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Published source:
Giuseppe Acciani, Filomena Di Modugno, Angelamaria Abrescia, Giuseppe C. Marano,
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Mathematics and Computers in Simulation, Volume 125, 2016, Pages 70-82, ISSN 0378-4754,
https://doi.org/10.1016/j.matcom.2015.11.006.

Integration algorithm for covariance nonstationary dynamic analysis using equivalent stochastic linearization

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Abstract

Deterministic mechanical systems subject to stochastic dynamic actions, such as wind or earthquakes, have to be properly evaluated using a stochastic approach. For nonlinear mechanical systems it is necessary to approximate solutions using mathematical tools, as the stochastic equivalent linearization. It is a simple approach from the theoretical point of view, but needs numerical techniques whose computational complexity increases in case of nonstationary excitations. In this paper a procedure to solve covariance analysis of stochastic linearised systems in the case of nonstationary excitation is proposed. The nonstationary Lyapunov differential matrix covariance equation for the linearised system is solved using a numerical algorithm which updates linearised system coefficient matrix at each step. The technique used is a predictor-corrector procedure based on backward Euler method. Accuracy and computational costs are analyzed showing the efficiency of the proposed procedure.

Keywords: Lyapunov Equation, Covariance analysis, stochastic dynamics, Bouc-Wen mechanical model, equivalent linearization

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1. Introduction

Many problems in structural dynamics deal with loads whose nature is intrinsically non deterministic, as earthquakes, wind pressures or sea waves. Due to intrinsic random nature of inputs random dynamic analysis is the best way to use. In the analysis of linear systems, the physical solution can be obtained by using standard analytical approaches that are able to describe response moments statistics considering input characteristics and system parameters, both in time and in frequency domains. Moreover there are many cases where structures exhibit a non linear behaviour, such as buildings subject to strong earthquakes. In those situations analytical approach does not work for many nonlinear systems because the theoretical solutions of nonlinear systems are known only in some special cases [29]. In the analysis of deterministic nonlinear systems, one of the most effective methods is the perturbation method, which was firstly proposed by Poincaré and has been extensively studied [28]. The counterpart in random analysis of nonlinear systems has also been studied (see [33]). An alternative approach is the statistical linearization method, also referred as stochastic (or equivalent) linearization [30, 31], proposed independently but simultaneously by Booton [3], Kazakov [16], Lu and Evan-Iwanowski [18]. The coefficient of linearized terms are obtained minimizing a stochastic measure of differences between the two solutions. This measure needs knowledge of state space (system displacement and velocity) and joint probability distribution, but they are often unknown. In many cases, a circular interdependency between linearization coefficients and structural stochastic response is made; therefore, an iterative algorithm is needed to break this loop. Accuracy, efficiency and limits of this approach has been widely studied both for stationary [30] and non stationary situations [10, 17, 21, 27]. The algorithm complexity increases dealing with nonstationary situations, where evolutive state space and joint probability density induce time variant stochastic equivalent linearization coefficients. A more complex problem is to consider nonstationary inputs as external forces like seismic actions. The inner evolutive nature of the input has serious consequences in structural responses, and this has been observed both in general cases, such as Duffing models, or real engineering problems, as buildings or bridges [32, 14]. To approach nonstationary situations is better to use multicorrelated inputs even if it produces numerical difficulties because input should be defined by using processes modulated in amplitude and/or in frequency [23]. In these problems analytical solutions of random responses are

known explicitly only in few cases [5, 15, 19]. Hence in applications it is necessary to approximate the solution. The most common way is to use Monte Carlo methods that consist in generating a suitable number of time histories of the excitations and integrating directly in order to produce a simulation of time history analysis. It is possible to obtain an accurate statistical analysis of responses, but with high computational cost that grows up as the number of direct integrations. Hence a simplified linearized covariance analysis shows a better appeal as a significant reduction of computational efforts. The main goal of the paper is to propose a computational method reducing the computational cost taking account of some covariance matrix properties [20]. A numerical approach for space state evolutive covariance matrix evaluation is proposed; the nonlinear behaviour is overcome using the equivalent statistical linearization technique, and the Lyapunov matrix differential equation is solved using backward Euler method (a modification of the one proposed for linear systems in [11, 24]). The algorithm is thus described and a predictor-corrector scheme for a step by step linearised coefficients update is proposed. The algorithm is applied to a nonstationary Gaussian filtered input, modulated in amplitude and frequency, acting on a generic SDoF base excited system, modelled on the Bouc-Wen model. The paper is organized as follows: in Section 2 the statistical linearization approach and the predictor-corrector numerical scheme are fully analyzed. An application is described in Section 3, while in Sections 4 and 5, respectively, the statistical properties together with some numerical issues are shown. Finally some remarks and further possible work are reported in Section 6.

