An existence result for perturber (p, q)-quasilinear elliptic problems

An existence result for perturbed $(p,q)-$quasilinear elliptic problems

Rossella Bartolo, Anna Maria Candela and Addolorata Salvatore

Abstract We investigate the existence of solutions of the $(p,q)-$quasilinear elliptic problem

$$\begin{cases} -\Delta_p u - \Delta_q u \ = \ g(x,u) + \varepsilon\, h(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is an open bounded domain in $\mathbb{R}^N$, $1 < p < q < +\infty$, the nonlinearity $g(x,u)$ behaves at infinity as $|u|^{q-1}$, $\varepsilon \in \mathbb{R}$ and $h \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$. In spite of the possible lack of a variational structure of this problem, appropriate procedures and estimates allow us to prove the existence of at least one nontrivial solution for small perturbations.

2020 Mathematics Subject Classification. 35J92, 35P30, 47J30, 58E05.

Keywords and phrases. $(p,q)-$quasilinear elliptic equation, asymptotically q–linear problem, q–Laplacian, variational methods, essential value, perturbed problem, linking.

Dedicated to Francesco Altomare,
great mathematician and good friend

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Acknowledgements The research that led to the present paper was partially supported by MIUR–PRIN Research Project 2017JPCAPN "Qualitative and quantitative aspects of nonlinear PDEs" and by *Fondi di Ricerca di Ateneo* 2017/18 "Problemi differenziali non lineari".
All the authors are members of the Research Group INdAM–GNAMPA

1 Introduction

Classical semilinear and quasilinear equations can be perturbed just by adding continuous functions, with no assumption on their growth or their symmetry, so that their structure may lose its variational nature. More precisely, here we consider the following class of quasilinear elliptic problems:

$$(P_{\alpha,\varepsilon}) \qquad \begin{cases} -\alpha\Delta_p u - \Delta_q u \ = \ g(x,u) + \varepsilon\, h(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with $\alpha \in \{0,1\}$, $1 < p < q < +\infty$, $\Delta_r u = \operatorname{div}(|\nabla u|^{r-2}\nabla u)$ if $r \in \{p,q\}$, $\varepsilon \in \mathbb{R}$, where Ω is an open bounded domain in $\mathbb{R}^N$ with Lipschitz boundary $\partial\Omega$, $N \geq 2$, while $g(x,u)$ and $h(x,u)$ are given functions on $\Omega \times \mathbb{R}$.

If $\alpha = 0$ and $q = 2$, results on multiple solutions of $(P_{0,\varepsilon})$ are stated in [23] for $g(x,\cdot)$ odd, superlinear at infinity, but subcritical (see also [18] for related results). On the other hand, if $g(x,\cdot)$ is asymptotically linear at infinity and both $g(x,\cdot)$ and $h(x,\cdot)$ are odd, a multiplicity theorem is proved in [24, Theorem 1.6] while, by means of the pseudo–index theory stated in [8], in [6, Theorems 1.1, 1.2] existence results are obtained even in presence of resonance as the number of distinct critical values of J is stable under small odd perturbations. Moreover, again in [6], more restrictive multiplicity results are obtained for non–odd functions $h(x,\cdot)$ (see [6, Theorems 1.3, 1.4]).

To our knowledge, for $q \neq 2$ problem $(P_{0,\varepsilon})$ has been studied only in [24, Theorem 1.8], assuming both $g(x,\cdot)$ and $h(x,\cdot)$ odd but $g(x,\cdot)$ "superlinear".

When the variational structure on $W_0^{1,q}(\Omega)$ of the equation in $(P_{0,\varepsilon})$ fails, we use the notion of essential value for perturbations of non–smooth functionals as introduced in [16, 17]; indeed, such values are preserved for small perturbations of a continuous functional. We note that essential values of functionals satisfying the Palais–Smale condition (or its variants) are also critical ones, while the reverse implication does not hold; furthermore, critical values arising from mini–max procedures are essential ones (we refer to Subsection 2.3 for more details).

On the other hand, for $q > p = 2$ problem $(P_{1,0})$ has been studied in [11, 20, 30]; while if $g(x,\cdot)$ is asymptotically "$(q-1)$–linear" at infinity, i.e., there exists

$$\lim_{|t|\to+\infty} \frac{g(x,t)}{|t|^{q-2}t} = \lambda_\infty \in \mathbb{R} \quad \text{uniformly in } \overline{\Omega},$$

and the problem is not resonant, i.e., $\lambda_\infty \notin \sigma(-\Delta_q)$, we refer to [12] for the existence of a nontrivial solution via Morse theory and to [4] for a multiplicity result. A further multiplicity result for $(P_{1,0})$ is contained in the recent paper [14].

At last, we recall that the asymptotically $(q-1)$–linear problem $(P_{0,0})$ has been widely investigated both for $q = 2$ (cf. [1, 3, 6] and references therein) and for $q \neq 2$ (for some existence results see [5, 15, 18, 23, 25, 28] while for some multiplicity ones see [5, 26, 28]). Moreover, for more recent related results we refer to [13].

In this paper, we want to investigate the existence of solutions for problem $(P_{1,\varepsilon})$ when $g(x,\cdot)$ is asymptotically $(q-1)$–linear at infinity and a perturbation term is allowed. More precisely, we consider $\alpha = 1$ and that there exist $\lambda_\infty \in \mathbb{R}$ and $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ such that

$$g(x,t) \;=\; \lambda_\infty |t|^{q-2} t + f(x,t) \quad \text{for all } (x,t) \in \overline{\Omega} \times \mathbb{R}; \tag{1}$$

hence, problem $(P_{1,\varepsilon})$ reduces to

$$(P_\varepsilon^\infty) \qquad \begin{cases} -\Delta_p u - \Delta_q u = \lambda_\infty |u|^{q-2} u + f(x,u) + \varepsilon\, h(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

On function $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ we assume the following conditions:

(f_1) $f \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$;
(f_2) there exists

$$\lim_{|t| \to +\infty} \frac{f(x,t)}{|t|^{q-1}} \;=\; 0 \quad \text{uniformly in } \overline{\Omega};$$

(f_3) there exists

$$\lim_{t \to 0} \frac{f(x,t)}{|t|^{q-2} t} \;=\; \lambda_0 \in \mathbb{R} \setminus \{0\} \quad \text{uniformly in } \overline{\Omega}.$$

We note that, if assumption (f_1) is satisfied, then we can define the C^1 real function

$$F(x,t) \;=\; \int_0^t f(x,s)\, \mathrm{d}s \quad \text{for all } (x,t) \in \overline{\Omega} \times \mathbb{R} \tag{2}$$

which is so that $F(x,0) = 0$ for all $x \in \overline{\Omega}$.

The behaviour of the nonlinearity as in (1) calls for a control of the interaction of $g(x,t)$ with the spectrum of $\sigma(-\Delta_q)$ which is mostly unknown for $q \neq 2$. Such a problem was overcome in [5] for $(P_{0,0})$ by taking into account two sequences of quasi–eigenvalues for $-\Delta_q$ in $W_0^{1,q}(\Omega)$ defined as in [10, 26], namely $(\eta_m^0)_m$ and $(\nu_m^0)_m$ (see Subsection 2.1 for their definitions), while here we prefer to use two sequences of quasi–eigenvalues for the (p,q)–Laplacian operator, denoted by $(\eta_m)_m$ and $(\nu_m)_m$, which are introduced in [14] along the lines of [10, 26] (see Subsection 2.1 for more details).

Firstly, we state an existence result which deals with the unperturbed case (P_0^∞).

Theorem 1 *Assume that $(f_1) - (f_3)$ hold and $\lambda_\infty \notin \sigma(-\Delta_q)$. Let $k \in \mathbb{N}$ be such that*

(Λ_1) *an integer* $\bar{k} \geq k$ *exists such that one of following assumptions holds:*

(i) *we have that*

$$\lambda_0 + \lambda_\infty < \eta_k, \qquad \frac{q}{p}\nu_k < \lambda_\infty \tag{3}$$

and

$$\nu_{k-1} < \nu_k = \nu_{k+1} = \ldots = \nu_{\bar{k}} < \eta_{\bar{k}+1}; \tag{4}$$

(ii) *we have that*

$$\lambda_\infty < \eta_k^0, \qquad \frac{q}{p}\nu_k < \lambda_0 + \lambda_\infty \tag{5}$$

and

$$\nu_{k-1} < \nu_k = \nu_{k+1} = \ldots = \nu_{\bar{k}} < \eta^0_{\bar{k}+1}; \tag{6}$$

(Λ_2) *a constant* $\eta > 0$ *exists such that*

$$\left[\frac{1}{p}(\nu_{k-1} + \eta) - \frac{\lambda_\infty}{q}\right] |t|^q \leq F(x,t) \quad \textit{for all } (x,t) \in \overline{\Omega} \times \mathbb{R}.$$

Then, problem (P_0^∞) *has at least a nontrivial solution*

Then, we are able to state the following result concerning the perturbed case.

Theorem 2 *Let* $(f_1) - (f_3)$ *hold and assume that* $\lambda_\infty \notin \sigma(-\Delta_q)$ *and* $k \in \mathbb{N}$ *exists such that* $(\Lambda_1) - (\Lambda_2)$ *are satisfied. If* $h \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ *then* $\bar{\varepsilon} > 0$ *exists such that for all* $|\varepsilon| \leq \bar{\varepsilon}$ *problem* (P_ε^∞) *has at least one nontrivial solution.*

Remark 1 Theorem 1 holds even if we replace assumption (f_1) with the weaker hypothesis

$(f_1)'$ f is a Carathéodory function (i.e., $f(\cdot,t)$ is measurable in Ω for all $t \in \mathbb{R}$ and $f(x,\cdot)$ is continuous in $\mathbb{R}$ for a.e. $x \in \Omega$) and

$$\sup_{|t|\leq a} |f(\cdot,t)| \in L^\infty(\Omega) \qquad \text{for all } a > 0;$$

but such a replacement does not work in Theorem 2 if a perturbation term is involved.

It is worth to point out that $q > p$ is not an assumption; indeed, the roles of p and q are interchangeable. Moreover, it is understood that by a solution we mean a weak solution, i.e., a function $u \in W_0^{1,q}(\Omega)$ solving the problems in the sense of distributions. We notice also that, under our assumptions, such weak solutions belong to $C^{1,\beta}(\overline{\Omega})$ for some $\beta \in]0,1]$ (e.g., see [22, Remark 1.3]).

