# Applications of Bar Code to Involutive Divisions and a “Greedy” Algorithm for Complete Sets

# Applications of Bar Code to involutive divisions and a greedy algorithm for complete sets.

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**Abstract.** Given a finite set of terms $U$ in $n$ variables, we describe an algorithm which finds – if it exists – an ordering on the variables such that $U$ is a complete set according to Janet involutive division. The algorithm, based on Bar Codes, is able to adjust the variables' ordering with a sort of backtracking technique, thus allowing to find the desired ordering without trying all the $n!$ possible ones.
**keywords:** Janet division, Bar Codes, Completeness.

*In memory of Vladimir Gerdt*

## 1. Introduction

Involutive divisions and involutive bases are a very important topic in Computer Algebra. Their theory dates back to the works by Janet [34, 35, 36, 37]. Given the polynomial ring $\mathcal{P} := \mathbf{k}[x_1, ..., x_n]$ in the $n$ variables $x_1, ..., x_n$ and coefficients in the field $\mathbf{k}$, the *semigroup of terms* $\mathcal{T} \subset \mathcal{P}$, is given by

$$\mathcal{T} := \{x_1^{\gamma_1} \cdots x_n^{\gamma_n} | \gamma_1, ..., \gamma_n \in \mathbb{N}\}.$$

In [34], Janet considers a semigroup/monomial ideal $J \subset \mathcal{T}$ and its minimal set of generators $\mathsf{G}(J)$. He associates to each generator a subset of variables, that are called *multiplicative*. Moreover, he decomposes $J$ in disjoint subsets called *cones* and describes a procedure (called *completion*) to construct this decomposition. For each term $v \in \mathcal{T}$, there is a *unique* way to write $v = tu$, with $t \in \mathsf{G}(J)$ and $u$ a product of powers of $t$'s multiplicative variables. In this context, while reducing a term $w$ modulo an ideal whose initial ideal is $J$, the polynomial to use is the only one whose leading term generates the cone containing $w$.

In [34], Janet aims to describe Riquier's formulation [42] of the description for the general solutions of a PDE problem, and for this aim he gives also an analogous decomposition for the escalier associated to $J$, namely $\mathsf{N}(J) := \mathcal{T} \setminus J$.

In his following works [35, 36, 37], he gives a new decomposition, named *involutive*, which is behind both Gerdt-Blinkov [21, 22, 23] procedure to compute Groebner bases and Seiler's involutivity theory [45]. His first aim is to give an interpretation by means of multiplicative variables of Cartan's solution to PDE problems [1, 2, 3] (whence the name *involutive*). The second aim is to evaluate in his theory's framework the notion of *generic initial ideal* introduced by Delassus [15, 16, 17] and the correction of his mistake by Robinson [44, 43] and Gunther [29, 30], who remarks that the notion requires $J$ to be Borel-fixed (an equivalent modern reformulation has been proposed by Galligo [20], who merges Hironaka and Grauert's ideas [33, 27]; see also [28, 18]).

In [36] Janet presents, as *nouvelle formes canoniques*, Delassus, Robinson and Gunther's results. Moreover, he gives a comparison with the canonical forms one can deduce from an involutive basis. In [37, p.62], given a homogeneous ideal $I$ of $\mathcal{P}$ in generic coordinates, he restates Riquier's completion in terms of a

Macaulay-like construction, iteratively computing the vector spaces $I_d := \{f \in I : \deg(f) = d\}$ until a precisely stated formula, called *Cartan test*, grants that Castelnuovo-Mumford regularity $D$ [39, pg.99] has been reached. Thus Castelnuovo-Mumford regularity was obtained for the first time by Janet via this explicit algorithm. This would allow him to consider the semigroup ideal $\mathsf{T}(I)$ of the leading terms with respect to deg-lex (in the sense of Groebner basis theory) and get the *involutive reduction* required by Riquier's procedure. The formal definition of involutive division is due to Gerdt-Blinkov [21, 22].

Bar Codes, introduced in [5, 6], are a compact, bidimensional representation for finite sets of terms $M \subset \mathcal{T}$ in any number of variables. In particular, if $M = \mathsf{N}(I)$ is the lexicographical Groebner escalier of a zerodimensional ideal $I$ of $\mathcal{P}$, many of the ideal's properties can be directly read from its Bar Code. As an example, in [9], Bar Codes are the main tool to develop a combinatorial algorithm which, given a finite set of simple points, computes the lexicographical Groebner escalier of its vanishing ideal. This algorithm is an alternative to those by Cerlienco-Mureddu [12, 13, 14] and by Felszeghy-Ráth-Rónyay [19], which keeps the former algorithm's iterativity, though reaching a complexity which is near to that of the latter one. In [5], we use Bar Codes to define and prove a bijection between zerodimensional (strongly) stable ideals in two or three variables and some partitions of their (constant) affine Hilbert polynomial.
Now, we are focusing on the properties of Bar Codes connected to involutive divisions, for which Bar Codes have already proved to be a good technology [7, 8]. For a general overview of Bar Codes' applications see [6].

In this paper, we discuss how the Bar Code associated to a finite set of terms (which is non-necessarily an order ideal) allows to decide whether that set is complete according to Janet's definition [34]. Moreover, we give an algorithm to check whether there is a variables' ordering such that a given set of terms is complete. We remark that the aim of this paper is not to build a completion, as it was for [22, 45], but to check whether there is need of a completion for every variables' ordering.

We need to remark that such a topic has some connections to the study of Stanley decompositions and Stanley depth. Indeed, Janet decomposition for a complete set is exactly a Stanley decomposition which can be easily read off from that set. Anyway, as stated by Herzog [32],

> *Janet decompositions from the viewpoint of Stanley depth are not optimal. They rarely give Stanley decompositions providing the Stanley depth of a monomial ideal. However one obtains the result that the Stanley depth of a monomial ideal is at least 1.*

and, actually, this paper places itself in the field of study mainly developed by Gerdt-Blinkov [21, 22, 23] and Seiler [45], which has aims and language that are different from those of Stanley depth.
After Section 2, devoted to notation (see Section 2.1) and to a recap on Bar Codes (Section 2.2), we describe Janet decomposition into multiplicative/non-multiplicative variables (Section 3), recalling how to use the Bar Code to get it from a finite set of terms. Moreover, we deal with complete sets, explaining how also completeness can be read from a suitable Bar Code. In Section 4, then, we explain an algorithm to detect a variable ordering (if it exists) such that a given set of terms is complete according to that ordering. The algorithm constructs a Bar Code from the maximal to the minimal variable, adjusting the variables' ordering with a sort of backtracking technique, and allowing to construct the desired ordering without trying all the $n!$ possible orderings.

## 2. Notation and preliminaries

### 2.1. Some general notation

The principal reference for the notation used in this paper are the books [38].
We start considering a field $\mathbf{k}$ and defining over it the polynomial ring $\mathcal{P} := \mathbf{k}[x_1, ..., x_n]$ in the $n$ variables $x_1, ..., x_n$; we also consider the *semigroup of terms* in the same variables $\mathcal{T} := \{x_1^{\gamma_1} \cdots x_n^{\gamma_n} | \gamma_1, ..., \gamma_n \in \mathbb{N}\}$. If $A \subseteq \{1, ..., n\}$ then $\mathcal{T}[A] := \{x_1^{\gamma_1} \cdots x_n^{\gamma_n} \in \mathcal{T} \mid \gamma_i \neq 0 \Rightarrow i \in A\}$. The degree of a term $t = x_1^{\gamma_1} \cdots x_n^{\gamma_n}$ is defined as $\deg(t) = \sum_{i=1}^n \gamma_i$, while its $h$-degree, for $h \in \{1, ..., n\}$, is $\deg_h(t) := \gamma_h$.

