

Research Article

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Connecting and Closed Geodesics of a Kropina Metric

<https://doi.org/10.1515/ans-2021-2133>

Received March 30, 2021; revised April 28, 2021; accepted April 29, 2021

Abstract: We prove some results about existence of connecting and closed geodesics in a manifold endowed with a Kropina metric. These have applications to both null geodesics of spacetimes endowed with a null Killing vector field and Zermelo's navigation problem with critical wind.

Keywords: Kropina Metric, Closed Geodesic, Lightlike Killing Field, Zermelo's Navigation

MSC 2010: 58E10, 53C22, 53C60

Communicated by: Silvia Cingolani

1 Introduction

Kropina metrics are homogeneous Lagrangians defined as the ratio of a Riemannian metric and a one-form, see [20]. Let S be a connected smooth manifold of dimension at least 2, and let g_0 , ω be, respectively, a Riemannian metric and a one-form on S . Let us assume that ω does not vanish at any point on S and, for each $x \in S$, let \mathcal{N}_x be the kernel of ω_x in $T_x S$. The Kropina metric on S associated to g_0 and ω is the Lagrangian $K : TS \setminus \mathcal{N} \rightarrow \mathbb{R} \setminus \{0\}$, defined as $K(v) := \frac{g_0(v,v)}{\omega(v)}$, where $\mathcal{N} := \bigcup_{x \in S} \mathcal{N}_x$.

For our purposes, it will be convenient to define K on $\mathcal{A} = \{v \in TS : -\omega(v) > 0\}$ as

$$K(v) = -\frac{g_0(v,v)}{2\omega(v)} \quad (1.1)$$

in such a way that K is a positive function on \mathcal{A} . On this domain K is a *conic Finsler metric* according to [16, Definition 3.1 (iii)], i.e. at each point $x \in S$, K_x is a Minkowski norm on \mathcal{A}_x , in particular for all $x \in S$ and for each $v \in \mathcal{A}_x$ its fundamental tensor

$$\mathbf{g}_v(u, w) := \frac{\partial^2}{\partial t \partial s} \frac{1}{2} K^2(v + tu + sw)|_{t=s=0},$$

$u, w \in T_x S$, is positive definite [16, Corollary 4.12]. We shall often call the couple (S, K) a *Kropina space*.

Notice that for all $x \in S$, $0 \notin \mathcal{A}_x$ though it is an accumulation point of the *indicatrix* \mathcal{I}_x of K at x , i.e. the set of vectors $\{v \in \mathcal{A}_x : K(v) = 1\}$. Thus, K is not extendible by continuity at 0. We point out that $\{0\} \cup \mathcal{I}_x$ is a compact strongly convex hypersurface in $T_x S$ (it is an ellipsoid, see [8, Proposition 2.57]).

The interest in the study of Kropina metrics (1.1) comes from some geometric and physical models.

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A first example where a Kropina space (S, K) appears is in general relativity. Let (S, K) be a Kropina space, let us consider the product manifold $S \times \mathbb{R}$ and let us denote by t the natural coordinate on \mathbb{R} and by $\pi : S \times \mathbb{R} \rightarrow S$ the canonical projection. Let g be the bilinear tensor field on $S \times \mathbb{R}$ defined by

$$g = \pi^* g_0 + \pi^* \omega \otimes dt + dt \otimes \pi^* \omega. \quad (1.2)$$

Since ω does not vanish on S , we have that g is a Lorentzian metric, t is a temporal function and $-\nabla t$ is timelike (see [8, Proposition 3.3]). Hence, $(S \times \mathbb{R}, g)$ is time-oriented by $-\nabla t$; moreover ∂_t is a lightlike Killing vector field. Observe now that a vector $(v, \tau) \in T_x S \times \mathbb{R}$ is future pointing and lightlike if and only if $v \in \mathcal{A}_x$ and $\tau = K(v)$, K in (1.1). Analogously, $(v, \tau) \in T_x S \times \mathbb{R}$ is past pointing and lightlike if and only if $v \in -\mathcal{A}_x$ and $\tau = -K(-v)$.

Thus, the future (resp. past) lightlike cones of the conformal class of $(S \times \mathbb{R}, g)$ are described by the flow lines of ∂_t and the graph of the function K on \mathcal{A} (resp. $-\mathcal{A}$).

This correspondence has been introduced in [8] extending that one between standard stationary Lorentzian metrics and Finsler metrics of Randers type ([6, 7]). Actually in [8], the class of spacetimes $S \times \mathbb{R}$ considered is larger and the Killing vector field ∂_t can also be spacelike in some points (but in this case, the associated Finsler geometry is not simply of Kropina or Randers type, see [8] for details).

A second model in which Kropina spaces appear is related to the *Zermelo's navigation problem* which consists in finding the paths between two points x_0 and x_1 that minimize the travel time of a ship or an airship moving in a wind in a Riemannian landscape (S, g_0) (see [9, 29, 32, 34]). If the wind is time-independent, then it can be represented by a vector field W on S . When $g_0(W, W) = 1$, called *critical wind* in [8], the solutions of the problem (if they exist) are the pregeodesics of the Kropina metric $K(v) := -\frac{g_0(v, v)}{2g_0(W, v)}$ associated to the Zermelo's navigation data g_0 and W which are minimizer of the length functional associated to K , see [8, Corollary 6.18 (i)]. This result and more general ones contained in [8] are strictly connected to the causality properties of the spacetime $S \times \mathbb{R}$ which is also associated to Zermelo navigation data ([8, Theorem 6.15]).

Recently, Kropina metrics have also been considered in relation to the so-called chains in a CR manifold M (see [10]). Indeed, these can be viewed as curves which are non-constant projections of null geodesics for an indefinite metric on a circle bundle over M whose action has infinitesimal generator which is a null Killing vector field. This interpretation leads to a very interesting relation of Kropina spaces with CR geometries and Lorentz geometry, see also [11, 18].

In [4, Question 2.5.1], the authors asked if a Kropina metric on a compact manifold admits a closed geodesic (this problem was posed in [8, Remark 6.29] as well). In the present work, we give some results in that direction plus results concerning the existence of multiple geodesics between two points.

Organization of the Paper. In Section 2, we introduce some notations, and we give some preliminary results; in particular we recall Theorem 2.3 from [8] where a connection between geodesics of a Kropina space (S, K) and lightlike geodesics of the spacetime $(S \times \mathbb{R}, g)$, g in (1.2) is established.

In Section 3, we introduce an approximation framework of the spacetime $(S \times \mathbb{R}, g)$, by a family of spacetimes $(S \times \mathbb{R}, g_\varepsilon)$, $\varepsilon > 0$, where ∂_t is a timelike Killing field (recall that ∂_t is a null Killing field for the spacetime $(S \times \mathbb{R}, g)$). These type of spacetimes are called in the literature *standard stationary* and variational methods for their geodesics and their causal properties are nowadays well-developed (cf. [2, 5–7, 12–14, 22, 23]). In particular, Theorem 3.1, obtained in [6], is the precursor of Theorem 2.3 and together with Lemma 3.2, it plays a fundamental role in proving existence of geodesics of the Kropina space associated to the limit spacetime. We emphasize that the same approximation has been profitably employed in [3] to study geodesic connectedness of a globally hyperbolic spacetime endowed with a null Killing vector field.