At last, we point out that the arguments we use for the proof of Theorem 2 still apply to the single q–Laplacian perturbed problem $(P_{0,\varepsilon})$; hence, being $\eta_m^0 \leq \nu_m^0$ for all $m \in \mathbb{N}$ (see [5, Proposition 2.9]), we obtain the following new existence result.

Corollary 1 *Assume that* $(f_1) - (f_3)$ *hold,* $\lambda_\infty \notin \sigma(-\Delta_q)$ *and* $h \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$. *Moreover, let* $k \in \mathbb{N}$ *be such that*

$(\Lambda_1)'$ *an integer* $\bar{k} \geq k$ *exists such that*

$$\min\{\lambda_0 + \lambda_\infty, \lambda_\infty\} \ < \ \eta_k^0 \ \leq \ \nu_k^0 \ < \ \max\{\lambda_0 + \lambda_\infty, \lambda_\infty\}$$

and

$$\nu_{k-1}^0 < \nu_k^0 = \nu_{k+1}^0 = \ldots = \nu_{\bar{k}}^0 < \eta_{\bar{k}+1}^0;$$

$(\Lambda_2)'$ *a constant* $\eta > 0$ *exists such that*

$$(\nu_{k-1}^0 + \eta - \lambda_\infty)\ \frac{|t|^q}{q} \ \leq \ F(x,t) \quad \textit{for all } (x,t) \in \overline{\Omega} \times \mathbb{R}.$$

Then, $\bar{\varepsilon} > 0$ *exists such that for all* $|\varepsilon| \leq \bar{\varepsilon}$ *problem* $(P_{0,\varepsilon})$ *has at least one nontrivial solution.*

Remark 2 (*a*) We note that, being $p < q$, in (3), respectively (5), it has to be $\eta_k < \frac{q}{p}\nu_k$, respectively $\eta_k^0 < \frac{q}{p}\nu_k$ (see Proposition 1). Therefore, the two conditions in (3) can be written as the chain of inequalities:

$$\lambda_0 + \lambda_\infty < \eta_k < \frac{q}{p}\nu_k < \lambda_\infty$$

so that it has to be $\lambda_0 < 0$. Similarly, (5) becomes

$$\lambda_\infty < \eta_k^0 < \frac{q}{p}\nu_k < \lambda_0 + \lambda_\infty,$$

and, then, it has to be $\lambda_0 > 0$. According to [1], we do not know whether an existence result holds for $\lambda_0 = 0$ or not.
(*b*) Instead of (3) and (5), we could state both Theorems 1 and 2 by requiring

$$\min\{\lambda_0 + \lambda_\infty, \lambda_\infty\} \ < \ \eta_k^0 \ < \ \frac{q}{p}\nu_k \ < \max\{\lambda_0 + \lambda_\infty, \lambda_\infty\}$$

but assuming

$$\eta_k^0 \ \leq \ \eta_k \quad \text{if } \lambda_0 < 0. \tag{7}$$

Anyway, even if there are not significant changes in the proof, the assumption (7) may be more restrictive since such inequality holds for $k = 1$, but we do not know if it is true also for other $k \geq 2$.
(*c*) All the existence results in Theorems 1, 2 and Corollary 1 hold also when the limit in (f_3) is infinite (see Propositions 3 and 4).

This paper is organized as follows: in Section 2 we present the tools we are going to use while in Sections 3 and 4 we prove Theorems 1 and 2, respectively.

Our strategy is the following: inspired by [23] (see also [6]) we prove the existence of at least one nontrivial solution of (P_0^∞) by using standard critical point theorems. Then, by means of cut–functions, we introduce perturbations of the functional associated to problem (P_0^∞) which have essential values near to the critical level of the solution for ε small enough. Such essential values turn out to be critical ones and suitable procedures - deeply different from those in [6], where the decomposition of $W_0^{1,2}(\Omega)$ by means of eigenvalues is exploited - allow us to prove that the L^∞–norm of the family of critical points of the cut–perturbed functionals is bounded. Hence, finally, solutions of (P_ε^∞) can be found.

2 Preliminary material

Throughout this paper, we will use the following notations:

- $\bar{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$;
- $(X, \|\cdot\|_X)$ Banach space with $(X', \|\cdot\|_{X'})$ its dual;
- $I : X \to \mathbb{R}$ C^1–functional;
- $I^b = \{u \in X : I(u) \le b\}$ sublevel of I corresponding to $b \in \mathbb{R}$;
- $I_b = \{u \in X : I(u) \ge b\}$ superlevel of I corresponding to $b \in \mathbb{R}$;
- $|\cdot|_s$ standard norm on the Lebesgue space $L^s(\Omega)$, $1 \le s \le +\infty$;
- $\|\cdot\|_q$ standard norm on $W_0^{1,q}(\Omega)$, i.e., $\|u\|_q = |\nabla u|_q$ for all $u \in W_0^{1,q}(\Omega)$;
- $(W^{-1,q'}(\Omega), \|\cdot\|_{W^{-1,q'}})$ dual space of $W_0^{1,q}(\Omega)$;
- $q^* = \frac{qN}{N-q}$ if $q < N$, $q^* = +\infty$ otherwise;
- $B_R = \{u \in W_0^{1,q}(\Omega) : \|u\|_q < R\}$ for any $R > 0$;
- $\overline{B}_R = \{u \in W_0^{1,q}(\Omega) : \|u\|_q \le R\}$ for any $R > 0$;
- $S_R = \{u \in W_0^{1,q}(\Omega) : \|u\|_q = R\}$ for any $R > 0$.

Moreover, by K_j, $j \in \mathbb{N}$, we denote any positive constant which appears in the proofs and, for simplicity, we denote by $(\beta_m)_m$ any infinitesimal sequence which depends only on a given sequence of functions and by $(\beta_m(\varphi))_m$ any infinitesimal sequence which depends also on a fixed function φ.

2.1 Quasi–eigenvalues for the operator $-\Delta_p - \Delta_q$

It is well known that, if $q = 2$, the spectrum $\sigma(-\Delta_2)$ of $-\Delta_2$ in $W_0^{1,2}(\Omega)$ consists of a diverging sequence $(\lambda_m)_m$ of eigenvalues, repeated according to their multiplicity, so that

$$0 < \lambda_1 < \lambda_2 \le \ldots \le \lambda_m \le \ldots,$$

which furnishes a decomposition of the Hilbert space $W_0^{1,2}(\Omega)$. Then, denoting by $(\varphi_m)_m$ the sequence of the corresponding eigenfunctions, for each $m \in \mathbb{N}$ the following inequalities hold:

$$\lambda_m |u|_2^2 \ge |\nabla u|_2^2 \quad \text{for all } u \in V_m$$

and

$$\lambda_m |u|_2^2 \le |\nabla u|_2^2 \quad \text{for all } u \in W_{m-1}$$

with

$$V_m = \text{span}\{\varphi_1, \dots, \varphi_m\}, \qquad W_m = V_m^\perp.$$

Instead, in the quasilinear case $q \neq 2$, the spectral properties of the q–Laplacian $-\Delta_q$ in the Sobolev space $W_0^{1,q}(\Omega)$ are still mostly unknown; indeed, when $N \geq 2$ it is not known whether the unbounded and increasing sequences of eigenvalues in [2, 21, 27, 28] cover the whole spectrum $\sigma(-\Delta_q)$ of $-\Delta_q$ in $W_0^{1,q}(\Omega)$ or not. Furthermore, unlike the case $q = 2$, the eigenvalues do not furnish a decomposition of the Banach space $W_0^{1,q}(\Omega)$. For these reasons in the $(q-1)$–asymptotically linear case we are studying, it is useful to consider two sequences of quasi–eigenvalues (cf. [5, Section 2]).

The first eigenvalue of $-\Delta_q$, denoted by $\lambda_1^{(q)}$, is characterized by

$$\lambda_1^{(q)} = \inf_{u \in W_0^{1,q}(\Omega)\setminus\{0\}} \frac{|\nabla u|_q^q}{|u|_q^q}$$

and is positive, simple, isolated with a unique positive eigenfunction $\varphi_1^{(q)}$ having unitary L^q–norm (cf., e.g., [27]).

In [10, Section 5], starting from $\eta_1^0 = \lambda_1^{(q)}$ and $\psi_1^0 \equiv \varphi_1^{(q)}$, it is shown the existence of an increasing diverging sequence $(\eta_m^0)_m$ of positive real numbers and a corresponding sequence of functions $(\psi_m^0)_m$, with $\psi_m^0 \neq \psi_n^0$ if $m \neq n$, such that

$$|\psi_m^0|_q = 1 \quad \text{and} \quad \eta_m^0 = |\nabla \psi_m^0|_q^q \qquad \text{for all } m \in \mathbb{N}. \tag{8}$$

Moreover, such a sequence generates the whole space $W_0^{1,q}(\Omega)$ and is such that

$$W_0^{1,q}(\Omega) = Y_m^0 \oplus Z_m^0 \qquad \text{for all } m \in \mathbb{N},$$

where $Y_m^0 = \text{span}\{\psi_1^0, \dots, \psi_m^0\}$ and its complement Z_m^0 can be explicitely described.

We recall that if $Y \subseteq X$ is a closed subspace of a Banach space X, a subspace $Z \subseteq X$ is a *topological complement* of Y, briefly $X = Y \oplus Z$, if Z is closed and every $x \in X$ can be uniquely written as $y + z$, with $y \in Y$ and $z \in Z$; furthermore, the projection operators onto Y and Z are (linear and) continuous and $L = L(Y, Z) > 0$ exists such that

$$\|y\| + \|z\| \leq L\|y + z\| \quad \text{for all } y \in Y,\, z \in Z \tag{9}$$

(see, e.g., [9, p. 38]).

Remarkably, for all $m \in \mathbb{N}$ on the infinite dimensional subspace Z_{m-1} the following inequality holds:

$$\eta_m^0 |u|_q^q \leq |\nabla u|_q^q \qquad \text{for all } u \in Z_{m-1}^0 \tag{10}$$

(cf. [10, Lemma 5.4]).