We call *semigroup ordering* on $\mathcal{T}$ a total ordering $<$ such that it holds

$$t_1 < t_2 \Rightarrow st_1 < st_2, \text{ for each } s, t_1, t_2 \in \mathcal{T}.$$

A semigroup ordering which has also the property to be a *well ordering* is called *term ordering*; the only term ordering we consider in this paper is the *lexicographical ordering* (also called Lex) with[1] $x_1 < ... < x_n$: $x_1^{\gamma_1} \cdots x_n^{\gamma_n} < x_1^{\delta_1} \cdots x_n^{\delta_n}$ if and only if there is $j$ such that $\gamma_j < \delta_j$, $\gamma_i = \delta_i$, for each $i > j$. Given $t \in \mathcal{T}$, we call $\max(t)$ (resp. $\min(t)$) the maximal (resp. minimal) variable dividing $t$. Once fixed a semigroup/term ordering $<$ on $\mathcal{T}$, for each $f \in \mathcal{P}$ we define its *leading term* $\mathsf{T}(f)$ to be its maximal term with respect to $<$.
We call *semigroup ideal* a subset $J \subseteq \mathcal{T}$ such that, if a term $t$ is in $J$, then the product $st$ is in $J$ as well, for each $s \in \mathcal{T}$; an *order ideal* is instead a subset $\mathsf{N} \subseteq \mathcal{T}$ such that, if $t \in \mathsf{N}$, then $s \in \mathsf{N}$, for each $s|t$. It is quite straightforward to show that $\mathsf{N} \subseteq \mathcal{T}$ is an order ideal if and only if $\mathcal{T} \setminus \mathsf{N} = J$ is a semigroup ideal.
The minimal generating set of a semigroup ideal $J \subset \mathcal{T}$ is called *monomial basis* and denoted by $\mathsf{G}(J)$. We associate to $J$ also the order ideal $\mathsf{N}(J) := \mathcal{T} \setminus J$.
Now, considered a set of polynomials $G \subset \mathcal{P}$, we define $\mathsf{T}\{G\} := \{\mathsf{T}(g),\ g \in G\}$ and $\mathsf{T}(G) := \{t\mathsf{T}(g),\ t \in \mathcal{T}, g \in G\}$; the latter is the semigroup ideal of leading terms. If $I$ is an ideal of $\mathcal{P}$, the monomial basis of $\mathsf{T}(I) = \mathsf{T}\{I\}$ is named *monomial basis* of $I$ and denoted again by $\mathsf{G}(I)$, while the order ideal $\mathsf{N}(I) := \mathcal{T} \setminus \mathsf{T}(I)$ is called *Groebner escalier* of $I$.

## 2.2. Bar Code for monomial ideals: a light recap

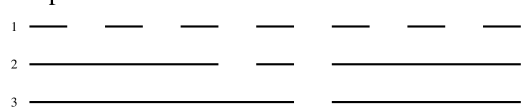
In this section we give a brief summary of the main definitions, facts and properties concerning Bar Codes. For more details, see [5, 6].

**Definition 2.1.** *A Bar Code* $\mathsf{B}$ *is a diagram composed by segments, (the* bars*), superimposed in horizontal rows, satisfying the Condition a. below. Denote by* $\mathsf{B}_j^{(i)}$ *the j-th bar (from left to right) of the i-th row (from top to bottom),* $1 \le i \le n$*, i.e. the* $j$-th $i$-bar *and by* $\mu(i)$ *the number of bars of the i-th row:*

a. $\forall i, j,\ 1 \le i \le n-1,\ 1 \le j \le \mu(i),\ \exists! \overline{j} \in \{1, ..., \mu(i+1)\}$ *s.t.* $\mathsf{B}_{\overline{j}}^{(i+1)}$ *lies under* $\mathsf{B}_j^{(i)}$.

We denote by $l_1(\mathsf{B}_j^{(1)}) := 1$, for each $j \in \{1, 2, ..., \mu(1)\}$, the $1-$*length* (or length for short) of the 1-bars and by $l_i(\mathsf{B}_j^{(k)})$, $2 \le k \le n$, $1 \le i \le k-1$, $1 \le j \le \mu(k)$ the *i-length* of $\mathsf{B}_j^{(k)}$, i.e. the number of $i$-bars lying over $\mathsf{B}_j^{(k)}$.

*Example* 2.2. The picture below represents a Bar Code




◇

Now we briefly sketch how to construct a Bar Code associated to a finite set of terms, following[2] [6].
We start considering a term $t = x_1^{\gamma_1} \cdots x_n^{\gamma_n} \in \mathcal{T} \subset \mathbf{k}[x_1, ..., x_n]$ and defining the term $\pi^i(t) := x_i^{\gamma_i} \cdots x_n^{\gamma_n} \in \mathcal{T}$, for each $i \in \{1, ..., n\}$. We can define $\pi^i(t)$, $i \in \{1, ..., n\}$, for each term $t$ of an ordered set of terms $M = [t_1, ..., t_m]$, in particular ordered increasingly with respect to Lex, getting $\overline{M}^{[i]} := \pi^i(M) := \{\pi^i(t) | t \in M\}$. We point out that, while in $M$ there are no repeated elements, they can occur in any of the lists $\overline{M}^{[i]}$. If it happens, the repeated elements are clearly adjacent, being each $\overline{M}^{[i]}$ a lexicographically ordered list. We continue computing the $n \times m$ matrix of terms $\mathcal{M}$, defined so that its $i$-th row is $\overline{M}^{[i]}$, $i = 1, ..., n$.

**Definition 2.3.** *The* Bar Code diagram $\mathsf{B}$ *associated to M (or, equivalently, to* $\overline{M}$*) is a* $n \times m$ *diagram, made by segments such that the i-th row of* $\mathsf{B}$, $1 \le i \le n$ *is constructed as follows:*

1. *take the i-th row of* $\mathcal{M}$*, i.e.* $\overline{M}^{[i]}$
2. *consider all the sublists of repeated terms, i.e.* $[\pi^i(t_{j_1}), \pi^i(t_{j_1+1}), ..., \pi^i(t_{j_1+h})]$ *such that* $\pi^i(t_{j_1}) = \pi^i(t_{j_1+1}) = ... = \pi^i(t_{j_1+h})$*, noting that*[3] $0 \le h < m$

[1] In this paper, the ordering on the variables will be fundamental. Unless otherwise specified we will consider $x_1 < ... < x_n$.
[2] An alternative construction has been given in detail in [5].
[3] Clearly if a term $\pi^i(t_{\overline{j}})$ is not repeated in $\overline{M}^{[i]}$, the sublist containing it will be only $[\pi_i(t_{\overline{j}})]$, i.e. $h = 0$.

3. *underline each sublist with a segment*
4. *delete the terms of* $\overline{M}^{[i]}$, $2 \le i \le n$, *leaving only the segments (i.e. the* $i$-bars*).*

*Each* 1*-bar* $\mathsf{B}_j^{(1)}$, $j \in \{1, ..., \mu(1)\}$ *remains labeled with the term* $t_j \in \overline{M}^{[1]}$.

We point out that a Bar Code diagram satisfies the condition of Definition 2.1 so it is a Bar Code.

*Example* 2.4. From the set $M = \{x_1, x_1^4, x_1^6, x_1^9 x_2, x_3, x_1x_3, x_1^4x_3\}$, we get the Bar Code of Example 2.2. ⋄

Bar Codes have been implemented in C and the implementation discussed in detail, also with a testing, in [11]. Essentially we use lists that are linked by pointers. Three different lists are needed: one for containing the monomials, one for the single bars and finally one for the levels of the Bar Code, which represent the variables.

