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Proposed Running Head: Non-linear Maximum Rank Distance Codes

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Abstract

By exploring some geometry of Segre varieties and Veronese varieties, new families of non linear maximum rank distance codes and optimal constant rank codes are provided.

KEYWORDS: Segre variety, Veronese variety, Maximum rank distance code, Constant rank distance code, Subspace codes, Singer cyclic group.
AMS MSC: 94B60; 51E20

1 Introduction

Subspace codes and maximum rank-distance codes (MRD) can be used to correct errors and erasures in networks with linear network coding. Network coding is a novel and efficient approach to transmitting data across a communication network. Both types of codes have been extensively studied in the last years. Subspace codes were introduced by Koetter and Kschischang in the inspiring article [22] to correct errors and erasures in networks with a randomized protocol where the topology is unknown (the non-coherent case). The codewords of a subspace code are vector subspaces of a fixed ambient vector space; thus the codes are collections of such subspaces and the natural measure of distance is defined by $d(A,B) = \dim(A)+\dim(B)-2\dim(A\cap B)$. An important subclass of subspace codes is represented by the constant-dimension codes (CDCs). CDCs have several interesting properties and in particular the decoding procedure is simplified, as a fixed number of linearly independent packets are required to perform the decoding.

Rank distance codes were introduced by Delsarte [8] and rediscovered in [14] and independently in [27] and are suitable for error correction in the case where the network topology and the underlying network code are known (the coherent case). A rank-distance code can be viewed as a set of matrices over a finite field where the distance between two codewords, referred to as the rank distance, is the rank of their difference. Gabidulin codes are a well-known class of algebraic rank-metric codes that meet the Singleton bound on the minimum rank-distance of a code.

A matrix can be lifted into a subspace of fixed dimension and hence a rank-distance code can be lifted into a CDC: a *lifted maximum rank distance (LMRD) code* is a subspace code obtained from an MRD code $\mathcal{A} \subseteq \mathcal{M}_{m\times n}(q)$ by the so–called *lifting construction* of [30], which associates to every matrix $\mathbf{A} \in \mathcal{M}_{m\times n}(q)$ the subspace $U = \langle(\mathbf{I}_m|\mathbf{A})\rangle$ of $\mathrm{GF}(q)^{m+n}$.

Since the injection and the subspace distances between two lifted matrices are related to their rank distance, the minimum distances of the lifting of a rank-distance code are related to that of the original rank-distance code. Thus error control in random linear network coding using CDCs can be turned into a rank metric problem. A recent survey on problems related to subspace coding can be found in [12], to which we also refer for more background on this topic. All these connections and ideas led to many new interesting problems in coding theory and in Galois geometries.

1.1 Preliminaries

The set $\mathcal{M}_{m\times n}(q)$ of $m\times n$ matrices over the finite field $\mathrm{GF}(q)$ forms a metric space with respect to the *rank distance* defined by $d_r(A,B) = \mathrm{rk}(A-B)$. The maximum size of a code of minimum distance d, $1 \le d \le \min\{m,n\}$, in $(\mathcal{M}_{m\times n}(q), d_r)$ is $q^{n(m-d+1)}$ for $m \le n$ and $q^{m(n-d+1)}$ for $m \ge n$. A code $\mathcal{A} \subset \mathcal{M}_{m\times n}(q)$ attaining this bound is said to be a q–ary (m,n,k) *maximum rank distance code* (*MRD*), where $k = m-d+1$ for $m \le n$ and $k = n-d+1$ for $m \ge n$. A rank distance code $\mathcal{A}$ is called $\mathrm{GF}(q)$–linear if $\mathcal{A}$ is a subspace of $\mathcal{M}_{m\times n}(q)$ considered as a vector space. We can always assume that $m \le n$. The $\mathrm{GF}(q)$–linear *Gabidulin codes* can be seen as the analogs of Reed-Solomon codes for rank metric and are defined as follows. Consider the vector space $V = \mathrm{End}(\mathrm{GF}(q^n), \mathrm{GF}(q))$ of all $\mathrm{GF}(q)$–linear operators of the field $\mathrm{GF}(q^n)$. Then V is also a vector space over the field $\mathrm{GF}(q^n)$ of dimension n and the vectors of V are uniquely represented as linearized polynomials of the form $x \mapsto a_0x + a_1x^q + a_2x^{q^2} + \cdots + a_{n-1}x^{q^{n-1}}$ with coefficients $a_i \in \mathrm{GF}(q^n)$ and q–degree less than n. The (n,n,k) Gabidulin code $\mathcal{G}$ consists of all such polynomials of q–degree less than k.

In terms of matrices, a codeword c in $\mathcal{G}$, can be represented by a vector $c = (c_1, \dots c_n)$, where $c_i \in \mathrm{GF}(q^n)$. Let $g_i \in \mathrm{GF}(q^n)$, $1 \le i \le n$, be linearly independent over $\mathrm{GF}(q)$. The generator matrix of a Gabidulin code is given by

$$\begin{pmatrix} g_1 & g_2 & \cdots & g_n \\ g_1^q & g_2^q & \cdots & g_n^q \\ g_1^{q^2} & g_2^{q^2} & \cdots & g_n^{q^2} \\ \vdots & \vdots & \vdots & \vdots \\ g_1^{q^k} & g_2^{q^k} & \cdots & g_n^{q^k} \end{pmatrix},$$

This matrix representation gives rise to an isomorphism between (V, d_r) and $(\mathcal{M}_{n\times n}(q), d_r)$ of metric spaces and the choice of the basis does not matter. Rectangular (m,n,k) Gabidulin codes (where $m < n$) are obtained by restricting the linear maps in $\mathcal{G}$ to an m–dimensional GF(q)–subspace W of GF(q^n).

A second family of MRD codes, referred to as *generalized Gabidulin codes* was introduced in [17]. These codes have a similar generator matrix to that of Gabidulin codes:

$$\begin{pmatrix} g_1 & g_2 & \cdots & g_n \\ g_1^{q^a} & g_2^{q^a} & \cdots & g_n^{q^a} \\ g_1^{q^{2a}} & g_2^{q^{2a}} & \cdots & g_n^{q^{2a}} \\ \vdots & \vdots & \vdots & \vdots \\ g_1^{q^{ka}} & g_2^{q^{ka}} & \cdots & g_n^{q^{ka}} \end{pmatrix},$$

where a is an integer such that $(n, a) = 1$.

