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Thermo-elastic waves in a model with nonlinear adhesion ☆

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ABSTRACT

In the context of thermo-elasticity we consider initial boundary value problems governed by parabolic and hyperbolic heat propagations. In particular, we describe the evolution of the temperature and displacement fields in a one dimensional string attached to a rigid substrate through an adhesive layer. This adhesive interaction is characterized by a nonlinear term describing the adhesion force exhibiting discontinuities when a critical value of the displacement is reached, in the limit of parabolic heat propagation. We study the well-posedness of the problem under Neumann boundary conditions in the two different regimes of heat propagation and investigate the long time dynamics.

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1. Introduction

Mathematical models describing thermal effects in continuum mechanics have found large application in engineering problems where heat conduction phenomena play a decisive role [13]. Typically, these models have been based on the classical Fourier law and, thus, on parabolic partial differential equations. On the other hand, the infinite speed of propagation associated to this choice has induced the community to overcome this approximation and consider the use of hyperbolic PDEs in order to obtain a more adequate description of heat conduction [21]. Indeed, some experiments since 1960's showed that a thermal disturbance could travel as a wave and thermal pulse propagation has been experimentally observed under certain conditions [10,15]. Appropriate mathematical theories have been widely developed in the context of the so-called *second sound* phenomenon and thermo-elasticity with finite wave speeds [14]. At the same time, models that include the interplay between thermal and elastic effects are fundamental in order to describe phenomena at different time and length scales in laminated materials. In particular, the study of temperature and displacement fields are fundamental to predict important phenomena such as delamination [22], a topic of large interest in the field of recycling processes of multilayered materials. The analysis of adhesion–decohesion, capillarity and wetting phenomena which have attracted a large interest because of their applications [1,12,16–20], is challenging since the evolution problem has to take into account both thermal and elastic effects.

In order to approach the problem of non-linear wave propagation in the context of thermo-elasticity for layered systems and to study the occurrence of singular phenomena like delamination, we consider a system of coupled PDEs describing the evolution of temperature and displacement in a string attached to a rigid substrate. The adhesive layer that mediates the interaction of the material with the substrate is represented through a general energetic source which is assumed to be smooth with a given growth during a first process modeled by hyperbolic heat propagation while it produces singularities in a second process, when the relaxation time tends to zero and the heat propagation becomes parabolic. In particular, the non-linear term models a discontinuous softening behavior of the adhesive material which can experience rupture phenomena if the displacement of the string is larger than a certain critical threshold (with the adhesive stress jumping to zero). Since the breaking of the material manifests itself at a macroscopic scale, we have chosen to analyze the two processes, the first one affecting the small scales and the second one affecting the large scale, in two different stages ruled by the relaxation time and the smoothness of the adhesive potential. In [6,7] authors have considered the same model for the glue layer in the context of elastodynamics while in [4,5] the model has been extended to flexural beams. The same type of adhesive potential has been used to model the phenomenon of temperature-induced melting with an associated phase transition in [11], and in [2] or in [9] to analyze the interaction between focal adhesions and extracellular matrix.

The paper is organized as follows. In Section 2 we expose the evolution problem in the framework of second sound theory of heat propagation under suitable regularity assumptions on the nonlinear source term representing the adhesive interaction. Section 3 is devoted to the proof of the main theorem stating well-posedness and stability for the previous problem. In Section 4 we study the asymptotics of the previous initial boundary value problem when the heat propagation becomes of parabolic type and the adhesive interaction becomes singular. Eventually, in Section 5 we address the problem of long time behavior of solutions proving that for bounded ones weak convergence to stationary states is achieved.

2. Thermoelastic evolution with adhesion interaction

Let us consider a one dimensional material body, i.e. a *string*, whose rest configuration at the initial time $t = 0$ coincides with the interval $[0, L]$ at a reference (absolute) temperature Θ_0. The quantities we are interested in are the displacement field u and the absolute temperature θ:

$$u, \theta : [0, \infty) \times [0, L] \to \mathbb{R}. \tag{2.1}$$

Fig. 1. Thermoelastic one-dimensional body interacting through and adhesive layer (in gray) with a rigid substrate.

We assume the material to be linear thermoelastic and the string to interact with an underlying rigid support through an infinitesimal layer of adhesive material characterized by an internal energy $\Psi_\tau(u)$ where the parameter $\tau > 0$ just emphasizes the relation between this constitutive assumption and the relaxation time affecting the hyperbolic heat propagation. This approach to thermal propagation was introduced by C. Cattaneo in 1949 in [3] and leads to the following equation relating the heat flux q and the temperature θ trough the relaxation time τ:

$$q(t+\tau, x) = -k\partial_x \theta(t, x),$$

where k is the thermal coefficient. The Cattaneo model is suitable for low temperatures and since the physical model addressed here is inspired by a technological problem related to cryogenic delamination [8], in the following we focus on the mathematical aspects related to this issue.

The interaction potential Ψ_τ satisfies the following conditions

$$\Psi_\tau(\cdot,\cdot) \geq 0, \qquad \Psi_\tau \in C^2(\mathbb{R}^2) \cap W^{2,\infty}(\mathbb{R}^2). \tag{2.2}$$

To underline the dependence of the variables on the relaxation time τ we will denote the displacement and the absolute temperature fields respectively as u_τ and θ_τ. Then we consider the longitudinal motion of a one-dimensional thermoelastic body, interacting with the rigid support through a tangential (shear) force experienced in a layer made of adhesive material (see Fig. 1).

Since we consider small relaxation times, the governing equations ruled by the linearized Cattaneo model consist in the following system of coupled equations [13,14]:

$$\begin{cases} \rho\partial_{tt}^2 u_\tau = K_e \partial_{xx}^2 u_\tau + \beta\partial_x\theta_\tau - \Psi_\tau'(u_\tau, \theta_\tau), & t > 0,\ 0 < x < L, \\ C_h(1+\tau\partial_t)\,\partial_t\theta_\tau = K_d\,\partial_{xx}^2\theta_\tau + (1+\tau\partial_t)\,(\beta\,\Theta_0\partial_{tx}^2 u_\tau + Q), & t > 0,\ 0 < x < L, \end{cases} \tag{2.3}$$

where $\rho > 0$ denotes the mass density, K_e is the stiffness coefficient of the string, $-\beta$ is the coupling thermal coefficient (proportional to the coefficient of thermal expansion through the Lamé coefficients), C_h is the heat capacity of the material, K_d is the heat conduction coefficient, Θ_0 is the reference (absolute) temperature, τ is the relaxation time, Q is the heat source, while the source term $\Psi_\tau'(u)$ represents the adhesion force experienced through the glue layer. Eqs. (2.3) are obtained in the context of isotropic linear thermoelasticity.

Since there are no external forces applied at the boundary of the string, we will consider the following initial and boundary conditions for the displacement function $u_\tau = u_\tau(t, x)$:

$$\begin{cases} \partial_x u_\tau(t, 0) = \partial_x u_\tau(t, L) = 0, & t > 0, \\ u_\tau(0, x) = u_{0,\tau}(x), & 0 < x < L, \\ \partial_t u_\tau(0, x) = u_{1,\tau}(x), & 0 < x < L, \end{cases} \tag{2.4}$$

where

$$u_{0,\tau} \in H^2(0, 1), \qquad u_{1,\tau} \in H^1(0, 1).$$

The initial and boundary conditions for the temperature θ_τ are given as follows

$$\begin{cases} \theta_\tau(0, x) = \theta_{0,\tau}(x), & 0 < x < L, \\ \partial_t\theta_\tau(0, x) = \theta_{1,\tau}(x), & 0 < x < L, \end{cases} \tag{2.5}$$

with

$$\theta_{0,\tau} \in H^1_0(0,L), \qquad \theta_{1,\tau} \in L^2(0,L).$$

In principle it is possible to choose two different sets of boundary conditions. In particular, Neumann boundary conditions

$$\partial_x \theta_\tau(t,0) = \partial_x \theta_\tau(t,L) = 0, \quad t > 0, \tag{2.6}$$

represent the situation in which the extremes of the string are thermally isolated. On the other hand, Dirichlet boundary conditions

$$\theta_\tau(t,0) = \theta_\tau(t,L) = \theta^*_\tau, \quad t > 0, \tag{2.7}$$

represent the situation in which the extremes of the string are kept at the fixed temperature θ^*_τ. We are interested in the latter one.

To improve the readability of the paper we set

$$\rho = K_e = C_h = K_d = L = 1, \qquad \theta^*_\tau = 0, \qquad \beta\,\Theta_0 = g.$$

Moreover, we shall denote by c and C any constant independent on the data of the problem.