## 3. Janet decomposition and completeness.

Janet, in [34], considered a monomial/semigroup ideal $J \subset \mathcal{T}$ and the related monomial basis $\mathsf{G}(J)$, introduced the concept of *multiplicative variable*, as well as the decomposition of $J$ into disjoint *cones*, characterizing what Gerdt-Blinkov would have called an *involutive division.*

**Definition 3.1.** [34, ppg.75-9] *Let* $U \subset \mathcal{T}$ *be a set of terms and* $t = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ *be an element of* $U$*. A variable* $x_j$ *is called* multiplicative *for* $t$ *with respect to* $U$ *if there is no term in* $U$ *of the form* $t' = x_1^{\beta_1} \cdots x_j^{\beta_j} x_{j+1}^{\alpha_{j+1}} \cdots x_n^{\alpha_n}$ *with* $\beta_j > \alpha_j$*. We denote by* $M(t, U)$ *the set of multiplicative variables for* $t$ *with respect to* $U$*.*
*The variables that are not multiplicative for* $t$ *with respect to* $U$ *are called* non-multiplicative *and we denote by* $NM(t, U)$ *the set containing them.*

Once stated this definition, we can explicit the divisibility relation characterizing the involutive division. Given $t \in U$ and $w \in \mathcal{T}$, $t \mid w$ with respect to Janet division, if and only if

$$w = tv \text{ and } \forall x_j \mid v,\ j \in \{1, ..., n\},\ x_j \in M(t, U).$$

The term $t$ is called *involutive divisor* of $w$ with respect to Janet division and we will write $t \mid_J w$.
Definition 3.1 is dependent on the variables' ordering, as shown in the following example.

*Example* 3.2. Consider the set $U = \{x_1, x_2^2\} \subset \mathbf{k}[x_1, x_2]$. If $x_1 < x_2$, then $M(x_1, U) = \{x_1\}$, $NM(x_1, U) = \{x_2\}$, $M(x_2^2, U) = \{x_1, x_2\}$, $NM(x_2, U) = \emptyset$. If, instead $x_2 < x_1$, then $M(x_1, U) = \{x_1, x_2\}$, $NM(x_1, U) = \emptyset$, $M(x_2^2, U) = \{x_2\}$, $NM(x_2^2, U) = \{x_1\}$. ⋄

**Definition 3.3.** *With the previous notation, the* cone *of* $t$ *with respect to* $U$ *is the set*

$$C(t, U) := \{t x_1^{\lambda_1} \cdots x_n^{\lambda_n} \mid \textit{where } \lambda_j \neq 0 \textit{ only if } x_j \in M(t, U)\}.$$

*Example* 3.4 ([7]). For the set $U = \{x_1x_3, x_2x_3\} \subseteq \mathbf{k}[x_1, x_2, x_3]$, we have $M(x_1x_3, U) = \{x_1, x_3\}$ and $M(x_2x_3, U) = \{x_1, x_2, x_3\}$, so $C(x_1x_3, U) = \{x_1^i x_3^j \ :\ i, j \in \mathbb{N} \setminus \{0\}\}$ and $C(x_2x_3, U) = \{tx_2x_3 \ :\ t \in \mathcal{T}\}$. ⋄

**Remark 3.5.** *Note that we have* $C(t, U) \cap U = \{t\}$*, by definition of multiplicative variable: if* $s \in \mathcal{T} \setminus \{1\}$ *and* $ts \in U$*, then* $\max(s) \notin M(t, U)$ *and so* $ts \notin C(t, U)$*.*

In [34], Janet introduced also the notion of *complete system*, together with the *completion* procedure, namely the procedure to produce a complete system with respect the decomposition in cones. Clearly, also the completion is dependent on the variables' ordering.

**Definition 3.6.** [34, ppg.75-9] *A set of terms* $U \subset \mathcal{T}$ *is called* complete *if for every* $t \in U$ *and* $x_j \in NM(t, U)$*, there exists* $t' \in U$ *such that* $x_jt \in C(t', U)$*, namely there is an involutive divisor of* $x_jt$ *with respect to Janet division.*

**Remark 3.7.** *If* $U \subseteq \mathbf{k}[x_1..., x_n]$ *has cardinality one, then it is always a complete set. Indeed, its unique element has no non-multiplicative variables.*

Also the order ideal $\mathbf{N}(J)$, associated to a monomial/semigroup ideal $J$, can be decomposed into disjoint cones. The related decomposition has been introduced in [34], where Janet was describing Riquier's way to represent the general solution of a PDE problem [42].

Given a finite set of terms $U \subset \mathcal{T} \subset \mathbf{k}[x_1, ..., x_n]$, $x_1 < x_2 < ... < x_n$, it is possible to associate to it a Bar Code $\mathsf{B}$. The diagram allows to read directly which are the Janet-multiplicative variables of the terms in $U$. First (see [6, 7]), we place a star symbol $*$ in the following positions[4]:

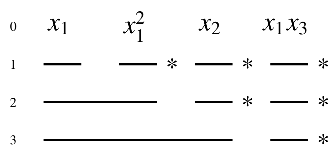
a) on the right of $\mathsf{B}^{(i)}_{\mu(i)}$, $\forall 1 \leq i \leq n$;

b) between two consecutive bars $\mathsf{B}^{(i)}_j$ and $\mathsf{B}^{(i)}_{j+1}$ not lying over the same $(i+1)$-bar, $\forall 1 \leq i \leq n-1$, $\forall 1 \leq j \leq \mu(i) - 1$.

Consider then a term $t \in U$; to detect its multiplicative variables we only have to check the presence/absence of stars just after bars over which $t$ lies, as stated in the following proposition (see [7] for its proof).

**Proposition 3.8.** *Let $U \subseteq \mathcal{T}$ be a finite set of terms and let us denote by $\mathsf{B}$ its Bar Code. For each $t \in U$ $x_i$, $1 \leq i \leq n$ is multiplicative for $t$ if and only if, in $\mathsf{B}$, the i-bar $\mathsf{B}^{(i)}_j$, over which $t$ lies, is followed by a star.*

*Example* 3.9. Let us consider the set $U = \{x_1, x_1^2, x_2, x_1x_3\} \subset \mathbf{k}[x_1, x_2, x_3]$, with $x_1 < x_2 < x_3$. The corresponding Bar Code is




According to Proposition 3.8, $x_1$ has no multiplicative variables. Indeed, following Definition 3.1, $x_1$ is not multiplicative since $x_1^2 \in U$, $x_2$ is not multiplicative since $x_2 \in U$ and $x_3$ is not multiplicative since $x_1x_3 \in U$. Therefore, $C(x_1, U) = \{x_1\}$.

◇

**Remark 3.10.** *In the context of detecting multiplicative variables, we can see the Bar Code as a reformulation of Gerdt-Blinkov-Yanovich* Janet tree [26], *with the characteristic of being more similar to the (equivalent) presentation given by Seiler* [45]. *However, for a given finite set of terms U, the algorithms to produce its Janet decomposition which can be deduced from both the formulations above of the Janet tree, are different from the algorithm naturally arising from the previous Proposition* 3.8 *(see* [7] *for more details).*

Completeness of a given finite set $U$ can be detected by means of the Bar Code, as stated in the following proposition (see also [8], for a very similar proposition for Janet-like division [24, 25]).

**Proposition 3.11.** *Let $U \subseteq \mathcal{T}$ be a finite set of terms and $\mathsf{B}$ be its Bar Code. Let $t \in U$, $x_i \in NM(t, U)$ and $\mathsf{B}^{(i)}_j$ the i-bar under t.*
*Let $s \in U$; it holds $s \mid_J x_it$ if and only if*

1. *$s \mid x_it$*
2. *$s$ lies over $\mathsf{B}^{(i)}_{j+1}$ and*
3. *for each variable $x_{j'}$ appearing with nonzero exponent in $\frac{x_it}{s}$ there is a star after the $j'$-bar under $s$.*

*Proof.* "⇐" We want to prove that an $s$ which satisfies the Conditions 1., 2., 3. is indeed an involutive divisor of $x_it$. By 1. $s$ divides $x_it$. Thanks to 3. and Proposition 3.8, all the variables in $\frac{x_it}{s}$ are multiplicative. The variable $x_i$ does not appear in $w := \frac{x_it}{s}$ and it does not need to be multiplicative for $s$, since, by 2., $s$ lies over $\mathsf{B}^{(i)}_{j+1}$, so $\deg_i(s) = \deg_i(t) + 1$. Therefore $sw = x_it$ and $w$ contains only multiplicative variables for $s$; therefore $s$ is the required involutive divisor for $x_it$.
"⇒" Let $s \in U$ be the involutive divisor of $x_it$. We prove that for $s$, the three conditions stated are verified. First of all, $s$ divides $x_it$ by definition of Janet division[5], this proving the first condition.