2. Statistical linearization approach

Consider the nonlinear SDoF system:

$$m\ddot{x} + g(x, \dot{x}) = \xi(t) \tag{1}$$

where m is the mass, $g(x, \dot{x})$ is the nonlinear force including the nonlinear damping and restoring forces and $\xi(t)$ is a stochastic excitation. The main idea behind a statistical linearization approximation is to replace equation (1) with an equivalent linear one:

$$m\ddot{y} + c_{\text{eq}}\dot{y} + k_{\text{eq}}y = \xi(t) \tag{2}$$

where c_{eq} and k_{eq} are the equivalent damping and stiffness respectively, such that the error between the solutions of the two systems is minimized with

the mean-square method. The difference between (1) and (2)

$$e_{\mathrm{ex}} = m\ddot{x} + g(x, \dot{x}) - (m\ddot{y} + c_{\mathrm{eq}}\dot{y} + k_{\mathrm{eq}}y)\,. \tag{3}$$

represents the error approximating the solution of (1) with that of (2). However, the response x of the nonlinear system is unknown hence the error so defined would be intractable. On the contrary, to obtain the equivalent response y in (2) is easier. Hence it is better to solve (2) instead of (1), defining the error by (3) where x is replaced by y;

$$e_{\mathrm{app}} = g(y, \dot{y}) - c_{\mathrm{eq}}\dot{y} - k_{\mathrm{eq}}y. \tag{4}$$

In order to choose the best equivalent damping c_{eq} and the equivalent stiffness k_{eq}, it is necessary to minimize the error with statistical procedure requiring the expectation of (4) to be zero,

$$E\left\langle e_{\mathrm{app}}\right\rangle = 0,$$

where $E\left\langle \bullet \right\rangle$ denotes the mathematical expectation, and minimizing the second-order moment, namely the mean-square error:

$$E\left\langle e_{\mathrm{app}}^2\right\rangle = E\left\langle (g(y, \dot{y}) - c_{\mathrm{eq}}\dot{y} - k_{\mathrm{eq}}y)^2\right\rangle. \tag{5}$$

This requires that:

$$\frac{\partial E\left\langle e_{\mathrm{app}}^2\right\rangle}{\partial c_{\mathrm{eq}}} = 0 \tag{6}$$

and

$$\frac{\partial E\left\langle e_{\mathrm{app}}^2\right\rangle}{\partial k_{\mathrm{eq}}} = 0 \tag{7}$$

Equations (6) and (7) yield two linear equations and, therefore, give the optimal values of c_{eq} and k_{eq}

$$c_{\mathrm{eq}} = \frac{E\left\langle g(y, \dot{y})\dot{y}\right\rangle E\left\langle y^2\right\rangle - E\left\langle g(y, \dot{y})y\right\rangle E\left\langle y\dot{y}\right\rangle}{E\left\langle \dot{y}^2\right\rangle E\left\langle y^2\right\rangle - \left(E\left\langle y\dot{y}\right\rangle\right)^2} \tag{8}$$

$$k_{\mathrm{eq}} = \frac{E\left\langle g(y, \dot{y})y\right\rangle E\left\langle \dot{y}^2\right\rangle - E\left\langle g(y, \dot{y})\dot{y}\right\rangle E\left\langle y\dot{y}\right\rangle}{E\left\langle \dot{y}^2\right\rangle E\left\langle y^2\right\rangle - \left(E\left\langle y\dot{y}\right\rangle\right)^2}. \tag{9}$$

It is important to note that to obtain the optimal values of c_{eq} and k_{eq}, in this procedure in (8) and (9) the joint probability density function (PDF)

