Periodic Solutions of the Degasperis-Procesi equation: well-posedness and asymptotics

PERIODIC SOLUTIONS OF THE THE DEGASPERIS-PROCESI EQUATION: WELPOSEDNESS AND ASYMPTOTICS

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Abstract. We prove the well-posedness of periodic entropy (discontinuous) solutions for the Degasperis-Procesi equation. Partly motivated by the bounded periodic solutions found by Vakhnenko and Parkes [20], we study the long-time asymptotic behavior of periodic entropy solutions.

1. Introduction

We investigate the well-posedness and long-time asymptotic behavior of periodic discontinuous solutions of the *Degasperis-Procesi* equation. It has the form

$$\partial_t u - \partial^3_{txx} u + 4u\partial_x u = 3\partial_x u \partial^2_{xx} u + u\partial^3_{xxx} u, \qquad (t,x) \in (0,\infty)\times\mathbb{R}, \tag{1.1}$$

and is augmented with the initial condition

$$u(0,x) = u_0(x), \qquad x \in \mathbb{R}, \tag{1.2}$$

We assume

$$u_0 \in L^\infty(\mathbb{R}), \qquad u_0 \text{ is 1-periodic.} \tag{1.3}$$

Degasperis and Procesi [7] deduced (1.1) from the following family of third order dispersive nonlinear equations, indexed over six constants $\alpha, \gamma, c_0, c_1, c_2, c_3 \in \mathbb{R}$:

$$\partial_t u + c_0 \partial_x u + \gamma \partial^3_{xxx} u - \alpha^2 \partial^3_{txx} u = \partial_x \left(c_1 u^2 + c_2 (\partial_x u)^2 + c_3 u \partial^2_{xx} u \right).$$

Using the method of asymptotic integrability, they found that only three equations within this family were asymptotically integrable up to the third order: the *KdV equation* ($\alpha = c_2 = c_3 = 0$), the *Camassa-Holm equation* ($c_1 = -\frac{3c_3}{2\alpha^2}$, $c_2 = \frac{c_3}{2}$), and one new equation ($c_1 = -\frac{2c_3}{\alpha^2}$, $c_2 = c_3$), which properly scaled reads

$$\partial_t u + \partial_x u + 6u\partial_x u + \partial^3_{xxx} u - \alpha^2 \left(\partial^3_{txx} u + \frac{9}{2} \partial_x u \partial^2_{xx} u + \frac{3}{2} u \partial^3_{xxx} u \right) = 0. \tag{1.4}$$

One can transform (1.4) into the form (1.1), see [8, 9] for details.

Degasperis, Holm, and Hone [9] proved the integrability of (1.1) by constructing a Lax pair. Moreover, they provided a relation to a negative flow in the Kaup-Kupershmidt hierarchy by a reciprocal transformation and derived two infinite

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sequences of conserved quantities along with a bi-Hamiltonian structure. Furthermore, they showed that the Degasperis-Procesi equation is endowed with weak (continuous) solutions that are superpositions of multipeakons and described the (finite-dimensional) integrable peakon dynamics. An explicit solution was also found in the perfectly anti-symmetric peakon-antipeakon collision case. Lundmark and Szmigielski [14], using an inverse scattering approach, computed n-peakon solutions to (1.1). Mustafa [16] proved that smooth solutions have infinite speed of propagation. Blow-up phenomena have been investigated for example in [23]. Regarding the Cauchy problem and the initial-boundary value problem for the Degasperis-Procesi equation (1.1), Escher, Liu, and Yin have studied its well-posedness within certain functional classes in a series of papers [11, 21, 22].

The approach taken in the papers just listed emphasizes the similarities between the Degasperis-Procesi equation and the Camassa-Holm equation, and consequently the main focus has been on (weak) continuous solutions. In a different direction, Coclite and Karlsen [3, 4, 5], Coclite, Karlsen, and Kwon [6], and Lundmark [13] initiated a study of discontinuous (shock wave) solutions to the Degasperis-Procesi equation (1.1). In particular, the existence, uniqueness, and stability of entropy solutions of the Cauchy problem for (1.1) is proved in [3, 4, 5] and for the initial-boundary value problem in [6].

When it comes to periodic solutions for the Degasperis-Procesi equation much less is known. The first results in that direction are those of Escher, Liu, and Yin [11, 21, 22], which apply to continuous solutions. To encompass discontinuous solutions we herein extend the approach of [3, 4, 5, 6].

Following [3] we rewrite (1.1) and (1.2) as a hyperbolic-elliptic system:

$$\begin{cases} \partial_t u + u\partial_x u + \partial_x P = 0, & (t,x) \in (0,\infty)\times\mathbb{R}, \\ -\partial_{xx}^2 P + P = \frac{3}{2}u^2, & (t,x) \in (0,\infty)\times\mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{1.5}$$

Since $e^{-|\xi|}/2$ is the Green's function of the differential operator $1-\partial_{xx}^2$ the function P has a convolution structure:

$$P(t,x) = P^u(t,x) := \frac{3}{4}\int_{\mathbb{R}} e^{-|x-y|}u^2(t,y)dy, \tag{1.6}$$

and (1.5) can be written as a conservation law with nonlocal flux

$$\partial_t u + \partial_x\left(\frac{u^2}{2} + \frac{3}{4}\int_{\mathbb{R}} e^{-|x-y|}u^2(t,y)dy\right) = 0, \qquad (t,x)\in(0,\infty)\times\mathbb{R}, \tag{1.7}$$

or as a conservation law with nonlocal source:

$$\partial_t u + \partial_x\left(\frac{u^2}{2}\right) + \frac{3}{4}\int_{\mathbb{R}} e^{-|x-y|}\mathrm{sign}\,(y-x)\,u^2(t,y)dy = 0, \quad (t,x)\in(0,\infty)\times\mathbb{R}. \tag{1.8}$$

Moreover, the spatial periodicity of u implies the spatial periodicity of P^u. Indeed,

$$\begin{aligned} \frac{3}{4}\int_{\mathbb{R}} e^{-|(x+1)-y|}u^2(t,y)dy &= \frac{3}{4}\int_{\mathbb{R}} e^{-|x-(y-1)|}u^2(t,y)dy \\ &= \frac{3}{4}\int_{\mathbb{R}} e^{-|x-y|}u^2(t,y+1)dy = \frac{3}{4}\int_{\mathbb{R}} e^{-|x-y|}u^2(t,y)dy, \end{aligned}$$

and so $P^u(t,x+1) = P^u(t,x)$.

Following [3] we use the following definition of a periodic entropy solution:

Definition 1.1. *We say that $u \in L^\infty((0,T)\times\mathbb{R})$, for any $T > 0$, is a periodic entropy solution of the Cauchy problem* (1.1), (1.2) *if*

i) for almost every $t \geq 0$, $u(t,\cdot)$ is 1-periodic;
ii) u is a distributional solution of (1.5)*;*
iii) for every convex function $\eta \in C^2(\mathbb{R})$ the entropy inequality

$$\partial_t \eta(u) + \partial_x q(u) + \eta'(u)\partial_x P^u \leq 0, \qquad q(u) = \int^u \xi \eta'(\xi)\, d\xi, \tag{1.9}$$

holds in the sense of distributions on $(0,\infty)\times\mathbb{R}$.

The aim of this paper is twofold. First of all, we study the well-posedness of periodic entropy solutions of the Degasperis-Procesi equation. Second, partly motivated by the explicit description of bounded periodic solutions found in [20] (see also [13, 14]), we study the long-time asymptotic behavior of periodic entropy solutions and prove that they decay to the mean value of the initial condition.

Our main result is the following theorem.