[4] In [6, 7] a set of terms is constructed with an analogous procedure; the paper [10] links this set with Pommaret bases [40, 41].
[5] And actually of involutive division, see [21, 22, 23].

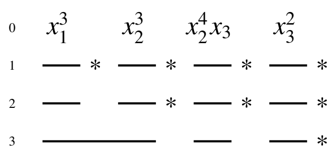
Let us consider Condition 2. If $s$ would lie over $\mathsf{B}^{(i)}_j$, then $\deg_l(s) = \deg_l(t)$ for $l = i, ..., n$. Then, being $s$ a divisor of $x_i t$, $x_i$ should be an element of $V_s := \{x_j, 1 \leq j \leq n : x_j \mid w := \frac{x_i t}{s}\}$, so $x_i$ should be multiplicative for $s$, this meaning having a star after $\mathsf{B}^{(i)}_j$, which is impossible by hypothesis, since in this case it would be $x_i t \in C(s, U) \cap C(t, U)$.
If $s$ lies over $\mathsf{B}^{(i)}_l$, $l > j + 1$, there exists $h \in \{i, ..., n\}$ such that $\deg_h(s) > \deg_h(x_i t)$, so $s$ would not divide $x_i t$, which is again a contradiction.
If $s$ lies over $\mathsf{B}^{(i)}_l$, $l < j$, then $s <_{Lex} t$ and it cannot happen that $\deg_{l'}(s) = \deg_{l'}(t)$ for $l' = i, ..., n$ (since otherwise $s$ would have been over $\mathsf{B}^{(i)}_j$). Let $x_k := \max\{x_h, h = 1, ..., n| \deg_h(s) < \deg_h(t)\}$; then, since $t \in U$ and $\deg_n(t) = \deg_n(s), ..., \deg_{k+1}(t) = \deg_{k+1}(s)$ and $\deg_k(t) > \deg_k(s)$, by definition of multiplicative variable according to Janet division, $x_k \in NM(s, U)$. Then $s$ must lie over $\mathsf{B}^{(i)}_{j+1}$. For being $s$ involutive divisor of $x_i t$, all the variables appearing with nonzero exponent in $\frac{x_i t}{s}$ must be multiplicative for $s$, and this implies that for each variable $x_{j'}$ appearing with nonzero exponent in $\frac{x_i t}{s}$ there is a star after the $j'$-bar under $s$, by Proposition 3.8, thus proving 3. □

From Proposition 3.11 we finally get the following theorem.

**Theorem 3.12.** *Let $U \subseteq \mathcal{T}$ be a finite set of terms and $\mathsf{B}$ be its Bar Code. Then $U$ is a complete set if and only if for each $t \in U$ and each $x_i \in NM(t, U)$, called $\mathsf{B}^{(i)}_j$ the i-bar under t, there exists a term $s \in U$ satisfying Conditions* 1, 2, 3 *of Proposition* 3.11.

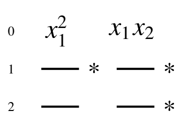
According to Proposition 3.11 and Theorem 3.12, given a finite set of term $U \subseteq \mathcal{T}$, to check its completeness we take, for each $t \in U$ and each $x_i \in NM(t, U)$, the $i$-bar $\mathsf{B}^{(i)}_j$, $1 \leq j \leq \mu(i)$ under $t$ and we look for an involutive divisor among the terms over $\mathsf{B}^{(i)}_{j+1}$, checking Conditions 1,3 above. We see now two simple examples of this procedure.

*Example* 3.13. Consider the set $U = \{x_1^3, x_2^3, x_2^4 x_3, x_3^2\} \subseteq \mathbf{k}[x_1, x_2, x_3]$ and its Bar Code



Take $t = x_1^3$ and $x_2 \in NM(t, U) = \{x_2, x_3\}$; $t$ lies over $\mathsf{B}^{(2)}_1$ and the only term over $\mathsf{B}^{(2)}_2$ is $x_2^3 \nmid x_1^3 x_2 = t x_2$, so $t x_2$ has no involutive divisor on $U$ and this implies that our set is actually non-complete. ◊

*Example* 3.14. Consider the set $U = \{x_1^2, x_1 x_2\} \subset \mathbf{k}[x_1, x_2]$, $x_1 < x_2$. Its Bar Code is as follows.



Looking at the stars, we can deduce $M(x_1^2, U) = \{x_1\}$, $NM(x_1^2, U) = \{x_2\}$, $M(x_1 x_2, U) = \{x_1, x_2\}$, $NM(x_1 x_2, U) = \emptyset$. Now, $t = x_1^2$ lies over $\mathsf{B}^{(2)}_1$ and over $\mathsf{B}^{(2)}_2$ there is only $x_1 x_2$ such that $x_1 x_2 \mid x_1^2 x_2$.

Since $x_1 \in M(x_1 x_2, U)$, $x_1 x_2 \mid_J x_1^2 x_2$ and we can conclude that $U$ is complete, with respect to the given ordering on the variables. ◊

## 4. A greedy algorithm for complete sets.

In this section, given a finite set of terms $U = \{t_1, ..., t_m\} \subseteq \mathcal{T}$, we try to find out whether there exists an *ordering on the variables* $x_1, ..., x_n$ such that $U$ is *complete*. As explained in Section 3, the Bar Code allows to detect the completeness of $U$. Clearly, such a construction depends on the variables' ordering, so if we want to solve the problem, in principle, we should draw and check $n!$ different Bar Codes, which turns out to be rather tedious and time consuming. We show now that we can look for the solution of our problem in a "greedy" way, so that most of the tests can be skipped. In order to do so, we first come back to [34].

Let $U = \{t_1, ..., t_m\} \subseteq \mathcal{T}$ be a finite set of terms, $t_i = x_1^{\alpha_1^{(i)}} \cdots x_n^{\alpha_n^{(i)}}$ and $t'_i = x_1^{\alpha_1^{(i)}} \cdots x_{n-1}^{\alpha_{n-1}^{(i)}} = t_i / x_n^{\alpha_n^{(i)}}$, for $i = 1, ..., m$. Let $\alpha = \max\{\alpha_n^{(i)}, 1 \leq i \leq m\}$. For each $\lambda \leq \alpha$, we define $I_\lambda := \{i : 1 \leq i \leq m | \alpha_n^{(i)} = \lambda\}$, the set indexing the terms in $U$ with $n$-th degree equal to $\lambda$, and $U'_\lambda := \{t'_i | i \in I_\lambda\} \subset \mathcal{T}[\{1, ..., n-1\}]$. Being $U'_\lambda$ a finite set of terms in $\mathcal{T}[\{1, ..., n-1\}]$, we can define Janet division on it and we can observe that, by Definition 3.1,

- for each $t' \in U'_\lambda$, $M(t', U) \cap \{x_1, ..., x_{n-1}\} = M(t', U'_\lambda)$;
- if $t = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \in U$, $U$ is complete and $\alpha_n < \alpha$, then $x_n \in NM(t, U)$ and the involutive divisor of $x_n t$ according to Janet division is a term $s \in U$ such that $s' \in U'_{\lambda+1}$.

These are the main observations leading to the following Proposition, first stated in [34] and then explicitly proved in [37].