A third family of MRD codes consists of cartesian products of a MRD code with length $n = m$, [16]. See also [32].

On the other hand, known non–linear rank distance codes are *coset codes* and one can construct examples of such codes for small lengths. As far as we know no infinite non–linear families of maximum rank distance codes are known. In this paper we are mainly interested into non–linear maximum rank distance codes of $(\mathcal{M}_{n\times n}(q), d_r)$, $n = 2, 3$ and $d = 2$. Our approach is based on the geometry of the Segre variety of PG$(n^2 - 1, q)$, $n = 2, 3$.

The *Segre map* may be defined as the map

$$\sigma : \mathrm{PG}(n-1, q) \times \mathrm{PG}(n-1, q) \to \mathrm{PG}(n^2 - 1, q),$$

taking a pair of points $x = (x_1, \dots x_n)$, $y = (y_1, \dots y_n)$ of PG$(n-1, q)$ to their product $(x_1y_1, x_1y_2, \dots x_ny_n)$ (the x_iy_j are taken in lexicographical order). The image of the Segre map is an algebraic variety called the *Segre variety* and denoted by $\mathcal{S}_{n-1,n-1}$.

When $n = 2$ the Segre variety $\mathcal{S}_{1,1}$ of PG$(3, q)$ is the non–degenerate hyperbolic quadric $\mathcal{Q}^+(3, q)$. This quadric is given as the zero locus of the quadratic polynomial given by the determinant of the matrix

$$\begin{pmatrix} x_1y_1 & x_1y_2 \\ x_2y_1 & x_2y_2 \end{pmatrix}.$$

In the case $n = 3$, the Segre variety $\mathcal{S}_{2,2}$ of PG$(8, q)$ is defined to be the zero locus of all quadratic polynomials given by the determinants of the 2×2 matrices of the matrix

$$\begin{pmatrix} x_1y_1 & x_1y_2 & x_1y_3 \\ x_2y_1 & x_2y_2 & x_2y_3 \\ x_3y_1 & x_3y_2 & x_3y_3 \end{pmatrix}.$$

In other terms, in the projective space $\mathrm{PG}(\mathcal{M}_{n\times n}(q))$, if $n=2$, the Segre variety $\mathcal{S}_{1,1}$ of $\mathrm{PG}(3,q)$ is represented by all 2×2 matrices of rank 1 and if $n=3$, the Segre variety $\mathcal{S}_{2,2}$ of $\mathrm{PG}(8,q)$ is represented by all 3×3 matrices of rank 1.

The set of matrices of $\mathcal{M}_{3\times 3}(q)$ of rank at most two gives rise to the so called secant variety $\Omega(\mathcal{S}_{2,2})$ of $\mathcal{S}_{2,2}$.

We introduce the following definition.

Definition 1.1. *An* **exterior set** *with respect to a Segre variety $\mathcal{S}_{n-1,n-1}$ of $\mathrm{PG}(n^2-1,q)$ is a set $\mathcal{E}$ of points of $\mathrm{PG}(n^2-1,q)\setminus\mathcal{S}_{n-1,n-1}$ of size $q^{n^2-n}-1/q-1$ such that the line joining any two points of $\mathcal{E}$ is disjoint from $\mathcal{S}_{n-1,n-1}$.*

This definition justifies the following proposition whose proof is immediate.

Proposition 1.2. *An exterior set with respect to $\mathcal{S}_{n-1,n-1}$ gives rise to an $(n,n,n-1)$ maximum rank distance code closed under $\mathrm{GF}(q)$–multiplication, and viceversa.*

Corollary 1.3. *An $(n,n,n-1)$ $\mathrm{GF}(q)$-linear Gabidulin code $\mathcal{G}$ is a certain subspace X of $\mathrm{PG}(n^2-1,q)$ of dimension n^2-n-1 which is an exterior set with respect to $\mathcal{S}_{n-1,n-1}$.*

Note that from [7] the maximum dimension of a subspace of $\mathrm{PG}(n^2-1,q)$ disjoint from $\mathcal{S}_{n-1,n-1}$ is exactly n^2-n-1.

In general, an exterior set $\mathcal{E}$ of $\mathrm{PG}(n^2-1,q)$ with respect to a Segre variety $\mathcal{S}_{n-1,n-1}$ of size $(q^{n^2-n}-1)/(q-1)$ gives rise to a MRD code: this is done by identifying a point of $\mathcal{E}$ and and its nonzero scalar multiples together with the zero matrix with members of $\mathcal{M}_{3\times 3}(q)$, and this is the key tool of our approach.

The paper is organized as follows.

In Section 2 we describe the case $n=2$ where there exists a complete classification of linear and non–linear MRD codes that are closed under $\mathrm{GF}(q)$–multiplication. This classification in turn relies on the classification of flocks of the hyperbolic quadric $\mathcal{Q}^+(3,q)$ of the projective space $\mathrm{PG}(3,q)$ which as already observed, represents the smallest example of Segre variety. A *flock* of the hyperbolic quadric $\mathcal{Q}^+(3,q)$ of the finite projective space $\mathrm{PG}(3,q)$ is a partition of the points of $\mathcal{Q}^+(3,q)$ into $q+1$ irreducible conics. Under the polarity of $\mathrm{PG}(3,q)$ induced by $\mathcal{Q}^+(3,q)$, a flock of $\mathcal{Q}^+(3,q)$ corresponds to an exterior set with respect to $\mathcal{Q}^+(3,q)$ producing a MRD code, and in particular a so called constant–rank code.

A *constant–rank code* (CRC) of constant rank r in $\mathcal{M}_{m\times n}(q)$ is a nonempty subset of $\mathcal{M}_{m\times n}(q)$ such that all elements have rank r. Much research has been done to investigate the maximum possible dimension of a constant rank r subspace of matrix vector spaces with particular attention to finite fields. The results and techniques differ greatly according to properties of the underlying field. See [29], [15] for more details and results.