Theorem 1.1. *Let u_0 satisfy* (1.3)*. The initial-boundary value problem* (1.1), (1.2) *possesses an unique periodic entropy solution $u \in L^\infty((0,T)\times\mathbb{R})$, $T > 0$. Moreover, if $v \in L^\infty((0,T)\times\mathbb{R})$, $T > 0$, is the unique periodic entropy solution of* (1.1) *with initial condition v_0 satisfying* (1.3)*, then*

$$\left\|u(t,\cdot) - v(t,\cdot)\right\|_{L^1(0,1)} \leq e^{M_T t}\left\|u_0 - v_0\right\|_{L^1(0,1)}, \tag{1.10}$$

for every $n \in \mathbb{N}$ and almost every $0 < t < T$, where

$$\begin{aligned} M_T := &3\Big[\left\|u_0\right\|_{L^\infty(\mathbb{R})} + \left\|v_0\right\|_{L^\infty(\mathbb{R})} \\ &+ 18T\left(\left\|u_0\right\|^2_{L^2(0,1)} + \left\|v_0\right\|^2_{L^2(0,1)}\right)\Big]. \end{aligned} \tag{1.11}$$

Finally, if $u \in L^\infty((0,\infty)\times\mathbb{R})$, we have that as $t \to \infty$,

$$u(t,\cdot) \longrightarrow \int_0^1 u_0(x)dx, \qquad \textit{a.e. and in } L^p_{loc}(\mathbb{R}),\ 1 \leq p < \infty. \tag{1.12}$$

The article is organized as follows. In Section 2, we introduce a class of regular approximate solutions and provide a series of uniform a priori estimates. Section 3 is devoted to the well-posedness of (1.1). Finally, in Section 4 we analyze the long-time asymptotic behavior of bounded periodic entropy solutions.

2. Approximate solutions and a priori estimates

We prove prove existence of a periodic solution to the Cauchy problem (1.1), (1.2) by analyzing the limiting behavior of the sequence of smooth functions $\{u_\varepsilon\}_{\varepsilon>0}$, where each function u_ε is the periodic solution of the viscous problem

$$\partial_t u_\varepsilon - \partial^3_{txx} u_\varepsilon + 4u_\varepsilon \partial_x u_\varepsilon = 3\partial_x u_\varepsilon \partial^2_{xx} u_\varepsilon + u_\varepsilon \partial^3_{xxx} u_\varepsilon + \varepsilon \partial^2_{xx} u_\varepsilon - \varepsilon \partial^4_{xxxx} u_\varepsilon, \tag{2.1}$$

endowed with the initial condition

$$u_\varepsilon(0,x) = u_{0,\varepsilon}(x), \quad x \in \mathbb{R},$$

or equivalently of the following parabolic-elliptic system:

$$\begin{cases} \partial_t u_\varepsilon + \partial_x\left(\frac{u_\varepsilon^2}{2}\right) + \partial_x P_\varepsilon = \varepsilon \partial^2_{xx} u_\varepsilon, & t > 0,\ x \in \mathbb{R}, \\ -\partial^2_{xx} P_\varepsilon + P_\varepsilon = \frac{3}{2} u_\varepsilon^2, & t > 0,\ x \in \mathbb{R}, \\ u_\varepsilon(0,x) = u_{0,\varepsilon}(x), & x \in \mathbb{R}, \end{cases} \tag{2.2}$$

where we assume that

$$
\begin{aligned}
&u_{0,\varepsilon} \in H^{\ell}_{loc}(\mathbb{R}), \text{ for some } \ell \geq 2, \quad u_{0,\varepsilon} \text{ is 1-periodic, for every } \varepsilon > 0,\\
&\|u_{0,\varepsilon}\|_{L^2(0,1)} \leq \|u_0\|_{L^2(0,1)}, \ \|u_{0,\varepsilon}\|_{L^\infty(\mathbb{R})} \leq \|u_0\|_{L^\infty(\mathbb{R})}, \text{ for every } \varepsilon > 0,\\
&u_{0,\varepsilon} \to u_0 \quad \text{in } L^2_{loc}(\mathbb{R}), \text{ as } \varepsilon \to 0.
\end{aligned} \tag{2.3}
$$

Using again the fact that $e^{-|\xi|}/2$ is the Green's function of the operator $1 - \partial_{xx}^2$ we have an explicit expression for P_ε in terms of u_ε:

$$
P_\varepsilon(t,x) = P^{u_\varepsilon}(t,x) = (1-\partial_{xx}^2)^{-1}\left(\frac{3}{2}u^2\right)(t,x) = \frac{3}{4}\int_{\mathbb{R}} e^{-|x-y|} u_\varepsilon^2(t,y)\,dy. \tag{2.4}
$$

The well posedness of the periodic solutions of the viscous problem (1.5) in $C([0,\infty); H^{\ell}_{loc}(R))$ for each fixed $\varepsilon > 0$ can be proved using an argument similar the one of [2, Theorem 2.3].

The first step in the analysis of (2.2) is an uniform L^2_{loc} bound on the approximate solution u_ε. The argument is based on a fundamental H^2_{loc} estimate on the quantity $v_\varepsilon = v_\varepsilon(t,x)$ defined by

$$
v_\varepsilon(t,x) = (4-\partial_{xx}^2)^{-1} u_\varepsilon(t,x) = \frac{1}{4}\int_{\mathbb{R}} e^{-\frac{|x-y|}{2}} u_\varepsilon(t,y)\,dy, \qquad t \geq 0,\ x \in \mathbb{R}. \tag{2.5}
$$

The use of the quantity v_ε is motivated by the fact that $\int_{\mathbb{R}} v\left(u - \partial_{xx}^2 u\right)\,dx$ is a conserved quantity of the Degasperis-Procesi equation, where $4v - \partial_{xx}^2 v = u$ and u solves (1.1) (see [8, 3]).

Lemma 2.1 (L^2 Estimate)**.** *The bounds*

$$
\begin{aligned}
\|u_\varepsilon(t,\cdot)\|_{L^2(0,1)} &\leq 2\|u_0\|_{L^2(0,1)},\\
\sqrt{\varepsilon}\,\|\partial_x u_\varepsilon\|_{L^2(\mathbb{R}_+\times(0,1))} &\leq \sqrt{2}\,\|u_0\|_{L^2(0,1)},
\end{aligned} \tag{2.6}
$$

hold for any $\varepsilon > 0$, $n \in \mathbb{N}$, and $t \geq 0$.

Proof. We multiply (2.2) by $v_\varepsilon - \partial_{xx}^2 v_\varepsilon$ and then integrate the result over $(0,1)$, obtaining

$$
\begin{aligned}
&\underbrace{\int_0^1 \partial_t u_\varepsilon (v_\varepsilon - \partial_{xx}^2 v_\varepsilon)dx}_{A_1}\\
&\qquad + \underbrace{\int_0^1 u_\varepsilon \partial_x u_\varepsilon (v_\varepsilon - \partial_{xx}^2 v_\varepsilon)dx + \int_0^1 \partial_x P_\varepsilon (v_\varepsilon - \partial_{xx}^2 v_\varepsilon)dx}_{A_2}\\
&\qquad\qquad = \varepsilon \underbrace{\int_0^1 \partial_{xx}^2 u_\varepsilon (v_\varepsilon - \partial_{xx}^2 v_\varepsilon)dx}_{A_3}.
\end{aligned} \tag{2.7}
$$