After these positions (2.3), (2.4), (2.5), (2.6) and (2.7) become

$$\begin{cases} \partial^2_{tt} u_\tau = \partial^2_{xx} u_\tau + \beta \partial_x \theta_\tau - \Psi'(u_\tau, \theta_\tau), & t > 0,\ 0 < x < 1, \\ (1+\tau\partial_t)\,\partial_t \theta = \partial^2_{xx}\theta_\tau + g(1+\tau\partial_t)\,\partial^2_{tx} u_\tau + (1+\tau\partial_t)Q, & t > 0,\ 0 < x < 1, \\ \partial_x u_\tau(t,0) = \partial_x u_\tau(t,1) = \theta_\tau(t,0) = \theta_\tau(t,1) = 0, & t > 0, \\ u_\tau(0,x) = u_{0,\tau}(x),\ \partial_t u_\tau(0,x) = u_{1,\tau}(x), & 0 < x < 1, \\ \theta_\tau(0,x) = \theta_{0,\tau}(x),\ \partial_t\theta_\tau(0,x) = \theta_{1,\tau}(x), & 0 < x < 1, \end{cases} \tag{2.8}$$

where we assume

$$\beta,\, g \in \mathbb{R}, \qquad \tau > 0, \qquad \Psi_\tau(\cdot) \geq 0, \qquad \Psi_\tau \in C^2(\mathbb{R}^2) \cap W^{2,\infty}(\mathbb{R}^2), \qquad Q \in C^2([0,\infty) \times \mathbb{R}), \tag{2.9}$$

$$u_{0,\tau} \in H^2(0,1), \quad u_{1,\tau} \in H^1(0,1), \quad \theta_{0,\tau} \in H^1_0(0,1), \quad \theta_{1,\tau} \in L^2(0,1). \tag{2.10}$$

We use the following definition of weak solution for problem (2.8).

Definition 2.1. Let u_τ, θ_τ : $[0,\infty) \times [0,1] \to \mathbb{R}$ be functions. We say that (u_τ, θ_τ) is a solution of the initial boundary value problem (2.8) if

(D.1) $u_\tau \in H^2((0,T) \times (0,1))$, $\theta_\tau \in H^1((0,T) \times (0,1))$, for every $T > 0$;
(D.2) the initial and boundary conditions are satisfied almost everywhere;
(D.3) the first equation is satisfied almost everywhere in $(0,\infty) \times (0,1)$;
(D.4) for every test function $\varphi \in C^\infty(\mathbb{R} \times (0,1))$ with compact support the following identity holds

$$\begin{aligned} &\int_0^\infty \int_0^1 \left(\theta_\tau (1-\tau\partial_t)\partial_t\varphi + \theta_\tau \partial^2_{xx}\varphi + g\partial^2_{tx} u_\tau (1-\tau\partial_t)\varphi + (1+\tau\partial_t)Q\varphi \right) dtdx \\ &\quad + \int_0^1 \left(\theta_{0,\tau}(x)(1-\tau\partial_t)\varphi(0,x) + \tau\theta_{1,\tau}(x)\varphi(0,x) - g\tau\partial_x u_{1,\tau}(x)\varphi(0,x)\right) dx = 0. \end{aligned} \tag{2.11}$$

We prove the following existence and stability result.

Theorem 2.1. *Assume (2.9). For every $u_{0,\tau}$, $u_{1,\tau}$, $\theta_{0,\tau}$, $\theta_{1,\tau}$ satisfying (2.10), the initial boundary value problem (2.8) admits a unique solution in the sense of Definition 2.1. Moreover, if (u_τ, θ_τ) and $(\widetilde{u_\tau}, \widetilde{\theta_\tau})$ are the solutions of (2.8) obtained in correspondence of the initial data $u_{0,\tau}$, $u_{1,\tau}$, $\theta_{0,\tau}$, $\theta_{1,\tau}$ and $\widetilde{u}_{0,\tau}$, $\widetilde{u}_{1,\tau}$, $\widetilde{\theta}_{0,\tau}$, $\widetilde{\theta}_{1,\tau}$ and the source terms Q and $\widetilde{Q}$ respectively, then the following stability estimate*

$$\Lambda(t) \leq \Lambda(0)e^{Ct} + C\int_0^t e^{C(t-s)}\left(\left\|(Q-\widetilde{Q})(s,\cdot)\right\|^2_{L^2(0,1)} + \left\|(\partial_t Q - \partial_t \widetilde{Q})(s,\cdot)\right\|^2_{L^2(0,1)}\right) ds \tag{2.12}$$

holds for every $t \geq 0$, where $C > 0$ is a constant and

$$\begin{aligned}\Lambda(t) = \int_0^1 &\Big((u_\tau - \widetilde{u_\tau})^2 + (\partial_t(u_\tau - \widetilde{u_\tau}))^2 + (\partial_x(u_\tau - \widetilde{u_\tau}))^2 + \left(\partial^2_{tt}(u_\tau - \widetilde{u_\tau})\right)^2 \\ &+ \left(\partial^2_{tx}(u_\tau - \widetilde{u_\tau})\right)^2 + \left(\theta_\tau - \widetilde{\theta}_\tau\right)^2 + \left(\partial_t\left(\theta_\tau - \widetilde{\theta}_\tau\right)\right)^2 + \left(\partial_x\left(\theta_\tau - \widetilde{\theta}_\tau\right)\right)^2\Big) dx.\end{aligned} \tag{2.13}$$

3. Proof of Theorem 2.1

This section is devoted to the proof of Theorem 2.1.

Let $\{u_{0,\tau,n}\}_{n\in\mathbb{N}}$, $\{u_{1,\tau,n}\}_{n\in\mathbb{N}}$, $\{\theta_{0,\tau,n}\}_{n\in\mathbb{N}}$, $\{\theta_{1,\tau,n}\}_{n\in\mathbb{N}} \subset C^\infty([0,1])$, $\{\Psi_{\tau,n}\}_{n\in\mathbb{N}} \subset C^\infty(\mathbb{R})$ be sequences of smooth approximations of $u_{0,\tau}$, $u_{1,\tau}$, $\theta_{0,\tau}$, $\theta_{1,\tau}$, and Ψ_τ respectively such that

$$\begin{aligned}
&u_{0,\tau,n} \to u_{0,\tau} \quad \text{in } H^2(0,1), \quad u_{1,\tau,n} \to u_{1,\tau} \quad \text{in } H^1(0,1),\\
&\theta_{0,\tau,n} \to \theta_{0,\tau} \quad \text{in } H^1(0,1), \quad \theta_{0,\tau,n} \to \theta_{0,\tau} \quad \text{in } L^2(0,1),\\
&\Psi_{\tau,n} \to \Psi_\tau \quad \Psi'_{\tau,n} \to \Psi'_\tau \quad \Psi''_{\tau,n} \to \Psi''_\tau \quad \text{uniformly in } \mathbb{R},\\
&\|u_{0,\tau,n}\|_{H^2(0,1)} \leq C, \quad \|u_{1,\tau,n}\|_{H^1(0,1)} \leq C, \qquad n \in \mathbb{N},\\
&\|\theta_{0,\tau,n}\|_{H^1(0,1)} \leq C, \quad \|\theta_{1,\tau,n}\|_{L^2(0,1)} \leq C, \qquad n \in \mathbb{N},\\
&0 \leq \Psi_{\tau,n},\ |\Psi'_{\tau,n}| \leq C, \qquad n \in \mathbb{N},\\
&\partial_x u_{0,\tau,n}(0) = \partial_x u_{0,\tau,n}(1) = u_{1,\tau,n}(0) = u_{1,\tau,n}(1) = 0, \qquad n \in \mathbb{N},\\
&\theta_{0,\tau,n}(0) = \theta_{0,\tau,n}(1) = \theta_{1,\tau,n}(0) = \theta_{1,\tau,n}(1) = 0, \qquad n \in \mathbb{N},
\end{aligned} \tag{3.1}$$

where $C > 0$ denotes some constant independent on n.

Let $(u_{\tau,n}, \theta_{\tau,n})$ be the unique classical solution of the initial boundary value problem

$$\begin{cases}
\partial^2_{tt}u_{\tau,n} = \partial^2_{xx}u_{\tau,n} + \beta\partial_x\theta_{\tau,n} - \Psi'_{\tau,n}(u_{\tau,n}), & t>0,\ 0<x<1,\\
(1+\tau\partial_t)\partial_t\theta_{\tau,n} = \partial^2_{xx}\theta_{\tau,n} + g(1+\tau\partial_t)\partial^2_{tx}u_{\tau,n} + (1+\tau\partial_t)Q, & t>0,\ 0<x<1,\\
\partial_x u_{\tau,n}(t,0) = \partial_x u_{\tau,n}(t,1) = \theta_{\tau,n}(t,0) = \theta_{\tau,n}(t,1) = 0, & t>0,\\
u_{\tau,n}(0,x) = u_{0,\tau,n}(x),\ \partial_t u_{\tau,n}(0,x) = u_{1,\tau,n}(x), & 0<x<1,\\
\theta_{\tau,n}(0,x) = \theta_{0,\tau,n}(x),\ \partial_t\theta_{\tau,n}(0,x) = \theta_{1,\tau,n}(x), & 0<x<1,
\end{cases} \tag{3.2}$$

The well-posedness of (3.2) is guaranteed for short time by the Cauchy–Kovalevskaja Theorem [24]. The solutions are indeed global in time thanks to the following a priori estimates.

