathcal{D}_i$ whose blocks are C_i. By [9, Corollary 71], for each i-dimensional cocircuit X of M, $X^\perp$ is a hyperplane of M and has dimension $n-i$. By Corollary 63, the set of hyperplanes of M of dimension $n-i$ form the blocks of a t-subspace design, i.e., the dual design of $\mathcal{D}_i$. With the same arguments as above, by Corollary 69 the analogous statements hold for the circuits of M and the hyperplanes of M^*. □

An element c of an $\mathbb{F}_{q^m}$-$[n,k,d]$ vector rank-metric code C is called *minimal* if for any $c' \in C$, $\sigma(c') \leq \sigma(c)$ implies $c' \in \langle c \rangle_{\mathbb{F}_{q^m}} := \{\nu c : \nu \in \mathbb{F}_{q^m}\}$. In this case, for $U = \sigma(c)$ and $M := M_C$, we have $p(M/U; q^m) = |C_{=U}| = q^m - 1$. If every codeword of rank i in C is minimal, then $A_{M^*}(i; q^m) = W_i(C^\perp) = (q^m - 1)|D_M(i; q^m)|$. If we apply this with Corollary 74, we retrieve the Assmus-Mattson Theorem for $\mathbb{F}_{q^m}$-$[n,k,d]$ codes (cf. [10]).

Corollary 75. *Let C be an $\mathbb{F}_{q^m}$-$[n,k,d]$ code. Let $t < d$ be a positive integer and let $C^\perp$ be an $\mathbb{F}_{q^m}$-$[n, n-k, d^\perp]$ code having no more than $d-t$ distinct rank weights in the set $\{1, \dots, n-t\}$. Let p be the greatest integer such that every codeword of C of rank at most p is minimal.*

1. *The supports of the words of rank weight d in C (respectively $d^\perp$ in $C^\perp$) form the blocks of a t-design over $\mathbb{F}_q$.*
2. *For each $i \in \{d, \dots, p\}$ (respectively, $\{d^\perp, \dots, p\}$) the supports of the minimal codewords of C (respectively $C^\perp$) of dimension $\min\{i, n-t\}$ form the blocks of a t-design over $\mathbb{F}_q$.*

Example 76. In [26, Theorem 12], it is shown that any non-degenerate $\mathbb{F}_{q^m}$-$[N, k > 1]$ rank-metric code with constant weight d satisfies $N = km$, $d = m$ and is generated by a matrix $G \in \mathbb{F}_{q^m}^{k \times N}$ whose N columns form a basis of $\mathbb{F}_{q^m}^k$ as an $\mathbb{F}_q$-vector space. Moreover, the dual code has minimum distance 2. Let $C^\perp$ be an $\mathbb{F}_{q^m}$-$[km, k, m]$ constant weight code constructed as above. Let $M = M_C$, so that $M^* = M_{C^\perp}$. For any $X \leq \mathbb{F}_q^{km}$, we have $p(M.X; q^m) = 0$ unless X is the support of a codeword of $C^\perp$, in which case $\dim(X) = m$. Therefore, $A_{M^*}(m; q^m) = q^{km} - 1$, $A_{M^*}(0; q^m) = 1$ and $A_{M^*}(i; q^m) = 0$ for $i \neq 0, m$. Then $d_M = d = 2$ and $R_M(2;\theta) = \{m\}$. Therefore, by Corollary 75 the cocircuits of M of dimension 2, which are the supports of codewords of rank 2, form a 1-design over $\mathbb{F}_q$. Similarly, the supports of the words of rank m in $C^\perp$ form the blocks of a 1-design over $\mathbb{F}_q$, in fact a 1-$(km, m, 1; q)$ design, which is a q-Steiner system whose blocks form a spread in $\mathbb{F}_q^{km}$.

While Theorem 66 has considerable potential for constructing weighted subspace designs, utilizing it requires constructions of a q-polymatroid M whose weight enumerator takes few non-zero values and whose cocircuits have large enough dimension. Most q-matroids and q-polymatroids are not representable, however those that are, i.e. those that can be represented by rank metric codes, offer more tangible constructions.