The periodicity of u_ε gives the periodicity of v_ε. Therefore, from (2.5),

$$
\begin{aligned}
A_1 &= \int_0^1 \partial_t(4v_\varepsilon - \partial_{xx}^2 v_\varepsilon)(v_\varepsilon - \partial_{xx}^2 v_\varepsilon)dx \\
&= \int_0^1 \left(4\partial_t v_\varepsilon v_\varepsilon - 4\partial_t v_\varepsilon \partial_{xx}^2 v_\varepsilon - \partial_{txx}^3 v_\varepsilon v_\varepsilon + \partial_{txx}^3 v_\varepsilon \partial_{xx}^2 v_\varepsilon\right) dx \\
&= \int_0^1 \left(4\partial_t v_\varepsilon v_\varepsilon + 5\partial_{tx}^2 v_\varepsilon \partial_x v_\varepsilon + \partial_{txx}^3 v_\varepsilon \partial_{xx}^2 v_\varepsilon\right) dx \\
&= \frac{1}{2}\frac{d}{dt}\int_0^1 \left(4v_\varepsilon^2 + 5(\partial_x v_\varepsilon)^2 + (\partial_{xx}^2 v_\varepsilon)^2\right) dx = \frac{1}{2}\frac{d}{dt}\left\|v_\varepsilon(t,\cdot)\right\|^2_{\widetilde{H}^2(0,1)},
\end{aligned}
\tag{2.8}
$$

where

$$
\|f\|_{\widetilde{H}^2(0,1)} = \sqrt{4\|f\|^2_{L^2(0,1)} + 5\|f'\|^2_{L^2(0,1)} + \|f''\|^2_{L^2(0,1)}}, \qquad \forall f \in H^2(0,1). \tag{2.9}
$$

Moreover, using again the periodicity of P_ε and v_ε,

$$
\begin{aligned}
A_2 &= \int_0^1 u_\varepsilon \partial_x u_\varepsilon (v_\varepsilon - \partial_{xx}^2 v_\varepsilon)dx + \int_0^1 \partial_x P_\varepsilon (v_\varepsilon - \partial_{xx}^2 v_\varepsilon)dx \\
&= \int_0^1 u_\varepsilon \partial_x u_\varepsilon (v_\varepsilon - \partial_{xx}^2 v_\varepsilon)dx + \int_0^1 \partial_x (P_\varepsilon - \partial_{xx}^2 P_\varepsilon) v_\varepsilon dx \\
&= \int_0^1 u_\varepsilon \partial_x u_\varepsilon (v_\varepsilon - \partial_{xx}^2 v_\varepsilon)dx + 3\int_0^1 u_\varepsilon \partial_x u_\varepsilon v_\varepsilon dx \\
&= \int_0^1 u_\varepsilon \partial_x u_\varepsilon (4v_\varepsilon - \partial_{xx}^2 v_\varepsilon)dx = \int_0^1 u_\varepsilon^2 \partial_x u_\varepsilon dx = 0, \\
A_3 &= \varepsilon \int_0^1 \partial_{xx}^2 (4v_\varepsilon - \partial_{xx}^2 v_\varepsilon)(v_\varepsilon - \partial_{xx}^2 v_\varepsilon)dx \\
&= \varepsilon \int_0^1 \left(4\partial_{xx}^2 v_\varepsilon v_\varepsilon - 4(\partial_{xx}^2 v_\varepsilon)^2 - \partial_{xxxx}^4 v_\varepsilon v_\varepsilon + \partial_{xxxx}^4 v_\varepsilon \partial_{xx}^2 v_\varepsilon\right) dx \\
&= \varepsilon \int_0^1 \left(-4(\partial_x v_\varepsilon)^2 - 4(\partial_{xx}^2 v_\varepsilon)^2 + \partial_{xxx}^3 v_\varepsilon \partial_x v_\varepsilon - (\partial_{xxx}^3 v_\varepsilon)^2\right) dx \\
&= -\varepsilon \int_0^1 \left(4(\partial_x v_\varepsilon)^2 + 5(\partial_{xx}^2 v_\varepsilon)^2 + (\partial_{xxx}^3 v_\varepsilon)^2\right) dx \\
&= -\varepsilon \left\|\partial_x v_\varepsilon(t,\cdot)\right\|^2_{\widetilde{H}^2(0,1)}.
\end{aligned}
$$

In view of (2.8), (2.7) becomes

$$
\frac{d}{dt}\left\|v_\varepsilon(t,\cdot)\right\|^2_{\widetilde{H}^2(0,1)} + 2\varepsilon \left\|\partial_x v_\varepsilon(t,\cdot)\right\|^2_{\widetilde{H}^2(0,1)} = 0, \tag{2.10}
$$

and hence, thanks to the Gronwall lemma,

$$
\left\|v_\varepsilon(t,\cdot)\right\|^2_{\widetilde{H}^2(0,1)} + 2\varepsilon \int_0^t \left\|\partial_x v_\varepsilon(s,\cdot)\right\|^2_{\widetilde{H}^2(0,1)} ds = \left\|v_\varepsilon(0,\cdot)\right\|^2_{\widetilde{H}^2(0,1)}. \tag{2.11}
$$

Squaring (2.5) and using (2.9) we have

$$
\left\|u_\varepsilon(t,\cdot)\right\|^2_{L^2(0,1)} = \int_0^1 \left(16v_\varepsilon^2 + 8(\partial_x v_\varepsilon)^2 + (\partial_{xx}^2 v_\varepsilon)^2\right) dx \le 4\left\|v_\varepsilon(t,\cdot)\right\|^2_{\widetilde{H}^2(0,1)},
$$

$$
\left\|\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(0,1)} = \int_0^1 \left(16(\partial_x v_\varepsilon)^2 + 8(\partial_{xx}^2 v_\varepsilon)^2 + (\partial_{xxx}^3 v_\varepsilon)^2\right) dx \le 4\left\|\partial_x v_\varepsilon(t,\cdot)\right\|^2_{\widetilde{H}^2(0,1)},
$$

$$
\begin{aligned}
\|u_0\|^2_{L^2(0,1)} &\geq \|u_{0,\varepsilon}\|^2_{L^2(0,1)} \\
&= \int_0^1 \left(16(v_\varepsilon(0,x))^2 + 8(\partial_x v_\varepsilon(0,x))^2 + (\partial^2_{xx} v_\varepsilon(0,x))^2\right) dx \\
&\geq \|v_\varepsilon(0,\cdot)\|^2_{\widetilde{H}^2(0,1)},
\end{aligned}
$$

therefore (2.11) says

$$
\|u_\varepsilon(t,\cdot)\|^2_{L^2(0,1)} + 2\varepsilon \int_0^t \|\partial_x u_\varepsilon(s,\cdot)\|^2_{L^2(0,1)} ds \leq 4\|u_0\|^2_{L^2(0,1)},
$$

that gives the claim. □

We continue with some a priori bounds on P_ε that come directly from the energy estimate stated in Lemma 2.1.

Lemma 2.2. *Assume* (1.3) *and* (2.3)*, and fix any* $\varepsilon > 0$*. Then*

$$
P_\varepsilon \geq 0, \tag{2.12}
$$

$$
\|P_\varepsilon(t,\cdot)\|_{L^1(0,1)}, \|\partial_x P_\varepsilon(t,\cdot)\|_{L^1(0,1)} \leq 6\|u_0\|^2_{L^2(0,1)}, \qquad t \geq 0, \tag{2.13}
$$

$$
\|P_\varepsilon\|_{L^\infty(\mathbb{R}_+\times\mathbb{R})} \leq 12\|u_0\|^2_{L^2(0,1)}, \tag{2.14}
$$

$$
\|\partial_x P_\varepsilon\|_{L^\infty(\mathbb{R}_+\times\mathbb{R})} \leq 18\|u_0\|^2_{L^2(0,1)}, \tag{2.15}
$$

$$
\left\|\partial^2_{xx} P_\varepsilon(t,\cdot)\right\|_{L^1(0,1)} \leq 12\|u_0\|^2_{L^2(0,1)}, \qquad t \geq 0. \tag{2.16}
$$

for every $n \in \mathbb{N}$*.*

Proof. Clearly, (2.12) follows from (2.4).