Lemma 3.1 (*Energy Estimate*)**.** *Let $T > 0$ be given. For every $0 \leq t \leq T$, the following inequality holds*

$$\begin{aligned}\mathcal{E}_n(t) + A_n(t) + D_n e^{D_n t}\int_0^t e^{-D_n s}A_n(s)ds + e^{D_n t}\int_0^t e^{-D_n s}B_n(s)ds \\ \leq e^{D_n t}\mathcal{E}_n(0) + C_n\frac{e^{D_n t}-1}{D_n} + e^{D_n t}A_n(0),\end{aligned} \tag{3.3}$$

where

$$
\begin{aligned}
\mathcal{E}_n(t) =& \int_0^1 \Bigg(\frac{(\partial_t u_{\tau,n})^2 + \tau^2(\partial_{tt}^2 u_{\tau,n})^2 + (\partial_x u_{\tau,n})^2 + \tau^2(\partial_{tx}^2 u_{\tau,n})^2}{2} \\
&\qquad + \frac{\kappa\theta_{\tau,n}^2 + \kappa\tau^2(\partial_t\theta_{\tau,n})^2 + \kappa\tau(\partial_x\theta_{\tau,n})^2}{2} + \Psi_{\tau,n}(u_{\tau,n}) \Bigg)(t,x)dx,\\
A_n(t) =& \tau \int_0^1 \left(\frac{(\partial_t u_{\tau,n})^2 + (\partial_x u_{\tau,n})^2 + \kappa\theta_{\tau,n}^2}{2} + 2\,\Psi_{\tau,n}(u_{\tau,n}) \right)(t,x)dx,\\
B_n(t) =& \kappa e^t \int_0^t \int_0^1 e^{-s}((\partial_x\theta_{\tau,n})^2 + \Psi_{\tau,n}(u_{\tau,n}))(s,x)dsdx,\\
C_n =& \mathcal{E}_n(0)e^T + 2\tau e^T \left\| \Psi'_{\tau,n} \right\|_{L^\infty(\mathbb{R})} \int_0^1 |u_{1,\tau,n}|dx + \frac{\kappa}{2}e^T \int_0^T \int_0^1 ((1+\tau\partial_t)Q)^2(s,x)dsdx,\\
D_n =& \max\left\{ \tau^2 \left\| \Psi''_{\tau,n} \right\|^2_{L^\infty(\mathbb{R})}, 1 \right\} e^T, \qquad \kappa = \frac{\beta}{g}.
\end{aligned}
$$

In particular, the sequences

$$
\begin{aligned}
&\{\partial_t u_{\tau,n}\}_{n\in\mathbb{N}},\ \{\partial_x u_{\tau,n}\}_{n\in\mathbb{N}},\ \{\partial_{tt}^2 u_{\tau,n}\}_{n\in\mathbb{N}},\ \{\partial_{tx}^2 u_{\tau,n}\}_{n\in\mathbb{N}},\\
&\{\theta_{\tau,n}\}_{n\in\mathbb{N}},\ \{\partial_t\theta_{\tau,n}\}_{n\in\mathbb{N}},\ \{\partial_x\theta_{\tau,n}\}_{n\in\mathbb{N}}
\end{aligned}
$$

are bounded in $L^\infty(0,T;L^2(0,1))$*, and the sequence*

$$\{\partial_x\theta_{\tau,n}\}_{n\in\mathbb{N}}$$

is bounded in $L^2((0,T)\times(0,1))$.

Proof. Consider the quantities

$$U_n = (1+\tau\partial_t)u_{\tau,n}, \qquad \Theta_n = (1+\tau\partial_t)\theta_{\tau,n}, \qquad H = (1+\tau\partial_t)Q,$$

(3.2) gives

$$
\begin{cases}
\partial_{tt}^2 U_n = \partial_{xx}^2 U_n + \beta\partial_x\Theta_n - (1+\tau\partial_t)\,\Psi'_{\tau,n}(u_{\tau,n}), & t>0,\ 0<x<1,\\
\partial_t\Theta_n = \partial_{xx}^2\theta_{\tau,n} + g\partial_{tx}^2 U_n + H, & t>0,\ 0<x<1,\\
\partial_x U_n(t,0) = \partial_x U_n(t,1) = \Theta_n(t,0) = \Theta_n(t,1) = 0, & t>0,\\
U_n(0,x) = u_{0,\tau,n}(x) + \tau u_{1,\tau,n}(x), & 0<x<1,\\
\Theta_n(0,x) = \theta_{0,\tau,n}(x) + \tau\theta_{1,\tau,n}(x), & 0<x<1.
\end{cases}
\tag{3.4}
$$

Multiplying the first equation in (3.4) by $\partial_t U_n$, the second one by $\kappa\Theta_n$, and integrating over $(0,1)$ we get

$$
\begin{aligned}
&\frac{d}{dt}\int_0^1 \frac{(\partial_t U_n)^2 + (\partial_x U_n)^2 + \kappa\Theta_n^2}{2}dx\\
&\quad= \int_0^1 \left(\partial_t U_n \partial_{tt}^2 U_n - \partial_{xx}^2 U_n \partial_t U_n + \kappa\Theta_n\partial_t\Theta_n\right)dx\\
&\quad= \beta\int_0^1 \partial_x\Theta_n\partial_t U_n dx - \int_0^1 (1+\tau\partial_t)\,\Psi'_{\tau,n}(u_{\tau,n})\partial_t U_n dx + \kappa\int_0^1 \partial_{xx}^2\theta_{\tau,n}\Theta_n dx\\
&\qquad + \underbrace{\kappa g}_{=\beta}\int_0^1 \partial_{tx}^2 U_n\Theta_n dx + \kappa\int_0^1 H\Theta_n dx\\
&\quad= -\int_0^1 (1+\tau\partial_t)\,\Psi'_{\tau,n}(u_{\tau,n})\partial_t U_n dx - \kappa\int_0^1 (\partial_x\theta_{\tau,n})^2 dx - \kappa\tau\frac{d}{dt}\int_0^1 \frac{(\partial_x\theta_{\tau,n})^2}{2}dx + \kappa\int_0^1 H\Theta_n dx
\end{aligned}
$$

$$
\begin{aligned}
=&-\int_0^1 \Psi'_{\tau,n}(u_{\tau,n})\partial_t u_{\tau,n}dx-\tau\int_0^1 \Psi'_{\tau,n}(u_{\tau,n})\partial^2_{tt}u_{\tau,n}dx-\tau\int_0^1 \Psi''_{\tau,n}(u_{\tau,n})(\partial_t u_{\tau,n})^2dx\\
&-\tau^2\int_0^1 \Psi''_{\tau,n}(u_{\tau,n})\partial_t u_{\tau,n}\partial^2_{tt}u_{\tau,n}dx-\kappa\int_0^1(\partial_x\theta_{\tau,n})^2dx-\kappa\tau\frac{d}{dt}\int_0^1\frac{(\partial_x\theta_{\tau,n})^2}{2}dx+\kappa\int_0^1 H\,\Theta_n dx\\
=&-\frac{d}{dt}\int_0^1 \Psi_{\tau,n}(u_{\tau,n})dx-\tau\frac{d}{dt}\int_0^1 \Psi'_{\tau,n}(u_{\tau,n})\partial_t u dx-\tau^2\int_0^1 \Psi''_{\tau,n}(u_{\tau,n})\partial_t u_{\tau,n}\partial^2_{tt}u_{\tau,n}dx\\
&-\kappa\int_0^1(\partial_x\theta_{\tau,n})^2dx-\kappa\tau\frac{d}{dt}\int_0^1\frac{(\partial_x\theta_{\tau,n})^2}{2}dx+\kappa\int_0^1 H\,\Theta_n dx\\
\leq&-\frac{d}{dt}\int_0^1 \Psi_{\tau,n}(u_{\tau,n})dx-\tau\frac{d}{dt}\int_0^1 \Psi'_{\tau,n}(u_{\tau,n})\partial_t u_{\tau,n} dx-\tau^2\int_0^1 \Psi''_{\tau,n}(u_{\tau,n})\partial_t u_{\tau,n}\partial^2_{tt}u_{\tau,n}dx\\
&-\kappa\int_0^1(\partial_x\theta_{\tau,n})^2dx-\kappa\tau\frac{d}{dt}\int_0^1\frac{(\partial_x\theta_{\tau,n})^2}{2}dx+\frac{\kappa}{2}\int_0^1 H^2dx+\frac{\kappa}{2}\int_0^1\Theta_n^2dx,
\end{aligned}
$$

that is

$$
\begin{aligned}
&\frac{d}{dt}\int_0^1\left(\frac{(\partial_t U_n)^2+(\partial_x U_n)^2+\kappa\Theta_n^2+\kappa\tau(\partial_x\theta_{\tau,n})^2}{2}+\Psi_{\tau,n}(u_{\tau,n})\right)dx+\kappa\int_0^1(\partial_x\theta_{\tau,n})^2dx\\
&\quad\leq-\tau\frac{d}{dt}\int_0^1\Psi'_{\tau,n}(u_{\tau,n})\partial_t u_{\tau,n}dx\\
&\quad\quad-\tau^2\int_0^1\Psi''_{\tau,n}(u_{\tau,n})\partial_t u_{\tau,n}\partial^2_{tt}u_{\tau,n}dx+\frac{\kappa}{2}\int_0^1 H^2dx+\frac{\kappa}{2}\int_0^1\Theta_n^2dx.
\end{aligned}
\tag{3.5}
$$