$p_Z(z)$ of the state space $z = [\ y\ \ \dot{y}\ ]^T$ is needed. This means that to solve the linear random vibration system (2) the values of c_{eq} and k_{eq} are needed. So a circular interdependency is produced for this problem and therefore, an iterative algorithm is useful to break the loop. This scheme is illustrated in Figure 1, where the superscripts represent the steps of iteration and $p_Z(z)$ represents the probabilistic information (e.g. the joint statistics or PDF). The iteration could reach the end if the error of c_{eq} and k_{eq} between two subsequent iterations is limited within the tolerance:

$$\left\| c_{\text{eq}}^{(j+1)} - c_{\text{eq}}^{(j)} \right\| < \varepsilon_{c_{\text{eq}}}$$

$$\left\| k_{\text{eq}}^{(j+1)} - k_{\text{eq}}^{(j)} \right\| < \varepsilon_{k_{\text{eq}}}$$

or the error of the probabilistic characteristics of the state space z are bounded by tolerances $\varepsilon_{E\langle y^2\rangle}$ and $\varepsilon_{E\langle \dot{y}^2\rangle}$:

$$\left\| \left(E\left\langle y^2 \right\rangle\right)^{(j+1)} - \left(E\left\langle y^2 \right\rangle\right)^{(j)} \right\| < \varepsilon_{E\langle y^2\rangle} \tag{10}$$

$$\left\| \left(E\left\langle \dot{y}^2 \right\rangle\right)^{(j+1)} - \left(E\left\langle \dot{y}^2 \right\rangle\right)^{(j)} \right\| < \varepsilon_{E\langle \dot{y}^2\rangle}. \tag{11}$$

In the case of a multi degree of freedom (MDoF) mechanical problem, it should be set as a linearised equation in the state space $\overline{z} = \left[\ \overline{y}\ \ \overline{\dot{y}}\ \right]^T$, where $\overline{y}$ and $\overline{\dot{y}}$ are, respectively, the displacement and the velocity vector, both with n degrees of freedom, in the following way:

$$\dot{\overline{z}} = \mathbf{A}_{\text{eq}}\overline{z} + \overline{\xi} \tag{12}$$

In Equation (12) $\mathbf{A}$ is the state space matrix, whose coefficients depending on the state space covariance matrix R_{zz}:

$$\mathbf{R}_{zz} = E\left\langle \overline{z}\overline{z}^T \right\rangle \tag{13}$$

that is the solution of the stationary Lyapunov matrix equation:

$$\left[\mathbf{A}_{\text{eq}}\left(\mathbf{R}_{zz}\right)\right]\mathbf{R}_{zz} + \mathbf{R}_{zz}\left[\mathbf{A}_{\text{eq}}^T\left(\mathbf{R}_{zz}\right)\right] + \mathbf{B} = 0 \tag{14}$$

and

$$\mathbf{A}_{\text{eq}} = \mathbf{A}_{\text{eq}}\left(\mathbf{R}_{zz}\right). \tag{15}$$

A similar scheme of that proposed in Figure 1 is applicable to solve the couple problem of (14) and (15): the problem so stated is well known in literature

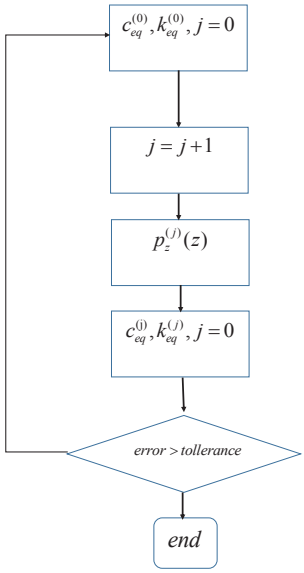

Figure 1: Scheme to evaluate the best equivalent damping c_{eq} and the equivalent stiffness k_{eq}.

and many standard and commercial codes to solve the Lyapunov equation in (14) exist, such as Lyap command in Matlab. The scheme so far proposed holds only for stationary problems, while nonlinear and nonstationary ones require the solution of a time depending Lyapunov Equation. It is important to note that determined optimal values of c_{eq} and k_{eq} are time variant if the response is a nonstationary process and therefore, the equivalent system in Equation (2) or in (14) for general nonlinear system is a time-variant linear system. The main question is that the state matrix has to be updated at each integration step because linearised coefficients depend on the same unknown covariance matrix that is time dependent in this case:

$$\mathbf{A}_{\text{eq}} = \mathbf{A}_{\text{eq}}\left(\mathbf{R}_{zz}(t), t\right). \tag{16}$$