From the P_ε equation in (2.2), the periodicity of P_ε, and (2.12) we have

$$
-\underbrace{\int_0^1 \partial^2_{xx} P_\varepsilon dx}_{=0} + \underbrace{\int_0^1 P_\varepsilon dx}_{=\|P_\varepsilon(t,\cdot)\|_{L^1(0,1)}} = \frac{3}{2} \underbrace{\int_0^1 u^2_\varepsilon dx}_{\|u_\varepsilon(t,\cdot)\|^2_{L^2(0,1)}},
$$

therefore Lemma 2.1 gives

$$
\|P_\varepsilon(t,\cdot)\|_{L^1(0,1)} \leq 6\|u_0\|^2_{L^2(0,1)}, \qquad t \geq 0. \tag{2.17}
$$

Moreover, by (2.4) and (2.12),

$$
\begin{aligned}
\|\partial_x P_\varepsilon(t,\cdot)\|_{L^1(0,1)} &= \int_0^1 \left|\frac{3}{4}\partial_x \int_{\mathbb{R}} e^{-|x-y|} u^2_\varepsilon(t,y)\, dy\right| dx \\
&= \int_0^1 \left|\frac{3}{4} \int_{\mathbb{R}} e^{-|x-y|} \operatorname{sign}(y-x)\, u^2_\varepsilon(t,y)\, dy\right| dx \\
&\leq \int_0^1 \underbrace{\frac{3}{4} \int_{\mathbb{R}} e^{-|x-y|} u^2_\varepsilon(t,y) dy}_{=P_\varepsilon} dx = \|P_\varepsilon(t,\cdot)\|_{L^1(0,1)},
\end{aligned}
$$

and thus, thanks to (2.17), we arrive at (2.13).

From the P_ε equation in (2.2) we obtain

$$
\left\|\partial^2_{xx} P_\varepsilon(t,\cdot)\right\|_{L^1(0,1)} \leq \|P_\varepsilon(t,\cdot)\|_{L^1(0,1)} + \frac{3}{2}\|u_\varepsilon(t,\cdot)\|^2_{L^2(0,1)}.
$$

Therefore (2.16) follows from (2.6) and (2.13).

Finally, (2.14) and (2.15) follow from (2.13), (2.16), the embedding $L^\infty(0,1) \subset W^{1,1}(0,1)$, and the spatial periodicity of the map P_ε. □

Using the $W^{1,\infty}$ bound on $\{P_\varepsilon\}_{\varepsilon>0}$ stated in Lemma 2.2, we show that the family $\{u_\varepsilon\}_{\varepsilon>0}$ is bounded in L^∞.

Lemma 2.3. *For every $t \geq 0$,*

$$\|u_\varepsilon(t,\cdot)\|_{L^\infty(\mathbb{R})} \leq \|u_0\|_{L^\infty(\mathbb{R})} + 18\,\|u_0\|^2_{L^2(0,1)}\,t. \tag{2.18}$$

Proof. Due to (2.2) and Lemma 2.2,

$$\partial_t u_\varepsilon + u_\varepsilon \partial_x u_\varepsilon - \varepsilon \partial^2_{xx} u_\varepsilon \leq \|\partial_x P_\varepsilon\|_{L^\infty((0,\infty)\times\mathbb{R})} \leq 18\,\|u_0\|^2_{L^2(0,1)}\,.$$

Since the map

$$f(t) := \|u_0\|_{L^\infty(\mathbb{R})} + 18\,\|u_0\|^2_{L^2(0,1)}\,t, \qquad t \geq 0,$$

solves the equation

$$\frac{df}{dt} = 18\,\|u_0\|^2_{L^2(0,1)}$$

and

$$u_\varepsilon(0,x) \leq f(0), \qquad x \in \mathbb{R},$$

the comparison principle for parabolic equations implies that

$$u_\varepsilon(t,x) \leq f(t), \qquad (t,x) \in (0,\infty) \times \mathbb{R}.$$

In a similar way we can prove that

$$u_\varepsilon(t,x) \geq -f(t), \qquad (t,x) \in (0,\infty) \times \mathbb{R}.$$

This concludes the proof of the lemma. □

As a consequence of Lemmas 2.2 and 2.3, the second equation in (2.2) yields

Lemma 2.4. *For every $t \geq 0$,*

$$\left\|\partial^2_{xx} P_\varepsilon(t,\cdot)\right\|_{L^\infty(\mathbb{R})} \leq \|u_0\|_{L^\infty(\mathbb{R})} + 12\,\|u_0\|^2_{L^2(0,1)} + 18\,\|u_0\|^2_{L^2(0,1)}\,t. \tag{2.19}$$

3. Well-posedness of the Degasperis-Procesi equation

Relying on the a priori estimates derived in Section 2, we prove in this section the existence, uniqueness, and L^1_{loc} stability of periodic entropy solutions. These claims are immediate consequence of Lemmas 3.1, 3.5, and Corollary 3.1 below.

We begin by proving that there exists at least one periodic entropy solution.

From KHK: Should this upgraded to a theorem?

Lemma 3.1 (Existence)**.** *Suppose* (1.3) *holds. There exists a periodic entropy solution to* (1.1) *and* (1.2).

We will construct a weak solution by passing to the limit in a sequence $\{u_\varepsilon\}_{\varepsilon>0}$ of viscosity approximations, see (2.1) or (2.2). We make the standing assumption that the approximate initial data $\{u_{0,\varepsilon}\}_{\varepsilon>0}$ are chosen such that they respect (1.3) and (2.3). We use the the compensated compactness method [19, 18] to obtain strong convergence of a subsequence of viscosity approximations.

Theorem 3.1. *Let $\{v_\nu\}_{\nu>0}$ be a family of functions defined on $(0,\infty)\times\mathbb{R}$ such that*

$$\|v_\nu\|_{L^\infty((0,T)\times\mathbb{R})} \le M_T, \qquad T,\, \nu > 0,$$

and the family

$$\{\partial_t \eta(v_\nu) + \partial_x q(v_\nu)\}_{\nu>0}$$

is compact in $H^{-1}_{loc}((0,\infty)\times\mathbb{R})$, for every convex $\eta \in C^2(\mathbb{R})$, where $q'(u) = u\eta'(u)$. Then there exist a sequence $\{\nu_n\}_{n\in\mathbb{N}} \subset (0,\infty)$, $\nu_n \to 0$, and $v \in L^\infty((0,T)\times\mathbb{R})$, for any $T>0$, such that

$$v_{\nu_n} \to v \quad \text{a.e. and in } L^p_{loc}((0,\infty)\times\mathbb{R}),\ 1 \le p < \infty.$$

The following compact enbedding of Murat [15] will be used.

From KHK: I changed this into a lemma. Ok?

Lemma 3.2. *Let Ω be a bounded open subset of $\mathbb{R}^N$, $N \ge 2$. Suppose the sequence $\{\mathcal{L}_n\}_{n\in\mathbb{N}}$ of distributions is bounded in $W^{-1,\infty}(\Omega)$. Suppose also that*

$$\mathcal{L}_n = \mathcal{L}_n^1 + \mathcal{L}_n^2,$$

where $\left\{\mathcal{L}_n^1\right\}_{n\in\mathbb{N}}$ lies in a compact subset of $H^{-1}_{\rm loc}(\Omega)$ and $\left\{\mathcal{L}_n^2\right\}_{n\in\mathbb{N}}$ lies in a bounded subset of $\mathcal{M}^1_{loc}(\Omega)$. Then $\{\mathcal{L}_n\}_{n\in\mathbb{N}}$ lies in a compact subset of $H^{-1}_{\rm loc}(\Omega)$.

From KHK: Lemma should be changed to Theorem??

We now turn to the proof of Lemma 3.1, which will be accomplished through two lemmas.