Introducing the energy

$$
E_n(t)=\int_0^1\left(\frac{(\partial_t U_n)^2+(\partial_x U_n)^2+\kappa\Theta_n^2+\kappa\tau(\partial_x\theta_{\tau,n})^2}{2}+\Psi_{\tau,n}(u_{\tau,n})\right)dx
$$

from (3.5) we gain

$$
\begin{aligned}
E_n'(t)+\kappa\int_0^1(\partial_x\theta_{\tau,n})^2dx\leq&-\tau\frac{d}{dt}\int_0^1\Psi'_{\tau,n}(u_{\tau,n})\partial_t u_{\tau,n}dx\\
&-\tau^2\int_0^1\Psi''_{\tau,n}(u_{\tau,n})\partial_t u_{\tau,n}\partial^2_{tt}u_{\tau,n}dx+\frac{\kappa}{2}\int_0^1 H^2dx+E_n(t).
\end{aligned}
\tag{3.6}
$$

Therefore the Gronwall Lemma gives

$$
\begin{aligned}
&E_n(t)+\kappa e^t\int_0^t\int_0^1 e^{-s}(\partial_x\theta_{\tau,n})^2(s,x)dsdx\\
&\leq E_n(0)e^t-\tau e^t\int_0^t e^{-s}\frac{d}{ds}\left(\int_0^1(\Psi'_{\tau,n}(u_{\tau,n})\partial_t u_{\tau,n})(s,x)dx\right)ds\\
&\quad-\tau^2e^t\int_0^t\int_0^1 e^{-s}(\Psi''_{\tau,n}(u_{\tau,n})\partial_t u_{\tau,n}\partial^2_{tt}u_{\tau,n})(s,x)dsdx+\frac{\kappa}{2}e^t\int_0^t\int_0^1 e^{-s}H^2(s,x)dsdx.
\end{aligned}
\tag{3.7}
$$

Since

$$
\begin{aligned}
e^{-s}\frac{d}{ds}\left(\int_0^1(\Psi'_{\tau,n}(u_{\tau,n})\partial_t u_{\tau,n})(s,x)dx\right)=&\frac{d}{ds}\left(e^{-s}\int_0^1(\Psi'_{\tau,n}(u_{\tau,n})\partial_t u_{\tau,n})(s,x)dx\right)\\
&+e^{-s}\int_0^1(\Psi'_{\tau,n}(u_{\tau,n})\partial_t u_{\tau,n})(s,x)dx,
\end{aligned}
$$

(3.7) becomes

$$
E_n(t)+\kappa e^t\int_0^t\int_0^1 e^{-s}(\partial_x\theta_{\tau,n})^2(s,x)dsdx
$$

$$
\begin{aligned}
&\le E_n(0)e^t - \tau\int_0^1 (\Psi'_{\tau,n}(u_{\tau,n})\partial_t u_{\tau,n})(t,x)dx \qquad (3.8)\\
&+ \tau e^t\int_0^1 (\Psi'_{\tau,n}(u_{\tau,n})\partial_t u_{\tau,n})(0,x)dx - \tau e^t\int_0^t\int_0^1 e^{-s}(\Psi'_{\tau,n}(u_{\tau,n})\partial_t u_{\tau,n})(s,x)dsdx\\
&- \tau^2 e^t\int_0^t\int_0^1 e^{-s}(\Psi''_{\tau,n}(u_{\tau,n})\partial_t u_{\tau,n}\partial^2_{tt}u_{\tau,n})(s,x)dsdx + \frac{\kappa}{2}e^t\int_0^t\int_0^1 e^{-s}H^2(s,x)dsdx,
\end{aligned}
$$

and using the initial data

$$
\begin{aligned}
&E_n(t) + \kappa e^t\int_0^t\int_0^1 e^{-s}(\partial_x\theta_{\tau,n})^2(s,x)dsdx\\
&\le E_n(0)e^t - \tau\frac{d}{dt}\int_0^1 \Psi_{\tau,n}(u_{\tau,n})(t,x)dx \qquad (3.9)\\
&+ \tau e^t\int_0^1 (\Psi'_{\tau,n}(u_{0,\tau,n})u_{1,\tau,n})(x)dx - \tau e^t\int_0^t\int_0^1 e^{-s}\,\Psi_{\tau,n}(u_{\tau,n})(s,x)dsdx\\
&- \tau^2 e^t\int_0^t\int_0^1 e^{-s}(\Psi''_{\tau,n}(u_{\tau,n})\partial_t u_{\tau,n}\partial^2_{tt}u_{\tau,n})(s,x)dsdx + \frac{\kappa}{2}e^t\int_0^t\int_0^1 e^{-s}H^2(s,x)dsdx.
\end{aligned}
$$

Moreover, since

$$
\begin{aligned}
&\int_0^1 \frac{(\partial_t U_n)^2 + (\partial_x U_n)^2 + \kappa\Theta_n^2 + \kappa\tau(\partial_x\theta_{\tau,n})^2}{2}dx\\
&= \int_0^1 \frac{(\partial_t u_{\tau,n})^2 + 2\tau\partial_t u_{\tau,n}\partial^2_{tt}u_{\tau,n} + \tau^2(\partial^2_{tt}u_{\tau,n})^2 + (\partial_x u_{\tau,n})^2 + 2\tau\partial_x u_{\tau,n}\partial^2_{tx}u_{\tau,n} + \tau^2(\partial^2_{tx}u_{\tau,n})^2}{2}dx\\
&+ \int_0^1 \frac{\kappa\theta^2_{\tau,n} + 2\kappa\tau\theta_{\tau,n}\partial_t\theta_{\tau,n} + \kappa\tau^2(\partial_t\theta_{\tau,n})^2 + \kappa\tau(\partial_x\theta_{\tau,n})^2}{2}dx\\
&= \int_0^1 \frac{(\partial_t u_{\tau,n})^2 + \tau^2(\partial^2_{tt}u_{\tau,n})^2 + (\partial_x u_{\tau,n})^2 + \tau^2(\partial^2_{tx}u_{\tau,n})^2 + \kappa\theta^2_{\tau,n} + \kappa\tau^2(\partial_t\theta_{\tau,n})^2 + \kappa\tau(\partial_x\theta_{\tau,n})^2}{2}dx\\
&+ \tau\frac{d}{dt}\int_0^1 \frac{(\partial_t u_{\tau,n})^2 + (\partial_x u_{\tau,n})^2 + \kappa\theta^2_{\tau,n}}{2}dx,
\end{aligned}
$$