In Section 4, we obtain some results about the existence of geodesics between two given points of a Kropina space (see Proposition 4.1, Corollary 4.3, Theorem 4.4 and Corollary 4.5). In particular, Corollary 4.5 implies that the Zermelo's navigation problem associated to the data (S, g_0) and W , $g_0(W, W) = 1$, has always a solution in each connected component of the space of curves between two points $x_0, x_1 \in S$ (see Corollary 4.6).

Section 5 is devoted to the closed geodesic problem. Existence results are given in Theorem 5.1, Corollary 5.2, Theorem 5.3. In Examples 5.4 and 5.5, we apply Theorem 5.3 to prove the existence of a closed geo-

desic in some particular type of compact Kropina space (in particular, Example 5.4 for an odd-dimensional sphere can be considered as the Kropina limit of a family of Finsler metrics of the type in Katok's example (see [33]). A couple of results for a compact manifold endowed with a periodic Killing vector field Y and a one-form invariant by the flow of Y are further given in Theorems 5.6 and 5.8. We notice that the latter holds for any compact Lie group endowed with a bi-invariant Riemannian metric and a left-invariant one-form (Corollary 5.9).

Finally, we point out that a fundamental and natural assumption for the existence of connecting or closed geodesics in a Kropina manifold is that the space of paths considered, according to the boundary conditions that the geodesics have to satisfy, must contain at least one admissible path, i.e. a curve γ such that $\omega(\dot{\gamma}(s)) < 0$ everywhere (under the point of view of the Zermelo's navigation problem, we can say that there must be a "navigable region"). The problem of the existence of such an admissible path is related to the existence of horizontal paths for the distribution of hyperplanes pointwise representing the kernel of ω and, indeed, a non-integrability condition for it ensures that there do exist admissible paths (see Corollaries 4.5 and 5.2). Nevertheless, when there exist points in S which are not reachable from a given point by an admissible path, we show in the Appendix that the boundary of the set of reachable points is a smooth hypersurface in S .

2 Some Notations and Known Results

The set of continuous and piecewise smooth, admissible curves from x_0 to x_1 will be denoted by $\Omega_{x_0x_1}(\mathcal{A})$, i.e.

$$\Omega_{x_0x_1}(\mathcal{A}) := \{\gamma : [0, 1] \rightarrow S : \gamma(0) = x_0, \gamma(1) = x_1, \dot{\gamma}^-(s), \dot{\gamma}^+(s) \in \mathcal{A} \text{ for all } s \in [0, 1]\}$$

(here $\dot{\gamma}^-(s)$ and $\dot{\gamma}^+(s)$ denote respectively the left and the right derivative of γ at the point s). A geodesic of (S, K) connecting a point $x_0 \in S$ to $x_1 \in S$ is a critical point of the energy functional

$$E(\gamma) = \frac{1}{2} \int_0^1 K^2(\dot{\gamma}) ds,$$

defined on $\Omega_{x_0x_1}(\mathcal{A})$. Notice that, as \mathcal{A} is an open subset of TS , variational vector fields along a curve $\gamma \in \Omega_{x_0x_1}(\mathcal{A})$ are well-defined and then it makes sense to define geodesics as critical points of E . Moreover, since the fundamental tensor of K is positive definite on \mathcal{A} , it can be proved that the Legendre transform of K is injective (see [8, Proposition 2.5.1]) and then a critical point γ of E is smooth and parametrized with $K(\dot{\gamma}) = \text{const}$ (see also [8, Lemma 2.5.2]).

Analogously, a closed geodesic is a critical point of E defined on the set

$$\Omega(\mathcal{A}) = \{\gamma : [0, 1] \rightarrow S : \gamma(0) = \gamma(1), \dot{\gamma}^-(s), \dot{\gamma}^+(s) \in \mathcal{A} \text{ for all } s \in [0, 1]\},$$

Remark 2.1. We observe that, also in the simplest cases, a Kropina space can be not geodesically connected. This essentially may happen because the set $\Omega_{x_0x_1}(\mathcal{A})$ is empty. For example, consider a constant one-form ω on \mathbb{R}^n endowed with the Euclidean metric $\langle \cdot, \cdot \rangle$. It can be easily seen that the geodesic of $(\mathbb{R}^n, \frac{\langle \cdot, \cdot \rangle}{\omega(\cdot)})$ are the straight lines which do not lie on the hyperplanes parallel to the kernel of ω . Hence, there is no geodesic (and no admissible curve) connecting two points belonging to one of such hyperplanes.

Remark 2.2. Notice also that since $K(-v) = -K(v)$, if $\gamma : [0, 1] \rightarrow S$ is a geodesic of (S, K) (hence, according to the above definition, $\dot{\gamma}([0, 1]) \subset \mathcal{A}$) the reverse curve $\tilde{\gamma}(s) = \gamma(1 - s)$ is a geodesic of $(S, -K)$ with $-K$ viewed as a conic Finsler metric on $-\mathcal{A}$.

Geodesics of a Kropina space (S, K) are related to lightlike geodesics of the product spacetime $(S \times \mathbb{R}, g)$, g as in (1.2). We recall that, since the coefficients of the metric g do not depend on the variable t , the vector field $\partial_t \equiv (0, 1)$ is a Killing vector field for $(S \times \mathbb{R}, g)$, hence if $z = z(s) = (\gamma(s), t(s))$ is a geodesic of $(S \times \mathbb{R}, g)$, then $g(\partial_t, \dot{z}) = \omega(\dot{x})$ must be constant. Then the following theorem can be proved.

Theorem 2.3 ([8, Corollary 5.6 (i)]). *Let γ be a piecewise smooth admissible curve in (S, K) . Then γ is a pregeodesic of the Kropina space (S, K) such that $\omega(\dot{\gamma}) = \text{const} < 0$ if and only if the curve z defined by*

$$z(s) = \left(\gamma(s), t(s) = t_0 + \int_0^s K(\dot{\gamma}) \, dr \right), \quad t_0 \in \mathbb{R},$$

is a future pointing lightlike geodesic of $(S \times \mathbb{R}, g)$ with non-constant component γ .

Observe that for a given $t_0 \in \mathbb{R}$, $z : [a, b] \rightarrow S \times \mathbb{R}$ connects the points $(\gamma(a), t_0)$ and $(\gamma(b), t_0 + L(\gamma))$, where $L(\gamma)$ is the Kropina length of γ , i.e.

$$L(\gamma) := \int_a^b K(\dot{\gamma}) \, ds. \tag{2.1}$$

In particular, by [8, Theorem 5.5 (i)], if $\gamma : [0, T] \rightarrow S$ is a non-trivial unit (i.e. $K(\dot{\gamma}) = \text{const} = 1$) closed geodesic of (S, K) , then $z(t) = (\gamma(t), t)$ is a future pointing lightlike pregeodesic such that its component γ is periodic with period $T = L(\gamma)$, $z(0) = (\gamma(0), 0)$ and $z(T) = (\gamma(0), T)$.

Remark 2.4. When considering lightlike geodesics connecting a point (x_0, t_0) to a flow line of ∂_t passing through a point different from (x_0, t_0) , Theorem 2.3 can be interpreted as a version, for spacetimes of the type $(S \times \mathbb{R}, g)$, g as in (1.2), of the Fermat’s principle in general relativity, stating that lightlike geodesics connecting a point to a timelike curve τ are the critical points of the arrival time at the curve τ (see [19, 25]). Indeed, in the class of spacetimes we are considering, the arrival time of a future pointing lightlike curve $z(s) = (\gamma(s), t(s))$ connecting a point (x_0, t_0) to an integral curve of the field ∂_t is $T(\gamma) = t_0 + L(\gamma)$. We point out that here τ , which is an integral line of ∂_t , is not timelike but lightlike.