Lemma 3.3. *There exists a subsequence $\{u_{\varepsilon_k}\}_{k\in\mathbb{N}}$ of $\{u_\varepsilon\}_{\varepsilon>0}$ and a limit function*

$$u \in L^\infty((0,T)\times\mathbb{R}),\ T>0,\ \text{1-periodic in the space variable,} \tag{3.1}$$

such that

$$u_{\varepsilon_k} \to u \ \text{a.e. and in } L^p_{loc}((0,\infty)\times\mathbb{R}),\ 1 \le p < \infty. \tag{3.2}$$

Proof. Let $\eta:\mathbb{R}\to\mathbb{R}$ be any convex C^2 entropy function, and let $q:\mathbb{R}\to\mathbb{R}$ be the corresponding entropy flux defined by $q'(u) = \eta'(u)\,u$. By multiplying the first equation in (2.2) with $\eta'(u_\varepsilon)$ and using the chain rule, we get

$$\partial_t \eta(u_\varepsilon) + \partial_x q(u_\varepsilon) = \underbrace{\varepsilon \partial_{xx}^2 \eta(u_\varepsilon)}_{=:\mathcal{L}^1_{\varepsilon,\alpha}} \underbrace{- \varepsilon \eta''(u_\varepsilon)\,(\partial_x u_\varepsilon)^2 + \eta'(u_\varepsilon)\partial_x P_\varepsilon}_{=:\mathcal{L}^2_{\varepsilon,\alpha}}, \tag{3.3}$$

where $\mathcal{L}^1_{\varepsilon,\alpha}$, $\mathcal{L}^2_{\varepsilon,\alpha}$ are distributions. We claim that

$$\begin{aligned} &\mathcal{L}^1_{\varepsilon,\alpha} \to 0 \text{ in } H^{-1}((0,T)\times(0,1)),\ T>0,\ n\in\mathbb{N},\\ &\mathcal{L}^2_{\varepsilon,\alpha} \text{ is uniformly bounded in } L^1((0,T)\times(0,1)),\ T>0,\ n\in\mathbb{N}. \end{aligned} \tag{3.4}$$

Indeed, (2.6), (2.18), and (2.13) imply

$$\|\varepsilon \partial_x \eta(u_\varepsilon)\|_{L^2((0,T)\times(0,1))} \le 2\sqrt{\varepsilon}\, \|\eta'\|_{L^\infty(\mathcal{I}_{T,n})} \|u_0\|_{L^2(0,1)} \to 0, \tag{3.5}$$

$$\left\|\varepsilon \eta''(u_\varepsilon)\,(\partial_x u_\varepsilon)^2\right\|_{L^1((0,T)\times(0,1))} \le 4\, \|\eta''\|_{L^\infty(\mathcal{I}_{T,n})} \|u_0\|^2_{L^2(0,1)}, \tag{3.6}$$

$$\|\eta'(u_\varepsilon)\partial_x P_\varepsilon\|_{L^1((0,T)\times(0,1))} \le 6T\, \|\eta'\|_{L^\infty(\mathcal{I}_{T,n})} \|u_0\|^2_{L^2(0,1)}, \tag{3.7}$$

where

$$\mathcal{I}_{T,n} = \left(-\left(\|u_0\|_{L^\infty(\mathbb{R})} + 18\|u_0\|^2_{L^2(0,1)} T\right), \|u_0\|_{L^\infty(\mathbb{R})} + 18\|u_0\|^2_{L^2(0,1)} T\right).$$

Hence, (3.4) follows. Therefore, Theorems 3.2 and 3.1 give the existence of a subsequence $\{u_{\varepsilon_k}\}_{k\in\mathbb{N}}$ and a limit function u satisfying (3.1) such that as $k\to\infty$ (3.2) holds.

Finally, the periodicity of u follows from the periodicity of the viscous approximants and the pointwise convergence stated in (3.2). □

A direct consequence of Lemmas 2.2 and 2.4 is the convergence of P_ε.

Lemma 3.4. *We have that*

$$P_{\varepsilon_k} \rightharpoonup P^u \text{ in } L^p((0,T);W^{2,p}_{loc}(\mathbb{R})),\ T>0,\ 1\le p<\infty, \tag{3.8}$$

where $\{\varepsilon_k\}_{k\in\mathbb{N}}$ and u are constructed in Lemma 3.3.

Proof of Lemma 3.1. Let $\varphi\in C^\infty(\mathbb{R}_+\times\mathbb{R})$ be a compactly supported test function. Due to (2.2),

$$\int_0^\infty\int_{\mathbb{R}}\left(u_\varepsilon\partial_t\phi+\frac{u_\varepsilon^2}{2}\partial_x\phi-\partial_xP_\varepsilon\phi+\varepsilon u_\varepsilon\partial^2_{xx}\phi\right)dx\,dt+\int_{\mathbb{R}}u_{0,\varepsilon}(x)\phi(0,x)\,dx=0.$$

Therefore, (2.3) and Lemma 3.3 imply that the function u constructed in Lemma 3.3 is a weak solution of (1.1), (1.2) in the sense of Definition 1.1.

Finally, we have to verify that u satisfies the entropy inequalities in Definition 1.1. Let $\eta\in C^2(\mathbb{R})$ be a convex entropy with flux q defined by $q'(u)=u\eta'(u)$. The convexity of η and (2.2) yield

$$\partial_t\eta(u_\varepsilon)+\partial_xq(u_\varepsilon)+\eta'(u_\varepsilon)\partial_xP_\varepsilon=\varepsilon\partial^2_{xx}\eta(u_\varepsilon)\underbrace{-\varepsilon\eta''(u_\varepsilon)(\partial_xu_\varepsilon)^2}_{\le0}\le\varepsilon\partial^2_{xx}\eta(u_\varepsilon).$$

Therefore, the entropy inequalities follow from Lemmas 3.3 and 3.4. □

Using Kruzkov's method [12] we can prove the L^1 stability (and thus uniqueness) of periodic entropy solutions.

From KHK: This should be turned into a Theorem??

Lemma 3.5 (L^1 stability)**.** *Let u and v be two periodic entropy solutions of* (1.1) *with initial data $u(0,\cdot)=u_0$ and $v(0,\cdot)=v_0$ satisfying* (1.3)*. Fix any $T>0$. Then*

$$\|u(t,\cdot)-v(t,\cdot)\|_{L^1(0,1)}\le e^{M_Tt}\|u_0-v_0\|_{L^1(0,1)},\qquad a.e.\ t\in(0,T), \tag{3.9}$$

where the positive constant M_T is defined in (1.11).

As an immediate consequence of this result is

Corollary 3.1 (Uniqueness)**.** *Suppose condition* (1.3) *holds. Then the Cauchy problem* (1.1), (1.2) *admits at most one periodic entropy solution.*

From KHK: No need for a separate result for this. Put the uniqueness result into the previous Theorem (Lemmma)?

Proof of Lemma 3.5. The doubling of variables argument of [12] gives

From KHK: Change Proof of Lemma to Proof of Theorem ...

$$\partial_t|u-v|+\partial_x\left(\operatorname{sign}(u-v)\frac{u^2-v^2}{2}\right)+\operatorname{sign}(u-v)\,\partial_x(P^u-P^v)\le0, \tag{3.10}$$

where

$$-\partial^2_{xx}P^u+P^u=\frac{3}{2}u^2,\qquad-\partial^2_{xx}P^v+P^v=\frac{3}{2}v^2. \tag{3.11}$$

Employing the periodicity of the map $u-v$ we get

$$
\begin{aligned}
\|u(t,\cdot)-v(t,\cdot)\|_{L^1(0,1)} &\le \|u_0-v_0\|_{L^1(0,1)} \\
&\quad -\int_0^t\int_0^1 \operatorname{sign}(u-v)\,\partial_x\left(P^u-P^v\right)dsdx \\
&\le \|u_0-v_0\|_{L^1(0,1)} + \|\partial_x\left(P^u-P^v\right)\|_{L^1((0,t)\times(0,1))},
\end{aligned}
\tag{3.12}
$$

for almost every t.

