Generalized Morrey regularity of 2b-parabolic systems

Generalized Morrey Regularity of $2b$-Parabolic Systems

Dian K. Palagachev[a], Lubomira G. Softova[b]

[a] *Polytechnic University of Bari, Department of Mechanics, Mathematics and Management; Bari, Italy*
[b] *University of Salerno, Department of Mathematics; Salerno, Italy*

Abstract

We derive the Calderón–Zygmund property in generalized Morrey spaces for the strong solutions to $2b$-order linear parabolic systems with discontinuous principal coefficients.

Keywords: Generalized Morrey space, Parabolic equations and systems, *VMO*, Discontinuous data, *A priori* estimates, Regularity
2010 MSC: 35K41, 35B65, 35G40, 35R05, 35B45, 49N60, 46E30

---

1. Introduction

The present note deals with local regularity in Morrey-type spaces of the strong solutions to $2b$-order non-divergence linear parabolic systems

$$\mathfrak{P}\mathbf{u} := D_t\mathbf{u}(x) - \sum_{|\alpha|=2b} \mathbf{A}_\alpha(x)D^\alpha\mathbf{u}(x) = \mathbf{f}(x) \tag{1.1}$$

with discontinuous coefficients. Here $x = (x', t) \in \mathbb{R}^n \times \mathbb{R}$ is a generic point lying in the cylinder $Q = \Omega \times (0, T)$, Ω is an n-dimensional domain with $n \geq 2$. Fixed two integers $b, m \geq 1$, $\mathbf{A}_\alpha(x)$ is the $m\times m$ matrix $\left\{a^{jk}_\alpha(x)\right\}^m_{j,k=1}$ of the measurable coefficients $a^{jk}_\alpha\colon Q \to \mathbb{R}$ with $\alpha = (\alpha_1, \dots, \alpha_n)$, $D_t := \partial/\partial t$ and $D^\alpha \equiv D^\alpha_{x'} := D^{\alpha_1}_{x_1} \dots D^{\alpha_n}_{x_n}$ with $D_i := \partial/\partial x_i$. Given a vector field $\mathbf{f}\colon Q \to \mathbb{R}^m$, we will consider *strong* solutions of (1.1), that is, a vector-valued function $\mathbf{u}\colon Q \to \mathbb{R}^m$ which belongs to a suitable parabolic Sobolev space and that satisfies (1.1) almost everywhere in Q.

The operator $\mathfrak{P}$ will be supposed to be *uniformly parabolic* in the sense of Petrovskii that means the p-roots of the m-degree polynomial $\det\left\{p\ \mathbf{Id}_m - \sum_{|\alpha|=2b} \mathbf{A}_\alpha(x)(i\nu)^\alpha\right\}$ satisfy, for some $\delta > 0$ and all $s = 1, \dots, m$, the inequality

$$\operatorname{Re} p_s(x, \nu) \leq -\delta|\nu|^{2b} \qquad \text{for a.a. } x \in Q,\ \forall \nu \in \mathbb{R}^n. \tag{1.2}$$

Indeed, the roots $p_s(x, \nu)$ are the eigenvalues of the $m \times m$ matrix $(-1)^b \sum_{|\alpha|=2b} \mathbf{A}_\alpha(x)(\nu)^\alpha$ and (1.2) ensures *uniform ellipticity* of the operator $(-1)^b \sum_{|\alpha|=2b} \mathbf{A}_\alpha(x)D^\alpha$ guaranteeing

Email addresses: dian.palagachev@poliba.it (Dian K. Palagachev), lsoftova@unisa.it (Lubomira G. Softova)

this way the representation formula (3.1). It turns out that (1.2) is not only *sufficient* but also *necessary* (cf. [9, Sect. 5]) for the validity of the a priori estimate (2.2) obtained.

In our previous paper [9] a *Calderón–Zygmund* type theory has been developed for (1.1) in the framework of the classical parabolic Morrey spaces $L^{p,\lambda}$, assuming the principal coefficients of the operator $\mathfrak{P}$ to be essentially bounded functions of *vanishing mean oscillation (VMO).* On the other hand, in the recent years an exhaustive Calderón–Zygmund theory has been elaborated both for elliptic and parabolic equations/systems in *divergence form* with VMO-coefficients in the framework of the *generalized Morrey spaces* $L^{p,\omega}$ (cf. [3, 4] and the survey [2]). These last spaces allow finer control of the local oscillation properties of a function near its singular points and that is why regularity results in $L^{p,\omega}$ of solutions to PDEs with discontinuous coefficients are of great importance in the applications to differential geometry, stochastic control, nonlinear optimization, adaptive discontinuous Galerkin FEMs, etc.