3 The Approximation Scheme with Standard Stationary Spacetimes

In this section we fix a Kropina space (S, K) with a Riemannian metric g_0 and a never vanishing one-form ω on the manifold S . For $\varepsilon > 0$, let us consider the standard stationary spacetime $(S \times \mathbb{R}, g_\varepsilon)$, where

$$g_\varepsilon = \pi^* g_0 + \pi^* \omega \otimes dt + dt \otimes \pi^* \omega - \varepsilon dt^2. \tag{3.1}$$

Notice that for each $\varepsilon > 0$, ∂_t is a timelike Killing vector field for g_ε . Let $(x_0, t_0) \in S \times \mathbb{R}$ and $\tau(t) = (x_1, t)$ be the integral line of ∂_t passing through the point $(x_1, 0)$. Let ∇ be the Levi-Civita connection of the Riemannian metric g_0 . Let us denote by d_0 the distance induced on S by g_0 . Moreover, let $\|\cdot\|_x$ be the norm on the space of linear operators on $T_x S$ endowed with the norm associated to the scalar product $(g_0)_x$, $x \in S$.

The geodesic equations for $(S \times \mathbb{R}, g_\varepsilon)$ are the following:

$$\begin{cases} \omega(\dot{x}_\varepsilon) - \varepsilon \dot{t}_\varepsilon = c_\varepsilon, \\ \nabla_{\dot{x}_\varepsilon} \dot{x}_\varepsilon = \dot{t}_\varepsilon \Omega^\sharp(\dot{x}_\varepsilon) - \omega^\sharp \dot{t}_\varepsilon, \end{cases} \tag{3.2}$$

while those of $(S \times \mathbb{R}, g)$ are

$$\begin{cases} \omega(\dot{x}) = c_0, \\ \nabla_{\dot{x}} \dot{x} = \dot{t} \Omega^\sharp(\dot{x}) - \omega^\sharp \dot{t}, \end{cases} \tag{3.3}$$

where $c_\varepsilon, c_0 \in \mathbb{R}$ and ω^\sharp and Ω^\sharp are the vector field and the $(1, 1)$ -tensor field g_0 -metrically equivalent respectively to ω and $\Omega = d\omega$.

The geodesic equations (3.2) for the standard stationary spacetime g_ε and (3.3) for the metric g can be obtained as the Euler–Lagrange equations of the respective energy functionals

$$I_\varepsilon(x, t) = \frac{1}{2} \int_0^1 g_\varepsilon(\dot{z}, \dot{z}) \, ds = \frac{1}{2} \int_0^1 [g_0(\dot{x}, \dot{x}) + 2\omega(\dot{x})\dot{t} - \varepsilon \dot{t}^2] \, ds$$

and

$$I_0(x, t) = \frac{1}{2} \int_0^1 g(\dot{z}, \dot{z}) \, ds = \frac{1}{2} \int_0^1 [g_0(\dot{x}, \dot{x}) + 2\omega(\dot{x})t] \, ds. \tag{3.4}$$

We point out that the constants c_ε and c_0 respectively in (3.2) and (3.3) derive from the fact that ∂_t is a Killing vector field both for g_ε and g , and so it gives rise to the conservation laws $g_\varepsilon(\dot{z}_\varepsilon, \partial_t) = c_\varepsilon$ and $g(\dot{z}, \partial_t) = c_0$, where $z_\varepsilon = (x_\varepsilon, t_\varepsilon)$ and $z = (x, t)$ are geodesics respectively for the metric g_ε and g .

Let us recall now the following:

Theorem 3.1 (Fermat principle in standard stationary spacetimes [6]). *A curve $z_\varepsilon : [0, 1] \rightarrow S \times \mathbb{R}$ given by $z_\varepsilon(s) = (x_\varepsilon(s), t_\varepsilon(s))$ is a future pointing lightlike geodesic of $(S \times \mathbb{R}, g_\varepsilon)$ if and only if x_ε is a pregeodesic of the Randers metric on S*

$$F_\varepsilon(v) := \frac{1}{\varepsilon} \left(\sqrt{\varepsilon g_0(v, v) + \omega^2(v)} + \omega(v) \right),$$

parametrized with $\varepsilon g_0(\dot{x}_\varepsilon, \dot{x}_\varepsilon) + \omega^2(\dot{x}_\varepsilon) = \text{const}$ and

$$t_\varepsilon(s) = t_0 + \int_0^s F_\varepsilon(\dot{x}_\varepsilon(r)) \, dr. \tag{3.5}$$

From Theorem 3.1, for a future pointing lightlike geodesic of $(S \times \mathbb{R}, g_\varepsilon)$, we have

$$t_\varepsilon(1) = t_0 + L_\varepsilon(x_\varepsilon), \tag{3.6}$$

where $L_\varepsilon(x_\varepsilon)$ is the length of x_ε w.r.t. the Randers metric F_ε ,

$$L_\varepsilon(x_\varepsilon) = \frac{1}{\varepsilon} \int_0^1 \left(\sqrt{\varepsilon g_0(\dot{x}_\varepsilon, \dot{x}_\varepsilon) + \omega^2(\dot{x}_\varepsilon)} + \omega(\dot{x}_\varepsilon) \right) \, dr. \tag{3.7}$$

In the next lemma we give a condition ensuring that a family of future pointing lightlike geodesics $(x_\varepsilon, t_\varepsilon)$ converges in the H^1 -topology to a future pointing lightlike geodesic of $(S \times \mathbb{R}, g)$. Let $H^1([0, 1], \mathbb{R})$ be the Sobolev space of absolute continuous functions on $[0, 1]$ with derivative in L^2 . Moreover, let $H_0^1([0, 1], \mathbb{R})$ be the subspace of $H^1([0, 1], \mathbb{R})$ consisting of functions $\tau(s)$ such that $\tau(0) = \tau(1) = 0$. Henceforth, we denote by $\mathcal{P}(S)$ both the Sobolev manifolds $\Lambda(S)$ of H^1 free loops on S or $\Lambda_{pq}(S)$ of H^1 paths between two points p and q (possibly equal) in S .

Lemma 3.2. *Assume that the Riemannian manifold (S, g_0) is complete and there exists a point \bar{x} and a positive constant $C_{\bar{x}}$ such that $\|\omega\|_x \leq C_{\bar{x}}(d_0(x, \bar{x}) + 1)$. For each $\varepsilon \in (0, 1)$, let $z_\varepsilon = (x_\varepsilon, t_\varepsilon) : [0, 1] \rightarrow S \times \mathbb{R}$ be a future pointing lightlike geodesic of $(S \times \mathbb{R}, g_\varepsilon)$. Let $\Delta_\varepsilon := t_\varepsilon(1) - t_\varepsilon(0)$ and let us assume that $\Delta := \sup_{\varepsilon \in (0,1)} \Delta_\varepsilon \in \mathbb{R}$. Then there exists a sequence $\varepsilon_n \rightarrow 0$ such that $(x_{\varepsilon_n}, t_{\varepsilon_n})$ converges in $\mathcal{P}(S) \times H^1([0, 1], \mathbb{R})$ to a future pointing lightlike geodesic $z = (x, t)$ of $(S \times \mathbb{R}, g)$.*