In Section 3 we concentrate on the case $n = 3$ and construct several families of non–linear MRD codes. Our starting point was a $(3, 3, 2)$ GF(q)–linear MRD code $\mathcal{G}$ represented as a 5–dimensional projective subspace W of PG$(8, q)$ and disjoint from the Segre variety $\mathcal{S}_{2,2}$ (rank one 3×3 matrices in the matrix model of PG$(8, q)$). In this case W is trivially an exterior set with respect to $\mathcal{S}_{2,2}$. We asked ourselves the following question. Is it possible to perturb the structure of W to obtain a non–linear MRD code? The answer is affirmative. Our goal was reached adopting a model of $\mathcal{S}_{2,2}$ in the projective plane PG$(2, q^3)$ where the Segre variety is represented by a subplane $\bar{\pi}$ of order q, the code $\mathcal{G}$ corresponds to a line ℓ disjoint from $\bar{\pi}$ and the new set is represented by the GF(q^3)–rational points of a suitable algebraic curve. More precisely, we will introduce a derivation technique by deleting from W a distinguished set of $q^2 + q + 1$ planes and by adding suitable Segre varieties. In the plane model this corresponds to deleting suitable subsets of ℓ of size $q^2 + q + 1$ and by adding suitable subplanes of order q. This procedure can be iterated a certain number of times (multiple derivation) producing several non equivalent non-linear MRD codes.

In the last section we will construct a family of optimal non–linear constant–rank codes. Again, our approach is based on the geometry of the Segre variety of PG$(8, q)$ and the Veronese surface of PG$(5, q)$. More precisely, we will show that there exists a Segre variety embedded in $\Omega(\mathcal{S}_{2,2})$ that is an exterior set with respect to the Segre variety $\mathcal{S}_{2,2}$).

We stress that in all our constructions Singer cyclic groups of PGL$(3, q)$ [21] and their liftings to collineation groups of higher dimensional projective spaces, fixing a Segre variety or a Veronese surface, play a crucial role.

2 The case $n = 2$

In this section we report for completeness the complete classification of linear and non–linear MRD codes that are closed under GF(q)–multiplication when $n = m = 2$.

A *maximal exterior set* (MES) with respect to $\mathcal{Q}^+(3, q)$ is a set of $q + 1$ points of PG$(3, q)$ such that the line joining any two of them has no point in common with $\mathcal{Q}^+(3,q)$. The polar planes, with respect to the polarity induced by $\mathcal{Q}^+(3,q)$, of the points of a MES, define a flock, and conversely.

A *flock* of the hyperbolic quadric $\mathcal{Q}^+(3,q)$ of the finite projective space $\mathrm{PG}(3,q)$ is a partition of $\mathcal{Q}^+(3,q)$ consisting of $q+1$ irreducible conics. In [31] Thas showed that all flocks of $\mathcal{Q}^+(3,q)$ are linear if q is even, and that $\mathcal{Q}^+(3,q)$ has non–linear flocks (called Thas flocks) if q is odd. Further, he showed that for $q = 3, 7$ and $q \equiv 1 \mod 4$ $\mathcal{Q}^+(3,q)$ has only (up to a projectivity) the linear flock and the Thas flock. For $q = 11, 23, 59$ other flocks of $\mathcal{Q}^+(3,q)$ were discovered, independently, by Bader, Baker and Ebert (for $q = 11, 23$), Bonisoli and Johnson. Since these three flocks are related to exceptional near fields, these flocks are called exceptional flocks, see [11] and the literature therein. Finally, flocks of $\mathcal{Q}^+(3,q)$, q odd, were classified by Bader and Lunardon [2] : Every flock of $\mathcal{Q}^+(3,q)$ q odd, is linear, a Thas flock or one of the exceptional flocks. Bonisoli and Korchmáros [5], Durante and Siciliano [11] presented other proofs of the above classification theorem.

The classification theorem is the following.

Theorem 2.1. *Let $\mathcal{E}$ be the MES defined by a flock F of $\mathcal{Q}^+(3,q)$ in the matrix model of* $\mathrm{PG}(3,q)$. *Then, either q is even and $\mathcal{E}$ is a line or q is odd and one of the following possibilities occur:*

1. *$\mathcal{E}$ is a line;*
2. *$\mathcal{E}$ consists of $(q+1)/2$ points on two lines ℓ, $\ell^\perp$, where $\perp$ is the polarity of $\mathcal{Q}^+(3,q)$;*
3. *$\mathcal{E}$ is one of the sporadic examples.*

In our setting the linear MES corresponds to a $(2,2,1)$ $\mathrm{GF}(q)$–linear MRD–code. In all the other instances the MES corresponds to a $(2,2,1)$ non–linear maximum rank distance code .

3 The case $n = 3$

In this section we construct several families of non–linear MRD–codes of $\mathcal{M}_{3\times 3}(q)$. As already mentioned in the Introduction our method is based on the geometry of a Segre variety of $\mathrm{PG}(8,q)$. A very useful model of $\mathcal{S}_{2,2}$ arises from the geometry of the Desarguesian projective plane $\pi := \mathrm{PG}(2,q^3)$. Indeed, each point P of $\mathrm{PG}(2,q^3)$ defines a projective plane $X(P)$ of the projective space $\mathrm{PG}(8,q)$ and the set $\mathcal{D} = \{X(P) : P \in \mathrm{PG}(2,q^3)\}$ is a *Desarguesian spread* of $\mathrm{PG}(8,q)$ ([28, Section 25]). The incidence structure

$\pi := (\mathcal{D}, \mathcal{L})$, whose points are the elements of $\mathcal{D}$ and whose line set $\mathcal{L}$ consists of the 5–dimensional projective subspaces of $\mathrm{PG}(8, q)$ joining any two distinct elements of $\mathcal{D}$, is isomorphic to $\mathrm{PG}(2, q^3)$. The pair $(\mathcal{D}, \mathcal{L})$ is called the $\mathrm{GF}(q)$–*linear representation* of $\mathrm{PG}(2, q^3)$ (with respect to the Desarguesian spread $\mathcal{D}$).

Let X_1, X_2, X_3 denote projective homogeneous coordinates in $\pi \simeq \mathrm{PG}(2, q^3)$ and let $\bar{\pi}$ be a subplane of π of order q. Let G denote the stabilizer of $\bar{\pi}$ in $\mathrm{PGL}(3, q^3)$.