**Proposition 4.1** ([34, 37])**.** *With the notation above, $U$ is complete if and only if the two conditions below hold:*

1. *For each $\lambda \in \{\alpha_n^{(i)},\ 1 \leq i \leq m\}$, $U'_\lambda$ is a complete set;*
2. *$\forall t'_i \in U'_\lambda$, $\lambda < \alpha$, there exists $j \in \{1, ..., m\}$ such that*
   a) *$t'_j \in U'_{\lambda+1}$.*
   b) *$t'_i \in C(t'_j, U'_{\lambda+1})$.*

Now, using the Bar Codes and the above Proposition 4.1, we can check whether there is an ordering on the variables making a given set $U$ complete.

The idea consists in constructing the Bar Code $\mathsf{B}$ of the set $U = \{t_1, ..., t_m\} \subset \mathcal{T}$ from the maximal variable to the minimal one, checking if, with the choice made up to the current point on the variables' ordering, the conditions of Proposition 3.11 hold for each term in $U$, and going back retracting our steps in case of failure, so modifying previous choices.

We describe now the algorithm, together with some examples; the whole list of procedures is displayed in Appendix A. In particular, the main procedure is Ordering (Algorithm 5) while Algorithms 1,2,3,4 are the subroutines on which it depends.

Let $X = \{x_1, ..., x_n\}$ be the set of all variables. First of all, the procedure Ordering looks for the subset $Y \subseteq X$ of good candidates for being the maximal variable, scanning the elements of $X$ (Algorithm 5, line 2). In particular, it relies on the subroutine Candidates (Algorithm 2) for this task.

The procedure Candidates considers the set of all variables and deletes all those that are not good candidates for being the maximal one.

It takes as input a list of terms $L$ (in this case $L = [U]$, so $|L| = 1$ and $L[1] = 1$) and a list of variables $C$ (in this case $C = [x_1, ..., x_n]$) and returns the list of good candidates for being the maximal one.

In order to decide whether a variable is a good candidate or not, Candidates scans $L$ and, for each list of terms in $L$, applies the subroutine CandidateVar (Algorithm 2, lines 2-4), returning the good candidates in Algorithm 2, line 5. For $i = 1, ..., |C|$, CandidateVar computes the sets $D_i := \{\beta \in \mathbb{N} | \exists t \in U, \deg_i(t) = \beta\}$ (Algorithm 1, lines 3-4) and excludes from the good candidates all variables $x_{\bar{i}}$, $\bar{i} \in \{1, ..., |C|\}$, for which there exists $\gamma \in D_{\bar{i}}$ such that $\gamma < \max(D_{\bar{i}})$ and $\gamma + 1 \notin D_{\bar{i}}$ (Algorithm 1, lines 5-6) . This procedure is justified by the following Lemma, strongly depending on Proposition 4.1.

**Lemma 4.2.** *Let $U \subseteq \mathcal{T}$ be a finite set of terms. For $i = 1, ..., n$, consider the sets*

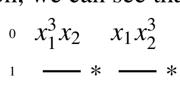
$$D_i := \{\deg_i(t) \ : \ t \in U\}.$$

*Suppose that for some $\bar{i}$, $\bar{i} \in \{1, ..., n\}$, there exists $\gamma \in D_{\bar{i}}$ such that $\gamma < \max(D_{\bar{i}})$ and $\gamma + 1 \notin D_{\bar{i}}$. Then $U$ is not a complete set for any ordering on the variables with $x_{\bar{i}}$ as maximal variable.*

*Proof.* In order to be complete, the set $U$ should satisfy the conditions of Proposition 4.1. Under our hypotheses, we have $\alpha > \gamma+1$, $I_\gamma \neq \emptyset$ and $I_{\gamma+1} = \emptyset$. To each $i \in I_\gamma$ corresponds a term $t'_i \in U'_\gamma$ and it must exist $j \in \{1, ..., |U|\}$ such that $t'_i \in C(t'_j, U)$ and $t'_j \in U'_{\gamma+1}$, but this is impossible since, being $I_{\gamma+1} = \emptyset$, also $U'_{\gamma+1} = \emptyset$. □

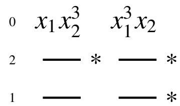
If $Y = \emptyset$, no variable is suitable for being the maximal one and making $U$ complete; this implies that $U$ is not complete for any variables' ordering (Algorithm 5, line 20).

*Example* 4.3. Consider $U = \{x_1 x_2^3, x_1^3 x_2\} \subset \mathbf{k}[x_1, x_2]$. Such a set is not complete since $D_1 = D_2 = \{1, 3\}$. As a confirmation, we can see that, if $x_1 < x_2$, we have




Then $M(x_1^3 x_2, U) = \{x_1\}$, $M(x_1 x_2^3, U) = \{x_1, x_2\}$ and $x_1^3 x_2^2$ does not belong neither to $C(x_1^3 x_2, U)$ nor to $C(x_1 x_2^3, U)$.

On the other hand, if $x_2 < x_1$, we have

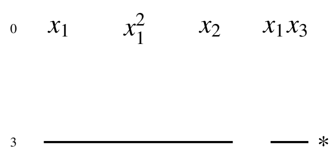



Thus $M(x_1x_2^3, U) = \{x_2\}$, $M(x_1^3x_2, U) = \{x_1, x_2\}$ and $x_1^2x_2^3$ does not belong neither to $C(x_1^3x_2, U)$ nor to $C(x_1x_2^3, U)$.

◊

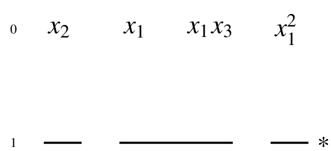
Suppose now $\emptyset \neq Y \subseteq X$; Ordering picks a variable $x_i \in Y$ and considers it as the maximal variable (Algorithm 5, line 3). Then it starts the construction of the last row[6] of the Bar Code associated to $U$ (Algorithm 5, line 4). In particular, the elements of $U$ are rearranged increasingly with respect to their $i$-degree, imposing, in addition, $t < t'$ when $t \mid t'$ for some $t, t' \in U$ with $deg_i(t) = deg_i(t')$. Denote the $i$-degrees of the terms in $U$ by $\lambda_1 < \lambda_2 < ... < \lambda_{\mu(i)}$, $1 \leq \mu(i) \leq |U|$. The $i$-bars are $\mathsf{B}_1^{(i)}, ..., \mathsf{B}_{\mu(i)}^{(i)}$ and they are drawn under the terms, grouping them according to their $i$-degree: the terms of $i$-degree $\lambda_1$ are underlined by $\mathsf{B}_1^{(i)}$, those of $i$-degree $\lambda_2$ by $\mathsf{B}_2^{(i)}$ and so on.
By construction, then, there is an obvious bijection $\phi_j^{(i)}$ between the set $A_j^{(i)}$ of terms over $\mathsf{B}_j^{(i)}$, $1 \leq j \leq \mu(i)$ and the set $U'_{\lambda_j}$ of Proposition 4.1.

*Example* 4.4. Let us consider the set $U = \{x_1, x_1^2, x_2, x_1x_3\} \subset \mathbf{k}[x_1, x_2, x_3]$; we first compute $D_1 = \{0, 1, 2\}$, $D_2 = D_2 = \{0, 1\}$. All the variables are good candidates for being the maximal one. We pick, for example, $x_3$, so we have




$\lambda_1 = 0, \lambda_2 = 1$, so $A_1^{(3)} = U_0 = \{x_1, x_1^2, x_2\}$ and $A_2^{(3)} = U_1 = \{x_1x_3\}$.

We remark that we could also have picked another variable, obtaining a different Bar Code; for example, picking $x_1$, we would have got:




◊

Afterwards, Ordering (Algorithm 5 line 5) launches the subroutine Friends (Algorithm 3), whose aim is essentially to mimic Condition 2 of Proposition 4.1. In particular, it adds requirements on what variables should be multiplicative for the terms to make Condition 2 hold and it checks whether the requirements on the variables imposed by its previous executions are met.