Proof. By [22, Theorem 3.3.2 and equations (3.5) and (3.17)], for each $\varepsilon > 0$, x_ε is a critical point of the following functional J_ε defined on $\mathcal{P}(S)$,

$$J_\varepsilon(x) = \frac{1}{2} \int_0^1 g_0(\dot{x}, \dot{x}) \, ds + \frac{1}{2\varepsilon} \int_0^1 \omega^2(\dot{x}) \, ds - \frac{\varepsilon}{2} \left(\Delta_\varepsilon - \frac{1}{\varepsilon} \int_0^1 \omega(\dot{x}) \, ds \right)^2$$

and $J_\varepsilon(x_\varepsilon) = 0$, for all $\varepsilon > 0$. Then

$$\int_0^1 g_0(\dot{x}_\varepsilon, \dot{x}_\varepsilon) \, ds \leq \varepsilon \Delta_\varepsilon^2 + 2\Delta_\varepsilon \int_0^1 |\omega(\dot{x}_\varepsilon)| \, ds \leq \Delta^2 + 2\Delta \int_0^1 |\omega(\dot{x}_\varepsilon)| \, ds$$

and, as in [2, Lemma 2.6], we obtain that the family of curves $(x_\varepsilon)_{\varepsilon \in (0,1)}$ is bounded in $\mathcal{P}(S)$. Moreover, from the second equation in (3.2), using the fact that Δ_ε is bounded, as in [3, Lemma 6.2] we get that the family $(\dot{t}_\varepsilon)_{\varepsilon \in (0,1)}$ is also bounded in $L^2([0, 1], \mathbb{R})$.

Now, for each ε , z_ε is a critical point of the energy functional I_ε of the Lorentzian metric g_ε , i.e.

$$\int_0^1 (g_0(\dot{x}_\varepsilon, \nabla_{\dot{x}_\varepsilon} \xi_\varepsilon) + \dot{t}_\varepsilon g_0(\nabla_{\dot{x}_\varepsilon} \omega^\sharp, \dot{x}_\varepsilon) + \dot{t}_\varepsilon g_0(\omega^\sharp, \nabla_{\dot{x}_\varepsilon} \xi_\varepsilon)) ds + \int_0^1 \omega(\dot{x}_\varepsilon) \dot{t} ds - \varepsilon \int_0^1 \dot{t}_\varepsilon \dot{t} ds = 0 \tag{3.8}$$

for any variational vector field ξ_ε , i.e. $\xi_\varepsilon \in T_{x_\varepsilon} \mathcal{P}(S)$, and for any $H^1_0([0, 1], \mathbb{R})$ function τ . As $(\dot{t}_\varepsilon)_{\varepsilon \in (0, 1)}$ is bounded in $L^2([0, 1], \mathbb{R})$, we get $\varepsilon \int_0^1 \dot{t}_\varepsilon \dot{t} ds \rightarrow 0$ as $\varepsilon \rightarrow 0$. Then, as in [22, Lemma 3.4.1], there exists a sequence $\varepsilon_n \rightarrow 0$ such that (x_{ε_n}) strongly converges to $x \in \mathcal{P}(S)$ as $n \rightarrow \infty$. Taking in (3.8) $\xi_{\varepsilon_n} = 0$, for each ε_n , we get $\int_0^1 \omega(\dot{x}_{\varepsilon_n}) \dot{t} ds - \varepsilon_n \int_0^1 \dot{t}_{\varepsilon_n} \dot{t} ds = 0$ and then passing to the limit on n , $\int_0^1 \omega(\dot{x}) \dot{t} ds = 0$. Therefore, $\omega(\dot{x})$ is constant. Now let t be the weak limit in $H^1([0, 1], \mathbb{R})$ of a sequence t_{ε_n} , $\varepsilon_n \in (0, 1)$, $\varepsilon_n \rightarrow 0$ as $n \rightarrow +\infty$. Then, since x_{ε_n} strongly converges to x , the curve $z(s) = (x(s), t(s))$ satisfies

$$\int_0^1 (g_0(\dot{x}, \nabla_{\dot{x}} \xi) + \dot{t} g_0(\nabla_{\dot{x}} \omega^\sharp, \dot{x}) + \dot{t} g_0(\omega^\sharp, \nabla_{\dot{x}} \xi)) ds = 0 \tag{3.9}$$

for any $\xi \in T_x \mathcal{P}(S)$. Therefore, z is a critical point of the energy functional I_0 in (3.4) of the Lorentzian metric g defined on $\mathcal{P}(S) \times (\{t\} + H^1_0([0, 1], \mathbb{R}))$.

By considering any H^1 variational vector field along z with compact support in a neighborhood J contained in $(0, 1)$ of any instant $s_0 \in (0, 1)$, and decomposing it in its components in $x^*_J(TS)$ and $H^1(J, \mathbb{R})$, this property of z remains true locally. Therefore, as the Lagrangian $(p, v) \in T(S \times \mathbb{R}) \mapsto g_p(v, v)$ is regular, it can be proved that z is smooth (see, e.g. [1, pp. 609–610]) and, therefore, it is a geodesic of $(S \times \mathbb{R}, g)$. (Notice also that if $\mathcal{P}(S) = \Lambda(S)$, as x satisfies (3.9) it must be a smooth 1-periodic curve.)

Since $I_{\varepsilon_n}(z_{\varepsilon_n}) \rightarrow I_0(z)$, as $n \rightarrow \infty$, and $I_\varepsilon(z_\varepsilon) = 0$, we have that z is lightlike. Finally, z is future pointing if, by definition, $\dot{t}(s) > 0$ for all $s \in [0, 1]$. Notice that $\dot{t}(s)$ cannot vanish at some $s \in [0, 1]$ because z is lightlike. Hence, \dot{t} cannot be negative otherwise $0 > t(1) - t(0) = \lim_{n \rightarrow \infty} (t_{\varepsilon_n}(1) - t_{\varepsilon_n}(0)) \geq 0$. \square

Remark 3.3. Notice that if $\mathcal{P}(S) = \Lambda(S)$ and the limit curve $z(s) = (x(s), t(s))$ has component x which is not constant, then $-\omega(\dot{x}) > 0$, i.e. x is admissible. This comes from the fact that, being z_{ε_n} future pointing in $(S \times \mathbb{R}, g_{\varepsilon_n})$,

$$0 > g_{\varepsilon_n}(\dot{z}_{\varepsilon_n}, \partial_t) = \omega(\dot{x}_{\varepsilon_n}) - \varepsilon_n \dot{t}_{\varepsilon_n} = \int_0^1 (\omega(\dot{x}_{\varepsilon_n}) - \varepsilon_n \dot{t}_{\varepsilon_n}) ds \rightarrow \int_0^1 \omega(\dot{x}) ds = \omega(\dot{x}) \quad \text{as } n \rightarrow \infty,$$

and the constant $\omega(\dot{x})$ cannot be 0 otherwise, as z is lightlike in $(S \times \mathbb{R}, g)$, we would have $g_0(\dot{x}, \dot{x}) = \text{const} = 0$.

4 The Existence of Geodesics Connecting Two Points

We refer to [24] for standard notations and notions about causality as, e.g., the subsets $I^+((x_0, t_0))$ and $J^+((x_0, t_0))$ which represent the set of points in a spacetime $(S \times \mathbb{R}, h)$ which can be connected to (x_0, t_0) by a future-pointing timelike or, respectively, causal curve.