Unluckily, it is not known whether, by making use of this sequence of quasi–eigenvalues, a reversed inequality holds on finite dimensional subspaces. Then, as in [26], we define another sequence of quasi–eigenvalues. More precisely, for all $m \in \mathbb{N}$ we consider the set of subspaces

$$\mathbb{W}_m^0 = \{Y \subset W_0^{1,q}(\Omega) : \ Y \text{ is a subspace of } W_0^{1,q}(\Omega), \varphi_1^{(q)} \in Y \text{ and } \dim Y \geq m\}$$

and define

$$\nu_m^0 = \inf_{Y \in \mathbb{W}_m^0} \sup_{u \in Y \setminus \{0\}} \frac{|\nabla u|_q^q}{|u|_q^q}.$$

The main properties of such a sequence are the following: $\nu_1^0 = \lambda_1^{(q)}$, $(\nu_m^0)_m$ is an increasing diverging sequence and, if $q = 2$, it agrees with $(\lambda_m)_m$ (cf. [26]). Furthermore, as already pointed out in Section 1,

$$\eta_m^0 \leq \nu_m^0 \quad \text{for all } m \in \mathbb{N}$$

(see [5, Proposition 2.9])

Moreover, since here we deal with (p, q)–Laplacian problems, it is convenient to use also two sequences of quasi–eigenvalues for the operator $-\Delta_p - \Delta_q$ with zero Dirichlet boundary conditions as introduced in [14, Subsection 2.3] where, overcoming the lack of homogeneity, the previous costructions are extended to the (p, q)–Laplacian operator. More precisely, starting from

$$\eta_1 \ := \inf_{\substack{u \in W_0^{1,q}(\Omega) \\ |u|_q = 1}} \left(|\nabla u|_p^p + |\nabla u|_q^q\right) \ \geq \ \lambda_1^{(q)},$$

attained by a function $\psi_1 \in W_0^{1,q}(\Omega)$ with $|\psi_1|_q = 1$, it is defined an increasing, diverging sequence $(\eta_m)_m$ of positive real numbers and a corresponding sequence of functions $(\psi_m)_m \subset W_0^{1,q}(\Omega)$ such that $\psi_m \neq \psi_n$ if $m \neq n$ and

$$|\psi_m|_q = 1 \quad \text{and} \quad \eta_m = |\nabla \psi_m|_p^p + |\nabla \psi_m|_q^q \qquad \text{for all } m \in \mathbb{N}. \tag{11}$$

As shown in [14, Lemma 2.6], these functions generate the whole space $W_0^{1,q}(\Omega)$ and for all $m \in \mathbb{N}$ it results

$$W_0^{1,q}(\Omega) = Y_m \oplus Z_m, \tag{12}$$

with $Y_m = \text{span}\{\psi_1, \dots, \psi_m\}$ and Z_m its topological complement, and the following inequalities hold:

$$\eta_m |u|_q^q \leq |\nabla u|_p^p + |\nabla u|_q^q \quad \text{ for all } u \in Z_{m-1} \cap \{u \in W_0^{1,q}(\Omega) : |u|_q \leq 1\} \tag{13}$$

and

$$\eta_m |u|_q^p \leq |\nabla u|_p^p + |\nabla u|_q^q \quad \text{ for all } u \in Z_{m-1} \setminus \{u \in W_0^{1,q}(\Omega) : |u|_q \leq 1\}.$$

On the other hand, in order to deal with finite dimensional spaces, for all $m \in \mathbb{N}$ we set

$$\mathbb{W}_m = \{Y \subset W_0^{1,q}(\Omega) : \ Y \text{ subspace of } W_0^{1,q}(\Omega), \ \psi_1 \in Y \text{ and } \dim Y \geq m\} \quad (14)$$

and

$$\nu_m = \inf_{Y \in \mathbb{W}_m} \sup_{u \in Y \setminus \{0\}} \frac{|\nabla u|_p^p + |\nabla u|_q^q}{|u|_q^q}. \quad (15)$$

Again, $(\nu_m)_m$ is increasing and a comparison between such a sequence and the previous ones can be established.

Proposition 1 *If $(\eta_m^0)_m$, $(\eta_m)_m$ and $(\nu_m)_m$ are sequences of quasi–eigenvalues defined as above, then it results*

$$\eta_m^0 \leq \nu_m, \quad \eta_m \leq \nu_m \qquad \textit{for all } m \in \mathbb{N}.$$

Proof Fixing $m \in \mathbb{N}$, inequality (10) holds on Z_{m-1}^0, with $\mathrm{codim}\ Z_{m-1}^0 = m - 1$, while taking any $\sigma > 0$ from (15) a subspace $Y \in \mathbb{W}_m$ exists such that

$$\sup_{u \in Y \setminus \{0\}} \frac{|\nabla u|_p^p + |\nabla u|_q^q}{|u|_q^q} < \nu_m + \sigma.$$

Therefore, since (14) implies $\dim Y \geq m$, an element $\bar{u} \in \left(Y \cap Z_{m-1}^0\right) \setminus \{0\}$ exists such that

$$\eta_m^0 \leq \frac{|\nabla \bar{u}|_q^q}{|\bar{u}|_q^q} < \frac{|\nabla \bar{u}|_p^p + |\nabla \bar{u}|_q^q}{|\bar{u}|_q^q} < \nu_m + \sigma.$$

Hence, being σ arbitrary, it has to be $\eta_m^0 \leq \nu_m$.
On the other hand, fixing any $\sigma > 0$, reasoning as before but from (13)–(15), an element $\bar{v} \in (Y \cap Z_{m-1}) \setminus \{0\}$ exists, with $|\bar{v}|_q = 1$, which gives $\eta_m < \nu_m + \sigma$ so that, again for the arbitrariness of σ, it follows $\eta_m \leq \nu_m$. □

2.2 Variational tools

In what follows we recall widely known definitions and results which apply to (P_0^∞) under our assumptions.

Firstly, we recall that a functional I satisfies the *Palais–Smale condition at level* c, $c \in \mathbb{R}$, briefly $(PS)_c$, if any sequence $(u_m)_m \subseteq X$ such that

$$\lim_{m \to +\infty} I(u_m) = c \quad \text{and} \quad \lim_{m \to +\infty} \|dI(u_m)\|_{X'} = 0$$

converges in X, up to subsequences.

If $-\infty \leq a < b \leq +\infty$, we say that I satisfies (PS) in $]a, b[$ if so is at each level $c \in]a, b[$.

Then, in order to state a classical existence critical point theorem, we recall the definition of sets which link as follows (e.g., see [31, Section II.8]).

Definition 1 Taking a subspace Y of X, let $S \subseteq X$ be a closed subset of X and consider $Q \subseteq Y$ with boundary ∂Q with respect to Y. Then, S and ∂Q *link* if

- $S \cap \partial Q = \emptyset$,
- $\phi(Q) \cap S \neq \emptyset$ for any $\phi \in C(X, X)$ such that $\phi\big|_{\partial Q} = \mathrm{id}$.

For further use, we recall two examples of linking sets (cf. [31, Examples II.8.2 and II.8.3] and also [3, Propositions 2.1 and 2.2] in the case of an Hilbert space).

***Example* 1** Let V, W be two closed subspaces of X such that $X = V \oplus W$ and $\dim V < +\infty$. Then, setting $Q = \overline{B}_R \cap V$ for $R > 0$ and $S = W$, we have that S and ∂Q link. □

***Example* 2** Let V, W be two closed subspaces of X such that $X = V \oplus W$, $\dim V < +\infty$, and fix $e \in W$ with $\|e\|_X = 1$. If R_1, R_2, $\rho > 0$ and

$$S = S_\rho \cap W, \qquad Q = \{te : t \in [0, R_1]\} \oplus \left(\overline{B}_{R_2} \cap V\right), \qquad Y = V \oplus \mathrm{span}\{e\},$$

then S and ∂Q link whenever $R_1 > \rho$. □

The following linking theorem holds (cf., e.g., [3, Theorem 2.3] with the weaker Cerami's variant of Palais–Smale condition or [32, Theorem 2.12]).

Theorem 3 *Consider* $a, b, \alpha, \beta \in \bar{\mathbb{R}}$ *such that* $a < \alpha < \beta < b$. *Assume that:*

(i) the functional I *satisfies* (PS) *in* $]a, b[$;
(ii) two subsets S *and* Q *exist such that* S *is closed in* X, $Q \subseteq Y$, *with* Y *subspace of* X *and* ∂Q *boundary of* Q *in* Y, *and the following assumptions are satisfied:*

(a) $I(u) \leq \alpha$ *for all* $u \in \partial Q$ *and* $I(u) \geq \beta$ *for all* $u \in S$;
(b) S *and* ∂Q *link;*
(c) $\sup_{u \in Q} I(u) < +\infty$.

Then, a critical level c *of* I *exists and is given by*

$$c = \inf_{\phi \in \Gamma} \sup_{u \in Q} I(\phi(u)), \quad \text{with} \quad \beta \leq c \leq \sup_{u \in Q} I(u),$$

where $\Gamma = \left\{\phi \in C(X, X) : \phi\big|_{\partial Q} = \mathrm{id}\right\}$.

2.3 Essential values

As already pointed out, we may deal with problems without a variational structure on $W_0^{1,q}(\Omega)$. Hence, following [23], we use the auxiliary notion of essential value as introduced in [17] for the study of perturbations of nonsmooth functionals (see also [16]). We note that: the notion of essential value is topological, an essential value is candidate to be a critical level and is stable under small perturbations, critical levels arising from standard mini–max procedures are essential ones.

Definition 2 Let $I : X \to \mathbb{R}$ be continuous and $a, b \in \bar{\mathbb{R}}$, with $a \leq b$. The pair (I^b, I^a) is *trivial* if, for each neighbourhood $[\alpha', \alpha'']$ of a and $[\beta', \beta'']$ of b in $\bar{\mathbb{R}}$, a continuous map $\varphi : I^{\beta'} \times [0,1] \to I^{\beta''}$ exists such that

(i) $\varphi(x,0) = x$ for each $x \in I^{\beta'}$;
(ii) $\varphi(I^{\beta'} \times \{1\}) \subseteq I^{\alpha''}$;
(iii) $\varphi(I^{\alpha'} \times [0,1]) \subseteq I^{\alpha''}$.

Since the lack of critical values for a smooth functional may give trivial pairs (see the proof of [17, Theorem 3.1]), the following definition allows one to locate possible critical levels.