In this note we obtain the Calderón–Zygmund property for the system (1.1) in the settings of the generalized Morrey spaces. Precisely, we prove that $\mathbf{f} \in L^{p,\omega}$ implies that the higher-order derivatives $D_t\mathbf{u}$ and $D^{2b}_{x'}\mathbf{u}$ of any strong solution belong to the same space $L^{p,\omega}$ once (1.2) and $\mathbf{A}_\alpha \in L^\infty \cap VMO$ hold true. The proof of the generalized Morrey regularity relies on the Calderón–Zygmund method ([5]) of expressing the highest order spatial derivatives of $\mathbf{u}$ in terms of singular integral operators and their commutators with the multiplication by the *VMO*-coefficients. Employing results on boundedness of these singular integrals in $L^{p,\omega}$ leads to an a priori estimate of Caccioppoli whence the desired regularity follows by interpolation.

2. Main result

For a given $r > 0$ consider the parabolic cylinders

$$\mathcal{C}_r(x) := \big\{y = (y', t) \in \mathbb{R}^{n+1}\colon \quad |x' - y'| < r,\ \tau \in (t - r^{2b}, t)\big\}$$

defined with respect to the parabolic metric $\rho(x) = \rho(x', t) := \max\{|x'|, |t|^{1/2b}\}$, and note that the Lebesgue measure $|\mathcal{C}_r|$ of $\mathcal{C}_r$ is comparable to r^{n+2b}.

We allow the coefficients a^{jk}_α of (1.1) to be *discontinuous* functions, with discontinuity measured in terms of *VMO,* in the sense of the following definition.

Definition 2.1. *For* $a \in L^1_{\mathrm{loc}}(\mathbb{R}^{n+1})$ *and any* $R > 0$ *set*

$$\gamma_a(R) := \sup_{\mathcal{C}_r, r\le R} \frac{1}{|\mathcal{C}_r|}\int_{\mathcal{C}_r} |a(y) - a_{\mathcal{C}_r}|\, dy, \quad a_{\mathcal{C}_r} := \frac{1}{|\mathcal{C}_r|}\int_{\mathcal{C}_r} a(y)\, dy,$$

where $\mathcal{C}_r$ *is any parabolic cylinder. We say that* $a \in BMO$ *if* $\|a\|_* = \sup_{R>0}\gamma_a(R) < \infty$, *while a function* a *is in VMO with VMO-modulus* γ_a *if* $a \in BMO$ *and* $\lim_{R\to 0}\gamma_a(R) = 0$.

In what follows, we consider a measurable function $\omega\colon\ \mathbb{R}^{n+1}\times\mathbb{R}_+ \to \mathbb{R}_+$ and assume there exist positive constants κ_1, κ_2 and κ_3 such that for any $x_0 \in \mathbb{R}^{n+1}$ the weight ω satisfies

$$\kappa_1 < \frac{\omega(x_0, s)}{\omega(x_0, r)} < \kappa_2 \quad \forall\ 0 < r \le s \le 2r; \qquad \int_r^\infty \frac{\omega(x_0, s)}{s^{n+2b+1}}\, ds \le \kappa_3 \frac{\omega(x_0, r)}{r^{n+2b}}. \tag{2.1}$$

Definition 2.2. see [7, 8] *A function $f \in L^p(\mathbb{R}^{n+1})$ with $1 \le p < \infty$ belongs to the generalized Morrey space $L^{p,\omega}(\mathbb{R}^{n+1})$ if the following norm is finite*

$$\|f\|_{p,\omega} = \left(\sup_{\mathcal{C}_r(x)} \frac{1}{\omega(x,r)} \int_{\mathcal{C}_r(x)} |f(y)|^p\, dy \right)^{1/p}.$$

For any bounded cylinder $Q \subset \mathbb{R}^{n+1}$ we define the space $L^{p,\omega}(Q)$ of functions $f \in L^p(Q)$ for which

$$\|f\|_{p,\omega;Q} = \left(\sup_{\mathcal{C}_r(x)} \frac{1}{\omega(x,r)} \int_{Q_r(x)} |f(y)|^p\, dy \right)^{1/p} < \infty,$$

where the supremum is taken over all cylinders centered at any $x \in Q$ and of radius $r \in (0, \operatorname{diam} Q]$ and $Q_r(x) := Q \cap \mathcal{C}_r(x)$.