Proposition 4.1. *Let $(S, -\frac{g_0(\cdot, \cdot)}{2\omega(\cdot)})$ be a Kropina space and let x_0, x_1 be two points on S such that $x_0 \neq x_1$ and $\Omega_{x_0 x_1}(\mathcal{A}) \neq \emptyset$. Assume that the Riemannian manifold (S, g_0) is complete and there exists a point \bar{x} and a positive constant $C_{\bar{x}}$ such that $\|\omega\|_x \leq C_{\bar{x}}(d_0(x, \bar{x}) + 1)$. Then there exists a geodesic γ of the Kropina space connecting x_0 to x_1 and which is a global minimizer of the Kropina length functional on $\Omega_{x_0 x_1}(\mathcal{A})$.*

Proof. Since g_0 is complete and $\|\omega\|_x \leq C_{\bar{x}}(d_0(x_0, x) + 1)$, from [28, Proposition 3.1 and Corollary 3.4] the spacetimes $(S \times \mathbb{R}, g_\varepsilon)$ are globally hyperbolic, for each $\varepsilon > 0$, with Cauchy hypersurfaces $S \times \{t\}$, $t \in \mathbb{R}$. As for any vector $w \in TS \times \mathbb{R}$, $g(w, w) \leq 0$ implies $g_\varepsilon(w, w) < 0$, also $(S \times \mathbb{R}, g)$ is globally hyperbolic with Cauchy hypersurfaces $S \times \{t\}$, $t \in \mathbb{R}$.

Let $Y = \{t \in (0, +\infty) : (x_1, t_0 + t) \in J^+((x_0, t_0))\}$. Now let $\gamma_0 \in \Omega_{x_0 x_1}(\mathcal{A})$ and consider the curve defined by $z(s) = (\gamma_0(s), t(s))$, with $t(s) = t_0 + \int_0^s K(\dot{\gamma}_0) dr$, which is lightlike and future pointing in $(S \times \mathbb{R}, g)$. Then

$L(\gamma_0) \in Y$, i.e. $Y \neq \emptyset$. Let $T = \inf Y$. The point $(x_1, t_0 + T)$ clearly belongs to $J^+((x_0, t_0)) \setminus I^+((x_0, t_0))$ hence there exists a future pointing lightlike geodesic $z(x) = (\gamma(s), t(s))$ connecting (x_0, t_0) to (x_1, T) (see [24, Proposition 10.46]). By Theorem 2.3, the projection γ is a pregeodesic of (S, K) which connects x_0 to x_1 and minimize the Kropina length functional. \square

Remark 4.2. As a globally hyperbolic spacetime is causally simple, Proposition 4.1 can be deduced by [8, Theorem 4.9 (i)] which concerns the more general case of a spacetime $S \times \mathbb{R}$ where ∂_t is a causal Killing vector field. For a related result see [17, Proposition 3.22].

Following [10, Section 5], we know that the condition of the existence of an admissible curve between two points in S is ensured provided a non-integrability condition for the kernel distribution \mathcal{N} of ω is satisfied. Precisely, if $\omega \wedge d\omega \neq 0$ in a connected, dense subset of S then there exists a smooth admissible curve between any two points p and q in S . Hence we get the following result about geodesic connectedness of a Kropina space that extends [10, Theorem 1.5] valid in the compact case:

Corollary 4.3. *Let (S, g_0) be a complete Riemannian manifold and ω be a nowhere vanishing one-form such that there exists a point \bar{x} and a positive constant $C_{\bar{x}}$ with $\|\omega\|_x \leq C_{\bar{x}}(d_0(x, \bar{x}) + 1)$ and $\omega \wedge d\omega \neq 0$ in a connected, dense subset of S . Then the Kropina space $(S, -\frac{g_0(\cdot, \cdot)}{2\omega(\cdot)})$ is geodesically connected.*

A multiplicity result holds if the fundamental group of S is non-trivial. This is based of the observation that the sequence of the lengths $\Delta_\varepsilon : t_\varepsilon(1) - t_\varepsilon(0)$ (or equivalently of the travel times as measured by observers at infinity in the standard stationary spacetimes $(S \times \mathbb{R}, g_\varepsilon)$), of the geodesics x_ε in Lemma 3.2 can be controlled from above provided these geodesics minimize the F_ε -length in some homotopy class $\mathcal{C} \subset \Lambda_{x_0x_1}(S)$ containing at least one admissible curve γ .

Theorem 4.4. *Let $(S, -\frac{g_0(\cdot, \cdot)}{2\omega(\cdot)})$ be a Kropina space and let x_0, x_1 be two points on S , $x_0 \neq x_1$. Assume that the Riemannian manifold (S, g_0) is complete and there exists a point \bar{x} and a positive constant $C_{\bar{x}}$ such that $\|\omega\|_x \leq C_{\bar{x}}(d_0(x, \bar{x}) + 1)$. Then for each connected component \mathcal{C} of $\Lambda_{x_0x_1}(S)$ there exists a geodesic of the Kropina space from x_0 to x_1 , which is a minimizer of the Kropina length functional on \mathcal{C} , provided that there exists an admissible curve $\gamma \in \mathcal{C}$. Moreover, if \mathcal{C} corresponds to a non-trivial element of the fundamental group of S , a geodesic loop in \mathcal{C} exists when x_0 and x_1 coincide.*

Proof. Let x_ε be a geodesic of (S, F_ε) with minimal F_ε -length in \mathcal{C} . From (3.6) we get

$$\Delta_\varepsilon := L_\varepsilon(x_\varepsilon) \leq L_\varepsilon(\gamma),$$

for each $\varepsilon > 0$, where L_ε is defined at (3.7). Since $\gamma \in \mathcal{C}$ is admissible, we obtain

$$L_\varepsilon(\gamma) = \frac{1}{\varepsilon} \int_0^1 -\omega(\dot{\gamma}) \left(\sqrt{\frac{\varepsilon g_0(\dot{\gamma}, \dot{\gamma})}{\omega^2(\dot{\gamma})} + 1} - 1 \right) dr \leq \frac{1}{\varepsilon} \int_0^1 \frac{1}{2} \frac{\varepsilon g_0(\dot{\gamma}, \dot{\gamma})}{-\omega(\dot{\gamma})} dr = L(\gamma) \tag{4.1}$$

for all $\varepsilon > 0$, where L is defined in equation (2.1). Thus, $\sup_{\varepsilon \in (0,1)} \Delta_\varepsilon \in \mathbb{R}$. After parametrizing x_ε with $\varepsilon g_0(\dot{x}_\varepsilon, \dot{x}_\varepsilon) + \omega^2(\dot{x}_\varepsilon) = \text{const}$ (recall Theorem 3.1), a sequence $\varepsilon_n \rightarrow 0$ exists such that the curves $(x_{\varepsilon_n}, t_{\varepsilon_n})$ converge to a future pointing lightlike geodesic (x, t) of (S, g) by Lemma 3.2. Therefore, by Remark 3.3, x is admissible and then by Theorem 2.3, it is a pregeodesic of (S, K) . In order to show that it minimizes the length in \mathcal{C} , let us assume that there exists an admissible curve $\gamma_1 \in \mathcal{C}$ such that $L(\gamma_1) < L(x)$. From (4.1), $L_{\varepsilon_n}(x_{\varepsilon_n}) \leq L_{\varepsilon_n}(\gamma_1) \leq L(\gamma_1)$; as $L_{\varepsilon_n}(x_{\varepsilon_n}) = t_{\varepsilon_n}(1) - t_{\varepsilon_n}(0) \rightarrow t(1) - t(0) = L(x)$ we get a contradiction. \square