Definition 3 Let $I : X \to \mathbb{R}$ be a continuous function. A real number c is an *essential value* of I if for each $\varepsilon > 0$ two values $a, b \in]c - \varepsilon, c + \varepsilon[$, $a < b$, exist such that the pair (I^b, I^a) is not trivial.

The following theorem states that small perturbations of a continuous functional preserve the essential values (cf. [17, Theorem 3.1] or also [16, Theorem 2.6]).

Theorem 4 *Let $c \in \mathbb{R}$ be an essential value of the continuous function $I : X \to \mathbb{R}$. Then, for every $\eta > 0$ a constant $\delta > 0$ exists such that every functional $G \in C(X, \mathbb{R})$ with*

$$\sup\{|I(u) - G(u)| : u \in X\} < \delta$$

admits an essential value in $]c - \eta, c + \eta[$.

Now, we focus on the setting of smooth functionals and recall some results which link critical and essential values, stating in particular that the critical values arising from mini–max procedures are essential, provided that all the involved deformations are of the "same kind" (see [17, Theorems 3.7 and 3.9]).

Theorem 5 *Let $c \in \mathbb{R}$ be an essential value of $I \in C^1(X, \mathbb{R})$. If $(PS)_c$ holds, then c is a critical value of I.*

Remark 3 In general, the reverse implication does not hold when the Palais–Smale condition is satisfied since a critical value is not necessarily an essential one (see, e.g., [17, Example 3.12]).

Theorem 6 *Taking $I \in C^1(X, \mathbb{R})$, assume that Γ, non empty family of non empty subsets of X, and $d \in \mathbb{R} \cup \{-\infty\}$ are such that*

$$\overline{\varphi(C \times \{1\})} \in \Gamma$$

for every $C \in \Gamma$ and for every continuous deformation $\varphi : X \times [0,1] \longrightarrow X$ with $\varphi(u,t) = u$ on $I^d \times [0,1]$. Then, setting

$$c = \inf_{C\in\Gamma} \sup_{u\in C} I(u),$$

if $d < c < +\infty$ *we have that* c *is an essential value of* I*.*

3 The unperturbed case

As announced in Section 1, at first we deal with the unperturbed problem; it has a variational structure and here we present the variational framework needed in order to study it. Let us consider

$$(P_0^\infty) \qquad \begin{cases} -\Delta_p u - \Delta_q u = \lambda_\infty |u|^{q-2}u + f(x,u) & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

To this aim, we note that from (f_1) and (f_2) for all $\sigma > 0$ a constant $K_\sigma > 0$ exists such that

$$|f(x,t)| \le \sigma |t|^{q-1} + K_\sigma \quad \text{for all } (x,t) \in \overline{\Omega}\times\mathbb{R}. \tag{16}$$

Hence, taking $F(x,t)$ as in (2), classical variational theorems imply that the weak solutions of problem (P_0^∞) are the critical points of the C^1–functional

$$J(u) = \frac{1}{p}\int_\Omega |\nabla u|^p \, dx + \frac{1}{q}\int_\Omega |\nabla u|^q \, dx - \frac{\lambda_\infty}{q}\int_\Omega |u|^q \, dx - \int_\Omega F(x,u)\, dx \tag{17}$$

on $W_0^{1,q}(\Omega)$, with

$$\begin{aligned} \langle dJ(u), v\rangle = &\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla v \, dx + \int_\Omega |\nabla u|^{q-2}\nabla u\cdot\nabla v \, dx\\ &- \lambda_\infty \int_\Omega |u|^{q-2}u\, v \, dx - \int_\Omega f(x,u)v \, dx \end{aligned} \tag{18}$$

for all u, $v \in W_0^{1,q}(\Omega)$ (see, e.g., [19, Theorem 9 and p. 355]).

Now, we prove that the functional J satisfies the Palais–Smale condition (cf. also [4, Proposition 3.1]). We point out that here assumption (f_3), i.e. the behaviour of f near 0, is not needed, while it will be crucial in order to obtain the geometric assumptions required in the linking theorem (we refer to [14, Lemma 3.2] for the proof in the resonant case under an additional assumption as in [26]).

Proposition 2 *Assume that* (f_1)*–*(f_2) *hold and* $\lambda_\infty \notin \sigma(-\Delta_q)$*. Then, the functional* J *in* (17) *satisfies* (PS) *in* $\mathbb{R}$*.*

Proof Taking $c \in \mathbb{R}$, let $(u_m)_m$ be a sequence in $W_0^{1,q}(\Omega)$ such that

$$\lim_{m\to+\infty} J(u_m) = c \quad \text{and} \quad \lim_{m\to+\infty} \|dJ(u_m)\|_{W^{-1,q'}} = 0. \tag{19}$$

Firstly, we note that from (18) and (19) taking any $\varphi \in W_0^{1,q}(\Omega)$ it has to be

$$\left|\int_\Omega |\nabla u_m|^{p-2}\nabla u_m \cdot \nabla\varphi \, dx + \int_\Omega |\nabla u_m|^{q-2}\nabla u_m \cdot \nabla\varphi \, dx \right.$$
$$\left. - \lambda_\infty \int_\Omega |u_m|^{q-2}u_m \,\varphi \, dx - \int_\Omega f(x, u_m)\varphi \, dx\right| \le \beta_m \|\varphi\|_q. \tag{20}$$

Then, since it is enough to show that $(\|u_m\|_q)_m$ is bounded (cf., e.g., [19, Lemma 2]), arguing by contradiction we assume that, up to subsequences, it is

$$\|u_m\|_q \to +\infty \quad \text{as } m \to +\infty. \tag{21}$$

Thus, without loss of generality, for all $m \in \mathbb{N}$ we can consider $\|u_m\|_q > 0$ and set

$$w_m = \frac{u_m}{\|u_m\|_q} \quad \text{with, clearly, } \|w_m\|_q = 1. \tag{22}$$

So, being $(w_m)_m$ bounded in $W_0^{1,q}(\Omega)$, an element $w \in W_0^{1,q}(\Omega)$ exists such that, up to subsequences, we have

$$w_m \rightharpoonup w \quad \text{weakly in } W_0^{1,q}(\Omega), \tag{23}$$

$$w_m \to w \quad \text{strongly in } L^q(\Omega). \tag{24}$$

Now, replacing φ in (20) with $\varphi_m = \frac{w_m - w}{\|u_m\|_q^{q-1}}$, as (21) and (22) imply $\|\varphi_m\|_q \to 0$, we get

$$\int_\Omega \frac{|\nabla w_m|^{p-2}}{\|u_m\|_q^{q-p}} \nabla w_m \cdot \nabla(w_m - w) \, dx + \int_\Omega |\nabla w_m|^{q-2}\nabla w_m \cdot \nabla(w_m - w) \, dx$$
$$= \lambda_\infty \int_\Omega |w_m|^{q-2}w_m \,(w_m - w) \, dx + \int_\Omega \frac{f(x, u_m)}{\|u_m\|_q^{q-1}}(w_m - w) \, dx + \beta_m,$$

where from Hölder inequality and (24) it follows that

$$\left|\int_\Omega |w_m|^{q-2}w_m(w_m - w) \, dx\right| \le |w_m|_q^{q-1}|w_m - w|_q = \beta_m,$$

while (16), (21) and, again, (24) imply that

$$\left|\int_\Omega \frac{f(x, u_m)}{\|u_m\|_q^{q-1}}(w_m - w) \, dx\right| \le \sigma|w_m|_q^{q-1}|w_m - w|_q + \frac{K_\sigma}{\|u_m\|_q^{q-1}}|w_m - w|_1 = \beta_m,$$

and, since $W_0^{1,q}(\Omega) \subset W_0^{1,p}(\Omega)$ being $q > p > 1$, from (21), (22) and direct computations we have that

$$\left|\int_\Omega \frac{|\nabla w_m|^{p-2}}{\|u_m\|_q^{q-p}}\ \nabla w_m \cdot \nabla(w_m - w)\ \mathrm{d}x\right| \ \le\ \frac{1}{\|u_m\|_q^{q-p}}\ |\nabla w_m|_p^{p-1} |\nabla(w_m - w)|_p$$
$$\le\ \frac{K_1}{\|u_m\|_q^{q-p}}\ \|w_m\|_q^{p-1} \|w_m - w\|_q\ \le\ \frac{K_2}{\|u_m\|_q^{q-p}}\ =\beta_m.$$

Hence, from all the previous estimates we obtain

$$\int_\Omega |\nabla w_m|^{q-2} \nabla w_m \cdot (\nabla w_m - \nabla w)\ \mathrm{d}x = \beta_m,$$

which, together with (23), implies

$$w_m \to w \qquad \text{strongly in } W_0^{1,q}(\Omega) \tag{25}$$

(see [19, Theorem 10]) with $w \neq 0$ from definition (22).
Now, taking any $\varphi \in W_0^{1,q}(\Omega)$ and dividing (20) by $\|u_m\|_q^{q-1}$, we have that

$$\begin{aligned}\int_\Omega \frac{|\nabla w_m|^{p-2}}{\|u_m\|_q^{q-p}} \nabla w_m \cdot \nabla\varphi\ \mathrm{d}x + \int_\Omega |\nabla w_m|^{q-2} \nabla w_m \cdot \nabla\varphi\ \mathrm{d}x \\ =\ \lambda_\infty \int_\Omega |w_m|^{q-2} w_m\ \varphi\ \mathrm{d}x\ + \int_\Omega \frac{f(x,u_m)}{\|u_m\|_q^{q-1}} \varphi\ \mathrm{d}x + \beta_m(\varphi),\end{aligned} \tag{26}$$

where, by reasoning as before, (21) and (22) imply

$$\left|\int_\Omega \frac{|\nabla w_m|^{p-2}}{\|u_m\|_q^{q-p}} \nabla w_m \cdot \nabla\varphi\ \mathrm{d}x\right| \le \frac{K_3}{\|u_m\|_q^{q-p}}\ \|\varphi\|_q\ =\ \beta_m\ \|\varphi\|_q. \tag{27}$$

We claim that

$$\lim_{m\to+\infty} \int_\Omega \frac{f(x,u_m)}{\|u_m\|_q^{q-1}} \varphi\ \mathrm{d}x = 0. \tag{28}$$