In order to search for examples of rank-metric codes satisfying the conditions in Corollary 75 we implemented in Magma [3] a search through random $\mathbb{F}_{2^m}$-$[n,k,d]$ rank metric codes, for different values of m, n, and k and $t=2$. We make some remarks on the parameters of potentially interesting codes.

A matrix code is called maximum rank distance (MRD) if it has parameters $\mathbb{F}_q$-$[n\times m,k,d]$ with $k=\max\{n,m\}(\min\{n,m\}-d+1)$. The MRD $\mathbb{F}_{q^m}$-linear codes have parameters $\mathbb{F}_{q^m}$-$[n,n-d+1,k]$. MRD codes do satisfy the criteria of Corollaries 72 and 75 and there are several constructions of them. However, the corresponding subspace designs associated with these codes are trivial. We therefore would exclude them from our search space. Note that, as either m or q grow asymptotically, the $\mathbb{F}_{q^m}$-linear MRD codes are dense in the space of linear codes (see for example [11,24]), just as MDS codes are dense as q becomes large. However, for small values of q and m, we do not necessarily have a high probability of a random code being MRD, and this was checked and confirmed for all sets of parameters chosen for experiments. We also have modest constraints on the sizes of m,n to exclude trivialities. For example with $t=2$, we require that $d\geq 3$. If $m\leq 4$ and $d=3$, then to find a suitable code C, we would require $C^\perp$ to be a one-weight code. However, by [10, Proposition 4.6], the dual code of a constant weight $\mathbb{F}_q$-$[n\times m,k]$ code has minimum distance at most 2. If $m=4=d$, then C must be a one-weight code and so $C^\perp$ has weight at most 2, which would not yield interesting results. We therefore set $m>4$ in order to meet the criteria of Corollary 75. This means that we search for linear codes over alphabets of size at least 2^5. In most experiments, we chose $m=n-1$ or $m=n$ to increase the probability of satisfying the criteria.

Each code is given by a generator matrix in standard form for a linear code, i.e. $(I_k|A)$, where A goes through the space of $k\times(n-k)$-matrices with entries in $\mathbb{F}_{2^m}$, up to equivalence under the action of the Galois group $\mathrm{Gal}(\mathbb{F}_{2^m}/\mathbb{F}_2)$. This yields a search space of size $2^{m(k(n-k)-1)}$ matrices, which is quickly out of reach of a computer, even for small values of k and n. Ideally, a single representative in each equivalence class for the underlying q-polymatroids should be computed, but it is not clear to us how to pre-compute these representatives such that running the search would be more time effective.

In our algorithm, we first compute the weight distribution of the rank metric code by going through all code words (up to a scalar) and then we deduce the weight distribution of the dual code by using the MacWilliams identities. The code is publicly available [7]. For each set of parameters, we ran the code on a different core of an 2.40 GHz Intel Xeon E5-2640 processor, and we set a timeout of 16 days for each run. The number of codes that we were able to check in this way are given in the fifth column of Table 1.

Table 1
Random search through $\mathbb{F}_{2^m}$-$[n, k, d]$ random rank metric codes.

m	n	k	no. of codes checked	Proportion of search space
5	6	2	405,285,656	0.017
6	6	2	146,666,189	3.334×10^{-5}
6	6	3	442,349	1.572×10^{-9}
6	8	2	44,700,000	6.058×10^{-13}
7	8	2	13,800,000	9.132×10^{-17}
8	8	2	3,800,000	1.228×10^{-20}

While our search is far from complete, these numbers suggest that a more systematic search for higher parameter values would be needed to effectively construct examples of rank metric codes yielding weighted subspace designs. More generally, what is really required is a theoretical approach to construct rank-metric codes and q-(poly)matroids with prescribed weight distributions.

Declaration of competing interest

We have no conflicts of interest to declare.

Data availability

No data was used for the research described in the article.

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