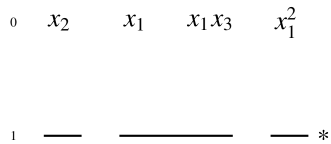
This is the first time in which Friends is run by the algorithm, so there are no previously imposed requirements on the multiplicative variables (we only have imposed $x_i$ as maximal variable) so the part of Friends that checks such requirements (Algorithm 3, lined 11 - 31) is skipped for now. It only lists the requirements imposed by the choice of $x_i$ as maximal variable, so that Condition 2. of Proposition 4.1 holds.

For each $1 \leq j < \mu(i)$, consider the bar $\mathsf{B}_j^{(i)}$ and the corresponding set of terms $A_j^{(j_1)}$.

For each $t \in A_j^{(i)}$, Friends computes the set $U(t, x_i) = \{(u, V) | u \in A_{j+1}^{(i)}$ and $V : tx_i = um, m \in \mathcal{T}[V]\}$. The elements in $U(t, x_i)$ are the candidates for being the involutive divisor of $tx_i$ (or, in other words, are the candidate terms for satisfying Condition 2 of Proposition 4.1) while $V$ is the set of variables – *smaller* than $x_i$ – that must belong to $M(u, U)$ for $u$ being the involutive divisor of $tx_i$ (Algorithm 3, lines 2–10). Indeed, due to the bijections $\phi_j^{(i)}, \phi_{j+1}^{(i)}, t' := t/x_i^{\lambda_j} \in U'_{\lambda_j}$ and $u' := u/x_i^{\lambda_{j+1}} \in U'_{\lambda_{j+1}}$ and, if the variables in $V$ are multiplicative for $u$, then $t' \in C(u', U'_{\lambda_{j+1}})$. If one of the sets $U(t, \{x_i\})$ is empty, then there is no candidate for being the involutive divisor of $tx_i$; therefore $x_i$ is not a good candidate for being the maximal variable, so we come back to $Y$ and we start again with a new maximal variable (Algorithm 5, lines 6 - 8).

[6] Remember that the last row of a Bar Code, namely that on the bottom, is the row associated to the maximal variable (see Section 2.2).

*Example* 4.5. Coming back to example 4.4, we have $U(x_1, x_3) = \{(x_1x_3, \emptyset)\}$, $U(x_1^2, x_3) = \{(x_1x_3, \{x_1\})\}$ and $U(x_2, x_3) = \emptyset$, so $x_3$ was a bad choice for being the maximal variable. We try with $x_1$, getting

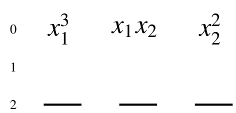



Now, $U(x_2, x_1) = \{(x_1, \{x_2\})\}$, $U(x_1, x_1) = \{(x_1^2, \emptyset)\}$ and $U(x_1x_3, x_1) = \{(x_1^2, \{x_3\})\}$, so $x_1$, at least for now, is a good choice for the maximal variable. ◊

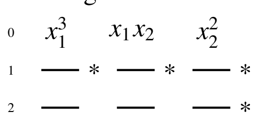
Suppose to be in the non-failure case. If, in addition, for $1 \leq j \leq \mu(i)$ there is only one term over $\mathsf{B}_j^{(i)}$, all the bars are *unitary*, so we say that we are in the *unitary case* (Algorithm 5, lines 9 - 12). In this special case, each variable ordering such that $x_i$ is the maximal variable makes $U$ a complete set of terms.

Indeed, in this case, for each choice on the ordering of the following (and so, smaller) variables, their corresponding bars will be unitary again and, by the construction of the stars, all of them will be followed by a star. In other words, for each $t \in U$, and for each $x_j \neq x_i$, $x_j \in M(t, U)$. Moreover, for each $t \in U$, $|U(t, \{x_i\})| = 1$, so let $(u, V)$ be the only element in $U(t, x_i)$, then $x_i \notin V$, so all variables in $V$ are multiplicative for $u$ and this makes $u$ the required involutive divisor of $x_it$, ensuring the completeness of $U$.

*Example* 4.6. For $U = \{x_1^3, x_1x_2, x_2^2\} \subset \mathbf{k}[x_1, x_2]$, $D_1 = \{0, 1, 3\}$ and $D_2 = \{0, 1, 2\}$, so $Y = \{x_2\}$. Chosing $x_2$ as maximal variable the Bar Code becomes:




and we have $U(x_1^3, x_2) = \{(x_1x_2, \{x_1\})\}$, $U(x_1x_2, x_2) = \{(x_2^2, \{x_1\})\}$. All the 2-bars are unitary, so, completing the Bar Code we get




We can easily check from the diagram that $(x_1^3) \cdot x_2 \in C(x_1x_2, U)$ and $(x_1x_2) \cdot x_2 \in C(x_2^2, U)$. Therefore $U$ is complete.

◊

If we are not in the unitary case (Algorithm 5 line 13), we have to choose the next variable and continue drawing the Bar Code, using the routine Common (Algorithm 4).

To get the candidates for being the next variable, we execute the procedure CandidateVar to each $i$-bar and (procedure Candidates) we intersect the results. Indeed, we want that the next variable is a good candidate to make all the sets $U_{\lambda_i}$ complete, according to Condition 1. of Proposition 4.1.

If the intersection is empty (Algorithm 4, lines 4 - 5) there are no such good candidates, then $x_i$ was not a good choice for being the maximal variable and we have to come back and repeat the whole procedure for another maximal variable (Algorithm 5 line 17).

Otherwise, we choose some $x_l$ among the variables in the intersection (Algorithm 4 line 6), and for each $1 \leq j \leq \mu(i)$, we order the terms over $\mathsf{B}_j^{(i)}$ in increasing order according to the $l$-degree, imposing in addition $t < t'$ when $t \mid t'$ for some $t, t' \in U$ with $deg_l(t) = deg_l(t')$ (exactly as we did for $x_i$). Then we draw all the $l$-bars (Algorithm 4 line 8).

Employing again the routine Friends, separately for each $i$-bar, we look for candidate involutive divisors when $x_l$ is not multiplicative (Algorithm 3, lines 2 - 10). Moreover, we check whether the choice of $x_l$ is a good one, by checking that $x_l$ is multiplicative for all terms we imposed it to be in the previous application of the routine Friends (Algorithm 3, lines 11 - 31).

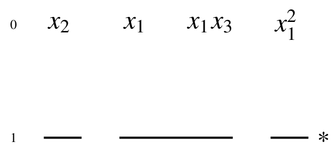
More precisely, for each $t$ over $\mathsf{B}_j^{(i)}$, $1 \leq j < \mu(i)$, we have constructed a set $U(t, \{x_i\})$ of candidate involutive divisors for $tx_i$. Given $(u, V) \in U(t, x_i)$, if $x_l \notin V$, then the multiplicativity of $x_l$ is irrelevant for $u$, so $(u, V)$ still remains a good candidate for being an involutive divisor. It is still a good candidate also if $x_l \in V$ and the $l$-bar of $u$ is in one of the conditions for being followed by a star (see section 2.2), since it means that

$x_l$ is multiplicative for $u$. Otherwise, we remove $(u, V)$ from the candidates. If for some $t$, its candidate list is empty, we have to revoke the choice of $x_l$ and come back to pick another variable in place (Algorithm 4 line 11).

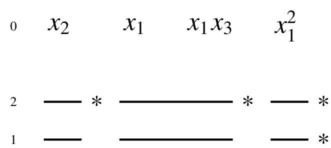
If the procedure Friends gives a positive outcome, then a new variable has been settled and the routine Common keeps calling itself until (Algorithm 5, lines 13 - 21)

- all variables have been placed (positive outcome)
- the unitary case is reached (positive outcome)
- continue revocations of choices lead to failure, in the sense that there are no more variables to pick (negative outcome, there is no ordering on the variables making the set complete).