We can always choose homogeneous coordinates in such a way that $\bar{\pi} := \{(1, x^{q+1}, x^q) : x \in \mathrm{GF}(q^3) \setminus \{0\}, N(x) = 1\}$, where here $N(\cdot)$ is the *norm function* from $\mathrm{GF}(q^3)$ over $\mathrm{GF}(q)$. Indeed, it turns out that $\bar{\pi}$ is fixed pointwise by the order three semilinear collineation of $\mathrm{PG}(2, q^3)$ given by $\phi : (X_1, X_2, X_3) \mapsto (X_3^q, X_1^q, X_2^q)$.

Let $\langle S \rangle$ be a Singer cyclic group of G [21]. We can assume that S is given by

$$\begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega^q & 0 \\ 0 & 0 & \omega^{q^2} \end{pmatrix},$$

where ω is a primitive element of $\mathrm{GF}(q^3)$.

Remark 3.1. *The subgroup $\langle S \rangle$ fixes the three points $E_1 = (1, 0, 0)$, $E_2 = (0, 1, 0)$ and $E_3 = (0, 0, 1)$ of π, and hence the lines E_iE_j, $1 \leq i, j \leq 3$. All the other orbits are subplanes of order q of π. Note that the line E_iE_j is partitioned into the two points E_i and E_j and into $q-1$ orbits of $\langle S \rangle$ of size $q^2 + q + 1$. The collineation ϕ above normalizes $\langle S \rangle$.*

The points of $\bar{\pi}$ correspond to the q^2+q+1 planes filling the system of a Segre variety $\mathcal{S}_{2,2}$ of $PG(8, q)$ contained in the Desarguesian spread $\mathcal{D}$. Also, the lines of π, arising from sublines of $\bar{\pi}$, yield a set of $(q^3 - q)(q^2 + q + 1)$ points of π that together with the points of $\bar{\pi}$ give rise to the points of the secant variety $\Omega(\mathcal{S}_{2,2})$ of $\mathcal{S}_{2,2}$ ([25], [23]).

Under the action of the stabilizer G of $\bar{\pi}$ in $\mathrm{PGL}(3, q^3)$ the point set of π is partitioned into three orbits corresponding to the points of $\bar{\pi}$, points of π on extended sublines of $\bar{\pi}$ and the complement. Under the same group, by duality, the line set of π is partitioned into three orbits corresponding to sublines of $\bar{\pi}$, lines meeting $\bar{\pi}$ in a point and lines external to $\bar{\pi}$.

Proposition 3.2. *In the linear representation of* $\mathrm{PG}(2, q^3)$ *any line of* π *disjoint from* $\bar{\pi}$ *corresponds to a* 5*–dimensional projective subspace of* $\mathrm{PG}(8, q)$ *disjoint from* $\mathcal{S}_{2,2}$.

Of course, any line of π disjoint from $\bar{\pi}$ gives rise to an exterior set with respect to $\mathcal{S}_{2,2}$ and hence, from a coding theory point of view, a $(3,3,2)$ GF(q)–linear MRD–code.

Now let $q > 2$ and consider the set $\mathcal{X}$ of points of π whose coordinates satisfy the equation $X_1X_2^q - X_3^{q+1} = 0$. The set $\mathcal{X}$ has size q^3+1 and it is fixed by $\langle S \rangle$. Also, it contains $q-1$ subplanes of order q, one of which is $\bar{\pi}$, and the points E_1, E_2. More precisely, the subplanes of order q embedded in $\mathcal{X}$ are the subsets of points of π given by

$$\pi_a := \{(1, x^{q+1}, x^q) : x \in \mathrm{GF}(q^3),\ N(x) = a\},$$

where a is a nonzero element of GF(q). In particular, $\pi_1 = \bar{\pi}$. From [10, Proposition 3.1] a line of π intersects $\mathcal{X}$ in 0, 1, 2 or $q+1$ points and the intersections of size $q+1$ are actually lines of subplanes of order q of π embedded in $\mathcal{X}$. We can assume that the Segre variety corresponding to $\bar{\pi} = \pi_1$ is the only Segre variety of PG$(8,q)$ corresponding to rank one matrices of order three.

We recall the following definition.

Definition 3.3. *[4] Let ℓ_∞ be a line of π disjoint from the subplane $\bar{\pi}$. The* **exterior splash** *of $\bar{\pi}$ is defined to be the set of q^2+q+1 points of ℓ_∞ that lie on an extended line of $\bar{\pi}$.*

The line E_1E_2 is disjoint from all the $q-1$ subplanes π_a's of π contained in $\mathcal{X}$. Also, for each subplane π_a, with $a \in \mathrm{GF}(q) \setminus \{0\}$, its exterior splash is the set of q^2+q+1 points of E_1E_2 given by

$$Z_a := \{(1, x, 0) : x \in \mathrm{GF}(q^3),\ N(x) = -a^2\}.$$

Such a set is a so–called $GF(q)$–linear set of pseudoregulus type. For further details on these linear sets see [25], [10] and [24]. All these subplanes and splashes are of course $\langle S \rangle$–orbits.

We need the following lemma.

Lemma 3.4. *Let T be the fundamental triangle $E_1E_2E_3$ of π. A line of π is either a side of T or it contains a vertex of T or it induces a subline of a unique subplane of order q of π invariant under $\langle S \rangle$.*

Proof. Assume that a line r of π induces sublines for two distinct subplanes of order q, say π_a and π_b of π, that are invariant under $\langle S \rangle$. Since π_a and π_b are both $\langle S \rangle$–orbits, the lines of π arising from the sublines of π_a and those arising from the sublines of π_b coincide. Hence π_a and π_b correspond to two disjoint Segre varieties $\mathcal{S}_{2,2}$ in $\mathrm{PG}(8,q)$ having the same secant variety $\Omega(\mathcal{S}_{2,2})$. Since a Segre variety is the singular locus X of its secant variety and X is uniquely determined, we have a contradiction. A counting argument completes the proof. $\square$

We are ready to prove one of the main results of this paper.