In fact, taking any $\varepsilon > 0$, since from (24) we have $|w_m|_q^{q-1} \le K_4$ for all $m \in \mathbb{N}$, for the arbitrariness of the possible choice of $\sigma > 0$ in (16), we can fix $\sigma = \frac{\varepsilon}{2K_4(\|\varphi\|_q+1)}$ and, for the corresponding K_σ in (16), from (21) an integer $\bar{m} \ge 1$ exists such that

$$\frac{K_\sigma |\varphi|_1}{\|u_m\|_q^{q-1}} < \frac{\varepsilon}{2} \quad \text{for all } m \ge \bar{m}.$$

Thus, from (16), Hölder inequality, all the previous estimates and direct computations it follows that

$$\left|\int_\Omega \frac{f(x,u_m)}{\|u_m\|_q^{q-1}} \varphi\ \mathrm{d}x\right|\ \le\ \sigma |w_m|_q^{q-1} |\varphi|_q + \frac{K_\sigma |\varphi|_1}{\|u_m\|_q^{q-1}} < \varepsilon \quad \text{for all } m \ge \bar{m}.$$

Therefore, by means of (27) and (28), passing to the limit in (26), from (24) and (25) it follows that

$$\int_\Omega |\nabla w|^{q-2}\nabla w \cdot \nabla\varphi \, \mathrm{d}x = \lambda_\infty \int_\Omega |w|^{q-2} w\varphi \, \mathrm{d}x.$$

For the arbitrariness of $\varphi \in W_0^{1,q}(\Omega)$ such an equality means that $\lambda_\infty \in \sigma(-\Delta_q)$, in contradiction with our nonresonant assumption. □

Now, in order to prove the existence result for the unperturbed problem (P_0^∞), we need the following technical lemma.

Lemma 1 *Assume that assumptions* (f_1)–(f_3) *are satisfied and consider* $F(x,t)$ *as in* (2). *Then, for any* $\sigma > 0$ *and* $s > 0$ *a constant* $k_0^\sigma > 0$, $k_0^\sigma = k_0^\sigma(s)$, *exists such that*

$$-k_0^\sigma |t|^{s+q} + \frac{\lambda_0 - \sigma}{q} |t|^q \ \le \ F(x,t) \ \le \ \frac{\lambda_0 + \sigma}{q} |t|^q + k_0^\sigma |t|^{s+q} \tag{29}$$

for all $(x,t) \in \overline{\Omega} \times \mathbb{R}$.

Proof From (f_3) it follows that

$$\lim_{t\to 0} \frac{F(x,t)}{|t|^q} \ = \ \frac{\lambda_0}{q} \quad \text{uniformly in } \overline{\Omega}.$$

Therefore, taking any $\sigma > 0$ a constant $\delta_\sigma > 0$ exists such that

$$\left| F(x,t) - \frac{\lambda_0}{q}|t|^q \right| \le \frac{\sigma}{q}|t|^q \qquad \text{for all } x \in \overline{\Omega} \text{ if } |t| < \delta_\sigma. \tag{30}$$

On the other hand, from (f_2) we have that

$$\lim_{|t|\to+\infty} \frac{F(x,t)}{|t|^q} \ = \ 0 \quad \text{uniformly in } \overline{\Omega},$$

so, taking any $s > 0$, it results

$$\lim_{|t|\to+\infty} \frac{\left| F(x,t) - \frac{\lambda_0}{q}|t|^q \right|}{|t|^{q+s}} = 0 \qquad \text{uniformly in } \overline{\Omega}.$$

From this last limit, (f_1) and direct computations a constant $k_0^\sigma > 0$ exists such that

$$\frac{\left| F(x,t) - \frac{\lambda_0}{q}|t|^q \right|}{|t|^{s+q}} \ \le \ k_0^\sigma \qquad \text{for all } x \in \overline{\Omega} \text{ if } |t| \ge \delta_\sigma. \tag{31}$$

Hence, from (30) and (31) it follows (29). □

Proof (Proof of Theorem 1.) From Proposition 2 functional J in (17) satisfies (PS) in $\mathbb{R}$. Next, we distinguish the proof according to hypothesis (Λ_1) if either case (i) or case (ii) occurs.

Case (i) Taking η as in (Λ_2), from $(\Lambda_1)(i)$ a constant $\sigma \in]0, \eta[$ exists such that

$$\lambda_0 + \lambda_\infty + \sigma < \eta_k, \quad \frac{q}{p}\,(\nu_k + 2\sigma) < \lambda_\infty, \quad \nu_{\bar{k}} + \sigma < \eta_{\bar{k}+1}. \tag{32}$$

Let us recall that in this setting it has to be $\lambda_0 < 0$ (see Remark 2(a)). Moreover, from (14) and (15) with $m = k-1$, a subspace $Y^\sigma_{k-1} \in \mathbb{W}_{k-1}$ exists such that

$$\sup_{u \in Y^\sigma_{k-1}\setminus\{0\}} \frac{|\nabla u|_p^p + |\nabla u|_q^q}{|u|_q^q} < \nu_{k-1} + \sigma. \tag{33}$$

Without loss of generality, it can be chosen so that dim $Y^\sigma_{k-1} = k-1$.
We claim that

$$J(u) \le 0 \qquad \text{for all } u \in Y^\sigma_{k-1}. \tag{34}$$

Indeed, from (33) and (Λ_2) we get

$$\begin{aligned} J(u) &\le \frac{1}{p}\left(|\nabla u|_p^p + |\nabla u|_q^q\right) - \frac{\lambda_\infty}{q}|u|_q^q - \int_\Omega F(x,u)\,\mathrm{d}x \\ &\le \left[\frac{1}{p}\,(\nu_{k-1} + \sigma) - \frac{\lambda_\infty}{q}\right]|u|_q^q - \int_\Omega F(x,u)\,\mathrm{d}x \\ &\le \left[\frac{1}{p}(\nu_{k-1} + \eta) - \frac{\lambda_\infty}{q}\right]|u|_q^q - \int_\Omega F(x,u)\,\mathrm{d}x \;\le\; 0 \qquad \text{for all } u \in Y^\sigma_{k-1}. \end{aligned}$$

On the other hand, from (12) with $m = k-1$, we have that $W_0^{1,q}(\Omega) = Y_{k-1} \oplus Z_{k-1}$, where $Y_{k-1} = \mathrm{span}\{\psi_1, \ldots, \psi_{k-1}\}$ and Z_{k-1} is its complement.
We prove that $\rho > 0$ and $\beta > 0$ exist such that

$$J(u) \ge \beta \qquad \text{for all } u \in Z_{k-1} \cap S_\rho. \tag{35}$$

Indeed, taking σ as above and fixing any $s > 0$ such that $s + q < q^*$, from (29) it follows that

$$\int_\Omega F(x,u)\,\mathrm{d}x \;\le\; \frac{\lambda_0 + \sigma}{q}|u|_q^q + k_0^\sigma |u|_{s+q}^{s+q} \quad \text{for all } u \in W_0^{1,q}(\Omega),$$

which, together with the Sobolev Embedding Theorem, implies the existence of a suitable $k_1^\sigma > 0$ such that

$$J(u) \ge \frac{1}{q}\left(|\nabla u|_p^p + |\nabla u|_q^q\right) - \frac{\lambda_\infty + \lambda_0 + \sigma}{q}|u|_q^q - k_1^\sigma |\nabla u|_q^{s+q} \quad \text{for all } u \in W_0^{1,q}(\Omega).$$

Now, from this last estimate and (13) with $m = k$ we obtain that

$$\begin{aligned} J(u) &\geq \frac{1}{q}\left(|\nabla u|_p^p + |\nabla u|_q^q\right) - \frac{1}{q}\frac{\lambda_\infty + \lambda_0 + \sigma}{\eta_k}\left(|\nabla u|_p^p + |\nabla u|_q^q\right) - k_1^\sigma |\nabla u|_q^{s+q} \\ &\geq \frac{1}{q}\left(1 - \frac{\lambda_\infty + \lambda_0 + \sigma}{\eta_k}\right)|\nabla u|_q^q - k_1^\sigma |\nabla u|_q^{s+q} \end{aligned}$$

for all $u \in Z_{k-1} \cap \{u \in W_0^{1,q}(\Omega) : \ |u|_q \leq 1\}$. Then, this last inequality together with (32), implies that

$$J(u) \geq k_2^\sigma \|u\|_q^q - k_1^\sigma \|u\|_q^{s+q} \quad \text{for all } u \in Z_{k-1} \cap \{u \in W_0^{1,q}(\Omega) : \ |u|_q \leq 1\}$$

for a suitable $k_2^\sigma > 0$. Hence, since $s > 0$, taking $\rho > 0$ small enough such that not only from the Sobolev Embedding Theorem $u \in S_\rho$ gives $|u|_q \leq 1$ but also $k_2^\sigma \rho^q - k_1^\sigma \rho^{s+q} > 0$, a constant $\beta > 0$ exists such that (35) holds.
Now, we claim that

$$W_0^{1,q}(\Omega) = Y_{k-1}^\sigma \oplus Z_{k-1}, \tag{36}$$

that is, Y_{k-1} actually is Y_{k-1}^σ. To this aim, firstly we prove that $Y_{k-1}^\sigma \cap Z_{k-1} = \{0\}$. Otherwise, $\bar{u} \in Z_{k-1} \cap Y_{k-1}^\sigma$ exists such that $\bar{u} \neq 0$ and, taking ρ as in (35), it has to be

$$u = \rho \frac{\bar{u}}{\|\bar{u}\|_q} \in Y_{k-1}^\sigma \cap (Z_{k-1} \cap S_\rho),$$

which yields a contradiction as the same u has to satisfy both (34) and (35). Then, $Y_{k-1}^\sigma \subset Y_{k-1}$ and, since the two subspaces have the same dimension, they coincide and (36) follows.
Furthermore, again from (15) but with $m = \bar{k}$, where $\bar{k}$ is as in (4), a subspace $Y^\sigma \in \mathbb{W}_{\bar{k}}$ exists such that $\dim Y^\sigma = \bar{k}$ and

$$\sup_{u \in Y^\sigma \setminus \{0\}} \frac{|\nabla u|_p^p + |\nabla u|_q^q}{|u|_q^q} < \nu_{\bar{k}} + \sigma. \tag{37}$$