The generalized Sobolev–Morrey space $W^{2b,1}_{p,\omega}(Q)$ consists of all functions $u \in L^p(Q)$ with generalized derivatives $D_t u,\ D^\alpha_{x'} u,\ |\alpha| \le 2b$, belonging to $L^{p,\omega}(Q)$ and endowed with the norm

$$\|\mathbf{u}\|_{W^{2b,1}_{p,\omega}(Q)} = \|D_t u\|_{p,\omega;Q} + \sum_{s=0}^{2b} \sum_{|\alpha|=s} \|D^\alpha_{x'} u\|_{p,\omega;Q}.$$

Similarly, $\mathbf{u} = \big(u_1, \dots, u_m\big) \in W^{2b,1}_{p,\omega}(Q)$ means $u_k \in W^{2b,1}_{p,\omega}(Q)$ and the norm $\|\mathbf{u}\|_{W^{2b,1}_{p,\omega}(Q)}$ is given by $\sum_{k=1}^m \|u_k\|_{W^{2b,1}_{p,\omega}(Q)}$.

Remark 2.3. It is clear that if $\omega(x,r) = r^\lambda$ with $\lambda \in (0, n+2b)$ then $L^{p,\omega}$ gives rise to the classical Morrey space $L^{p,\lambda}$, while $L^{p,1} \equiv L^p$ and $W^{2b,1}_{p,1}$ reduces to the classical parabolic Sobolev space $W^{2b,1}_p$ (cf. [9]) when $\omega \equiv 1$. In what follows, we will use also a localized version $W^{2b,1}_{p,\omega,\text{loc}}(Q)$ of $W^{2b,1}_{p,\omega}(Q)$, where *local* means local only in the spatial variable x' but global in time t, that is $\mathbf{u} \in W^{2b,1}_{p,\omega,\text{loc}}(Q)$ if $\mathbf{u} \in W^{2b,1}_{p,\omega}(\Omega' \times (0,T))$ for each $\Omega' \Subset \Omega$.

It should be noted that the first part in (2.1) is a sort of "doubling condition" satisfied by the weight ω, while the second one generalizes the original reguirement on ω due to Nakai (cf. [8]) that ensures boundedness of the Hardy–Littlewood maximal operator. Thus (2.1) takes into account the specific homogeneity $n + 2b$ of the kernels of the singular integrals involved in (3.1) and guarantees their boundedness in $L^{p,\omega}$ as shown in [10, 11].

The main result of the note is contained in the next theorem.

Theorem 2.4. *Suppose* (1.2), *$\mathbf{A}_\alpha = \{a^{jk}_\alpha\} \in VMO(Q) \cap L^\infty(Q)$ and let $\mathbf{u} \in W^{2b,1}_{p,\text{loc}}(Q)$ be a strong solution to* (1.1) *with $p \in (1,\infty)$ such that $\mathbf{u}(x,0) = \mathbf{0}$. Let $\mathbf{f} \in L^{p,\omega}(Q)$ with ω satisfying* (2.1).

Then $\mathbf{u} \in W^{2b,1}_{p,\omega,\text{loc}}(Q)$ and

$$\|\mathbf{u}\|_{W^{2b,1}_{p,\omega}(Q')} \le C\big(\|\mathbf{f}\|_{p,\omega;Q} + \|\mathbf{u}\|_{p,\omega;Q''}\big) \tag{2.2}$$

for all $Q' = \Omega \times (0,T)$, $Q'' = \Omega'' \times (0,T)$ with $\Omega' \Subset \Omega'' \Subset \Omega$, and where the constant C depends on $n, p, m, b, \delta, \omega, \|\mathbf{A}_\alpha\|_{\infty;Q}$, the VMO-moduli $\gamma_{\mathbf{A}_\alpha}$ and on $\operatorname{dist}(\Omega', \partial\Omega'')$.

The proof of Theorem 2.4 relies on some real analysis results regarding boundedness of Calderón–Zygmund type singular integral operators and their commutators.