(3.8) becomes

$$
\begin{aligned}
&\int_0^1 \Bigg(\frac{(\partial_t u_{\tau,n})^2 + \tau^2(\partial^2_{tt}u_{\tau,n})^2 + (\partial_x u_{\tau,n})^2 + \tau^2(\partial^2_{tx}u_{\tau,n})^2}{2}\\
&\qquad + \frac{\kappa\theta^2_{\tau,n} + \kappa\tau^2(\partial_t\theta_{\tau,n})^2 + \kappa\tau(\partial_x\theta_{\tau,n})^2}{2} + \Psi_{\tau,n}(u_{\tau,n})\Bigg)(t,x)dx\\
&+ \tau\frac{d}{dt}\int_0^1 \left(\frac{(\partial_t u_{\tau,n})^2 + (\partial_x u_{\tau,n})^2 + \kappa\theta^2_{\tau,n}}{2} + 2\,\Psi_{\tau,n}(u_{\tau,n})\right)(t,x)dx\\
&+ \kappa e^t\int_0^t\int_0^1 e^{-s}((\partial_x\theta_{\tau,n})^2 + \Psi_{\tau,n}(u_{\tau,n}))(s,x)dsdx\\
&\le E_n(0)e^t + 2\tau e^t\int_0^1 (\Psi'_n(u_{0,\tau,n})u_{1,\tau,n})(x)dx \qquad (3.10)\\
&- \tau^2 e^t\int_0^t\int_0^1 e^{-s}(\Psi''_{\tau,n}(u_{\tau,n})\partial_t u_{\tau,n}\partial^2_{tt}u_{\tau,n})(s,x)dsdx + \frac{\kappa}{2}e^t\int_0^t\int_0^1 e^{-s}H^2(s,x)dsdx\\
&\le E_n(0)e^t + 2\tau e^t\int_0^1 (\Psi'_{\tau,n}(u_{0,\tau,n})u_{1,\tau,n})(x)dx\\
&+ \frac{\tau^2\left\|\Psi''_{\tau,n}\right\|^2_{L^\infty(\mathbb{R})}}{2}e^t\int_0^t\int_0^1 e^{-s}(\partial_t u_{\tau,n}(s,x))^2dsdx
\end{aligned}
$$

$$+\frac{\tau^2}{2}e^t\int_0^t\int_0^1 e^{-s}(\partial_{tt}^2 u_{\tau,n}(s,x))^2 dsdx+\frac{\kappa}{2}e^t\int_0^t\int_0^1 e^{-s}H^2(s,x)dsdx.$$

Then for $0\le t\le T$, (3.10) gives

$$\mathcal{E}_n(t)+A_n'(t)+B_n(t)\le C_n+D_n\int_0^t \mathcal{E}_n(s)ds.$$

Using again the Gronwall Lemma and integrating by parts we get (3.3). □

Lemma 3.2 (L^2 *Estimate*). *The sequence $\{u_{\tau,n}\}_{n\in\mathbb{N}}$ is bounded in $L^\infty(0,T;L^2(0,1))$, for every $T>0$.*

Proof. Since

$$\begin{aligned}\int_0^1 u_{\tau,n}^2(t,x)dx=&\int_0^1\left(u_{0,\tau,n}(x)+\int_0^t \partial_s u_{\tau,n}(s,x)ds\right)^2 dx\\ \le&2\int_0^1 u_{0,\tau,n}^2(x)dx+2\int_0^1\left(\int_0^t|\partial_s u_{\tau,n}(s,x)|ds\right)^2 dx\\ \le&2\int_0^1 u_{0,\tau,n}^2(x)dx+2t\int_0^t\int_0^1(\partial_s u_{\tau,n}(s,x))^2 dsdx\\ \le&2\int_0^1 u_{0,\tau,n}^2(x)dx+2t^2\sup_{s\ge0}\int_0^1(\partial_s u_{\tau,n}(s,x))^2dx,\end{aligned}$$

the claim follows from Lemma 3.1. □

Lemma 3.3 (L^∞ *Estimate*). *The sequence $\{u_{\tau,n}\}_{n\in\mathbb{N}}$ is bounded in $L^\infty((0,T)\times(0,1))$, for every $T>0$.*

Proof. Fix $0<t<T$ and $0<x<1$. Lemmas 3.1 and 3.2 imply that $\{u_{\tau,n}\}_{n\in\mathbb{N}}$ is bounded in $L^\infty(0,T;H^1(0,1))$. Since $H^1(0,1)\subset L^\infty(0,1)$ we have

$$|u_{\tau,n}(t,x)|\le\|u_{\tau,n}(t,\cdot)\|_{L^\infty(0,1)}\le c\|u_{\tau,n}(t,\cdot)\|_{H^1(0,1)}\le c\|u_{\tau,n}\|_{L^\infty(0,T;H^1(0,1))},$$

for some constant $c>0$. Therefore

$$\|u_{\tau,n}\|_{L^\infty((0,T)\times(0,1))}\le c\|u_{\tau,n}\|_{L^\infty(0,T;H^1(0,1))},$$

that gives the claim. □

Lemma 3.4 (H^2 *Estimate*). *The sequence $\{\partial_{xx}^2 u_{\tau,n}\}_{n\in\mathbb{N}}$ is bounded in $L^\infty(0,T;L^2(0,1))$, for every $T>0$.*

Proof. Since by (3.2)

$$\partial_{xx}^2 u_{\tau,n}=\partial_{tt}^2 u_{\tau,n}-\beta\partial_x\theta_{\tau,n}+\Psi_{\tau,n}'(u_{\tau,n}),$$

the claim follows by using Lemmas 3.1 and 3.2. □

Proof of Theorem 2.1. Thanks to Lemmas 3.1, 3.2, 3.4 and [23, Theorem 5] there exist two functions u_τ and θ_τ satisfying **(D.1)** and **(D.2)** of Definition 2.1 such that, passing to a subsequence,

$$\begin{aligned}u_{\tau,n}&\rightharpoonup u_\tau &&\text{weakly in } H^2((0,T)\times(0,1)),\text{ for each } T\ge0,\\ u_{\tau,n}&\to u_\tau &&\text{uniformly in } (0,T)\times(0,1),\text{ for each } T\ge0,\\ \theta_{\tau,n}&\rightharpoonup\theta_\tau &&\text{weakly in } H^1((0,T)\times(0,1)),\text{ for each } T\ge0,\\ \partial_x\theta_{\tau,n}&\rightharpoonup\partial_x\theta_\tau &&\text{weakly in } L^2((0,T)\times(0,1)),\text{ for each } T\ge0.\end{aligned}\tag{3.11}$$

We have to verify that $(u_\tau,\, \theta_\tau)$ is a weak solution of (2.8) in the sense of Definition 2.1. To this aim we fix a test function with compact support $\varphi \in C^\infty(\mathbb{R}\times(0,1))$. The following identity holds

$$\begin{aligned}&\int_0^\infty\int_0^1 \left(\theta_{\tau,n}(1-\tau\partial_t)\partial_t\varphi + \theta_{\tau,n}\partial^2_{xx}\varphi + g\partial^2_{tx}u_{\tau,n}(1-\tau\partial_t)\varphi + (1+\tau\partial_t)Q\varphi\right)dtdx \\ &\quad+\int_0^1\left(\theta_{0,\tau,n}(x)(1-\tau\partial_t)\varphi(0,x)+\tau\theta_{1,\tau,n}(x)\varphi(0,x)-g\tau u'_{1,\tau,n}(x)\varphi(0,x)\right)dx=0.\end{aligned} \tag{3.12}$$

Using (3.1) and (3.11), sending $n\to\infty$ in (3.12) we get (2.11).

We continue by proving the stability estimate (2.12). Define

$$v = u_\tau - \widetilde{u_\tau}, \qquad w = \theta_\tau - \widetilde{\theta}_\tau \tag{3.13}$$

Thanks to (2.8), $(v,\, w)$ solve the problem

$$\begin{cases}\partial^2_{tt}v = \partial^2_{xx}v + \beta\partial_x w - \left(\Psi'(u_\tau,\theta_\tau) - \Psi'(\widetilde{u_\tau},\widetilde{\theta}_\tau)\right), & t>0,\ 0<x<1,\\ (1+\tau\partial_t)\,\partial_t w = \partial^2_{xx}w + g(1+\tau\partial_t)\,\partial^2_{tx}v + (1+\tau\partial_t)\left(Q-\widetilde{Q}\right), & t>0,\ 0<x<1,\\ \partial_x v(t,0) = \partial_x v(t,1) = w(t,0) = w(t,1) = 0, & t>0,\\ v(0,x) = u_{0,\tau}(x) - \widetilde{u}_0(x),\ \partial_t v(0,x) = u_{1,\tau}(x) - \widetilde{u}_{1,\tau}(x), & 0<x<1,\\ w(0,x) = \theta_{0,\tau}(x) - \widetilde{\theta}_{0,\tau}(x),\ \partial_t w(0,x) = \theta_{1,\tau}(x) - \widetilde{\theta}_{1,\tau}(x), & 0<x<1,.\end{cases} \tag{3.14}$$

Arguing as in Lemma 3.1 we can prove

$$\begin{aligned}\frac{d}{dt}\int_0^1 &\left((\partial_t v)^2 + (\partial_x v)^2 + (\partial^2_{tt}v)^2 + (\partial^2_{tx}v)^2 + w^2 + (\partial_t w)^2 + (\partial_x w)^2\right)dx\\ &\le c\int_0^1\left(\left(\Psi'(u_\tau,\theta_\tau) - \Psi'(\widetilde{u_\tau},\widetilde{\theta}_\tau)\right)^2 + \left((1+\tau\partial_t)\left(Q-\widetilde{Q}\right)\right)^2\right)dx\\ &\le c\int_0^1\left(v^2 + w^2 + \left(Q-\widetilde{Q}\right)^2 + \left(\partial_t Q - \partial_t\widetilde{Q}\right)^2\right)dx.\end{aligned}$$