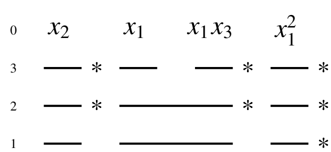
*Example* 4.7. We conclude now examples 4.4 and 4.5. From



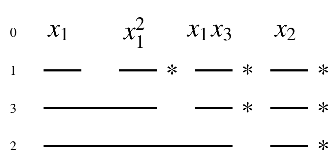

we choose now $x_2$ as following variable and we get




With Friends we do not impose any condition on the terms over the 2-bars. Indeed, over each 1-bar there is only one 2-bar and so, that 2-bar is followed by a star, this implying that $x_2$ is multiplicative for all terms. Moreover, we have only to check $U(x_2, x_1) = \{(x_1, \{x_2\})\}$; being $x_2 \in M(x_1, U)$, the procedure gives a positive outcome. Finally choosing $x_3$, we get




Now, $x_3$ is multiplicative for $x_1^2$ as required by $U(x_1x_3, x_1)$ and we have $U(x_1, x_3) = \{(x_1x_3, \emptyset)\}$, so $U$ turns out to be complete with the variables' ordering $x_3 < x_2 < x_1$.
We point out that this is not the only ordering making $U$ complete, in particular, for $x_1 < x_3 < x_2$ $U$ is complete again:




Indeed

- $M(x_1, U) = \emptyset$, $NM(x_1, U) = \{x_1, x_2, x_3\}$, with $x_1^2 \in C(x_1^2, U)$, $x_1x_2 \in C(x_2, U)$, $x_1x_3 \in C(x_1x_3, U)$ ;
- $M(x_1^2, U) = \{x_1\}$, $NM(x_1^2, U) = \{x_2, x_3\}$, with $x_1^2x_2 \in C(x_2, U)$, $x_1^2x_3 \in C(x_1x_3, U)$;
- $M(x_1x_3, U) = \{x_1, x_3\}$, $NM(x_1x_3, U) = \{x_2\}$, with $x_1x_2x_3 \in C(x_2, U)$;
- $M(x_2, U) = \{x_1, x_2, x_3\}$, $NM(x_2, U) = \emptyset$.

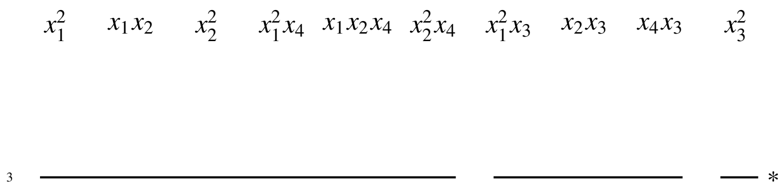
$\diamond$

We see now a complete example for the execution of the whole procedure.

*Example* 4.8. Consider the set

$$M = \{x_2x_3, x_1^2, x_3^2, x_2^2, x_1x_2, x_1x_2x_4, x_1^2x_4, x_4x_3, x_2^2x_4, x_1^2x_3\} \subset \mathbf{k}[x_1, x_2, x_3, x_4].$$

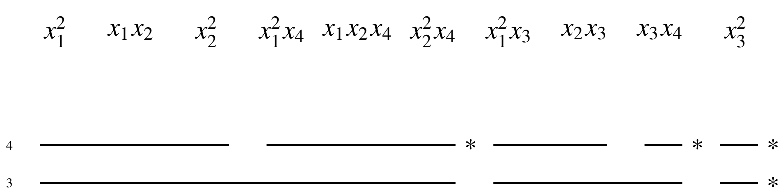
First, we compute $D_1 = D_2 = D_3 = \{0, 1, 2\}$, $D_4 = \{0, 1\}$, deducing that each variable is a good candidate for being the maximal one, so $Y = \{x_1, x_2, x_3, x_4\}$. We choose, for example, $x_3$, getting




Now, running Friends for the first time, we get

- $U(x_1^2, x_3) = \{(x_1^2x_3, \emptyset)\}$;
- $U(x_1x_2, x_3) = \{(x_2x_3, \{x_1\})\}$;
- $U(x_2^2, x_3) = \{(x_2x_3, \{x_2\})\}$;
- $U(x_1^2x_4, x_3) = \{(x_1^2x_3, \{x_4\}), (x_3x_4, \{x_1\})\}$;
- $U(x_1x_2x_4, x_3) = \{(x_2x_3, \{x_1, x_4\}), (x_3x_4, \{x_1, x_2\})\}$;
- $U(x_2^2x_4, x_3) = \{(x_2x_3, \{x_2, x_4\}), (x_3x_4, \{x_2\})\}$;
- $U(x_1^2x_3, x_3) = \{(x_3^2, \{x_1\})\}$;
- $U(x_2x_3, x_3) = \{(x_3^2, \{x_2\})\}$;
- $U(x_3x_4, x_3) = \{(x_3^2, \{x_4\})\}$.

The procedure gives a positive outcome, so, since we are not in the unitary case, we apply Common. All the variables are good candidates for being the second in order of magnitude and, for example, we choose $x_4$, getting:




We have:

- $U(x_1^2, x_4) = \{(x_1^2x_4, \emptyset), (x_4, \{x_1\})\}$;
- $U(x_1x_2, x_4) = \{(x_1x_2x_4, \emptyset)\}$;
- $U(x_2^2, x_4) = \{(x_2^2x_4, \emptyset)\}$;
- $U(x_2, x_4) = \{(x_4, \{x_2\})\}$.

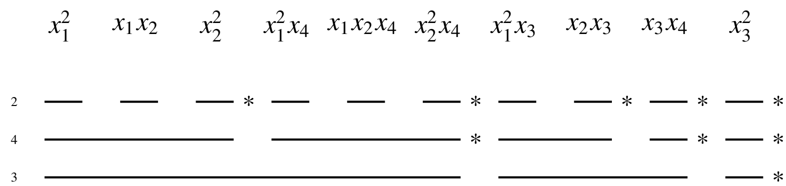
We check that the choice of $x_4$ is suitable for the conditions imposed in the previous step:

- for $U(x_1^2x_4, x_3) = \{(x_1^2x_3, \{x_4\}), (x_3x_4, \{x_1\})\}$ note that $x_1^2x_3$ does not lie on the rightmost 4-bar, so $x_4$ is not multiplicative. Since we have more than one term associated to $x_1^2x_4$, we only delete $x_1^2x_3$ and keep $x_3x_4$. The same argument holds for $x_1x_2x_4, x_2^2x_4$.
- For $U(x_3x_4, x_3) = \{(x_3^2, \{x_4\})\}$, since $x_3^2$ lies on the rightmost 4-bar, $x_3^2$ passes the test, remaining a good candidate for being an involutive divisor.

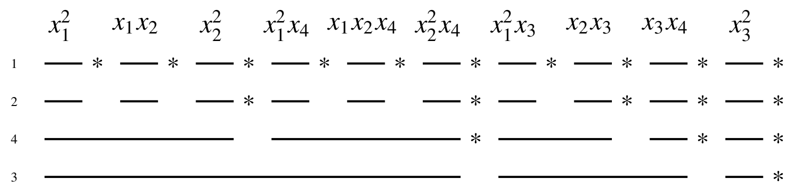
So we have

- $U(x_1^2, x_3) = \{(x_1^2x_3, \emptyset)\}$;
- $U(x_1x_2, x_3) = \{(x_2x_3, \{x_1\})\}$;
- $U(x_2^2, x_3) = \{(x_2x_3, \{x_2\})\}$;
- $U(x_1^2x_4, x_3) = \{(x_3x_4, \{x_1\})\}$;
- $U(x_1x_2x_4, x_3) = \{(x_3x_4, \{x_1, x_2\})\}$;
- $U(x_2^2x_4, x_3) = \{(x_3x_4, \{x_2\})\}$;
- $U(x_1^2x_3, x_3) = \{(x_3^2, \{x_1\})\}$;
- $U(x_2x_3, x_3) = \{(x_3^2, \{x_2\})\}$;
- $U(x_3x_4, x_3) = \{(x_3^2, \{x_4\})\}$;
- $U(x_1^2, x_4) = \{(x_1^2x_4, \emptyset), (x_4, \{x_1\})\}$;
- $U(x_1x_2, x_4) = \{(x_1x_2x_4, \emptyset)\}$;
- $U(x_2^2, x_4) = \{(x_2^2x_4, \emptyset)\}$;
- $U(x_2, x_4) = \{(x_4, \{x_2\})\}$.