We have now to estimate the last term. We cannot argue as in Lemma 2.2 because P^u-P^v may change sign. From (3.11), we have

$$
-\partial_{xx}^2\left(P^u-P^v\right)+\left(P^u-P^v\right)=\frac{3}{2}\left(u^2-v^2\right). \tag{3.13}
$$

Since $(P^u-P^v)(t,\cdot)\in C^1(\mathbb{R})$ and is periodic, there exists $x(t)\in(0,1)$ such that

$$
\partial_x(P^u-P^v)(t,x(t))=0.
$$

Therefore, integrating (3.13) in $(x,x(t))$ we get

$$
\begin{aligned}
\partial_x(P^u-P^v)(t,x) &= \int_{x(t)}^x (P^u-P^v)(t,y)dy-\frac{3}{2}\int_{x(t)}^x\left(u^2-v^2\right)(t,y)dy \\
&= \int_{x(t)}^x (P^u-P^v)(t,y)dy-\frac{3}{2}\int_{x(t)}^x\left(u+v\right)(t,y)\left(u-v\right)(t,y)dy.
\end{aligned}
$$

Then

$$
\begin{aligned}
|\partial_x(P^u-P^v)(t,x)| &\le \|P^u(t,\cdot)-P^v(t,\cdot)\|_{L^1(0,1)} \\
&\quad +\frac{3}{2}\|u+v\|_{L^\infty((0,T)\times(0,1))}\|u(t,\cdot)-v(t,\cdot)\|_{L^1(0,1)},
\end{aligned}
$$

and thanks to Lemma 2.3

$$
\begin{aligned}
\|\partial_xP^u(t,\cdot)-\partial_xP^v(t,\cdot)\|_{L^1(0,1)} &\le \|P^u(t,\cdot)-P^v(t,\cdot)\|_{L^1(0,1)} \\
&\quad + M_T\|u(t,\cdot)-v(t,\cdot)\|_{L^1(0,1)}.
\end{aligned}
\tag{3.14}
$$

We conclude estimating $\|P^u(t,\cdot)-P^v(t,\cdot)\|_{L^1(0,1)}$. From (3.13) we have

$$
\begin{aligned}
|P^u-P^v| &= \frac{3}{2}\left(u^2-v^2\right)\operatorname{sign}\left(P^u-P^v\right)+\partial_{xx}^2\left(P^u-P^v\right)\operatorname{sign}\left(P^u-P^v\right) \\
&= \frac{3}{2}\left(u^2-v^2\right)\operatorname{sign}\left(P^u-P^v\right)+\partial_{xx}^2|P^u-P^v|\underbrace{-\left(\partial_x\left(P^u-P^v\right)\right)^2\delta_{\{P^u=P^v\}}}_{\le 0} \\
&\le \frac{3}{2}|u+v||u-v|+\partial_{xx}^2|P^u-P^v|,
\end{aligned}
$$

where $\delta_{\{P^u=P^v\}}$ is the Dirac delta concentrated on the set $\{P^u=P^v\}$. An integration on $(0,1)$ and Lemma 2.3 give

$$
\|\partial_xP^u(t,\cdot)-\partial_xP^v(t,\cdot)\|_{L^1(0,1)} \le M_T\|u(t,\cdot)-v(t,\cdot)\|_{L^1(0,1)}, \tag{3.15}
$$

where we used the periodicity of the map $\partial_x|P^u-P^v|$, namely

$$
\partial_x|P^u-P^v|(t,0)=\partial_x|P^u-P^v|(t,1), \qquad \text{a.e. } t>0.
$$

Using (3.14) and (3.15) in (3.12) we get

$$
\|u(t,\cdot)-v(t,\cdot)\|_{L^1(0,1)} \le \|u_0-v_0\|_{L^1(0,1)} + M_T\int_0^t\|u(s,\cdot)-v(s,\cdot)\|_{L^1(0,1)}\,ds,
$$

for almost every t. The claim follows from the Gronwall Lemma. □

4. Asymptotic behavior

Let u be the periodic entropic solution of (1.1) and (1.2). Assume that

$$u \in L^\infty((0,\infty)\times\mathbb{R}). \tag{4.1}$$

Following [1] we introduce the functions

From KHK: Explain why these functions are relevant for studying long-time behavior of solutions, increases the readability!

$$u_T(t,x) := u(Tt,Tx), \quad P_T(t,x) := P(Tt,Tx), \qquad T,\, t \geq 0,\ x \in \mathbb{R}.$$

Clearly, u_T and P_T are $1/T$ periodic in the space variable.

Since $(u,\, P)$ solves (1.5), $(u_T,\, P_T)$ satisfies

$$\begin{cases} \partial_t u_T + u_T\partial_x u_T + \partial_x P_T = 0, & (t,x)\in(0,\infty)\times\mathbb{R},\\ -\frac{1}{T^2}\partial_{xx}^2 P_T + P_T = \frac{3}{2}u_T^2, & (t,x)\in(0,\infty)\times\mathbb{R},\\ u_T(0,x) = u_0(Tx), & x\in\mathbb{R}. \end{cases} \tag{4.2}$$

Moreover, u_T formally solves

$$\partial_t u_T + 4u_T\partial_x u_T = \frac{\partial_{txx}^3 u_T + 3\partial_x u_T\partial_{xx}^2 u_T + u_T\partial_{xxx}^3 u_T}{T^2}.$$

Due to (2.6) and (4.1) the estimates

$$\|u_T(t,\cdot)\|_{L^2(0,1)} \leq c\sqrt{\frac{[T]+1}{T}}\,\|u_0\|_{L^2(0,1)}\,, \tag{4.3}$$

$$\|u_T\|_{L^\infty((0,\infty)\times\mathbb{R})} \leq \|u\|_{L^\infty((0,\infty)\times\mathbb{R})}\,, \tag{4.4}$$

hold for any $T>0$, $t\geq 0$, and some constant $c>0$ independent on t, and T, where $[T]$ is the integer part of T. Indeed

$$\begin{aligned}\int_0^1 u_T^2(t,x)dx &= \int_0^1 u^2(Tt,Tx)dx = \frac{1}{T}\int_0^T u^2(Tt,x)dx\\ &\leq \frac{1}{T}\int_0^{[T]+1} u^2(Tt,x)dx\\ &\leq \frac{1}{T}\int_0^{[T]+1} u_0^2(x)dx = 2\frac{[T]+1}{T}\int_0^1 u_0^2(x)dx.\end{aligned}$$

Arguing in the same way and using (2.13), (2.15), (4.4) we have

$$\begin{aligned}\|P_T(t,\cdot)\|_{L^1(0,1)} &\leq c\frac{[T]+1}{T}\,\|u_0\|_{L^2(0,1)}^2\,,\\ \|\partial_x P_T(t,\cdot)\|_{L^2(0,1)} &\leq c\frac{[T]+1}{T}\,\|u_0\|_{L^2(0,1)}^4\,,\\ \|P_T\|_{L^\infty((0,\infty)\times\mathbb{R})} &\leq \|u\|_{L^\infty((0,\infty)\times\mathbb{R})}\,,\end{aligned} \tag{4.5}$$

for very $T>0$, $n\in\mathbb{N}$, $t\geq 0$, and some constant $c>0$ independent on n, t, and T.

Let $\eta\in C^2(\mathbb{R})$ be a convex entropy with flux q defined by $q'(u)=u\eta'(u)$. It is not restrictive to assume $\eta''\in L^\infty(\mathbb{R})$. We claim that

$$\partial_t\eta(u_T) + \partial_x q(u_T) + \eta'(u_T)\partial_x P_T = -\mu_T, \tag{4.6}$$

for some nonnegative Radon measure μ_T on $(0,\infty)\times\mathbb{R}$ such that

$$\mu_T((0,\infty)\times(0,1)) \leq c\,\|\eta''\|_{L^\infty(\mathbb{R})}\,\frac{[T]+1}{T}\,\|u_0\|_{L^2(0,1)}^2\,, \tag{4.7}$$

for every $T > 0$ and some constant $c > 0$ independent on T.