Adding $\int_0^1 v\,\partial_t v\,dx$ to both sides we get

$$\begin{aligned}\frac{d}{dt}\int_0^1 &\left(v^2 + (\partial_t v)^2 + (\partial_x v)^2 + (\partial^2_{tt}v)^2 + (\partial^2_{tx}v)^2 + w^2 + (\partial_t w)^2 + (\partial_x w)^2\right)dx\\ &\le 2\int_0^1 v\partial_t v dx + c\int_0^1\left(v^2 + w^2 + \left(Q-\widetilde{Q}\right)^2 + \left(\partial_t Q - \partial_t\widetilde{Q}\right)^2\right)dx\\ &\le c\int_0^1\left(v^2 + (\partial_t v)^2 + w^2 + \left(Q-\widetilde{Q}\right)^2 + \left(\partial_t Q - \partial_t\widetilde{Q}\right)^2\right)dx.\end{aligned}$$

Therefore, using (2.13) and (3.13),

$$\Lambda(t) = \int_0^1\left(v^2 + (\partial_t v)^2 + (\partial_x v)^2 + (\partial^2_{tt}v)^2 + (\partial^2_{tx}v)^2 + w^2 + (\partial_t w)^2 + (\partial_x w)^2\right)dx$$

we have

$$\frac{d}{dt}\Lambda(t) \le c\Lambda(t) + c\int_0^1\left(\left(Q-\widetilde{Q}\right)^2 + \left(\partial_t Q - \partial_t\widetilde{Q}\right)^2\right)dx.$$

Using the Gronwall Lemma we gain

$$\Lambda(t) \le \Lambda(0)e^{ct} + c\int_0^t e^{c(t-s)}\left(\left\|(Q-\widetilde{Q})(s,\cdot)\right\|^2_{L^2(0,1)} + \left\|(\partial_t Q - \partial_t\widetilde{Q})(s,\cdot)\right\|^2_{L^2(0,1)}\right)ds,$$

that is (2.12). Then, the uniqueness of the solutions trivially follows. □

4. Singular limit: Breakable adhesion and parabolic propagation

In our intuition the occurrence of an instantaneous phenomenon like delamination could be well captured in a model with infinite speed of heat propagation, hence $\tau = 0$. Hence in this section we study the asymptotic limit of a family of differential problems obtained by relaxing the regularity properties of the adhesion potential in relation to the relaxation time affecting the hyperbolic heat propagation. More precisely, we assume that $\Psi_\tau \to \Psi$ locally uniformly on $\mathbb{R}$ as $\tau \to 0$, where a prototypical example is given by

$$\Psi(u) = \begin{cases} \mu u^2, & \text{if } |u| \le u^*, \\ \mu(u^*)^2, & \text{if } |u| > u^*, \end{cases} \tag{4.1}$$

where u^* denotes the threshold beyond which the glue cannot sustain further stress and μ is a (positive) constitutive parameter.

Then we consider the family of differential problems depending on the parameter $\tau > 0$:

$$\begin{cases} \partial_{tt}^2 u_\tau = \partial_{xx}^2 u_\tau + \beta\partial_x\theta_\tau - \Psi_\tau'(u_\tau), & t > 0,\ 0 < x < 1, \\ (1 + \tau\partial_t)\partial_t\theta_\tau = \partial_{xx}^2\theta_\tau + g(1 + \tau\partial_t)\partial_{tx}^2 u_\tau + (1 + \tau\partial_t)Q, & t > 0,\ 0 < x < 1, \\ \partial_x u_\tau(t, 0) = \partial_x u_\tau(t, 1) = \theta_\tau(t, 0) = \theta_\tau(t, 1) = 0, & t > 0, \\ u_\tau(0, x) = u_{0,\tau}(x),\ \partial_t u_\tau(0, x) = u_{1,\tau}(x), & 0 < x < 1, \\ \theta_\tau(0, x) = \theta_{0,\tau}(x),\ \partial_t\theta_\tau(0, x) = \theta_{1,\tau}(x), & 0 < x < 1, \end{cases} \tag{4.2}$$

where

$$\text{for every } \tau > 0,\ \Psi_\tau,\ u_{0,\tau},\ u_{1,\tau},\ \theta_{0,\tau},\ \theta_{1,\tau} \text{ satisfy (2.9) and (2.10), respectively;} \tag{4.3}$$

$$\Psi_\tau \to \Psi \text{ uniformly on every compact of } \mathbb{R} \text{ as } \tau \to 0; \tag{4.4}$$

$$u_{0,\tau} \to u_0 \text{ a.e. in } (0, 1) \text{ and in } H^1(0, 1) \text{ as } \tau \to 0; \tag{4.5}$$

$$u_{1,\tau} \to u_1 \text{ and } \theta_{0,\tau} \to \theta_0 \text{ a.e. in } (0, 1) \text{ and in } L^2(0, 1) \text{ as } \tau \to 0. \tag{4.6}$$

The study of the convergence of this family of problems will lead us to single out a limit problem characterized by parabolic heat propagation. More precisely we have to face with the following hyperbolic–parabolic system:

$$\begin{cases} \partial_{tt}^2 u = \partial_{xx}^2 u + \beta\partial_x\theta - \Psi'(u), & t > 0,\ 0 < x < 1, \\ \partial_t\theta = \partial_{xx}^2\theta + g\partial_{tx}^2 u + Q, & t > 0,\ 0 < x < 1, \\ \partial_x u(t, 0) = \partial_x u(t, 1) = \theta(t, 0) = \theta(t, 1) = 0, & t > 0, \\ u(0, x) = u_0(x),\ \partial_t u(0, x) = u_1(x), & 0 < x < 1, \\ \theta(0, x) = \theta_0(x), & 0 < x < 1, \end{cases} \tag{4.7}$$

where we assume that (see Figure 2)

$$\beta,\ g \in \mathbb{R}, \quad \Psi(\cdot) \ge 0, \quad \Psi \in C^\infty(\mathbb{R} \setminus \{1, -1\}), \quad \Psi' \in L^\infty(\mathbb{R}), \tag{4.8}$$

$$u_0 \in H^1(0, 1), \quad u_1 \in L^2(0, 1), \quad \theta_0 \in L^2(0, 1), \tag{4.9}$$

In this case we use the following definition of solution (u, θ) to problem (4.7).

Definition 4.1. Let $u,\ \theta : [0, \infty) \times [0, 1] \to \mathbb{R}$ be functions. We say that $(u,\ \theta)$ is a solution of the initial boundary value problem (4.7) if

(D.1) $u \in H^1((0, T) \times (0, 1))$, $\theta \in L^2(0, T; H_0^1(0, 1))$, for every $T > 0$;

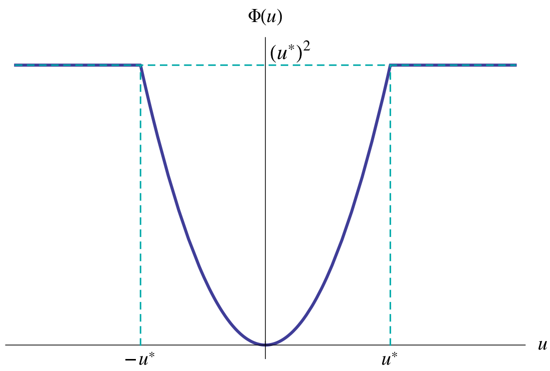

Fig. 2. Potential $\Psi(u)$ in Eq. (4.1) with $\mu = 1$.

(D.2) the initial and boundary conditions are satisfied almost everywhere;
(D.3) for every test function $\varphi \in C^\infty(\mathbb{R}^2)$ with compact support the following identity holds

$$\begin{aligned}\int_0^\infty \int_0^1 \left(u\partial_{tt}^2\varphi + \partial_x u \partial_x \varphi - \beta \partial_x \theta \varphi + h_u \varphi\right) dtdx \\ - \int_0^1 u_1(x)\varphi(0,x)dx + \int_0^1 u_0(x)\partial_t\varphi(0,x)dx = 0,\end{aligned} \tag{4.10}$$

for some $h_u \in \partial\Psi(u)$, where $\partial\Psi(u)$ is the subdifferential of Ψ at u;
(D.4) for every test function $\varphi \in C^\infty(\mathbb{R} \times (0,1))$ with compact support the following identity holds

$$\int_0^\infty \int_0^1 (\theta\partial_t\varphi - \partial_x\theta\partial_x\varphi - g\partial_t u\partial_x\varphi + Q\varphi)\, dtdx + \int_0^1 \theta_0(x)\varphi(0,x)dx = 0. \tag{4.11}$$

To exploit the precise convergence result as $\tau \to 0$ we prove the following statement (see Fig. 2).