Let $\mathcal{K}(x;\xi)\colon\ \mathbb{R}^{n+1}\times(\mathbb{R}^{n+1}\setminus\{0\})\to\mathbb{R}$ be a *variable Calderón-Zygmund kernel of parabolic type* (see [9, Definition 3.1]). Given a function $f\in L^1(\mathbb{R}^{n+1})$, define the singular integral operator

$$\mathfrak{K}f(x):=P.V.\int_{\mathbb{R}^{n+1}}\mathcal{K}(x;x-y)f(y)\,dy$$

and its commutator with multiplication by a function $a\in L^\infty(\mathbb{R}^{n+1})$ as

$$\mathfrak{C}[a,f](x):=P.V.\int_{\mathbb{R}^{n+1}}\mathcal{K}(x;x-y)[a(y)-a(x)]f(y)\,dy=\mathfrak{K}(af)(x)-a(x)\mathfrak{K}f(x).$$

The L^p- and $L^{p,\omega}$-boundedness of the operators $\mathfrak{K}$ and $\mathfrak{C}$ have been obtained in [6, 1] and [10, 11], respectively. For the sake of completeness, we summarize these results below.

Proposition 2.5. *Let the weight ω satisfy* (2.1).

Then there exists a positive constant $C=C(p,\omega,\mathcal{K})$ such that

$$\|\mathfrak{K}f\|_{p,\omega}\le C\|f\|_{p,\omega},\qquad \|\mathfrak{C}[a,f]\|_{p,\omega}\le C\|a\|_*\|f\|_{p,\omega}$$

for any $f\in L^{p,\omega}(\mathbb{R}^{n+1})$ with $p\in(1,\infty)$.

3. Proof of Theorem 2.4

Let $\mathbf{v}\in W^{2b,1}_p(\mathbb{R}^{n+1})$ be compactly supported in x' and such that $\mathbf{v}(x',0)=\mathbf{0}$. Extending $\mathbf{v}$ as zero for $t<0$, [9, (3.9)] implies that for each multiindex α with $|\alpha|=2b$ one has

$$\begin{aligned}D^\alpha\mathbf{v}(x)={}&P.V.\int_{\mathbb{R}^{n+1}}D^\alpha\mathbf{\Gamma}(x;x-y)\mathfrak{P}\mathbf{v}(y)\,dy\\&+\sum_{|\alpha'|=2b}P.V.\int_{\mathbb{R}^{n+1}}D^\alpha\mathbf{\Gamma}(x;x-y)\big(\mathbf{A}_{\alpha'}(y)-\mathbf{A}_{\alpha'}(x)\big)D^{\alpha'}_{y'}\mathbf{v}(y)\,dy\\&+\int_{\mathbb{S}^n}D^{\beta^s}\mathbf{\Gamma}(x;y)\nu_s\,d\sigma_y\mathfrak{P}\mathbf{v}(x)\\=:{}&\mathfrak{K}_\alpha(\mathfrak{P}\mathbf{v})+\sum_{|\alpha'|=2b}\mathfrak{C}_\alpha[\mathbf{A}_{\alpha'},D^{\alpha'}\mathbf{v}]+\mathbf{F}(x)\mathfrak{P}\mathbf{v}(x),\end{aligned}\tag{3.1}$$

where $\mathbf{\Gamma}(x',t;y',\tau)$ is the fundamental matrix of the operator $\mathbf{Id}_mD_t-\sum_{|\alpha=2b}\mathbf{A}_\alpha(x',t)D^\alpha$, and the derivatives $D^\alpha\mathbf{\Gamma}$ are taken with respect to y'. Each entry of the $m\times m$ matrix $D^\alpha_{y'}\mathbf{\Gamma}(x;y)$, $|\alpha|=2b$, is a Calderón-Zygmund kernel in the sense of [9, Definition 3.1], while the second integral in (3.1) is its commutator with the multiplication by the coefficients matrix $\mathbf{A}_{\alpha'}$. As for the last term, $\mathbf{F}\in L^\infty$ as consequence of the boundedness properties of the Gauss kernel. In particular, if $\mathbf{v}\in W^{2b,1}_p(\mathcal{C}_r(x_0))$ with $\mathbf{v}(x,t_0-r^{2b})=0$, then (3.1), Proposition 2.5 and $\mathbf{A}_\alpha\in VMO(Q)$ imply that for each $\varepsilon>0$ there exists $r_0=r_0(\varepsilon,\gamma_{\mathbf{A}_\alpha})$ such that

$$\|D^{2b}\mathbf{v}\|_{p,\omega;\mathcal{C}_r}\le C\big(\|\mathfrak{P}\mathbf{v}\|_{p,\omega;\mathcal{C}_r}+\varepsilon\|D^{2b}\mathbf{v}\|_{p,\omega;\mathcal{C}_r}\big)$$

whenever $r < r_0$. Choosing ε small enough we obtain

$$\|D^{2b}\mathbf{v}\|_{p,\omega;\mathcal{C}_r} \le C\|\mathfrak{P}\mathbf{v}\|_{p,\omega;\mathcal{C}_r}\,. \tag{3.2}$$