Let us show that

$$Y^\sigma = \operatorname{span}\{\psi_1, \ldots, \psi_{\bar{k}}\} = Y_{k-1}^\sigma \oplus \operatorname{span}\{\psi_k, \ldots, \psi_{\bar{k}}\}, \tag{38}$$

with $(\psi_m)_m$ which generates the whole space $W_0^{1,q}(\Omega)$ and is so that (11) holds. As a matter of fact, if some $j \geq \bar{k} + 1$ exists so that $\psi_j \in Y^\sigma$, from (11), (32), (37) and the monotonicity of the sequence $(\eta_m)_m$ we get

$$\eta_j \geq \eta_{\bar{k}+1} > \nu_{\bar{k}} + \sigma > \sup_{u \in Y^\sigma \setminus \{0\}} \frac{|\nabla u|_p^p + |\nabla u|_q^q}{|u|_q^q} \geq |\nabla \psi_j|_p^p + |\nabla \psi_j|_q^q = \eta_j,$$

which is a contradiction. Thus, (38) is proved.
At last, if we consider (16) with the constant σ as in (32), a suitable $k_3^\sigma > 0$ exists such that

$$J(u) \leq \frac{1}{p}\left(|\nabla u|_p^p + |\nabla u|_q^q\right) - \frac{\lambda_\infty}{q}|u|_q^q + \frac{\sigma}{q}|u|_q^q + k_3^\sigma |u|_q \qquad \text{for all } u \in W_0^{1,q}(\Omega).$$

Hence, from (37) it follows that

$$J(u) \leq \left[\frac{1}{p}(\nu_{\bar{k}} + 2\sigma) - \frac{\lambda_\infty}{q}\right] |u|_q^q + k_3^\sigma |u|_q \qquad \text{for all } u \in Y^\sigma.$$

As $\nu_k = \nu_{\bar{k}}$, from (32) we have that

$$J(u) \to -\infty \quad \text{as } |u|_q \to +\infty, u \in Y^\sigma,$$

then, since all the norms are equivalent on the finite dimensional space Y^σ, a constant $R_2 > 0$ exists, large enough, such that

$$J(u) \leq 0 \quad \text{if } u \in Y^\sigma, \|u\|_q \geq R_2. \tag{39}$$

Finally, setting $V := Y_{k-1}^\sigma$, $W := Z_{k-1}$, $e := \frac{\psi_k}{\|\psi_k\|_q}$, $Y := Y_{k-1}^\sigma \oplus \operatorname{span}\{e\}$ and

$$S = Z_{k-1} \cap S_\rho, \quad Q = \{te : t \in [0, R_1]\} \oplus (\overline{B}_{R_2} \cap Y_{k-1}^\sigma), \tag{40}$$

from Example 2, (36) and (38), it results that S and ∂Q, boundary of Q in Y, link just taking $R_1 > \rho$. Then, if we assume also $R_1 \geq R_2$, from (34), (35) and (39) we have that Theorem 3 applies and a critical level c exists, with

$$\sup_{u \in Q} J(u) \ \geq \ c \ \geq \ \beta \ > \ 0$$

corresponding to a nontrivial solution of (P_0^∞).

Case (ii) Taking η as in (Λ_2), from $(\Lambda_1)(ii)$ a constant $\sigma \in]0, \eta[$ exists such that

$$\lambda_\infty + \sigma < \eta_k^0, \quad \frac{q}{p}(\nu_k + 2\sigma) < \lambda_0 + \lambda_\infty, \quad \nu_{\bar{k}} + \sigma < \eta_{\bar{k}+1}^0, \tag{41}$$

where it has to be $\lambda_0 > 0$ (see Remark 2(a)).

Now, from (16) with such a σ, and the Sobolev inequality, by using (10) with $m = k$, a suitable $k_4^\sigma > 0$ exists such that

$$J(u) \ \geq \ \frac{1}{q}\left(1 - \frac{\lambda_\infty + \sigma}{\eta_k^0}\right) |\nabla u|_q^q - k_4^\sigma |\nabla u|_q \quad \text{for all } u \in Z_{k-1}^0$$

and from (41) a constant $\beta < 0$ exists such that

$$J(u) \geq \beta \qquad \text{for all } u \in Z_{k-1}^0. \tag{42}$$

On the other hand, by reasoning as in the previous case, a subspace $Y_{k-1}^\sigma \in \mathbb{W}_{k-1}$ exists such that $\dim\, Y_{k-1}^\sigma = k-1$ and (33) holds. Then, since for any $u \in W_0^{1,q}(\Omega)$ we can write

$$J(u) \ = \ \frac{1}{p}\left(|\nabla u|_p^p + |\nabla u|_q^q\right) - \left(\frac{1}{p} - \frac{1}{q}\right) |\nabla u|_q^q - \frac{\lambda_\infty}{q} |u|_q^q - \int_\Omega F(x,u)\, \mathrm{d}x,$$

from (33) (recall that $\sigma < \eta$) it follows that

$$J(u) \;\le\; \int_\Omega \left[\left(\frac{1}{p}(\nu_{k-1}+\eta) - \frac{\lambda_\infty}{q}\right)|u|^q - F(x,u)\right]\,\mathrm{d}x - \left(\frac{1}{p}-\frac{1}{q}\right)|\nabla u|_q^q$$

for all $u \in Y^\sigma_{k-1}$. Hence, setting $\delta_1 = \frac{1}{p} - \frac{1}{q} > 0$, from (Λ_2) this last estimate implies that

$$J(u) \le -\delta_1 \|u\|_q^q \qquad \text{for all } u \in Y^\sigma_{k-1}. \tag{43}$$

Thus, it results not only that

$$\sup_{u \in Y^\sigma_{k-1}} J(u) \;=\; 0, \tag{44}$$

but also that a radius $R > 0$, large enough, and a constant $\alpha < \beta$ exist such that

$$J(u) \le \alpha \qquad \text{for all } u \in Y^\sigma_{k-1} \cap S_R. \tag{45}$$

At last, being $\alpha < \beta$, by reasoning as for the proof of (36) but by means of (42) and (45), we have that

$$W_0^{1,q}(\Omega) = Y^\sigma_{k-1} \oplus Z^0_{k-1}. \tag{46}$$

Hence, setting

$$S = Z^0_{k-1} \qquad \text{and} \qquad Q = Y^\sigma_{k-1} \cap \overline{B}_R, \tag{47}$$

from Example 1 and estimates (42), (44), (45), it follows that Theorem 3 applies and J has a critical level c such that

$$\beta \;\le\; c \;=\; \inf_{\phi\in\Gamma}\sup_{u\in Q} J(\phi(u)) \;\le\; \sup_{u\in Q} J(u) \;=\; 0,$$

where $\Gamma = \left\{\phi \in C(W_0^{1,q}(\Omega), W_0^{1,q}(\Omega)) :\ \phi\big|_{\partial Q} = \mathrm{id}\right\}$.

Next, we want to show that $c < 0$; so, (P_0^∞) admits a nontrivial solution.

To this aim, it is enough to prove that a function $\bar\phi \in C(W_0^{1,q}(\Omega), W_0^{1,q}(\Omega))$ exists, with $\bar\phi\big|_{\partial Q} = \mathrm{id}$, such that

$$\sup_{u\in Q} J(\bar\phi(u)) < 0. \tag{48}$$

At first, we observe that from (15) with $m = \bar k$ with $\bar k$ as in (6), a subspace $Y^\sigma \in \mathbb{W}_{\bar k}$, with $\dim Y^\sigma = \bar k$, exists such that (37) holds. Hence, we have that

$$\sup_{u\in Y^\sigma\setminus\{0\}} \frac{|\nabla u|_q^q}{|u|_q^q} \;<\; \nu_{\bar k} + \sigma. \tag{49}$$

We claim that

$$Y^\sigma \;=\; \mathrm{span}\{\psi_1^0,\dots,\psi_{\bar k}^0\} = Y^\sigma_{k-1} \oplus \mathrm{span}\{\psi_k^0,\dots,\psi_{\bar k}^0\}, \tag{50}$$

where $(\psi_m^0)_m$ generates the whole space $W_0^{1,q}(\Omega)$ and is such that (8) holds. Indeed, if $j \geq \bar{k}+1$ exists such that $\psi_j^0 \in Y^\sigma$, then (8) and estimates (41), (49), together with the monotonicity of the sequence $(\eta_m^0)_m$, imply that

$$\eta_j^0 \geq \eta_{\bar{k}+1}^0 > \nu_{\bar{k}} + \sigma > |\nabla \psi_j^0|_q^q = \eta_j^0,$$

which gives a contradiction.

Now, taking $L > 0$ such that (9) is verified with respect to the decomposition (46), without loss of generality we can suppose $L > 1$. Then, $\rho \in]0, R[$ and $\delta_2 > 0$ exist such that

$$J(u) \leq -\delta_2 \qquad \text{for all } u \in Y^\sigma \text{ with } \tfrac{\rho}{L} \leq \|u\|_q \leq 2\rho. \tag{51}$$

Indeed, fixing any $s > 0$ and taking $u \in Y^\sigma$, from (29) and (37) it results that

$$\begin{aligned} J(u) &\leq \frac{1}{p}\left(|\nabla u|_p^p + |\nabla u|_q^q\right) - \frac{\lambda_\infty}{q}|u|_q^q - \frac{\lambda_0 - \sigma}{q}|u|_q^q + k_0^\sigma |u|_{s+q}^{s+q} \\ &\leq \frac{1}{q}\left[\frac{q}{p}(\nu_{\bar{k}} + 2\sigma) - \lambda_\infty - \lambda_0\right]|u|_q^q + k_0^\sigma |u|_{s+q}^{s+q}. \end{aligned}$$

Hence, since all the norms are equivalent on the finite dimension subspace Y^σ, from this last estimates and (41) with $\nu_k = \nu_{\bar{k}}$, we have that two constants $c_1, c_2 > 0$ exist such that

$$J(u) \leq -c_1\|u\|_q^q + c_2\|u\|_q^{s+q} \qquad \text{for all } u \in Y^\sigma.$$

Thus, $s > 0$ and direct computations allow us to prove that (51) holds if $\rho > 0$ is small enough, in particular $\rho < R$.