Theorem 4.1. *Let us assume that (4.8), (4.9), (4.3), (4.4), (4.5), (4.6) hold. For every $\tau > 0$, let (u_τ, θ_τ) be a solution of (4.2) in the sense of Definition 2.1. Assume also that*

$$\tau \,\|\Psi_\tau''\|_{L^\infty(\mathbb{R})} \le C, \qquad \tau > 0, \tag{4.12}$$

for some constants $C > 0$. Then there exist a sequence $\{\tau_n\}_{n\in\mathbb{N}} \subset (0,\infty)$, $\tau_n \to 0$, and a solution (u, θ) of the initial boundary value problem (4.7) in the sense of Definition 4.1 such that

$$\begin{aligned} &u_{\tau_n} \to u \text{ and } \theta_{\tau_n} \to \theta \text{ a.e. in } (0,\infty)\times(0,1),\\ &u_{\tau_n} \to u \text{ and } \theta_{\tau_n} \to \theta \text{ in } L^p((0,T)\times(0,1)) \text{ for every } T>0 \text{ and } 1 \le p < \infty,\\ &u_{\tau_n} \rightharpoonup u \text{ weakly in } H^1((0,T)\times(0,1)) \text{ for every } T>0,\\ &\theta_{\tau_n} \rightharpoonup \theta \text{ weakly in } L^2(0,T;H_0^1(0,1)) \text{ for every } T>0,\end{aligned} \tag{4.13}$$

as $n \to \infty$. In addition, if

$$\beta g > 0, \qquad Q \equiv 0, \tag{4.14}$$

then (u, θ) satisfies the following energy dissipation inequality

$$E(t) + \frac{1}{|g|}\int_0^t \int_0^1 (\partial_x\theta)^2 dsdx \le E(0), \tag{4.15}$$

for every t, where

$$E(t) = \int_0^1 \left(\frac{(\partial_t u)^2 + (\partial_x u)^2}{2|\beta|} + \frac{\theta^2}{2|g|} + \frac{\Psi(u)}{2|\beta|}\right) dx,$$

$$E(0) = \int_0^1 \left(\frac{u_1^2 + (\partial_x u_0)^2}{2|\beta|} + \frac{\theta_0^2}{2|g|} + \frac{\Psi(u_0)}{2|\beta|} \right) dx.$$

Proof. Arguing as in Lemmas 3.1, 3.2, 3.3 we have

- $\{\partial_t u_\tau\}_{\tau>0}$, $\{\partial_t u_\tau\}_{\tau>0}$, $\{\partial_x u_\tau\}_{\tau>0}$, $\{\tau\partial_{tt}^2 u_\tau\}_{\tau>0}$, $\{\tau\partial_{tx}^2 u_\tau\}_{\tau>0}$, $\{\theta_\tau\}_{\tau>0}$, $\{\tau\partial_t\theta_\tau\}_{\tau>0}$, $\{\sqrt{\tau}\partial_x\theta_\tau\}_{\tau>0}$ are bounded in $L^\infty(0,T;L^2(\mathbb{R}))$, for every $T>0$;
- $\{\partial_x\theta_\tau\}_{\tau>0}$ is bounded in $L^2((0,T)\times\mathbb{R})$, for every $T>0$;
- $\{u_\tau\}_{\tau>0}$ is bounded in $L^\infty(0,T;L^2(0,1))$, for every $T>0$;
- $\{u_\tau\}_{\tau>0}$ is bounded in $L^\infty((0,T)\times(0,1))$, for every $T>0$.

Thanks to [23, Theorem 5] there exist three functions u, θ and $h_u \in L^\infty((0,T)\times(0,1))$, $h_u \in \partial\Psi(u)$, satisfying **(D.1)** and **(D.2)** of Definition 4.1 such that, passing to a subsequence,

$$\begin{aligned}
u_\tau &\rightharpoonup u && \text{weakly in } H^1((0,T)\times(0,1)), \text{ for each } T\geq 0,\\
u_\tau &\to u && \text{strongly in } L^p((0,T)\times(0,1)), \text{ for each } T\geq 0,\ 1\leq p<\infty,\\
\theta_\tau &\rightharpoonup \theta && \text{weakly in } L^2(0,T;H_0^1(0,1)), \text{ for each } T\geq 0,\\
\partial_x\theta_\tau &\rightharpoonup \partial_x\theta && \text{weakly in } L^2((0,T)\times(0,1)), \text{ for each } T\geq 0,\\
\Psi_\tau'(u_\tau) &\rightharpoonup h_u && \text{weakly in } L^p((0,T)\times(0,1)), \text{ for each } T\geq 0 \text{ and } 1\leq p<\infty.
\end{aligned} \tag{4.16}$$

We have to verify that $(u,\,\theta)$ is a weak solution of (4.7) in the sense of Definition 4.1. To this aim let $\varphi\in C^\infty(\mathbb{R}^2)$ be a test function with compact support. By (4.2), for every τ we have

$$\begin{aligned}
&\int_0^\infty\int_0^1 \left(u_\tau\partial_{tt}^2\varphi + \partial_x u_\tau\partial_x\varphi - \beta\partial_x\theta_\tau\varphi + \Psi_\tau'(u_\tau)\varphi\right)dtdx\\
&\qquad -\int_0^1 u_{1,\tau}(x)\varphi(0,x)dx + \int_0^1 u_{0,\tau}(x)\partial_t\varphi(0,x)dx = 0.
\end{aligned} \tag{4.17}$$

By virtue of (4.4), (4.5), (4.6), (4.12), (4.16) sending $\tau\to 0$ in (4.17) we get (4.10). Moreover, for every test function $\varphi\in C^\infty(\mathbb{R}\times(0,1))$ with compact support the following identity holds

$$\begin{aligned}
&\int_0^\infty\int_0^1 \left(\theta_\tau(1-\tau\partial_t)\partial_t\varphi + \theta_\tau\partial_{xx}^2\varphi + g\partial_{tx}^2 u_\tau(1-\tau\partial_t)\varphi + (1+\tau\partial_t)Q\varphi\right)dtdx\\
&\quad + \int_0^1 \left(\theta_{0,\tau}(x)(1-\tau\partial_t)\varphi(0,x) + \tau\theta_{1,\tau}(x)\varphi(0,x) - g\tau u_{1,\tau}'(x)\varphi(0,x)\right)dx = 0.
\end{aligned} \tag{4.18}$$

Using (4.4), (4.5), (4.6), (4.12), (4.16) sending $\tau\to 0$ in (4.18) we get (4.11). Eventually, (4.15) follows from (3.3) and (4.16). $\square$

5. Long time behavior

There are several interesting questions concerning the long time behavior of the dynamics ruled by (4.7). In particular, one is interested in characterizing the limit states of the system since it is not clear *a priori* if the cohesion–decohesion evolution collapses in a single stationary state as $t\to\infty$. Such (equilibrium) states consist of a constant value for the displacement $a\in(-\infty,-1]\cup\{0\}\cup[1,\infty)$ and 0 for the temperature.

The present section is devoted to prove the following result.

Theorem 5.1 (*Long Time Behavior*). *Assume (4.14). Let $(u,\,\theta)$ be a weak solution of (4.7) satisfying (4.15) and $\{t_n\}_{n\in\mathbb{N}}\subset(0,\infty)$ such that $t_n\to\infty$. If*

$$u,\,\theta\in L^\infty((0,\infty)\times(0,1)), \tag{5.1}$$

then there exist a subsequence $\{t_{n_k}\}_{k\in\mathbb{N}}$ *and a constant* $a \in \mathbb{R}$ *such that*

$$a \in (-\infty, -1] \cup \{0\} \cup [1, \infty), \tag{5.2}$$

$$u(t_{n_k}, \cdot) \rightharpoonup a \quad \textit{weakly in } H^1(0,1) \textit{ as } k \to \infty, \tag{5.3}$$

$$\theta(t_{n_k}, \cdot) \rightharpoonup 0 \quad \textit{weakly in } L^2(0,1) \textit{ as } k \to \infty. \tag{5.4}$$

Proof. Let (u, θ) be a weak solution of (4.7) satisfying (4.15) and (5.1).

STEP 1. We begin by deducing the effective asymptotic problem.