Corollary 4.5. *Under the assumption of Theorem 4.4, assume also that $\omega \wedge d\omega \neq 0$ in a connected, dense subset of S . Then for each connected component \mathcal{C} of $\Lambda_{x_0x_1}(S)$ there exists a geodesic of the Kropina space from x_0 to x_1 which is a minimizer of the Kropina length functional on \mathcal{C} . Moreover, if \mathcal{C} corresponds to a non-trivial element of the fundamental group of S , a geodesic loop in \mathcal{C} exists when x_0 and x_1 coincide.*

Proof. By Theorem 4.4, it is enough to show that \mathcal{C} contains an admissible curve. In fact, if γ_0 is any curve in \mathcal{C} , by [10, Theorem 1.5 (A)], we can select a finite number of points p_j belonging to the support of γ_0 and an equal finite number of convex neighborhood U_j (convex with respect to the metric g_0) covering γ_0 such that

any point in U_j can be joined to p_j by an admissible smooth curve (actually a length minimizing geodesic for the Kropina metric). In such a way we obtain a piecewise smooth admissible curve γ belonging to the same class \mathcal{C} of γ_0 . \square

As a consequence, from Corollary 4.5, under the non-integrability assumption for the kernel distribution of the one-form g_0 -metrically equivalent to W , we get that the Zermelo's navigation problem on the complete Riemannian manifold (S, g_0) with critical wind W has a solution in each homotopy class of curves between x_0 and x_1 in S .

Corollary 4.6. *Let (S, g_0) be a complete Riemannian metric and W be a vector field on S such that $g_0(W, W) = 1$. Then the Zermelo's navigation problem (with data g_0 and W) between two points $x_0, x_1 \in S, x_0 \neq x_1$, has a solution in each connected component \mathcal{C} of $\Lambda_{x_0 x_1}(S)$ provided that there exists at least one admissible curve in \mathcal{C} . In particular, this happens if $\omega \wedge d\omega \neq 0$ in a connected, dense subset of S . Moreover, if \mathcal{C} corresponds to a non-trivial element of the fundamental group of S , a solution exists when x_0 and x_1 coincide.*

Proof. It is enough to observe that the one-form metrically equivalent to W has constant g_0 -norm equal to 1 and therefore the assumption on the growth of $\|\omega\|_x$ in Theorem 4.4 holds. Thus, the curves which minimize the length functional of the Kropina metric $-\frac{g_0(\cdot, \cdot)}{2g_0(W, \cdot)}$ are solutions of the Zermelo's navigation problem with data g_0 and W , see [8, Proposition 2.57 (ii) and Corollary 6.18 (i)]. \square

5 The Existence of Closed Geodesics

We first consider the case when the fundamental group of S is non-trivial. It is well known that any Finsler metric F on a compact manifold S admits a closed geodesic in each connected component of the free loop space $\Lambda(S)$ which is a minimizer of the energy functional of F and also of its length functional. As in the statement of Lemma 3.2, given a closed geodesic x_ε in (S, F_ε) , we denote by Δ_ε the time travel of the corresponding future pointing lightlike pregeodesic $z_\varepsilon(s) = (x_\varepsilon(s), t_\varepsilon(s))$ in $(S \times \mathbb{R}, g_\varepsilon)$, $t_\varepsilon = t_\varepsilon(s)$ given by (3.5), which also coincides with the F_ε -length of x_ε . Therefore, arguing as in the proof of Theorem 4.4 we obtain the following result that can be interpreted, by the viewpoint of Zermelo's navigation problem, as the possibility of round trips which minimize the navigation time if the topology of the sea is non-trivial.

Theorem 5.1. *Let S be a compact manifold having zero Euler characteristic and non-trivial fundamental group endowed with a Kropina metric $K := -\frac{g_0(\cdot, \cdot)}{2\omega(\cdot)}$. Then (S, K) admits a (non-trivial) closed geodesic with minimal Kropina length in each connected component \mathcal{C} of $\Lambda(S)$ which does not correspond to a non-trivial conjugacy class of the fundamental group provided that \mathcal{C} contains at least one admissible closed curve.*

Analogously to Corollary 4.5, the following also holds:

Corollary 5.2. *Under the assumptions of Theorem 5.1, assume further that $\omega \wedge d\omega \neq 0$ in a connected, dense subset of S . Then (S, K) admits a (non-trivial) closed geodesic with minimal Kropina length in each connected component \mathcal{C} of $\Lambda(S)$ which does not correspond to a non-trivial conjugacy class of the fundamental group.*

We give now an existence result in a setting including possibly the cases that S is simply connected or non-compact. We also allow ω vanishing somewhere in S , and we denote by S_0 the set of points $x \in S$ where $\omega_x = 0$ (S_0 being possibly empty).

Theorem 5.3. *Let (S, g_0) be a Riemannian manifold and let ω be a one-form on S . Let $(\varepsilon_n)_n$ be an infinitesimal sequence of positive numbers and, for each $n \in \mathbb{N}$, and let x_{ε_n} be a closed geodesic of the Randers metric F_{ε_n} . Assume that $\Delta := \sup_n \Delta_{\varepsilon_n} \in \mathbb{R}$ and the images of the curves x_n are contained in a compact set C included in an open subset $U \subset S \setminus S_0$ with compact closure such that x_n are non-contractible in U . Then the Kropina space $(S \setminus S_0, -\frac{g_0(\cdot, \cdot)}{2\omega(\cdot)})$ admits a (non-trivial) closed geodesic.*

Proof. We can apply Lemma 3.2 to the sequence of standard stationary spacetimes $((S \setminus S_0) \times \mathbb{R}, g_{\varepsilon_n})$. As the images of x_{ε_n} are contained in the compact subset \bar{U} , we can assume both completeness of g_0 and

boundedness of $\|\omega\|_x$. Hence, up to reparametrization, the sequence of future-pointing lightlike geodesics $(x_{\varepsilon_n}, t_{\varepsilon_n})$ converges in $\Lambda(S \setminus S_0) \times H^1([0, 1], \mathbb{R})$ to a future pointing lightlike geodesic (x, t) of the spacetime $((S \setminus S_0) \times R, g)$. Since $x_n \rightarrow x$ in the C^0 -topology, if x was a constant curve in \tilde{C} then, for ε_n small enough, x_{ε_n} would be contractible in U . Therefore, x is a closed pregeodesic of the Kropina space $(S \setminus S_0, -\frac{g_0(\cdot, \cdot)}{2\omega(\cdot)})$. \square

In some cases it is possible to control from above the lengths Δ_ε of the prime closed geodesics of the Randers metrics associated to the approximating stationary spacetimes $(S \times \mathbb{R}, g_\varepsilon)$.