Here the state space covariance matrix is the solution of nonstationary Lyapunov equation in the form:

$$\dot{\mathbf{R}}_{zz}(t) = \mathbf{A}_{\text{eq}}\left(\mathbf{R}_{zz}, t\right)\mathbf{R}_{zz}(t) + \mathbf{R}_{zz}(t)\mathbf{A}_{\text{eq}}^{T}\left(\mathbf{R}_{zz}, t\right) + \mathbf{B}(t) \tag{17}$$

where $\dot{\mathbf{R}}_{zz} = \frac{d}{dt}\mathbf{R}_{zz}$. To solve this specific problem a simple numerical predictor-corrector algorithm is developed in this paper, using an iterative procedure to update the linearised system matrix at each step as its time varying coefficients. The total duration $[0, T_{\text{tot}}]$ is divided in m equal sub-interval of length $\Delta t = t_{h+1} - t_h$, $h = 0, 1, 2, \ldots, m-1$, and in each sub-interval a linear variation of the time derivative covariance matrix $\dot{\mathbf{R}}(t)$ is assumed. The backward Euler method is used as predictor:

$$\text{Predictor}: \qquad \mathbf{R}^p_{(h+1)} = \mathbf{R}_{(h)} + \Delta t \dot{\mathbf{R}}_{(h)} \tag{18}$$

while the trapezoidal rule is used as corrector:

$$\text{Corrector}: \qquad \mathbf{R}^c_{(h+1)} = \mathbf{R}_{(h)} + \frac{\Delta t}{2}\left(\dot{\mathbf{R}}_{(h)} + \dot{\mathbf{R}}^p_{(h+1)}\right) \tag{19}$$

where the symbol $a_{(h)}$ denotes the generic quantity a evaluated at time $t = h\Delta t$. For each integration step h is evaluated as prediction of covariance matrix with (18); than it is used to obtain a corrected value using the implicit scheme in (19). Moreover the coefficients of linearised state matrix A depend on covariance matrix so there is an implicit evolution of A, so the system matrix A_{eq} has to be updated at each time depending on the covariance matrix at the same time:

$$\mathbf{A}_{\text{eq}(h)} = \mathbf{A}_{\text{eq}}\left(\mathbf{R}_{(h)}(t), t_h\right). \tag{20}$$

This equation is still implicit because the evaluation of generic unknown covariance $\mathbf{R}_{(h+1)}$ at step $h+1$, needs still unknown state space matrix $\mathbf{A}_{\text{eq}(h+1)}$ at step $h+1$ while $\mathbf{A}_{\text{eq}(h)}$ is known because it depends only on $\mathbf{R}_{(h)}$. A first evaluation of $\mathbf{A}_{\text{eq}(h)}$ is obtained using (20) with predictor $\mathbf{R}^p_{(h+1)}$ of (18). Hence a second more accurate evaluation is obtained using (20) with corrector $\mathbf{R}^c_{(h+1)}$ of (19). For a more accurate evaluation of $\mathbf{A}_{\text{eq}(h+1)}$ and $\mathbf{R}_{(h+1)}$ an iterative scheme based on a modification of (19) is then possible. Starting from the corrector evaluation of the covariance matrix $\mathbf{R}^1_{(h+1)} = \mathbf{R}^c_{(h+1)}$, it is updated with the following scheme:

$$\mathbf{R}^{j+1}_{(h+1)} = \mathbf{R}_{(h)} + \frac{\Delta t}{2}\left(\dot{\mathbf{R}}_{(h)} + \dot{\mathbf{R}}^{j}_{(h+1)}\right) \tag{21}$$

$$\mathbf{A}^{j+1}_{\text{eq}(h)} = \mathbf{A}_{\text{eq}}\left(\mathbf{R}^{j+1}_{(h+1)}(t), t_h\right). \tag{22}$$

The convergence of the iterative process is reached when the following stopping criterion is satisfied:

$$\varepsilon_{A_{\rm eq}}^{(j+1)} = \| \mathbf{A}_{\rm eq}\left(\mathbf{R}_{(h+1)}^{j+1}\right) - \mathbf{A}_{\rm eq}\left(\mathbf{R}_{(h+1)}^{j}\right) \| \leq \varepsilon_{\max}. \tag{23}$$

It is a way to achieve the given accuracy as the implicit prevision of matrix $\mathbf{A}_{\rm eq}$ will differs smoothly from its explicit prediction. In this way there are two iterative processes, the first for time integration and the latter to satisfy (23). In this way reducing the time step will produce a greater number of iterations to evaluate the system matrix as in (23). As it is clear in general a greater value of time step will reduce the total steps for time integration, but will increase the number of iterations as stopping criterion (23) is reached. In order to obtain knowledge of covariance matrix evolution, the scheme proposed has to be repeated for each time step and must be solved in sequence for each time value t_h, starting from $t_0 = 0$ (whose covariance matrix value $\mathbf{R}_{(0)} = 0$). In this way the m unknown matrices $\mathbf{R}_{(h)}$, should be determined, defining statistics of structural response.
The integration scheme, written in pseudo-code, is the following:

INITIALIZE THE OUTER LOOP:
$h = 0,\ t_0 = 0,\ \mathbf{R}_1 = \mathbf{R}(t=0),\ \mathbf{A}_1 = \mathbf{A}_{\rm eq}(\mathbf{R}_1)$
`while` $t_h \leq T_{\rm tot}$
  $t_{h+1} = t_h + \Delta t$
  COMPUTE $\mathbf{R}_{(h+1)}^{p}$ AND $\dot{\mathbf{R}}_{h+1}^{p}$ USING (18) AND (17)
  COMPUTE $\mathbf{R}_{(h+1)}^{c}$ AND $\dot{\mathbf{R}}_{h+1}^{c}$ USING (19) AND (17)
  INITIALIZE THE INNER LOOP:
  $j = 1,\ \mathbf{R}_{(h+1)}^{1} = \mathbf{R}_{(h+1)}^{c},\ \mathbf{A}_{{\rm eq}(h+1)}^{1} = \mathbf{A}_{\rm eq}\left(\mathbf{R}_{(h+1)}^{1}\right)$
  `while` $\left\| \mathbf{A}_{h+1}^{(j+1)} - \mathbf{A}_{h+1}^{(j)} \right\| \leq \varepsilon_{A_{\rm eq}}$
    COMPUTE $\mathbf{R}_{(h+1)}^{j+1}$ USING (21)
    COMPUTE $\mathbf{A}_{h+1}^{(j+1)}$ AND $\dot{\mathbf{R}}_{h+1}^{j+1}$ USING (22) AND (17)
    $j = j + 1$
  `end`
  $h = h + 1$
`end`

3. An application: Bouc-Wen system subject to nonstationary base random input

The Bouc-Wen model is a nonlinear differential model often taken on to assess the structural response of buildings subjected to earthquakes (see

[2, 4, 34]). The model in Figure 2 with the following motion equation

$$m\ddot{x}(t) + F(x, \dot{x}, z, t) = ma(t) \tag{24}$$

describes a single degree of freedom system, having a mass m, subject to a generic base acceleration $a(t)$ and characterized by a hysteretic constitutive law. In (24) the first term represents the inertia force whereas $F(x, \dot{x}, z)$ is

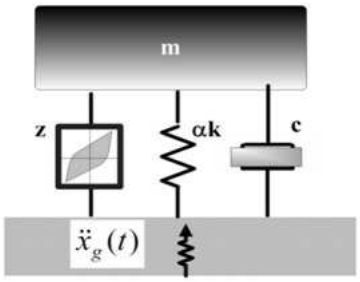

Figure 2: SDoF nonlinear system Bouc-Wen model.

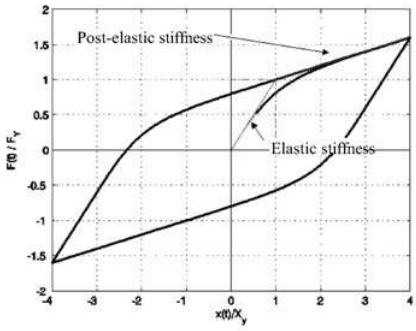

Figure 3: Bouc-Wen Hysteresic force-displacement model.

the restoring force. This term can be seen as the sum of two terms:

$$F(x, \dot{x}, z; t) = L(x, \dot{x}; t) + H(x, \dot{x}, z; t). \tag{25}$$

The first one $L(x, \dot{x}; t)$ is due to the linear viscous-elastic contribution while the second $H(x, \dot{x}, z; t)$ is due to the hysteretic one:

$$L(x, \dot{x}; t) = c\dot{x}(t) + \alpha k x(t) \tag{26}$$

$$H(x, \dot{x}, z; t) = (1 - \alpha)kz(t) \tag{27}$$

where c is the damping and k is the initial elastic stiffness. The function $z(t)$ satisfies the following nonlinear differential equation:

$$\dot{z}(t) = G(z, \dot{x}) = \dot{x}(t) \left[\lambda - |z(t)|^{\eta} \left(\beta + \gamma \text{sgn}\{z(t)\}\text{sgn}\{\dot{x}(t)\}\right)\right]. \tag{28}$$