Consider the functions

$$u_\sigma(t,x) = u(\sigma t, x), \quad \theta_\sigma(t,x) = \theta(\sigma t, x), \qquad \sigma > 0,\ t \geq 0,\ x \in [0,1].$$

$(u_\sigma, \theta_\sigma)$ is a weak solution of the following initial boundary value problem

$$\begin{cases} \dfrac{\partial^2_{tt} u_\sigma}{\sigma^2} = \partial^2_{xx} u_\sigma + \beta \partial_x \theta_\sigma - \Psi'(u_\sigma), & t > 0,\ 0 < x < 1, \\ \dfrac{\partial_t \theta_\sigma}{\sigma} = \partial^2_{xx} \theta_\sigma + \dfrac{g}{\sigma} \partial^2_{tx} u_\sigma, & t > 0,\ 0 < x < 1, \\ \partial_x u_\sigma(t,0) = \partial_x u_\sigma(t,1) = \theta_\sigma(t,0) = \theta_\sigma(t,1) = 0, & t > 0, \\ u_\sigma(0,x) = u_0(x),\ \partial_t u_\sigma(0,x) = \sigma u_1(x), & 0 < x < 1, \\ \theta_\sigma(0,x) = \theta_0(x), & 0 < x < 1, \end{cases} \tag{5.5}$$

in the sense of Definition 4.1, namely for every test function $\varphi \in C^\infty(\mathbb{R}^2)$ with compact support

$$\begin{aligned} &\int_0^\infty \int_0^1 \left(-\frac{\partial_t u_\sigma \partial_t \varphi}{\sigma^2} + \partial_x u_\sigma \partial_x \varphi - \beta \partial_x \theta_\sigma \varphi + h_\sigma \varphi \right) dt dx \\ &\qquad - \int_0^1 \frac{u_1(x)}{\sigma} \varphi(0,x) dx = 0, \\ &\int_0^\infty \int_0^1 \left(\frac{\theta_\sigma \partial_t \varphi}{\sigma} - \partial_x \theta_\sigma \partial_x \varphi + \frac{g u_\sigma \partial^2_{tx} \varphi}{\sigma} \right) dt dx \\ &\qquad + \int_0^1 \frac{\theta_0(x) \varphi(0,x)}{\sigma} dx + \int_0^1 \frac{g u_0(x) \partial_x \varphi(0,x)}{\sigma} dx = 0. \end{aligned} \tag{5.6}$$

where $h_\sigma \in \partial\Psi(u)$, that is the subdifferential of Ψ at u_σ. In addition $(u_\sigma, \theta_\sigma)$ may *dissipate energy*, i.e. for almost every $t > 0$:

$$\begin{aligned} &\int_0^1 \left(\frac{(\partial_t u_\sigma)^2}{2|\beta|\sigma^2} + \frac{(\partial_x u_\sigma)^2}{2|\beta|} + \frac{\theta_\sigma^2}{2|g|} + \frac{\Psi(u_\sigma)}{2|\beta|} \right) dx + \frac{1}{|g|} \int_0^t \int_0^1 (\partial_x \theta_\sigma)^2 ds dx \\ &\leq \int_0^1 \left(\frac{u_1^2 + (\partial_x u_0)^2}{2|\beta|} + \frac{\theta_0^2}{2|g|} + \frac{\Psi(u_0)}{2|\beta|} \right) dx. \end{aligned} \tag{5.7}$$

Thanks to (4.8), (5.1), and (5.7),

$$\begin{aligned} &\{u_\sigma\}_{\sigma>0} \text{ is bounded in } L^\infty(0,\infty; H^1(0,1)), \\ &\{\theta_\sigma\}_{\sigma>0} \text{ is bounded in } L^\infty(0,\infty; L^2(0,1)), \\ &\{h_\sigma\}_{\sigma>0} \text{ is bounded in } L^\infty((0,\infty) \times (0,1)), \end{aligned}$$

so there exist three functions $U \in L^\infty(0,\infty; H^2(0,1))$, $\Theta \in L^\infty(0,\infty; L^2(0,1))$, $H \in L^\infty((0,\infty) \times (0,1))$ such that, passing to a subsequence,

$$u_\sigma \overset{\star}{\rightharpoonup} U \qquad \text{weakly-}\star \text{ in } L^\infty_{loc}((0,\infty) \times (0,1)) \text{ as } \sigma \to \infty,$$

$$\begin{aligned} \theta_\sigma &\overset{\star}{\rightharpoonup} \Theta && \text{weakly-}\star \text{ in } L^\infty_{loc}((0,\infty)\times(0,1)) \text{ as } \sigma\to\infty,\\ h_\sigma &\overset{\star}{\rightharpoonup} H && \text{weakly-}\star \text{ in } L^\infty_{loc}((0,\infty)\times(0,1)) \text{ as } \sigma\to\infty. \end{aligned}$$

Using (5.7)

$$\{\partial_t u_\sigma/\sigma\}_{\sigma>0},\ \{u_\sigma\}_{\sigma>0},\ \{\theta_\sigma\}_{\sigma>0} \text{ are bounded in } L^\infty(0,\infty;L^2(0,1)),$$

therefore as $\sigma\to\infty$ in (5.6) we get

$$\begin{aligned} &\int_0^\infty\int_0^1 (\partial_x U\partial_x\varphi - \beta\partial_x\Theta\varphi + H\varphi)\,dtdx = 0,\\ &\int_0^\infty\int_0^1 \partial_x\Theta\partial_x\varphi dtdx = 0. \end{aligned} \tag{5.8}$$

Since $U=U(x)$, $\Theta=\Theta(x)$, $H=H(x)$, the effective asymptotic problem is

$$\begin{cases} \partial^2_{xx}U + \beta\partial_x\Theta = H, & 0<x<1,\\ \partial^2_{xx}\Theta = 0, & 0<x<1,\\ \partial_x U(0) = \partial_x U(1) = 0, & \\ \Theta(0)=\Theta(1)=0. & \end{cases} \tag{5.9}$$

STEP 2. We exploit more subtle characterizations of the limit functions U and H. To this aim we fix a sequence $\{t_n\}_{n\in\mathbb{N}}\subset(0,\infty)$ such that $t_n\to\infty$ and study the convergence of the sequence

$$\{(u(t_n,\cdot),\ \theta(t_n,\cdot))\}_{n\in\mathbb{N}}.$$

Since we have the dissipation inequality (5.7) and assumption (5.1), we gain

$$\begin{aligned} &\{u(t_n,\cdot)\}_{n\in\mathbb{N}} \text{ is bounded in } H^1(0,1),\\ &\{\theta(t_n,\cdot)\}_{n\in\mathbb{N}} \text{ is bounded in } L^2(0,1),\\ &\{h_u(t_n,\cdot)\}_{n\in\mathbb{N}} \text{ is bounded in } L^\infty(0,1). \end{aligned}$$

Therefore there exist three functions $u_\infty\in H^1(0,1)$, $\theta_\infty\in L^2(0,1)$, $h_\infty\in L^\infty(0,1)$ such that passing to a subsequence

$$\begin{aligned} u(t_n,\cdot) &\rightharpoonup u_\infty && \text{weakly in } H^1(0,1) \text{ as } n\to\infty,\\ u(t_n,\cdot) &\to u_\infty && \text{a.e. in } (0,1) \text{ as } n\to\infty,\\ \theta(t_n,\cdot) &\rightharpoonup \theta_\infty && \text{weakly in } L^2(0,1) \text{ as } n\to\infty,\\ h_u(t_n,\cdot) &\overset{\star}{\rightharpoonup} h_\infty && \text{weakly-}\star \text{ in } L^\infty(0,1) \text{ as } n\to\infty. \end{aligned} \tag{5.10}$$

Due to the result in **STEP 1**, we know that the functions u_∞, θ_∞ and h_∞ must satisfy the effective problem

$$\begin{cases} \partial^2_{xx}u_\infty + \beta\partial_x\theta_\infty = h_\infty, & 0<x<1,\\ \partial^2_{xx}\theta_\infty = 0, & 0<x<1,\\ \partial_x u_\infty(0) = \partial_x u_\infty(1) = 0, & \\ \theta_\infty(0)=\theta_\infty(1)=0. & \end{cases} \tag{5.11}$$

Moreover, by (5.10) we have also that

$$h_\infty \in \partial\Phi'(u_\infty). \tag{5.12}$$

By multiplying the second equation in (5.11) by θ_∞, integrating over $(0,1)$, we get

$$\int_0^1 (\partial_x\theta_\infty)^2 dx = 0,$$

so

$$\partial_x \theta_\infty \equiv 0.$$

Thanks to the boundary conditions

$$\theta_\infty \equiv 0, \tag{5.13}$$

that proves (5.4). By multiplying the first equation in (5.11) by u_∞ and by integrating over $(0,1)$, and recalling (5.12) and (5.13), we get

$$\int_0^1 (\partial_x u_\infty)^2 dx = -\int_0^1 h_\infty u_\infty dx \leq 0,$$

so

$$\partial_x u_\infty \equiv 0.$$

Therefore, we can conclude that (5.3) holds. Using (5.3) in (5.11) we have also that

$$h_\infty \equiv 0,$$

hence, due to (5.12), (5.2) occurs. □

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