Theorem 3.5. *The set $K := \mathcal{X} \setminus \{\pi_1\} \cup Z_1$ is such that every line defined by any two of its points is disjoint from π_1.*

Proof. As already mentioned, $\mathcal{X}$ is of type $(0,1,2,q+1)$ with respect to the lines of π. Any line meeting $\mathcal{X}$ in $q+1$ points is a subline of some order q subplane embedded in $\mathcal{X}$ invariant under $\langle S \rangle$ [10, Proposition 3.1]. Therefore, from Lemma 3.4 a line meeting $\mathcal{X} \setminus \{\pi_1\}$ in $q+1$ points is disjoint from π_1. Assume now that a line r of π is 2-secant to $\mathcal{X}$ and that r is 1–secant to π_1 at a point P. Without loss of generality, we can assume that $P = (1,1,1)$ since $\langle S \rangle$ acts transitively on points of π_1. Let $Q = (1, x^{q+1}, x^q)$ be a point on an order q subplane embedded in $\mathcal{X}$ distinct from π_1. Then $N(x) = a \neq 1$. A straightforward calculation shows that the line PQ meets E_1E_2 in the point $(1, x^q(1-x)/(x^q-1), 0)$. Since $N(-x(x-1)/(x^q-1)) = -N(x) = -a \neq -1$ it follows that the line PQ is disjoint from Z_1. Then, a line joining a point of Z_1 with a point of an order q subplane of $\mathcal{X}$ distinct from π_1 is disjoint from π_1. From [10, Proposition 3.1] a line of π through a vertex of T is either 1–secant or 2–secant to $\mathcal{X}$. In the latter case if such a line contains a point of π_1 then it intersects $\mathcal{X}$ in exactly one point; otherwise, it is disjoint from π_1. $\square$

The corresponding result in $\mathrm{PG}(8,q)$ is as follows.

Theorem 3.6. *The set K' corresponding to K in $\mathrm{PG}(8,q)$, $q > 2$, is an exterior set of size $(q^3+1)(q^2+q+1)$ with respect to the Segre variety $\mathcal{S}_{2,2}$ corresponding to π_1.*

Proof. As observed before, every line of π corresponds to a projective 5-subspace of $\mathrm{PG}(8,q)$ partitioned into q^3+1 planes of the Desarguesian spread $\mathcal{D}$. Hence any 5–dimensional projective subspace corresponding to a secant of K is disjoint from $\mathcal{S}_{2,2}$. It follows that a secant line to K' is either contained in a plane of $\mathcal{D}$ or meets $q+1$ disjoint planes of $\mathcal{D}$ of a 5–dimensional projective subspace of $\mathrm{PG}(8,q)$ in which $\mathcal{D}$ induces members of a plane–spread corresponding to the points of a line of π. $\square$

In terms of coding theory we have the following result.

Theorem 3.7. *There exists a* $(3,3,2)$ *maximum rank distance non-linear code admitting a Singer cyclic group of* $\mathrm{PGL}(3,q)$ *as an automorphism group.*

We end this section by showing that our geometric approach allows us to construct several non–equivalent $(3,3,2)$ maximum rank distance non-linear codes. We will consider a derivation technique of a $(3,3,2)$ $\mathrm{GF}(q)$–linear maximum rank distance code.

Let us consider the partition of the line E_1E_2, which is disjoint from π_1, into the points E_1, E_2 and the $q-1$ orbits Z_a, with $a \in \mathrm{GF}(q) \setminus \{0\}$, of $\langle S \rangle$ of size q^2+q+1. Note that the line E_1E_2 corresponds in $\mathrm{PG}(8,q)$ to a 5–dimensional projective subspace which is an exterior set with respect to the Segre variety $\mathcal{S}_{2,2}$ determined by π_1 and hence leads to a $(3,3,2)$ $\mathrm{GF}(q)$–linear maximum rank distance code.

Our derivation technique works as follows. Let us start from the partition $(Z_1,\ldots,Z_{q-1})$ of the line E_1E_2 introduced above.

Proposition 3.8. *The set* $E_1E_2 \setminus (\bigcup_{a\in Y} Z_a) \cup (\bigcup_{a\in Y} \pi_a)$*, where* Y *is a subset of* $\mathrm{GF}(q) \setminus \{0,1\}$*, is such that every line defined by any two of its points is disjoint from* π_1*.*

Proof. From the proof of Theorem 3.5 a line of π joining a point of π_1 with a point of π_a, $a \neq 1$, meets E_1E_2 in a point of Z_a. □

Corollary 3.9. *There exist* $\sum_{k=1}^{q-2}\binom{q-2}{k}$ $(3,3,2)$ *non–linear maximum rank distance codes of which at least* $q-2$ *are not equivalent.*

Remark 3.10. In [13] R. Figueroa presented a new class of non–desarguesian projective planes of order q^3, q a prime power with $q \not\equiv 1 \mod 3$, $q>2$. C. Hering and H.-J. Schaffer in [19] improved and simplified the construction for all prime powers q. From [26, Corollary 3] the set K constructed in Theorem 3.5 represents a line in the Figueroa plane. Moreover, any two sets constructed as in Theorem 3.5 are equivalent [9].

Remark 3.11. When $q=2$ some computer tests performed with MAGMA [6] give that all subsets of $\mathrm{PG}(2,8)$ yielding exterior sets with respect to a Segre variety $\mathcal{S}_{2,2}$ are just the 24 lines disjoint from $\bar{\pi}$. When $q=2$ no non–linear maximum rank distance codes arise from our construction.

4 Optimal Constant rank distance codes

In this section we will construct a family of optimal non–linear constant–rank codes. Again, our approach is based on the geometry of Segre varieties and Veronese varieties of projective spaces.

We recall the definition of constant–rank code.

Definition 4.1. *A* constant–rank code *(CRC) of constant rank r in $\mathcal{M}_{m\times n}(q)$ is a nonempty subset of $\mathcal{M}_{m\times n}(q)$ such that all elements have rank r.*

We denote a constant–rank code with length n, minimum rank distance d, and constant–rank r by (q,m,n,d,r). The term $A_R(q,m,n,d,r)$ denotes the maximum cardinality of a (q,m,n,d,r) constant–rank code over $\mathrm{GF}(q)$. If C is a (q,m,n,d,r) constant–rank code, then the code C^T obtained by transposing all the expansion matrices of codewords in C is a (q,n,m,d,r) constant–rank code with the same cardinality. Therefore $A_R(q,m,n,d,r) = A_R(q,n,m,d,r)$, and henceforth we can assume $n \le m$ without loss of generality. From [15, Proposition 8] we have that $A_R(q,n,m,d,r) \le \begin{bmatrix} n \\ r \end{bmatrix} \prod_{i=0}^{r-d}(q^m - q^i)$ and if this upper bound is attained the CRC is said to be optimal.