The parameters β, γ, η, α and λ in (28) are the shape factors of the hysteretic cycle. Mechanical quantities can be related to analytical parameters in order to model real structural elements. When is necessary the analysis of mechanical systems exhibiting a softening behavior (a reduction of stiffness for great deformations), as for buildings when exposed to strong earthquakes, it is possible to define the initial and the post-elastic stiffness, k_i and k_f, as

$$k_i = \left(\frac{\partial G}{\partial x}\right)_{Z=0} = \alpha k + (1 - \alpha)k\lambda$$

and

$$k_f = \left(\frac{\partial G}{\partial x}\right)_{Z=z_{\max}} = \alpha k.$$

The maximum asymptotic value of internal variable z is

$$z_{\max} = \left(\frac{\lambda}{\beta + \gamma}\right)^{\frac{1}{\eta}}.$$

The ultimate hysteretic restoring force is given by:

$$F_h^{\max} = (1 - \alpha)kz_{\max}$$

In addition it is possible to define the elastic limit displacement X_Y:

$$X_Y = \frac{F_h^{\max}}{(1 - \alpha)k\lambda} = \frac{z_{\max}}{\lambda} = \lambda^{\frac{1-\eta}{\eta}} (\beta + \gamma)^{-\frac{1}{\eta}} \tag{29}$$

The corresponding yielding resistance is given by:

$$F_Y = k_i X_Y = (\alpha k + (1 - \alpha)k\lambda)\lambda^{\frac{1-\eta}{\eta}} (\beta + \gamma)^{-\frac{1}{\eta}}. \tag{30}$$

From (29) and (30) it is possible to observe that if $\lambda = 1$ then $k = k_i$, α becomes the ratio between the post-elastic and initial elastic stiffness (for $\beta = \gamma$):

$$X_Y = \left(\frac{1}{2\beta}\right)^{\frac{1}{\eta}} \tag{31}$$

and

$$F_Y = kX_Y = k\left(\frac{1}{2\beta}\right)^{\frac{1}{\eta}}. \tag{32}$$

hence

$$\beta = \frac{1}{2X_Y^{\eta}}.$$

It has been found [22] that the parameters of the Bouc-Wen model are functionally redundant. Removing this redundancy is best achieved by setting $\lambda = 1$. Moreover if the unloading stiffness is equal to the elastic one (as it happens in many cases) then $\beta = \gamma$. In order to describe the mechanical constitutive law only three mechanical parameters are needed:
• initial elastic stiffness k_i;
• post-elastic stiffness k_f;
• maximum hysteretic restoring force F_Y (or maximum elastic displacement X_Y).
Considering now a seismic excitation $F(t) = -m\ddot{x}_g(t)$, the motion equations are:

$$\begin{cases} m\ddot{x}(t) + c\dot{x}(t) + \alpha kx(t) + (1-\alpha)kz(t) = -m\ddot{x}_g(t) \\ \\ \dot{z}(t) = \dot{x}(t)\left[1 - z(t)\beta\left(1 + sgn\{z(t)\}sgn\{\dot{x}(t)\}\right)\right] \end{cases} \tag{33}$$

where the parameter η can be properly evaluated according to the relative smoothness in transition between elastic and post-elastic phases. In the following it will be assumed $\eta = 1$ as a common approach, with a smooth transition between elastic and post-elastic phase, as done in many works. Introducing $\omega_0^2 = k/m$, $f_Y = F_Y/m$ and $2\xi_0\omega_0 = c/m$, and considering (31) when $\eta = 1$, equations (33) become:

$$\begin{cases} \ddot{x}(t) + 2\xi_0\omega_0\dot{x}(t) + \alpha\omega_0^2 x(t) + (1-\alpha)\omega_0^2 z(t) = -\ddot{x}_g(t) \\ \\ \dot{z}(t) = \dot{x}(t)\left[1 - \frac{1}{2}\left(\frac{z(t)}{X_Y}\right)\left(1 + sgn\{z(t)\}sgn\{\dot{x}(t)\}\right)\right]. \end{cases}$$