Example 5.4 (Kropina Limit of Katok Metrics). In order to show a class of examples we need to change a bit the approximation scheme. Let us replace the one-form ω in (3.1) by $\sqrt{1 - \varepsilon}\omega$, so that the stationary Lorentzian metric g_ε is now given by

$$g_\varepsilon := \pi^*g_0 + \sqrt{1 - \varepsilon}\pi^*\omega \otimes dt + dt \otimes \sqrt{1 - \varepsilon}\pi^*\omega - \varepsilon dt^2.$$

The Randers metrics

$$F_\varepsilon(v) = \frac{1}{\varepsilon} \left(\sqrt{\varepsilon g_0(v, v) + (1 - \varepsilon)\omega^2(v)} + \sqrt{1 - \varepsilon}\omega(v) \right)$$

defined by the modified g_ε is also associated to the Zermelo data

$$h_\varepsilon = \frac{g_0}{\varepsilon + (1 - \varepsilon)\|\omega\|^2} \quad \text{and} \quad W_\varepsilon = -\sqrt{1 - \varepsilon}\omega^\sharp,$$

see [8, Proposition 3.6] and [26, p. 1634]. Let $S = \mathbb{S}^n$, $n \geq 2$ and g_0 be the round metric on \mathbb{S}^n . Let us assume that ω^\sharp is a Killing vector field for g_0 . Then the Randers metric F_ε obtained in this case is one of the celebrated examples considered by Katok. It is well known that the resulting Randers metrics admit at least $2m$, $n = 2m$ or $n = 2m - 1$, closed geodesics which correspond to the m great circles C_i invariant by the flow of ω^\sharp , each of them considered twice according to the orientation (see [26, 33]). Let us assume that $\|\omega^\sharp\|_x = 1$ for all $x \in C_i$ and let us parametrize the F_ε -geodesics with unit velocity with respect to the metric g_0 . Then the F_ε length of these geodesics is given by

$$\Delta_\varepsilon = \frac{2\pi}{\varepsilon} (1 - \sqrt{1 - \varepsilon}) = \frac{2\pi}{1 + \sqrt{1 - \varepsilon}}$$

if they are parametrized in the same direction of the rotation, and

$$\Delta_\varepsilon = \frac{2\pi}{\varepsilon} (1 + \sqrt{1 - \varepsilon}) = \frac{2\pi}{1 - \sqrt{1 - \varepsilon}}$$

in the other case. Thus, the first family of geodesics has uniformly bounded length and since for each i , their support is C_i , they converge to a geodesic of the Kropina space $(\mathbb{S}^{2m-1}, -\frac{g_0(\cdot, \cdot)}{2\omega(\cdot)})$ if $n = 2m - 1$ and of $(\mathbb{S}^{2m} \setminus \{p, q\}, -\frac{g_0(\cdot, \cdot)}{2\omega(\cdot)})$ if $n = 2m$ and p, q are two antipodal points on the sphere where ω^\sharp vanishes. Thus, both these Kropina space admits at least m distinct closed geodesics of length π and support the m circles C_i .

Example 5.5. The above example can be generalized as follows. Let (S, g_0) be a compact Riemannian manifold endowed with a non-trivial periodic Killing vector field W (i.e. all orbits of W are closed). It is well known that (S, g_0) admits at least one non-constant closed geodesic which is one of these orbits (it corresponds to the geodesic with initial conditions p and W_p where $g_0(W_p, W_p) = \max\{x \in S : g_0(W_x, W_x)\}$). Let γ be such a geodesic. Since $g_0(W_{\gamma(t)}, W_{\gamma(t)})$ is constant along γ we call it c^2 and we consider the Killing vector field $\frac{W}{c}$. Hence, $g_0(\frac{W}{c}, \frac{W}{c}) \leq 1$. Let us then consider a perturbation parameter $\alpha < 1$. Then the Randers metrics on S defined by the Zermelo navigation data g_0 and $\frac{\sqrt{\alpha}}{c}W$, for each $\alpha \in (0, 1)$, admit γ as a closed geodesic by [26, Theorem 2], counted twice considering as a different geodesic the one obtained by reversing the orientation of γ . The shortest of these two geodesics has length

$$\Delta_\alpha = \frac{T}{1 - \alpha} (1 - \sqrt{\alpha}) = \frac{T}{1 + \sqrt{\alpha}},$$

where T is the g_0 -length of γ . As Δ_α are bounded, from Theorem 5.3 by passing to the limit as $\alpha \rightarrow 1$, we conclude that the Kropina metric $K(v) = \frac{g_0(v, v)}{2g_0(W, v)}$ on $S \setminus S_0$ admits γ as closed geodesic with Kropina length $\frac{T}{2}$.

Assuming that the one-form ω is invariant by the flow of a Killing vector field Y (i.e. the Lie derivative $L_Y\omega$ vanishes) and constant on it, gives a result about existence of at least two closed geodesics even when ω is

not the one-form metrically associated to Y ; in this case the approximation with stationary spacetimes can be bypassed.

Theorem 5.6. *Let (S, g_0) be a compact Riemannian manifold, endowed with a non-trivial periodic Killing vector field Y , and let ω be a one-form such that $\omega(Y) < 0$ is constant and $L_Y\omega = 0$. Then there exist at least two closed (non-trivial) geodesics of the Kropina metric $K = -\frac{g_0(\cdot, \cdot)}{2\omega(\cdot)}$.*

Proof. Notice that being $\omega(Y) < 0$, Y does not vanish at any point of S and then its orbits are non-constant. Then the conclusion follows by observing that there are at least two orbits, passing through a minimizer and a maximizers of the function $p \in S \mapsto g_0(Y_p, Y_p)$ which are geodesics for the Riemannian metric g_0 and, under our assumptions, they are geodesics of the Kropina metric as well. In fact, let $\gamma : [0, T] \rightarrow S$ be one of these two orbits (the period T depending on γ). Since Y is Killing and $L_Y\omega = 0$, both $g_0(\dot{\gamma}, \dot{\gamma})$ and $\omega(\dot{\gamma})$ are constant along γ . Moreover, being $\omega(Y) < 0$, γ is a smooth admissible curve. Then the first variation of the length functional of K with respect to any smooth periodic vector field ξ along γ is well-defined and given by

$$-\frac{1}{2} \int_0^T \left(\frac{2g_0(\dot{\gamma}, \nabla_{\dot{\gamma}}^0 \xi)}{\omega(\dot{\gamma})} - \frac{g_0(\dot{\gamma}, \dot{\gamma}) \left(d\omega(\xi, \dot{\gamma}) + \frac{d}{ds}(\omega(\xi)) \right)}{\omega^2(\dot{\gamma})} \right) ds, \tag{5.1}$$

where $\nabla_{\dot{\gamma}}^0$ is the covariant derivative along γ induced by the Levi-Civita connection of the metric g_0 . Hence, integrating by parts and using that $d\omega(Y, \xi) = (L_Y\omega)(\xi) - \xi(\omega(Y)) = 0$ (recall that γ is an orbit of Y), we get that the above integral reduces to

$$\int_0^T \frac{g_0(\nabla_{\dot{\gamma}}^0 \dot{\gamma}, \xi)}{\omega(\dot{\gamma})} ds,$$

which is 0 for all ξ . □

Remark 5.7. If Y has constant length too, then $(S, -\frac{g_0(\cdot, \cdot)}{2\omega(\cdot)})$ admits infinitely many closed geodesics.

Nevertheless, the assumption Y has constant length can replace $\omega(Y)$ constant.

Theorem 5.8. *Let (S, g_0) be a compact Riemannian manifold, endowed with a non-trivial periodic Killing vector field Y of constant length, and let ω be a one-form such that $\omega(Y) < 0$ everywhere and $L_Y\omega = 0$. Then there exist at least two closed (non-trivial) geodesics of the Kropina metric $K = -\frac{g_0(\cdot, \cdot)}{2\omega(\cdot)}$.*

Proof. As in the previous proof consider the first variation (5.1) of the Kropina length of an orbit $\gamma : [0, T] \rightarrow S$ of Y in the direction of a periodic vector field ξ along γ . Note that $\omega(\dot{\gamma})$ is constant and therefore can be removed from under the integral. Then the first term vanishes since the length of Y with respect to g_0 is constant and therefore γ is a g_0 -geodesic.