The compactness argument of the previous section guarantees that

$$u_\varepsilon \to u \text{ a.e. and in } L^p_{loc}((0,\infty)\times\mathbb{R}),\ 1 \le p < \infty \text{ as } \varepsilon \to 0, \tag{4.8}$$

where u_ε and u solve (2.2) and (1.1), respectively. Defining

$$u_{\varepsilon,T}(t,x) := u_\varepsilon(Tt, Tx),$$

we have

$$u_{\varepsilon,T} \to u_T \text{ a.e. and in } L^p_{loc}((0,\infty)\times\mathbb{R}),\ 1 \le p < \infty \text{ as } \varepsilon \to 0, \tag{4.9}$$

and $u_{\varepsilon,T}$ solves

$$\begin{cases} \partial_t u_{\varepsilon,T} + \partial_x \left(\frac{u^2_{\varepsilon,T}}{2}\right) + \partial_x P_{\varepsilon,T} = \frac{\varepsilon}{T}\partial^2_{xx} u_{\varepsilon,T}, & t > 0,\ x \in \mathbb{R},\\ -\frac{1}{T^2}\partial^2_{xx} P_{\varepsilon,T} + P_{\varepsilon,T} = \frac{3}{2}u^2_{\varepsilon,T}, & t > 0,\ x \in \mathbb{R},\\ u_{\varepsilon,T}(0,x) = u_{0,\varepsilon}(Tx), & x \in \mathbb{R}. \end{cases} \tag{4.10}$$

We have

$$\begin{aligned} &\partial_t \eta(u_{\varepsilon,T}) + \partial_x q(u_{\varepsilon,T}) + \eta'(u_{\varepsilon,T})\partial_x P_{\varepsilon,T} \\ &\qquad\qquad = \frac{\varepsilon}{T}\partial^2_{xx}\eta(u_{\varepsilon,T}) - \frac{\varepsilon}{T}\eta''(u_{\varepsilon,T})(\partial_x u_{\varepsilon,T})^2. \end{aligned} \tag{4.11}$$

Since (see (2.6))

$$\begin{aligned} \frac{\varepsilon}{T}\int_0^\infty\int_0^1 \eta''(u_{\varepsilon,T})(\partial_x u_{\varepsilon,T})^2 dtdx &= \frac{\varepsilon}{T^3}\int_0^\infty\int_0^T \eta''(u_\varepsilon)(\partial_x u_\varepsilon)^2 dtdx \\ &\le \frac{\varepsilon}{T^3}\frac{1}{T}\int_0^{[T]+1} \eta''(u_\varepsilon)(\partial_x u_\varepsilon)^2 dtdx \\ &\le c\left\|\eta''\right\|_{L^\infty(\mathbb{R})}\frac{[T]+1}{T^3}\left\|u_0\right\|^2_{L^2(0,1)}, \end{aligned}$$

as $\varepsilon \to 0$ we get (4.6) and (4.7).

We now use again the argument of the proof of Lemma 3.3 for the family $\{u_T\}_{T>0}$. Thanks to (4.4), (4.5), and (4.7), we have that $\{\partial_t\eta(u_T) + \partial_x q(u_T)\}_{T>0}$ is bounded in $\mathcal{M}^1_{loc}((0,\infty)\times\mathbb{R})$. Therefore, Theorems 3.2 and 3.1 give the existence of a subsequence $\{u_{T_k}\}_{k\in\mathbb{N}}$, $T_k \to \infty$, and a limit function $u^* \in L^\infty((0,\infty)\times\mathbb{R})$ such that as $k \to \infty$

$$u_{T_k} \to u^* \text{ a.e. and in } L^p_{loc}((0,\infty)\times\mathbb{R}),\ 1 \le p < \infty. \tag{4.12}$$

Moreover,

$$P_{T_k} \rightharpoonup P^* \text{ in } L^p((0,T);W^{2,p}_{loc}(\mathbb{R})),\ T > 0,\ 1 \le p < \infty. \tag{4.13}$$

Using the P^T equation in (4.2) and (2.16) we gain

$$P_{T_k} \to \frac{3}{2}(u^*)^2 \text{ in } L^1_{loc}((0,\infty)\times\mathbb{R}) \text{ and a.e. in } (0,\infty)\times\mathbb{R}. \tag{4.14}$$

In particular

$$P^* = \frac{3}{2}(u^*)^2. \tag{4.15}$$

Therefore, u^* is a weak solution of

$$\begin{cases} \partial_t u^* + \partial_x\left(2(u^*)^2\right) = 0, & (t,x) \in (0,\infty)\times\mathbb{R},\\ u^*(0,x) = \int_0^1 u_0(x)dx, & x \in \mathbb{R}, \end{cases} \tag{4.16}$$

where we used the convergence of the periodic functions to the mean value as the oscillations diverge. We claim that u^* is the unique entropy solution of (4.16), namely

$$u^* = \int_0^1 u_0(x)dx. \tag{4.17}$$

From KHK: Replace "where we used the convergence of the periodic functions to the mean value as the oscillations diverge" by a reference to the precise result.

Let $\varphi \in C^\infty((0,\infty)\times\mathbb{R})$ be a nonnegative test function with compact support. Thanks to [10, Corollary 2.5] and [17], it suffices to consider the convex entropy

$$\eta(u) = \frac{1}{3}|u|^3, \tag{4.18}$$

and its corresponding flux (see (4.16))

$$q(u) = u^4 \mathrm{sign}\,(u)\,. \tag{4.19}$$

We have to prove that

$$\int_0^\infty \int_{\mathbb{R}} \left(\frac{|u^*|^3}{3}\partial_t\varphi + (u^*)^4 \mathrm{sign}\,(u^*)\,\partial_x\varphi \right) dtdx \geq 0. \tag{4.20}$$

Let $\delta > 0$ and define

$$\eta_\delta(u) = \int^u \frac{\xi^3}{\sqrt{\xi^2+\delta}}d\xi, \qquad q_\delta(u) = \int^u \frac{4\xi^4}{\sqrt{\xi^2+\delta}}d\xi. \tag{4.21}$$

Using the entropy η_δ and q_δ in (4.2) we get

$$\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}} \left(\eta_\delta(u_{T_k})\partial_t\varphi + \frac{1}{4}q_\delta(u_{T_k})\partial_x\varphi \right) dtdx \\
&\quad \geq \int_0^\infty \int_{\mathbb{R}} \partial_x P_{T_k}\eta_\delta'(u_{T_k})\varphi dtdx = \int_0^\infty \int_{\mathbb{R}} \partial_x P_{T_k}\frac{(u_{T_k})^3}{\sqrt{u_{T_k}^2+\delta}}\varphi dtdx \\
&\quad = \frac{2}{3}\int_0^\infty \int_{\mathbb{R}} \partial_x P_{T_k}\left(P_{T_k} - \frac{1}{T^2}\partial_{xx}^2 P_{T_k}\right)\frac{u_{T_k}}{\sqrt{u_{T_k}^2+\delta}}\varphi dtdx \\
&\quad = \frac{2}{3}\int_0^\infty \int_{\mathbb{R}} \partial_x P_{T_k} P_{T_k}\frac{u_{T_k}}{\sqrt{u_{T_k}^2+\delta}}\varphi dtdx \\
&\quad - \frac{2}{3T_k^2}\int_0^\infty \int_{\mathbb{R}} \partial_x P_{T_k}\partial_{xx}^2 P_{T_k}\frac{u_{T_k}}{\sqrt{u_{T_k}^2+\delta}}\varphi dtdx \\
&\quad = -\frac{1}{3}\int_0^\infty \int_{\mathbb{R}} (P_{T_k})^2\frac{u_{T_k}}{\sqrt{u_{T_k}^2+\delta}}\partial_x\varphi dtdx - \frac{1}{3T_k^2}\int_0^\infty \int_{\mathbb{R}} (P_{T_k})^2\frac{\delta}{(u_{T_k}^2+\delta)^{3/2}}\varphi dtdx \\
&\quad + \frac{1}{3T_k^2}\int_0^\infty \int_{\mathbb{R}} (\partial_x P_{T_k})^2\frac{u_{T_k}}{\sqrt{u_{T_k}^2+\delta}}\partial_x\varphi dtdx \\
&\quad + \frac{1}{3T_k^2}\int_0^\infty \int_{\mathbb{R}} (\partial_x P_{T_k})^2\frac{\delta}{(u_{T_k}^2+\delta)^{3/2}}\varphi dtdx \\
&\quad \geq -\frac{1}{3}\int_0^\infty \int_{\mathbb{R}} (P_{T_k})^2\frac{u_{T_k}}{\sqrt{u_{T_k}^2+\delta}}\partial_x\varphi dtdx - \frac{1}{3T_k^2}\int_0^\infty \int_{\mathbb{R}} (P_{T_k})^2\frac{\delta}{(u_{T_k}^2+\delta)^{3/2}}\varphi dtdx
\end{aligned}$$