3.1. The Bouc-Wen model equivalent linearization

The approximate linearised form of the original nonlinear equation is then achieved minimizing the difference between the nonlinear equation and the linearised one. Then, the equation governing the internal variable $z(t)$ is replaced with the following:

$$\dot{z}(t) = G(z, \dot{x}) = -c_{\text{eq}}(t)\dot{x}(t) - k_{\text{eq}}(t)z(t) \tag{34}$$

where the linearised evolutive coefficients $c_{\mathrm{eq}}(t)$ and $k_{\mathrm{eq}}(t)$ are nonlinear functions of covariance response elements. Atalik and Utku [1] provided these equivalent coefficients, which appear in equations (8) and (9): for the most common case where $\eta = 1$, $\beta = \gamma$ and $\lambda = 1$, in the hypothesis of processes z and $\dot{x}$ jointly Gaussian, the time variant equivalent coefficients are:

$$c_{\mathrm{eq}}(t) = \sqrt{\frac{2}{\pi}}\beta \left[\sigma_z(t) + \frac{E\left\langle \dot{x}(t)z(t)\right\rangle}{\sigma_{\dot{x}}(t)}\right] - 1 \tag{35}$$

$$k_{\mathrm{eq}}(t) = \sqrt{\frac{2}{\pi}}\beta \left[\sigma_{\dot{x}}(t) + \frac{E\left\langle \dot{x}(t)z(t)\right\rangle}{\sigma_z(t)}\right] \tag{36}$$

where terms $\sigma_z(t)$ and $\sigma_{\dot{x}}(t)$ are the standard deviations of z and $\dot{x}$, respectively, and $E\left\langle \dot{x}(t)z(t)\right\rangle$ is their cross covariance.

3.2. Modelling the input ground motion

It is widely known that seismic accelerograms are often modelled as a zero mean stochastic nonstationary processes. A widely used description with constant content both in amplitude and in frequency could be considered. An extensively applied stochastic approach is that proposed by Clough and Penzien [8] which considers a linear fourth-order filter, obtained by a series of two linear oscillators, forced by a modulated white noise ([9, 12, 25]). Ground acceleration $\ddot{X}_g(t)$ is therefore given by:

$$\begin{cases} \ddot{X}_g(t) = -\omega_p^2 X_p(t) - 2\xi_p\omega_p\dot{X}_p(t) + \omega_f^2 X_f + 2\xi_f\omega_f\dot{X}_f(t) \\ \\ \ddot{X}_p(t) + \omega_p^2 X_p(t) - 2\xi_p\omega_p\dot{X}_p(t) = \omega_f^2 X_f + 2\xi_f\omega_f\dot{X}_f(t) \\ \\ \ddot{X}_f(t) + 2\xi_f\omega_f\dot{X}_f(t) + \omega_f^2 X_f = -\phi(t)W(t) \end{cases} \tag{37}$$

where $X_f(t)$ is the response of the first filter, having frequency ω_f and damping coefficient ξ_f, $X_p(t)$ is the response of the second filter characterized by frequency ω_p and damping ratio ξ_p. Moreover $W(t)$ is the white noise stochastic process, whose constant bilateral power spectral density function is S_0 and $\phi(t)$ is the modulation function. In this study the function proposed by Jennings et al. is adopted [13]. Several expressions relate the maximum value of ground acceleration to S_0: this maximum value is called the peak ground acceleration (PGA) and is indicated with $\max\left(|\ddot{X}_g(t)|\right)$. A common way to express PGA is:

$$PGA = 3\sigma_{\ddot{X}_g}. \tag{38}$$

In this way after some rearrangements it is obtained [26]:

$$S_0 = \frac{0.222}{\pi} \frac{PGA^2 \xi_f \xi_p \left(\omega_f^4 + 4\xi_f \xi_p \omega_f^3 \omega_p + 2(-1 + 2\xi_f^2 + 2\xi_p^2)\omega_f^2 \omega_p^2 + 4\xi_f \xi_p \omega_f \omega_p^3 + \omega_p^4\right)}{\omega_f^2 \left((1 + 4\xi_f^2)\xi_p \omega_f^3 + \xi_f (1 + 16\xi_f^2 \xi_p^2)\omega_f^2 \omega_p + 16\xi_f^4 \xi_p \omega_f \omega_p^2 + 4\xi_f^3 \omega_p^3\right)}.$$

4. Stochastic analysis

The linearised motion equations for the system plus filter are given by:

$$\begin{cases} \ddot{X}(t) + 2\xi_s \omega_s \dot{X}(t) + \alpha \omega_s^2 X(t) + (1-\alpha) Z(t) \omega_s^2 = \omega_p^2 X_p(t) + \\ \qquad + 2\xi_p \omega_p \dot{X}_p(t) - \omega_f^2 X_f(t) - 2\xi_f \omega_f \dot{X}_f(t) \\ \ddot{X}_p(t) + \omega_p^2 X_p(t) + 2\xi_p \omega_p \dot{X}_p(t) = \omega_f^2 X_f + 2\xi_f \omega_f \dot{X}_f(t) \\ \ddot{X}_f(t) + 2\xi_f \omega_f \dot{X}_f(t) + \omega_f^2 X_f = -\phi(t) W(t) \\ \dot{Z}(t) = -c_{\text{eq}} \dot{X}(t) - k_{\text{eq}} Z(t). \end{cases} \tag{39}$$