Remembering that $\mathbf{u} \in W^{2b,1}_{p,\mathrm{loc}}(Q)$ with $\mathbf{u}(x,0) = \mathbf{0}$, we extend $\mathbf{u}$ as $\mathbf{0}$ for $t < 0$, and fix an arbitrary point $x_0 \in \operatorname{supp}\mathbf{u}$ as a center of the parabolic cylinder $\mathcal{C}_r(x_0)$. Let $r \in (0, r_0)$, $\theta \in (0,1)$, $\theta' = \theta(3-\theta)/2 > 0$ and define the cut-off function

$$\varphi(x) = \varphi_1(x')\varphi_2(t) \quad \text{with} \quad \varphi_1(x') \in C_0^\infty(\mathcal{B}_r(x_0')),\ \varphi_2 \in C^\infty(\mathbb{R})$$

such that

$$\varphi_1(x') = \begin{cases} 1 & x' \in \mathcal{B}_{\theta r}(x_0') \\ 0 & x' \notin \mathcal{B}_{\theta' r}(x_0') \end{cases} \qquad \varphi_2(t) = \begin{cases} 1 & t \in (t_0 - (\theta r)^{2b}, t_0] \\ 0 & t < t_0 - (\theta' r)^{2b}\,. \end{cases}$$

Since $\theta' - \theta = \theta(1-\theta)/2$, it is clear that

$$|D^s\varphi| \le C(s)[\theta(1-\theta)r]^{-s}, \quad |D_t\varphi| \le C[\theta(1-\theta)r]^{-2b}, \quad \forall\ s = 1, 2, \dots, 2b.$$

Setting $\mathbf{v} = \varphi\mathbf{u}$, (3.2) yields

$$\begin{aligned} \|D^{2b}\mathbf{u}\|_{p,\omega;\mathcal{C}_{\theta r}} &\le \|D^{2b}\mathbf{v}\|_{p,\omega;\mathcal{C}_{\theta' r}} \le C\|\mathfrak{P}\mathbf{v}\|_{p,\omega;\mathcal{C}_{\theta' r}} \\ &\le C\left(\|\mathbf{f}\|_{p,\omega;\mathcal{C}_{\theta' r}} + \sum_{s=1}^{2b-1}\frac{\|D^{2b-s}\mathbf{u}\|_{p,\omega;\mathcal{C}_{\theta' r}}}{[\theta(1-\theta)r]^s} + \frac{\|\mathbf{u}\|_{p,\omega;\mathcal{C}_{\theta' r}}}{[\theta(1-\theta)r]^{2b}}\right), \end{aligned}$$

whence

$$\Theta_{2b} \le C\left(r^{2b}\|\mathbf{f}\|_{p,\omega;\mathcal{C}_r} + \sum_{s=1}^{2b-1}\Theta_s + \Theta_0\right) \tag{3.3}$$

as consequence of $\theta(1-\theta) \le 2\theta'(1-\theta')$ and the choice of θ', and where Θ_s stands for the seminorm

$$\sup_{0<\theta<1}[\theta(1-\theta)r]^s\|D^s\mathbf{u}\|_{p,\omega;\mathcal{C}_{\theta r}} \qquad \forall s \in \{0, \dots, 2b\}.$$

To get the claim, we need the following *interpolation inequality.*

Lemma 3.1. *There is a constant C, independent of r, such that*

$$\Theta_s \le \varepsilon\Theta_{2b} + \frac{C}{\varepsilon^{s/(2b-s)}}\Theta_0 \qquad \textit{for each } \varepsilon \in (0,2). \tag{3.4}$$

Proof. We have

$$\Theta_s \le 2[\theta_0(1-\theta_0)r]^s\|D\mathbf{u}\|^s_{p,\omega;\mathcal{C}_{\theta_0 r}}$$

for some $\theta_0 \in (0,1)$ and therefore

$$\Theta_s \le 2[\theta_0(1-\theta_0)r]^s\delta^{2b-s}\|D^{2b}\mathbf{u}\|_{p,\omega;\mathcal{C}_{\theta_0 r}} + \frac{2C'[\theta_0(1-\theta_0)r]^s}{\delta^s}\|\mathbf{u}\|_{p,\omega;\mathcal{C}_{\theta_0 r}}$$

in view of the standard interpolation inequalities and suitable scaling arguments. The bound (3.4) follows by taking $\delta = [\theta_0(1-\theta_0)r](\frac{\varepsilon}{2})^{1/(2b-s)}$ above. □