Next consider the critical points of the function $f := p \in S \mapsto \omega_p(Y_p)$. Since $L_Y\omega = 0$, it follows that f is constant along orbits of Y . Let then γ be an orbit whose points are all critical for f . With the formula $d\omega(Y, \xi) = (L_Y\omega)(\xi) - \xi(\omega(Y)) = 0$ and the periodicity of ξ , we see that the first variation of the Kropina length of γ vanishes. As S is compact, there exist at least two critical points of f whose values are both negative. The orbits through these two points are the required closed geodesic of K . □

Notice that in particular Theorem 5.8 holds with the assumptions of a *strong Kropina metric with quasi-regular Killing field* of [27].

We finish this section with a result for a Kropina metric on a compact Lie group endowed with a bi-invariant Riemannian metric.

Corollary 5.9. *Let S be a compact Lie group endowed with a bi-invariant Riemannian metric and a non-trivial left-invariant one-form ω . Then the Kropina metric $K = -\frac{g_0(\cdot, \cdot)}{2\omega(\cdot)}$ on S admits at least two closed (non-trivial) geodesic.*

Proof. We will construct a right-invariant periodic vector field Y such that $\omega(Y) < 0$ everywhere. As g_0 is bi-invariant, the vector field Y is a Killing vector field of g_0 with constant length. It further preserves ω since the form is left-invariant. The claim then follows from Theorem 5.8.

For the construction of Y we start with an arbitrary right-invariant vector field X on S such that $\omega(X) < 0$ everywhere. This is possible since ω is non-trivial and left-invariant. Let H be closure of the subgroup generated by X . Since S is compact, it follows that H is a compact abelian group, i.e. $H \cong T^k$ for some k . We can now choose a compact 1-dimensional subgroup of H whose generator Y_e lies arbitrary close to X_e , especially $\omega_e(Y_e) < 0$. The right-invariant vector field Y associated to Y_e is then the desired right-invariant periodic vector field. \square

A On the Boundary of a Reachable Set

In light of the problem of existence of paths with finite Kropina length between given points the question begs itself what can be said about the set of reachable points. Recall from the introduction that $\mathcal{A} := \{v \in TS \mid -\omega(v) > 0\}$, and an absolutely continuous curve $\gamma : I \rightarrow S$ is admissible if $\dot{\gamma}(t) \in \mathcal{A}$ for almost all $t \in I$. For $x \in S$, define then $I_\omega^\pm(x)$ to be the sets of terminal or initial points of admissible curves with starting or ending at x . It is well known that these sets are open for all $x \in S$ and the induced relation is transitive. Under the assumption that $\omega \wedge d\omega \neq 0$ on a dense and connected set, the Chow–Rashevsky Theorem (see [15, Section 1]) implies that any pair of points is connected by an admissible curve, i.e. $I_\omega^\pm(x) = S$ for all $x \in S$.

In analogy to spacetime geometry we will prove the following analog, see [24, Proposition 14.25]. Note that this is related to the integrability of distributions, see [30, 31].

Theorem A.1. *Let S be connected and $x \in S$ such that $I_\omega^+(x) \neq S$. Then the boundary $\Sigma := \partial I_\omega^+(x)$ is a (non-empty) smooth hypersurface which separates S .*

Remark A.2. We make the following observations.

- (a) The opposite set $\partial I_\omega^-(x)$ has the same properties under the appropriate assumptions.
- (b) By the Chow–Rashevsky Theorem we know that at every point $p \in \partial I_\omega^+(x)$ we have $\omega_p \wedge d\omega_p = 0$.

Proof. The result is local in nature therefore the proof is a local argument.

First Step: Σ is a topological hypersurface. Let $p \in \Sigma$ and let us choose coordinates $\phi = (\phi^1, \dots, \phi^n)$ in a neighborhood U of p such that $\phi(p) = 0$ and $-\omega_p = (d\phi^n)_p$. By restricting ϕ if necessary, we can assume that the intersections of $\ker \omega_q, q \in U$, with the double cones $\{v \in TU : |d\phi^n(v)| \geq \sum_{i=1}^{n-1} |d\phi^i(v)|\}$ only contain the zero section. This follows from the continuity of ϕ and ω . Since any neighborhood of p contains points in $I_\omega^+(x)$, we obtain

$$C^+ := \left\{ q \in U : \phi^n(q) > \sum_{i=1}^{n-1} |\phi^i(q)| \right\} \subset I_\omega^+(x).$$

Analogously it follows from the fact that every neighborhood of p contains points in $\overline{I_\omega^+(x)}^c$ that

$$C^- := \left\{ q \in U : -\phi^n(q) > \sum_{i=1}^{n-1} |\phi^i(q)| \right\} \subset \overline{I_\omega^+(x)}^c.$$

So every line $t \mapsto (q_1, \dots, q_{n-1}, t)$ will pass Σ exactly once. Like in the case of achronal surfaces (see [24, Proposition 14.25]) we see that the intersection point depends Lipschitz continuously on (q_1, \dots, q_{n-1}) . Therefore, near p the set Σ is the graph of a Lipschitz function σ .

Second Step: Σ is a C^1 -hypersurface. We claim that Σ has a tangent space everywhere and $T\Sigma_q = \ker \omega_q$ for all $q \in \Sigma$. The claim readily implies the continuous differentiability of Σ . To prove the claim, consider coordinates ϕ around $p \in \Sigma$ and function σ as before. Further, let $\eta : (-\varepsilon, \varepsilon) \rightarrow \{\phi^n = 0\}$ be a Lipschitz curve with $\eta(0) = q$. Then $t \mapsto (\eta(t), \sigma \circ \eta(t))$ defines a Lipschitz curve in Σ . If $\limsup_{t \rightarrow 0} \frac{|\sigma \circ \eta(t)|}{|\eta(t)|}$ is positive, up to consider larger cones than the ones defined in the first step, we conclude that $(\eta(t), \sigma \circ \eta(t)) \in I_\omega^+(x)$ or $(\eta(t), \sigma \circ \eta(t)) \in \overline{I_\omega^+(x)}^c$ for $|t|$ sufficiently small, a contradiction. Therefore we have

$$\limsup_{t \rightarrow 0} \frac{|\sigma \circ \eta(t)|}{|\eta(t)|} = 0$$

for every Lipschitz curve η , i.e. $T\Sigma_q = \ker(d\phi^n)_q = \ker \omega_q$.

Third Step: Σ is smooth. Without loss of generality, we can assume that the codomain of the local coordinates ϕ is the cube $[-1, 1]^n$. Denote with π_n the projection $[-1, 1]^n \rightarrow [-1, 1]^{n-1}$ which forgets the last coordinate. Under this projection we can uniquely lift the coordinate fields $\partial_1^\phi, \dots, \partial_{n-1}^\phi$ on $[-1, 1]^{n-1}$ to sections X_1, \dots, X_{n-1} spanning $\ker \omega$ such that $d\pi_n(X_i) = \partial_i^\phi$. Note that the sections are smooth.

Since Σ is everywhere tangent to $\ker \omega$, the flow lines of any X_i starting in Σ will remain in Σ . This implies that the flows of the sections X_i commute along Σ and can therefore be used to parameterize the hypersurface Σ around p (see for instance [21, Chapter 19]). \square

Acknowledgment: Part of this work has been developed while Stefan Suhr was a visiting professor at Politecnico di Bari; he thanks Politecnico di Bari and the Department of Mechanics, Mathematics and Management for the support.

Funding: Erasmo Caponio and Antonio Masiello are partially supported by PRIN 2017JPCAPN *Qualitative and quantitative aspects of nonlinear PDEs*. Stefan Suhr is partially supported by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – Project-ID 281071066 – TRR 191.

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