At last, we can define $\bar{\phi} : W_0^{1,q}(\Omega) \to W_0^{1,q}(\Omega)$ as the continuous extension to $W_0^{1,q}(\Omega)$ of function $\bar{\phi} : Y_{k-1}^\sigma \to \mathbb{R}$ such that

$$\bar{\phi}(u) = \begin{cases} u & \text{if } \|u\|_q > \rho \\ \dfrac{\sqrt{\rho^2 - \|u\|_q^2}}{(\eta_k^0)^{\frac{1}{q}}}\,\psi_k^0 + u & \text{if } \|u\|_q \leq \rho \end{cases}.$$

We notice that $\bar{\phi}$ satisfies the required assumptions. Indeed, by definition we have that $\bar{\phi} \in C(W_0^{1,q}(\Omega), W_0^{1,q}(\Omega))$ and $\bar{\phi}(u) = u$ for all $u \in \partial Q = Y_{k-1}^\sigma \cap S_R$ as $R > \rho$. Moreover, if $u \in Y_{k-1}^\sigma \cap \bar{B}_R$ two cases may occur: either $\|u\|_q > \rho$ or $\|u\|_q \leq \rho$. If $\|u\|_q > \rho$, from the definition of $\bar{\phi}(u)$ and (43) we have that

$$J(\bar{\phi}(u)) = J(u) \leq -\delta_1\rho^q < 0.$$

On the other hand, if $\|u\|_q \leq \rho$, then (50) implies that $\bar{\phi}(u) \in Y^\sigma$. Furthermore, (8), (9) and direct computations imply that

$$\frac{1}{L}\left(\|u\|_q + \sqrt{\rho^2 - \|u\|_q^2}\right) \leq \|\bar{\phi}(u)\|_q \leq \|u\|_q + \sqrt{\rho^2 - \|u\|_q^2}$$

with $\rho \leq \|u\|_q + \sqrt{\rho^2 - \|u\|_q^2} \leq 2\rho$; hence, from (51) it follows that $J(\bar{\phi}(u)) \leq -\delta_2$. Thus, summing up, (48) holds and the proof is complete. □

Finally, we note that an existence result still holds for the unperturbed problem (P_0^∞) if λ_0 as defined in hypothesis (f_3), is infinite. More precisely, the following statements can be proved.

Proposition 3 *Suppose that conditions (f_1) and (f_2) hold and $\lambda_\infty \notin \sigma(-\Delta_q)$. If, moreover, we have that*

(f_3) $\lim\limits_{t\to 0} \dfrac{f(x,t)}{|t|^{q-2}t} = -\infty$ *uniformly in* $\overline{\Omega}$;

(Λ_1) $\frac{\nu_1}{p} < \frac{\lambda_\infty}{q}$;

then (P_0^∞) has at least a nontrivial solution.

Proof Taking any $\lambda > 0$ and $\sigma > 0$ such that

$$\lambda > \lambda_\infty, \qquad \frac{\nu_1 + 2\sigma}{p} < \frac{\lambda_\infty}{q}, \tag{52}$$

from $(f_3)'$ we have that $\delta_\lambda > 0$ exists so that

$$F(x,t) < -\frac{\lambda}{q}|t|^q \quad \text{for all } x \in \overline{\Omega} \text{ if } |t| \leq \delta_\lambda.$$

On the other hand, from (f_2) a radius $R_\sigma \geq \max\{1, \delta_\lambda\}$ exists such that

$$|F(x,t)| \leq \sigma|t|^q \quad \text{for all } x \in \overline{\Omega} \text{ if } |t| \geq R_\sigma.$$

Then, fixing any $s > 0$ so that $q + s < q^*$, the continuity of $\frac{F(x,t)}{|t|^{q+s}}$ on the compact set $\overline{\Omega} \times [\delta_\lambda, R_\sigma]$ and direct computations allow us to find some constants $k_{\lambda,i} > 0$ large enough so that

$$F(x,t) \leq k_{\lambda,1}|t|^{q+s} \leq -\frac{\lambda}{q}|t|^q + k_{\lambda,2}|t|^{q+s} \quad \text{for all } x \in \overline{\Omega} \text{ if } |t| \geq \delta_\lambda.$$

Hence, summing up it results

$$F(x,t) \leq -\frac{\lambda}{q}|t|^q + k_{\lambda,2}|t|^{q+s} \quad \text{for all } (x,t) \in \overline{\Omega} \times \mathbb{R}$$

which implies

$$J(u) \geq \frac{1}{q}|\nabla u|_q^q + \frac{\lambda - \lambda_\infty}{q}|u|_q^q - k_{\lambda,2}|u|_{q+s}^{q+s} \quad \text{for all } u \in W_0^{1,q}(\Omega),$$

thus, from (52) and the Sobolev Embedding Theorem we obtain that

$$J(u) \geq \beta \quad \text{for all } u \in S_\rho,$$

for suitable constants $\rho > 0$ and $\beta > 0$.
Now, from (15) with $m = 1$, a subspace $Y \in \mathbb{W}_1$ exists such that

$$\sup_{u\in Y\setminus\{0\}} \frac{|\nabla u|_p^p + |\nabla u|_q^q}{|u|_q^q} \; < \; \nu_1 + \sigma,$$

where, without loss of generality, we can take $Y = \operatorname{span}\{\psi_1\}$. Thus, from (16) and direct computations it results

$$J(t\psi_1) \le t^q \left(\frac{\nu_1 + 2\sigma}{p} - \frac{\lambda_\infty}{q}\right) + tK_\sigma \int_\Omega |\psi_1| dx \qquad \text{for all } t > 0,$$

which implies $J(t\psi_1) \to -\infty$ if $t \to +\infty$ as (52) holds.
At last, from Proposition 2 and the previous geometrical estimates, the classical Mountain Pass Theorem applies (cf. [29, Theorem 2.2]) and the existence of a nontrivial solution corresponding to a critical level $c \ge \beta > 0$ is proved. □

Proposition 4 *Suppose that conditions* (f_1) *and* (f_2) *hold and* $\lambda_\infty \notin \sigma(-\Delta_q)$. *Moreover, assume that*

$(f_3)''$ $\displaystyle\lim_{t\to 0} \frac{f(x,t)}{|t|^{q-2}t} = +\infty$ *uniformly in* $\overline{\Omega}$;
$(\Lambda_1)'''$ *some integers* $1 \le k \le \bar{k}$ *exist such that*

$$\lambda_\infty < \eta_k^0, \qquad \nu_{\bar{k}} < \eta_{\bar{k}+1}^0.$$

If (Λ_2) *holds for the same* k *in* $(\Lambda_1)'''$, *then* (P_0^∞) *has at least a nontrivial solution.*

Proof From $(\Lambda_1)'''$ a constant $\sigma > 0$ exists so that

$$\sigma < \eta, \qquad \lambda_\infty + \sigma < \eta_k^0, \qquad \nu_{\bar{k}} + \sigma < \eta_{\bar{k}+1}^0. \tag{53}$$

Firstly, reasoning as in the proof of (42), from (10), (16) and (53) a constant $\beta < 0$ exists such that $J(u) \ge \beta$ for all $u \in Z_{k-1}^0$.
On the other hand, reasoning as in the proof of *Case* (ii) of Theorem 1, from (Λ_2) a subspace $Y_{k-1}^\sigma \in \mathbb{W}_{k-1}$ exists such that (43) holds and for a large enough radius $R > 0$ inequality (45) is satisfied with a suitable $\alpha < \beta$.
Hence, (46) is verified and from Example 1, Proposition 2 and Theorem 3 applied to $S = Z_{k-1}^0$ and $Q = Y_{k-1}^\sigma \cap \overline{B}_R$, we get the existence of a critical level $c \le 0$.
At last, by considering again $\bar{\phi}$ for a suitable $\rho \in]0, R[$ as in the proof of *Case* (ii) of Theorem 1, we get that (48) holds, thus $c < 0$ and the corresponding solution is non trivial. Indeed, from (53) both (49) and (50) hold. Furthermore, from $(f_3)''$, taking any $\lambda > \frac{\nu_{\bar{k}}+\sigma}{p} - \frac{\lambda_\infty}{q}$ a constant $\delta_\lambda > 0$ exists such that

$$F(x,t) \ge \lambda |t|^q \quad \text{for all } x \in \overline{\Omega} \text{ if } |t| \le \delta_\lambda,$$

while from (f_2) and direct computations (as in the proof of Lemma 1) taking any $s > 0$ a constant $k_\lambda > 0$ exists such that

$$\frac{|F(x,t)-\lambda|t|^q|}{|t|^{q+s}} \le k_\lambda \quad \text{for all } x\in\overline{\Omega} \text{ if } |t|\ge\delta_\lambda.$$

Hence,

$$F(x,t)\ge \lambda|t|^q - k_\lambda|t|^{q+s} \quad \text{for all } (x,t)\in\overline{\Omega}\times\mathbb{R}$$

which, together with (53), implies (51) which allows us to prove (48). □

4 The perturbed case

Now, we are able to deal with the perturbed problem (P^∞_ε).