To our aim we need to recall some facts about Veronese surfaces of $\mathrm{PG}(5,q)$.

The Veronese surface of all conics of $\mathrm{PG}(2,q)$ is the variety $\mathcal{V}$ of $\mathrm{PG}(5,q)$ with parametric equations

$$(X_{00}, X_{11}, X_{22}, X_{01}, X_{02}, X_{12}) = (x_0^2, x_1^2, x_2^2, x_0x_1, x_0x_2, x_1x_2) \qquad (1)$$

where $x_0, x_1, x_2 \in \mathrm{GF}(q)$ and $(x_0, x_1, x_2) \ne (0,0,0)$. The mapping

$$\mu : (x_0, x_1, x_2) \in \mathrm{PG}(2,q) \mapsto (x_0^2, x_1^2, x_2^2, x_0x_1, x_0x_2, x_1x_2) \in \mathrm{PG}(5,q)$$

is called the *Veronese embedding* of $\mathrm{PG}(2,q)$. The variety $\mathcal{V}$ consists of q^2+q+1 points. We stress some important properties of the Veronese surface $\mathcal{V}$ (for further details see [20]). To the conics of $\mathrm{PG}(2,q)$ there correspond all hyperplane sections of $\mathcal{V}$. The hyperplane is uniquely determined by a conic if and only if the latter is not a single point. If the conic $\mathcal{C}$ of $\mathrm{PG}(2,q)$ is a repeated line, then the corresponding hyperplane H of $\mathrm{PG}(5,q)$ meets $\mathcal{V}$ at a non–degenerate conic. If $\mathcal{C}$ is a pair of distinct lines of $\mathrm{PG}(2,q)$, then H meets $\mathcal{V}$ at two non–degenerate conics with exactly one point in common. If $\mathcal{C}$ is a non–degenerate conic of $\mathrm{PG}(2,q)$, then H meets $\mathcal{V}$ along a rational quartic curve. Hence, $\mathcal{V}$ contains q^2+q+1 non–degenerate conics and any two points of $\mathcal{V}$ are contained in a unique conic. Since the conics of

$\mathcal{V}$ correspond to the lines of PG$(2,q)$, any two of these conics have a unique point in common. The planes of PG$(5,q)$ meeting $\mathcal{V}$ at a conic are called the *conic planes* of $\mathcal{V}$. Moreover, any two conic planes of $\mathcal{V}$ have exactly one point in common, and this common point belongs to $\mathcal{V}$.

Identifying the points of PG$(5,q)$ with all 3×3 symmetric matrices over GF(q), i.e.

$$(X_{00},X_{11},X_{22},X_{01},X_{02},X_{12}) \longleftrightarrow \begin{pmatrix} X_{00} & X_{01} & X_{02} \\ X_{01} & X_{11} & X_{12} \\ X_{02} & X_{12} & X_{22} \end{pmatrix},$$

the Veronese surface corresponds to the matrices

$$\begin{pmatrix} x_0^2 & x_0x_1 & x_0x_2 \\ x_0x_1 & x_1^2 & x_1x_2 \\ x_0x_2 & x_1x_2 & x_2^2 \end{pmatrix} = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} \cdot \begin{pmatrix} x_0 & x_1 & x_2 \end{pmatrix}.$$

The 3×3 symmetric matrices over GF(q) correspond to the conics of PG$(2,q)$, hence there is an identification of the points of PG$(5,q)$ with the conics of PG$(2,q)$. The points of PG$(5,q)$ which correspond to the degenerate conics of PG$(2,q)$ are those represented by the set Ω_1 of 3×3 symmetric matrices $\begin{pmatrix} X_{00} & X_{01} & X_{02} \\ X_{01} & X_{11} & X_{12} \\ X_{02} & X_{12} & X_{22} \end{pmatrix}$ over GF(q) with determinant zero and it turns out to be the union of the conic planes of $\mathcal{V}$. Moreover, Ω_1 consists of the GF(q)–rational points of the cubic hypersurface $\mathcal{M}_4^3$ of PG$(5,q)$ with equation $F=0$, where

$$F = \begin{vmatrix} X_{00} & X_{01} & X_{02} \\ X_{01} & X_{11} & X_{12} \\ X_{02} & X_{12} & X_{22} \end{vmatrix}.$$

The hypersurface $\mathcal{M}_4^3$ has $(q^2+q+1)(q^2+1)$ points and it has the Veronesean $\mathcal{V}$ as double surface.

The tangent lines of the conics of $\mathcal{V}$ are called the *tangents* or *tangent lines* of $\mathcal{V}$. Since no point of the surface $\mathcal{V}$ is singular, all tangents of $\mathcal{V}$ at the point P of $\mathcal{V}$ are contained in a plane $\pi(P)$. This plane $\pi(P)$ is called the *tangent plane* of $\mathcal{V}$ at P. Since P is contained in exactly $q+1$ conics of $\mathcal{V}$ and since no two conic planes through P have a line in common, the tangent plane $\pi(P)$ is the union of the $q+1$ tangent lines of $\mathcal{V}$ through P. Also $\pi(P)\cap\mathcal{V}=\{P\}$. Clearly, all tangent lines to $\mathcal{V}$ and all tangent planes to $\mathcal{V}$ belong to the hypersurface $\mathcal{M}_4^3$. Since $\mathcal{M}_4^3$ is the union of the conic planes of $\mathcal{V}$, any point of $\mathcal{M}_4^3$ is on at least one tangent or bisecant of $\mathcal{V}$. As any two points of $\mathcal{V}$ are contained in a conic of $\mathcal{V}$, each bisecant of $\mathcal{V}$ is a line of $\mathcal{M}_4^3$. Hence $\mathcal{M}_4^3$ can be also described as the union of all tangents and bisecants of $\mathcal{V}$ and it is also said to be the *secant variety* of $\mathcal{V}$.