Turning back to (3.3) and choosing suitably $\varepsilon \in (0,2)$, we invoke (3.4) in order to get

$$\Theta_{2b} \leq C\big(r^{2b}\|\mathbf{f}\|_{p,\omega;\mathcal{C}_r} + \Theta_0\big).$$

Fixing $\theta = 1/2$ at the seminorm Θ_s leads to the following Caccioppoli-type estimate

$$\|D^{2b}\mathbf{u}\|_{p,\omega;\mathcal{C}_{r/2}} \leq C\big(\|\mathbf{f}\|_{p,\omega;Q} + Cr^{-2b}\|\mathbf{u}\|_{p,\omega;\mathcal{C}_r}\big). \tag{3.5}$$

A similar bound holds also for $D_t\mathbf{u}$, exploiting the parabolic structure of the equation and the boundedness of the coefficients

$$\|D_t\mathbf{u}\|_{p,\omega;\mathcal{C}_{r/2}} \leq \|\mathbf{A}\|_{\infty;\mathcal{C}_{r/2}}\|D^{2b}\mathbf{u}\|_{p,\omega;\mathcal{C}_{r/2}} + \|\mathbf{f}\|_{p,\omega;\mathcal{C}_{r/2}} \leq C\big(\|\mathbf{f}\|_{p,\omega;Q} + Cr^{-2b}\|\mathbf{u}\|_{p,\omega;\mathcal{C}_r}\big).$$

The desired estimate (2.2) follows now by means of standard covering arguments and partition of unity over the cylinder Q''. □

Acknowledgments

The authors are indebted to the referees for the valuable remarks. Both authors are members of INdAM-GNAMPA. The work of D.K. Palagachev was supported by the Italian Ministry of Education, University and Research under the Programme "Department of Excellence" L. 232/2016 (Grant No. CUP - D94I18000260001). The research of L.G. Softova was partially supported by the Project GNAMPA 2020 "Elliptic operators with unbounded and singular coefficients on weighted L^p spaces".

References

[1] M. Bramanti, M.C. Cerutti, Commutators of singular integrals on homogeneous spaces, *Boll. Un. Mat. Ital. B (VII)* **10** (1996), 843–883.

[2] S.-S. Byun, D.K. Palagachev, L. Softova, Survey on gradient estimates for nonlinear elliptic equations in various function spaces, *St. Petersburg Math. J.,* **31** (2020), No. 3, 401–419.

[3] S.-S. Byun, L. Softova, Gradient estimates in generalized Morrey spaces for parabolic operators, *Math. Nachr.,* **288** (2015), No. 14–15, 1602–1614.

[4] S.-S. Byun, L. Softova, Asymptotically regular operators in generalized Morrey spaces, *Bull. London Math. Soc.,* **52** (2020), No. 2, 64–76.

[5] A. P. Calderón, A. Zygmund, On the existence of certain singular integrals, *Acta Math.,* **88** (1952), 85–139.

[6] E.B. Fabes, N. Rivière, Singular integrals with mixed homogeneity, *Studia Math.* **27** (1966), 19–38.

[7] Mizuhara, T., Boundedness of some classical operators on generalized Morrey spaces, *Harmonic Analysis,* Proc. Conf., Sendai/Jap. 1990, ICM-90 Satell. Conf. Proc., (1991), 183–189.

[8] E. Nakai, Hardy-Littlewood maximal operator, singular integral operators and the Riesz potentials on generalized Morrey spaces, *Math. Nachr.,* **166** (1994), 95–103.

[9] D.K. Palagachev, L. Softova, A priori estimates and precise regularity for parabolic systems with discontinuous data, *Discrete Contin. Dyn. Syst.*, **13** (2005), No. 3, 721–742.

[10] L. Softova, Singular integrals and commutators in generalized Morrey spaces, *Acta Math. Sin., Engl. Ser.,* **22** (2006), No. 3, 757–766.

[11] L. Softova, Singular integral operators in functional spaces of Morrey type, In: Nguyen Minh Chuong (Ed.) et al., *Advances in Deterministic and Stochastic Analysis,* pp. 33–42, World Sci. Publ., Hackensack, NJ, 2007.