Proof (Proof of Theorem 2) Following [23], for any $j\in\mathbb{N}$ we consider a continuous cut function $\gamma_j:\mathbb{R}\to\mathbb{R}$ such that

$$\gamma_j(t) = \begin{cases} 0 \text{ if } |t|\ge j+1 \\ 1 \text{ if } |t|\le j \end{cases},$$

and $0<\gamma_j(t)<1$ if $j<|t|<j+1$, and set

$$h_j(x,t)=\gamma_j(t)h(x,t), \qquad H_j(x,t)=\int_0^t h_j(x,s)\,\mathrm{d}s.$$

Since for any $j\in\mathbb{N}$ there exists $\varepsilon_1(j)>0$ such that

$$\varepsilon_1(j)|h_j(x,t)|<1, \qquad \varepsilon_1(j)|H_j(x,t)|<1 \quad \text{for all } (x,t)\in\overline{\Omega}\times\mathbb{R}, \tag{54}$$

for any ε, with $|\varepsilon|\le\varepsilon_1(j)$, we can consider the functionals

$$J_{j,\varepsilon}(u)=J(u)-\varepsilon\int_\Omega H_j(x,u)\,\mathrm{d}x \quad \text{on } W^{1,q}_0(\Omega).$$

Now, taking Q as in the proof of Theorem 1 (namely, as in (40) in *Case* (i) or as in (47) in *Case* (ii)), from Theorem 1 we have that $c\in[\beta,\sup_{u\in Q}J(u)]$ is a critical level of J in $W^{1,q}_0(\Omega)$ with

$$c = \inf_{\phi\in\Gamma}\sup_{u\in Q} J(\phi(u)),$$

where $\Gamma=\left\{\phi\in C(W^{1,q}_0(\Omega),W^{1,q}_0(\Omega)) : \phi\big|_{\partial Q}=\mathrm{id}\right\}$. Then, from Theorem 6 we have that such a level c has to be essential for J; thus, Theorem 4 implies the existence of a constant $\varepsilon_2(j)\in]0,\varepsilon_1(j)[$ such that if $|\varepsilon|\le\varepsilon_2(j)$ then $J_{j,\varepsilon}$ has at least one essential value $d^{j,\varepsilon}$ with

$$\frac{\beta}{2}<d^{j,\varepsilon}<\sup_{u\in Q}J(u)+1.$$

We note that, since for each ε, j the nonlinear term $f(x,t) + \varepsilon h_j(x,t)$ satisfies assumptions (f_1) and (f_2), then from the same arguments in Proposition 2 we have that each $J_{j,\varepsilon}$ satisfies the (PS) condition in $\mathbb{R}$. Hence, from Theorem 5 it follows that if $|\varepsilon| \le \varepsilon_2(j)$ the level $d^{j,\varepsilon}$ is also critical for $J_{j,\varepsilon}$ and $u^{j,\varepsilon} \in W_0^{1,q}(\Omega)$ exists such that

$$\begin{aligned}\int_\Omega |\nabla u^{j,\varepsilon}|^{p-2}\nabla u^{j,\varepsilon}\cdot\nabla\varphi\,\mathrm{d}x + \int_\Omega |\nabla u^{j,\varepsilon}|^{q-2}\nabla u^{j,\varepsilon}\cdot\nabla\varphi\,\mathrm{d}x \\ = \lambda_\infty \int_\Omega |u^{j,\varepsilon}|^{q-2}u^{j,\varepsilon}\,\varphi\,\mathrm{d}x + \int_\Omega f(x,u^{j,\varepsilon})\varphi\,\mathrm{d}x + \varepsilon\int_\Omega h_j(x,u^{j,\varepsilon})\varphi\,\mathrm{d}x\end{aligned} \tag{55}$$

for all $\varphi \in W_0^{1,q}(\Omega)$.
We claim that a constant $K_1 > 0$ exists such that

$$\|u^{j,\varepsilon}\|_q \le K_1 \quad \text{for all } j \in \mathbb{N}, |\varepsilon| \le \varepsilon_2(j). \tag{56}$$

Indeed, arguing by contradiction, let us assume that the set

$$A := \{\|u^{j,\varepsilon}\|_q : j \in \mathbb{N}, |\varepsilon| \le \varepsilon_2(j)\}$$

is unbounded. Then, a sequence $(u^{j_m,\varepsilon_m})_m \subset W^{1,q}(\Omega)$ exists, with $|\varepsilon_m| \le \varepsilon_2(j_m)$, such that

$$\|u^{j_m,\varepsilon_m}\|_q \to +\infty \quad \text{as } m \to +\infty. \tag{57}$$

Setting $w_{j_m,\varepsilon_m} = \frac{u^{j_m,\varepsilon_m}}{\|u^{j_m,\varepsilon_m}\|_q}$, we have that $(w_{j_m,\varepsilon_m})_m$ is a bounded sequence in $W_0^{1,q}(\Omega)$, so $w \in W_0^{1,q}(\Omega)$ exists such that, up to subsequences, it results

$$w_{j_m,\varepsilon_m} \rightharpoonup w \quad \text{weakly in } W_0^{1,q}(\Omega), \tag{58}$$

$$w_{j_m,\varepsilon_m} \to w \quad \text{strongly in } L^q(\Omega). \tag{59}$$

Now, taking

$$\varphi_{j_m,\varepsilon_m} = \frac{w_{j_m,\varepsilon_m} - w}{\|u^{j_m,\varepsilon_m}\|_q^{q-1}}$$

in (55) with $j = j_m$ and $\varepsilon = \varepsilon_m$, we obtain that

$$
\begin{aligned}
&\int_\Omega \frac{|\nabla w_{j_m,\varepsilon_m}|^{p-2}}{\|u^{j_m,\varepsilon_m}\|_q^{q-p}} \nabla w_{j_m,\varepsilon_m} \cdot \nabla (w_{j_m,\varepsilon_m} - w)\, \mathrm{d}x \\
&\quad + \int_\Omega |\nabla w_{j_m,\varepsilon_m}|^{q-2} \nabla w_{j_m,\varepsilon_m} \cdot \nabla (w_{j_m,\varepsilon_m} - w)\, \mathrm{d}x \\
&\quad = \lambda_\infty \int_\Omega |w_{j_m,\varepsilon_m}|^{q-2} w_{j_m,\varepsilon_m}\, (w_{j_m,\varepsilon_m} - w)\, \mathrm{d}x \\
&\quad + \int_\Omega \frac{f(x, u^{j_m,\varepsilon_m})}{\|u^{j_m,\varepsilon}\|_q^{q-1}} (w_{j_m,\varepsilon_m} - w)\, \mathrm{d}x \\
&\qquad + \varepsilon_m \int_\Omega \frac{h_j(x, u^{j_m,\varepsilon_m})}{\|u^{j_m,\varepsilon_m}\|_q^{q-1}} (w_{j_m,\varepsilon_m} - w)\, \mathrm{d}x.
\end{aligned}
$$

Then, from (54), (57) and (59) we have that

$$
\varepsilon_m \int_\Omega \frac{h_j(x, u^{j_m,\varepsilon_m})}{\|u^{j_m,\varepsilon_m}\|_q^{q-1}} (w_{j_m,\varepsilon_m} - w)\, \mathrm{d}x = \beta_m,
$$

and also, by reasoning as in the proof of Proposition 2,

$$
\begin{aligned}
&\int_\Omega \frac{|\nabla w_{j_m,\varepsilon_m}|^{p-2}}{\|u^{j_m,\varepsilon_m}\|_q^{q-p}} \nabla w_{j_m,\varepsilon_m} \cdot \nabla (w_{j_m,\varepsilon_m} - w)\, \mathrm{d}x = \beta_m, \\
&\int_\Omega |w_{j_m,\varepsilon_m}|^{q-2} w_{j_m,\varepsilon_m}\, (w_{j_m,\varepsilon_m} - w)\, \mathrm{d}x = \beta_m, \\
&\int_\Omega \frac{f(x, u^{j_m,\varepsilon_m})}{\|u^{j_m,\varepsilon}\|_q^{q-1}} (w_{j_m,\varepsilon_m} - w)\, \mathrm{d}x = \beta_m,
\end{aligned}
$$

which imply that

$$
\int_\Omega |\nabla w_{j_m,\varepsilon_m}|^{q-2} \nabla w_{j_m,\varepsilon_m} \cdot \nabla (w_{j_m,\varepsilon_m} - w)\, \mathrm{d}x = \beta_m.
$$

Hence, from this last limit and (58) it follows that

$$
w_{j_m,\varepsilon_m} \ \to\ w \quad \text{strongly in } W_0^{1,q}(\Omega), \tag{60}
$$

which gives also $w \neq 0$.
Finally, taking any $\varphi \in W_0^{1,q}(\Omega)$ and applying again (55) with $j = j_m$ and $\varepsilon = \varepsilon_m$ on $\frac{\varphi}{\|u^{j_m,\varepsilon_m}\|_q^{q-1}}$, we obtain

$$\begin{aligned} &\int_\Omega \frac{|\nabla w_{j_m,\varepsilon_m}|^{p-2}}{\|u^{j_m,\varepsilon_m}\|_q^{q-p}} \nabla w_{j_m,\varepsilon_m} \cdot \nabla\varphi \,\mathrm{d}x + \int_\Omega |\nabla w_{j_m,\varepsilon_m}|^{q-2} \nabla w_{j_m,\varepsilon_m} \cdot \nabla\varphi \,\mathrm{d}x \\ &= \lambda_\infty \int_\Omega |w_{j_m,\varepsilon_m}|^{q-2} w_{j_m,\varepsilon_m}\, \varphi \,\mathrm{d}x \; + \; \int_\Omega \frac{f(x,u^{j_m,\varepsilon_m})}{\|u^{j_m,\varepsilon_m}\|_q^{q-1}}\, \varphi \,\mathrm{d}x \\ &\quad + \varepsilon_m \int_\Omega \frac{h_j(x,u^{j_m,\varepsilon_m})}{\|u^{j_m,\varepsilon_m}\|_q^{q-1}}\, \varphi \,\mathrm{d}x. \end{aligned} \tag{61}$$

Thus, since from (54) and (57) we have that

$$\left| \varepsilon_m \int_\Omega \frac{h_j(x,u^{j_m,\varepsilon_m})}{\|u^{j_m,\varepsilon_m}\|_q^{q-1}} \varphi \,\mathrm{d}x \right| \;\le\; \beta_m \|\varphi\|_q,$$

by reasoning again as in the proof of Proposition 2 by means of (57) we are able to prove that

$$\left| \int_\Omega \frac{|\nabla w_{j_m,\varepsilon_m}|^{p-2}}{\|u^{j_m,\varepsilon_m}\|_q^{q-p}} \nabla w_{j_m,\varepsilon_m} \cdot \nabla\varphi \,\mathrm{d}x \right| \;\le\; \beta_m \|\varphi\|_q,$$

$$\lim_{m\to+\infty} \int_\Omega \frac{f(x,u^{j_m,\varepsilon_m})}{\|u^{j_m,\varepsilon_m}\|_q^{q-1}}\, \varphi \,\mathrm{d}x = 0.$$

Hence, from (59), (60) and passing to the limit in (61), for the arbitrariness of φ we get that $\lambda_\infty \in \sigma(-\Delta_q)$, against our assumption. Thus, the claim (56) is proved.
Finally, from [24, Lemmas 4.5 and 4.6] (see [7] for more details) a constant $K_2 > 0$ exists such that

$$|u^{j,\varepsilon}|_\infty \le K_2 \qquad \text{for all } j \in \mathbb{N}, |\varepsilon| \le \varepsilon_2(j);$$

thus, for $j > K_2$ problem (P_ε^∞) has at least a nontrivial solution. □

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