Denote by Ω_1^e the points of $\mathcal{M}_4^3$ corresponding to the line pairs of $\mathrm{PG}(2,q)$, and similarly denote by Ω_1^i the points of $\mathcal{M}_4^3$ corresponding to those degenerate conics which are made up of two imaginary lines intersecting in a real point. The repeated line conics correspond to the Veronese surface $\mathcal{V}$, hence $\mathcal{M}_4^3 = \mathcal{V} \cup \Omega_1^e \cup \Omega_1^i$.
Note that

$$X := \begin{pmatrix} X_{00} & X_{01} & X_{02} \\ X_{01} & X_{11} & X_{12} \\ X_{02} & X_{12} & X_{22} \end{pmatrix}$$

is of rank 1 ($X_{ij} \in \mathrm{GF}(q)$ and not all the X_{ij}'s are zero), if and only if $X_{11}X_{22} - X_{01}^2 = X_{00}X_{22} - X_{02}^2 = X_{00}X_{11} - X_{12}^2 = X_{00}X_{01} - X_{02}X_{12} = X_{01}X_{12} - X_{11}X_{02} = X_{22}X_{12} - X_{01}X_{02} = 0$ if and only if $X_{00} : X_{11} : X_{22} : X_{01} : X_{02} : X_{12} = x_0^2 : x_1^2 : x_2^2 : x_0x_1 : x_0x_2 : x_1x_2$ for some x_0, x_1, x_2 in $\mathrm{GF}(q)$, if and only if $X \in \mathcal{V}$. Thus the points of $\mathcal{V}$ correspond to the 3×3 symmetric matrices over $\mathrm{GF}(q)$ of rank 1 and the points of $\Omega_1^e \cup \Omega_1^i$ correspond to those of rank 2. Ω_1^e is called the set of *external points* of $\mathcal{M}_4^3$ and Ω_1^i is called the set of *interior points* of $\mathcal{M}_4^3$. Simple counting arguments show that

$$|\Omega_1^e| = (q^2+q+1)(q^2+q)/2, \qquad |\Omega_1^i| = (q^2+q+1)(q^2-q)/2.$$

Then, $|\mathrm{PG}(5,q) \backslash \mathcal{M}_4^3| = q^5 - q^2$ corresponds to the number of non–degenerate conics in $\mathrm{PG}(2,q)$. Call $\mathcal{N}$ the orbit of non–degenerate conics.

When q is even, the hypersurface $\mathcal{M}_4^3$ is the set of points of $\mathrm{PG}(5,q)$ whose coordinates satisfies the equation $X_{00}X_{11}X_{22} + X_{00}X_{12}^2 + X_{11}X_{02}^2 + X_{22}X_{01}^2 = 0$. In this case, $\mathcal{M}_4^3$ contains the plane $\pi : X_{00} = X_{11} = X_{22} = 0$, which is disjoint from $\mathcal{V}$. Such a plane is called the *nucleus* of $\mathcal{V}$, and consists of all nuclei of conics of $\mathcal{V}$.

Let J be the automorphism group of $\mathcal{V}$. From [20, Theorem 25.1.1.0], J is an isomorphic copy of the group $\mathrm{PGL}(3,q)$, and so each linear collineation of $\mathrm{PG}(2,q)$ can be "lifted" to a collineation of $\mathrm{PG}(5,q)$ leaving $\mathcal{V}$ invariant.

Let $S = \langle \sigma \rangle$ be the Singer cyclic group of $\mathrm{PG}(2,q)$. From [3] the lifting of σ to a collineation group of $\mathrm{PG}(5,q)$ fixing $\mathcal{V}$ has the following rational form

$$M = \begin{pmatrix} T_1 & O_3 \\ O_3 & T_2 \end{pmatrix},$$

where $T_1 = S^2$ and $T_2 = S^{q+1}$ both induce Singer cycles on $\mathrm{PG}(2,q)$. The group $\langle M \rangle$ has order q^2+q+1. Geometrically, $\langle M \rangle$ fixes two planes, π_1, π_2, and partition the remaining points of $\mathrm{PG}(5,q)$ into Veronese surfaces, one of which is $\mathcal{V}$, [3, Corollary 5]. In particular, the planes π_1 and π_2 are both full orbits of $\langle M \rangle$. Note that when q is even one of the two planes π_1 and π_2 is the nucleus for all the q^3-1 Veronese surfaces in the partition.

Under the action of $\langle M \rangle$ the variety $\mathcal{M}_4^3$ is partitioned into q^2+1 orbits. When q is odd all such orbits are Veronese surfaces whereas if q is even one of such orbits is the nucleus of $\mathcal{V}$.

Definition 4.2. *An* exterior set *with respect to a Veronese surface $\mathcal{V}$ of* $\mathrm{PG}(5,q)$ *is a set $\mathcal{E}$ of points of* $\mathrm{PG}(5,q) \setminus \mathcal{V}$ *such that the line joining any two points of $\mathcal{E}$ is disjoint from $\mathcal{V}$.*

Since any two tangent planes to $\mathcal{V}$ meet in a point not on $\mathcal{V}$, the tangent lines to $\mathcal{V}$ cover a subset, say T, of $\mathcal{M}_4^3$ consisting of $(q^2+q+1)(q^2+q+2)/2$ points and T is invariant under $\langle M \rangle$. It follows that for all q, $\mathcal{M}_4^3$ contains a Veronese surface, that is an $\langle M \rangle$–orbit not belonging to T. More precisely $\mathcal{M}_4^3 \setminus T$ contains $(q^2-q)/2$ Veronese surfaces, different from $\mathcal{V}$, that are $\langle M \rangle$–orbits.

Let $\mathcal{V}_1 \neq \mathcal{V}$ be any Veronese surface of $\mathcal{M}_4^3 \setminus T$ which is an $\langle M \rangle$–orbit.