$$+\frac{1}{3T_k^2}\int_0^\infty\int_{\mathbb{R}}(\partial_x P_{T_k})^2\frac{u_{T_k}}{\sqrt{u_{T_k}^2+\delta}}\partial_x\varphi dtdx.$$

The bounds in (4.5) and the compactness of the support of φ give

$$\begin{aligned}\Bigg| -\frac{1}{3T_k^2}\int_0^\infty\int_{\mathbb{R}}(P_{T_k})^2\frac{\delta}{(u_{T_k}^2+\delta)^{3/2}}\varphi dtdx \\ +\frac{1}{3T_k^2}\int_0^\infty\int_{\mathbb{R}}(\partial_x P_{T_k})^2\frac{u_{T_k}}{\sqrt{u_{T_k}^2+\delta}}\partial_x\varphi dtdx\Bigg| \\ \leq \frac{1}{3T_k^2}\int_0^\infty\int_{\mathbb{R}}\big((P_{T_k})^2\varphi+(\partial_x P_{T_k})^2|\partial_x\varphi|dtdx\big|\leq c\frac{1}{3T_k^2}\frac{[T_k]+1}{T_k},\end{aligned}$$

for some constant independent of k. Therefore as $k\to\infty$, thanks to (4.12), (4.13), and (4.15) we get

$$\int_0^\infty\int_{\mathbb{R}}\left(\eta_\delta(u^*)\partial_t\varphi+\frac{1}{4}q_\delta(u^*)\partial_x\varphi\right)dtdx\geq-\frac{3}{4}\int_0^\infty\int_{\mathbb{R}}\frac{(u^*)^5}{\sqrt{(u^*)^2+\delta}}\partial_x\varphi dtdx.$$

As $\delta\to 0$ we get (4.20).

Thanks to [10, Corollary 2.5] and [17], u^* is the unique entropy solution of (4.16). Therefore, (4.17) and (1.12) hold.

Proof of Theorem 1.1. The existence, stability, and uniqueness of entropy solutions are stated in Lemmas 3.3, 3.5 and Corollary 3.1. The asymptotic behavior is proved in this section. □

References

[1] G.-Q. Chen and H. Frid. Decay of entropy solutions of nonlinear conservation laws. *Arch. Ration. Mech. Anal.*, 146(2):95–127, 1999. 4

[2] G. M. Coclite, H. Holden, and K. H. Karlsen. Well posedness for a parabolic-elliptic system. *Discrete Contin. Dyn. Syst.*, 13(3):659–682, 2005. 2

[3] G. M. Coclite and K. H. Karlsen. On the well-posedness of the Degasperis-Procesi equation. *J. Funct. Anal.*, 233(1):60–91, 2006. 1, 1, 2

[4] G. M. Coclite and K. H. Karlsen. On the uniqueness of discontinuous solutions to the Degasperis-Procesi equation. *J. Differential Equations*, 233(1):142-160, 2007. 1

[5] G. M. Coclite and K. H. Karlsen. Bounded solutions for the Degasperis-Procesi equation. *Boll. Unione Mat. Ital. (9)*, 1(2):439–453, 2008. 1

[6] G. M. Coclite, K. H. Karlsen, and Y.-S. Kwon. Initial-boundary value problems for conservation laws with source terms and the Degasperis-Procesi equation. *J. Funct. Anal.*, 257(12):3823–3857, 2009. 1

[7] A. Degasperis and M. Procesi. Asymptotic integrability. In *Symmetry and perturbation theory (Rome, 1998)*, pages 23–37. World Sci. Publishing, River Edge, NJ, 1999. 1

[8] A. Degasperis, D. D. Holm, and A. N. W. Hone. Integrable and non-integrable equations with peakons. In *Nonlinear physics: theory and experiment, II (Gallipoli, 2002)*, pages 37–43. World Sci. Publishing, River Edge, NJ, 2003. 1, 2

[9] A. Degasperis, D. D. Holm, and A. N. I. Khon. A new integrable equation with peakon solutions. *Teoret. Mat. Fiz.*, 133(2):170–183, 2002. 1

[10] C. De Lellis, F. Otto, and M. Westdickenberg. Minimal entropy conditions for Burgers equation. *Quart. Appl. Math.*, 62(4):687–700, 2004. 4, 4

[11] J. Escher, Y. Liu, and Z. Yin. Shock waves and blow-up phenomena for the periodic Degasperis-Procesi equation. *Indiana Univ. Math. J.*, 56(1):87–117, 2007. 1

[12] S. N. Kružkov. First order quasi-linear equations in several independent variables. *Math. USSR Sbornik*, 10(2):217–243, 1970. 3, 3

[13] H. Lundmark. Formation and dynamics of shock waves in the Degasperis-Procesi equation. *J. Nonlinear Sci.*, 17(3):169–198, 2007. 1, 1
[14] H. Lundmark and J. Szmigielski. Multi-peakon solutions of the Degasperis-Procesi equation. *Inverse Problems*, 19(6):1241–1245, 2003. 1, 1
[15] F. Murat. L'injection du cône positif de H^{-1} dans $W^{-1,q}$ est compacte pour tout $q < 2$. *J. Math. Pures Appl. (9)*, 60(3):309–322, 1981. 3
[16] O. G. Mustafa. A note on the Degasperis-Procesi equation. *J. Nonlinear Math. Phys.*, 12(1):10–14, 2005. 1
[17] E. Y. Panov. Uniqueness of the Cauchy problem for a first order quasilinear equation with one admissible strictly convex entropy. *Mat. Zametki*, 55:116–129, 1994. 4, 4
[18] M. E. Schonbek. Convergence of solutions to nonlinear dispersive equations. *Comm. Partial Differential Equations,* 7(8):959–1000, 1982. 3
[19] L. Tartar. Compensated compactness and applications to partial differential equations. In *Nonlinear analysis and mechanics: Heriot-Watt Symposium, Vol. IV*, pages 136–212. Pitman, Boston, Mass., 1979. 3
[20] V. O. Vakhnenko and E. J. Parkes. Periodic and solitary-wave solutions of the Degasperis-Procesi equation. *Chaos Solitons Fractals*, 20(5):1059–1073, 2004. (document), 1
[21] Z. Yin. Global existence for a new periodic integrable equation. *J. Math. Anal. Appl.*, 283(1):129–139, 2003. 1
[22] Z. Yin. Global weak solutions for a new periodic integrable equation with peakon solutions. *J. Funct. Anal.*, 212(1):182–194, 2004. 1
[23] Y. Zhou. Blow-up phenomenon for the integrable Degasperis-Procesi equation. *Phys. Lett. A*, 328(2–3):157–162, 2004. 1

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