We continue choosing $x_2$ as next variable and we get:




This way, all the 2-bars are unitary. We check on the 2-bars to have nonincreasing exponents for $x_1$ and this is true. Moreover, we check that $x_2$ is multiplicative where it is marked, i.e. for $x_2x_3, x_3x_4$ but it clearly holds. The set $M$ is complete for $x_1 < x_2 < x_4 < x_3$ and its final Bar Code with respect to the chosen ordering is




◊

The algorithm terminates since there is a finite number of variables and each time we pick a variable as candidate, we remove it from the candidates' list, so we do not choose a variable in some position of the ordering more than once.

The correctness, instead is an easy consequence of Proposition 4.1, since the algorithm executes the tests imposed by that proposition.

**Remark 4.9.** *We point out that, even in the case in which the given set U is not complete for any variables' ordering, it is possible to store the state in which most of the variables have been settled, before retracting due to some failure condition. This - though not being a warranty of minimality for the terms one has to add in order to get the completion - can reduce the number of tests one has to do in the first step of completion.*

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## Appendix A. List of all procedures

Before listing all procedure, we recall that a Bar Code is given by a concatenation of lists via pointers. We have one list for the levels (i.e. the variables), one for the bars within any level and finally one for the terms.
Constructing a bar, then, means adding a new element to the list of bars, connecting it in the right position by means of pointers.
When we put a star at the end of a bar, we are putting a star symbol at the end of the corresponding bar, therefore we suppose known a procedure Star($x_i, t$), which takes a variable $x_i$ and a monomial $t$ as input and returns true if at level $i$, the bar under $t$ (therefore placed in correspondence to its exponents from level $n$ to level $i$) has a star as its last entry, and false otherwise.

**Algorithm 1** Procedure to generate the candidate list for the current maximal variable (subroutine).

**procedure** CandidateVar($M, C$) ▹ $M$ is a list of terms; $C$ is a list of variables.
    $Y := C$
    **for** $i = 1, ..., |C|$ **do**
    $D_i := \{\beta \in \mathbb{N} | \exists t \in M, \deg_{C[i]}(t) = \beta\}$
    **if** for some $\gamma_1 \in D_i, \gamma_1 < \max(D_i), \gamma_1 + 1 \notin D_i$ **then**
        Delete $C[i]$ from $Y$
    **end if**
    **end for**
    **return** $Y$
**end procedure**

**Algorithm 2** Procedure to generate the candidate list for the current maximal variable.

**procedure** Candidates($L, C$) ▹ $L$ is a list of lists of terms; $C$ is a list of variables.
    **for** $i = 1, ..., |L|$ **do**
    $Y[i]$ :=CandidateVar($L[i], C$);
    **end for**
    **return** $\bigcap_i Y[i]$
**end procedure**

**Algorithm 3** Friends

**procedure** FRIENDS$(A, Y, x_i, T)$ ▷ $T$ is the output of a previous execution of Friends (or it is empty), so it is formed by sets of the form $T(t, x_j)$, $t$ terms in the given set, and $x_j$ variables.
    **for** $j = 1, ..., \mu(i) - 1$ **do**
    $B' = A^{(i)}_{j+1}$ ▷ We directly consider the set of terms associated to a bar.
    $U = \emptyset$
    **for** $t \in A^{(i)}_j$ **do**
    $U(t, x_i) = \{(u, V) | u \in B'$ and $V : tx_i = um,\ m \in \mathcal{T}[V]\}$
    **if** $U(t, x_i) = \emptyset$ **then return** $\emptyset$
    **end if**
    **end for**
    **end for**
    **if** $T \neq \emptyset$ **then**
        **for** $j = 1, ..., \mu(i)$ **do**
        **for** $t \in A^{(i)}_j$ **do**
        **for** $y \in X \setminus Y$ **do** ▷ These are the variables that are already been ordered.
        $U(t, y) = \emptyset$
        **for** $(u, V) \in T(t, y)$ **do**
        **if** $x_i \notin V$ **then**
            $U(t, y) = \{(u, V)\} \cup U(t, y)$, $U = U \cup U(t, y)$
        **else**
            **if** $x_i \in V$ and Star$(x_i, t)$ =true **then** ▷ Star$(x_i, t)$ =true means that there is a star after the $i$-bar under $t$.
                $U(t, y) = \{(u, V)\} \cup U(t, y)$, $U = U \cup U(t, y)$
            **end if**
        **end if**
        **end for**
        **if** $U(t, y) = \emptyset$ **then**
            **return** $\emptyset$
        **end if**
        **end for**
        **end for**
        **end for**
    **end if**
    **return** $U$
**end procedure**

**Algorithm 4** Common

**procedure** COMMON($A, X, x_i, T$) ▷ $T$ is the output of a previous execution of Friends (or it is empty), so it is formed by sets of the form $T(t, x_j)$, $t$ terms in the given set, and $x_j$ variables.
    $Y = X \setminus \{x_i\}$
    $Y'$ =CANDIDATES($A, Y$)
    **if** $|Y'| = 0$ **then return** $\emptyset$
    **end if**
    **for** $x_l \in Y'$ **do**
    **for** $j = 1, ..., \mu(i)$ **do**
    construct the $l$-bars $C = \{A_m^{(l)}\}$ over $A_j^{(i)}$;
    **end for**
    $U$ =Friends($C, Y, x_j, T$)
    **if** $U = \emptyset$ **then continue**
    **end if**
    **if** $|A_m^{(l)}| = 1, \ \forall 1 \le m \le \mu(l)$ **then** $ord = Y \cup \{x_l\}$ **return** $ord$
    **end if**
    **if** $Y \neq \emptyset$ **then**
        $C$ =COMMON($C, Y, x_l, U$)
    **else**
        **if** $Y = \emptyset$ **then** $ord = ord \cup \{x_l\}$
            **return** ord
        **end if**
    **end if**
    **if** $ord \neq \emptyset$ **then** $ord = ord \cup \{x_l\}$
    **else** **continue**
    **end if**
    **end for**
    **return** $\emptyset$
**end procedure**

**Algorithm 5** Ordering

**procedure** ORDERING($M, X$) ▷ $M$ is a given list of terms; $X$ is the list of all variables
    $Y$ =CANDIDATES($M, X$)
    **for** $x_i \in Y$ **do**
    $A = \{A_j^{(i)}\}, 1 \leq j \leq \mu(i)$ ▷ Construct the sets of terms associated to the $i$-bars.
    $T$ =FRIENDS($A, X \setminus \{x_i\}, x_i, \emptyset$)
    **if** $T = \emptyset$ **then**
        **continue**
    **end if**
    **if** $|A_j^{(i)}| = 1, 1 \leq j \leq \mu(i)$ **then** ▷ Unitary case.
        $ord$ = Append($X \setminus \{x_i\}, x_i$) ▷ We append $x_i$ at the end of the list $X \setminus \{x_i\}$, namely the list $X$ from which $x_i$ has previously been pruned. This way $x_i$ is the last (and so maximal) variable.
        **return** $ord$
    **end if**
    $C$ =COMMON($A, X, x_i, T$)
    **if** $C \neq \emptyset$ **then**
        $ord$ = Append($C, x_i$)
        **return** $ord$
    **else** **continue**
    **end if**
    **end for**
    **return** $\emptyset$
**end procedure**

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