Proposition 4.3. *The Veronese surface $\mathcal{V}_1$ is an exterior set with respect to $\mathcal{V}$.*

Proof. Two points P_1 and P_2 of $\mathcal{V}_1$ correspond to two degenerate conics C_1 and C_2 of $\mathrm{PG}(2,q)$ not consisting of a repeated line. The line P_1P_2 corresponds to the pencil $\mathcal{P}$ of conics generated by C_1 and C_2. From [18, Table 7.7, p. 175] the case in which the base locus of $\mathcal{P}$ consists of $q+1$ points is excluded from our previous argument on tangent lines to $\mathcal{V}$: indeed in such a case P_1, P_2 should lie on a tangent line to $\mathcal{V}$. In all the other cases, the base locus of $\mathcal{P}$ is a single point P. In our setting, P_1 and P_2 are images one each other of a suitable collineation in $\langle M \rangle$. This means that in S there is a collineation τ sending C_1 in C_2. Assuming that $C_1 = L_1L_1'$ and $C_2 = L_2L_2'$ we have that $L_1^\tau = L_2$ and $L_1'^\tau = L_2'$. Then $P^\tau = P' \in L_2$ and $P^\tau = P'' \in L_2'$. It follows that $P' = P'' = P$, a contradiction since S acts semi regularly on points of $\mathrm{PG}(2,q)$. $\square$

Now, let us consider the lifting of S to a collineation of $\mathrm{PG}(8,q)$ fixing a Segre variety $\mathcal{X}_1 = \mathcal{S}_{2,2}$. It has the following rational form

$$N = \begin{pmatrix} T_1 & O_3 & O_3 \\ O_3 & T_2 & O_3 \\ O_3 & O_3 & T_2 \end{pmatrix},$$

where $T_1 = S^2$ and $T_2 = S^{q+1}$. The group $\langle N \rangle$ has order q^2+q+1. Geometrically, $\langle N \rangle$ fixes three planes, π_1, π_2 and π_3 and the projective 5–dimensional subspaces generated by any two of them. In particular the 5–dimensional projective subspace where $\langle N \rangle$ induces the group generated by

$$\begin{pmatrix} O_3 & O_3 & O_3 \\ O_3 & T_2 & O_3 \\ O_3 & O_3 & T_2 \end{pmatrix},$$

is partitioned in turn into q^3+1 planes forming a Desarguesian spread D. Also, it gives rise to a partition, say $\mathcal{F}$, of points of $\mathrm{PG}(8,q)$ not on the three 5–dimensional projective subspaces generated by π_i, π_j, $i \neq j, i, j = 1, 2, 3$, into $(q-1)(q^3-1)$ Segre varieties $\mathcal{S}_{2,2}$ which, in turn, are partitioned into Veronese surfaces (the so called flock of $\mathcal{S}_{2,2}$) [1, Theorem 3]. A proof of the fact that $\mathrm{PG}(8,q)$ can be partitioned into Segre varieties (apart from a number of subspaces) comes from Remark 3.1, by applying the $\mathrm{GF}(q)$–linear representation of $\mathrm{PG}(2,q^3)$. Another proof of this fact comes from a slight modification of [3]. Note that the projective space $\mathrm{PG}(8,q)$ is the union of the q^3+1 $\langle N \rangle$–invariant 5–dimensional projective subspaces sharing the plane invariant under the group generated by

$$\begin{pmatrix} T_1 & O_3 & O_3 \\ O_3 & O_3 & O_3 \\ O_3 & O_3 & O_3 \end{pmatrix}$$

and a plane in the spread D. By construction there are $q-1$ sets of 5–dimensional projective subspaces each of size q^2+q+1 inducing a flock for q^3-1 Segre varieties in $\mathcal{F}$. Let L be a projective 5–dimensional projective subspace of $\mathrm{PG}(8,q)$ fixed by $\langle N \rangle$ and intersecting $\mathcal{X}_1$ into a Veronese surface $\mathcal{V}$, and choose $\mathcal{V}_1$ to be another Veronese surface in the secant variety $\mathcal{M}_4^3$ of $\mathcal{V}$ that is an exterior set with respect to $\mathcal{V}$. Of course $\mathcal{V}_1$ belongs to a unique Segre variety, say $\mathcal{X}_2$, in $\mathcal{F}$.

Theorem 4.4. *The Segre variety $\mathcal{X}_2$ is an exterior set of* $\mathrm{PG}(8,q)$ *with respect to $\mathcal{X}_1$.*

Proof. First of all note that the secant variety of $\mathcal{V}_1$ is the intersection between the secant variety of $\mathcal{X}_2$ with L. We have to show that a secant line to $\mathcal{X}_2$ at the points P_1 and P_2 is disjoint from $\mathcal{X}_1$. If P_1 and P_2 are on $\mathcal{V}_1$ then from Proposition 4.3 there is nothing to prove since the line P_1P_2 lies on L. The previous argument holds true for any of the q^2+q+1 5–dimensional projective subspaces inducing the flock of $\mathcal{X}_1$ (and also the flock of $\mathcal{X}_2$). Assume that P_1 and P_2 lie on distinct Veronese surfaces of the flock of $\mathcal{X}_2$. Then the line $\ell = P_1P_2$ shares at most one point with the other 5–dimensional projective subspaces inducing the flock of $\mathcal{X}_2$. If ℓ met another Veronese surface of the flock of $\mathcal{X}_2$ then ℓ would lie on $\mathcal{X}_2$ and we are done. Otherwise, let P be a point on ℓ distinct from P_1 and P_2 and belonging to a 5–dimensional projective subspace of the flock, say L', and let $\mathcal{V}_2'$ be the Veronese surface obtained by sectioning $\mathcal{X}_2$ with L'. Then, it turns out that P lies on the secant variety of $\mathcal{V}_2'$. Let $\mathcal{V}_1'$ be the Veronese surface $L' \cap \mathcal{X}_1$. It follows that $\mathcal{V}_2'$ is an exterior set of L' with respect to $\mathcal{V}_1'$ and hence P cannot lie on $\mathcal{V}_1'$ and hence on $\mathcal{X}_1$ as well. This completes the proof. □

Theorem 4.5. *There exists a family of* $(q, 3, 3, 2, 2)$ *optimal non-linear constant–rank codes admitting a Singer cyclic group of* $\mathrm{PGL}(3, q)$ *as an automorphism group.*

Proof. The points of $\mathcal{X}_2$ correspond in the matrix model of $\mathrm{PG}(8, q)$ to matrices of rank 2. By scaling such matrices by nonzero scalars we get the desired codes. □

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