Introducing the state vector $\overline{Y}(t) = \left[X(t), X_p(t), X_f(t), Z(t), \dot{X}(t), \dot{X}_p(t), \dot{X}_f(t)\right]^T$ equation (39) becomes:

$$\dot{\overline{Y}}(t) = \mathbf{A}_{\text{eq}}(\mathbf{R}, t) \overline{Y}(t) + \overline{F}(t) \tag{40}$$

where $\overline{F}(t) = [\ 0\ 0\ 0\ 0\ 0\ 0\ -\phi(t)W(t)\ ]$ is the forcing vector and

$$\mathbf{A}_{\text{eq}}(\mathbf{R}, t) = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -k_{\text{eq}}(\mathbf{R}, t) & -c_{\text{eq}}(\mathbf{R}, t) & 0 & 0 \\ -\alpha\omega_s^2 & +\omega_p^2 & -\omega_f^2 & -(1-\alpha)\omega_s^2 & -2\xi_s\omega_s & +2\omega_p\xi_p & -2\omega_f\xi_f \\ 0 & -\omega_p^2 & +\omega_f^2 & 0 & 0 & -2\omega_p\xi_p & +2\omega_f\xi_f \\ 0 & 0 & -\omega_f^2 & 0 & 0 & 0 & -2\omega_f\xi_f \end{bmatrix}$$

is the equivalent system matrix.
It is possible to perform the covariance analysis starting from equation (40) and solving the matrix differential Lyapunov equation:

$$\dot{\mathbf{R}}(t) = \mathbf{A}_{\text{eq}}(\mathbf{R}, t)\, \mathbf{R}(t) + \mathbf{R}(t) \mathbf{A}_{\text{eq}}^T(\mathbf{R}, t) + \mathbf{B}(t)$$

where $\mathbf{R}(t) = \langle \overline{Y} \overline{Y}^T \rangle$ is the covariance matrix, while $\mathbf{B}(t)$ is a square matrix having all zero entries except the last one equal to $2\pi S_0 \phi(t)^2$. Coefficients k_{eq}

and c_{eq} in matrix depend on the elements of $\mathbf{R}(t)$, and therefore an iterative procedure is required to solve equation (38) as for equation (17). This is solved using a Matlab code where the coefficients of the equivalent linear system are updated at each step and the solution of response covariance matrix equation is searched for.

5. Numerical analysis

Using a Bouc-Wen model, in Figure 4 a nonlinear SDoF system is analysed under a nonstationary base acceleration, modelled by a filtered separable white noise process. To evaluate numerical efficiency of proposed algorithm, nonstationary covariance response is evaluated under different nonlinear mechanical parameters and using different time step value Δt. The modulation function used is the following [13]:

$$\phi(t) = \alpha_\phi t e^{-\beta_\phi t} \tag{41}$$

where the maximum intensity is reached at $T_{\max} = 10$ (sec), $\phi(T_{\max}) = 1$ (see Figure 4) and α_ϕ and β_ϕ are so defined:

$$\alpha_\phi = \frac{e}{T_{\max}}, \qquad \beta_\phi = \frac{1}{T_{\max}}.$$

The filtered white noise in (37) is modelled with the following parameters: $\omega_f = 20$ rad/sec, $\omega_p = 5$ rad/sec, $\xi_f = 0.6$ and $\xi_p = 0.7$.
The PGA is assumed as 0.15 (g), while the other elastic mechanical parameters are a natural period $T_0 = 1.0$ (sec), $\omega_0 = 6.28$ (rad/sec) and a damping $\xi_0 = 0.05$. Three different values of post-elastic/elastic ratio are used ($\alpha = 0.1, 0.5, 0.8$) to compare numerical performances under different levels of nonlinearity. For each case numerical results have been evaluated under different time step values Δt, in the range $0.005 - 0.045$ (sec.). Finally in (23) it is assumed $\varepsilon_{\max} = 10^{-7}$. In Figure 5 are reported main system structural responses from covariance equation, in terms of state space vector (displacement and velocity vectors) $\overline{z}$ standard deviation, compared with the pure elastic response (only for displacement and velocity). In Figures 6 and 7 are reported evaluated linearised coefficients for each case.

The results are stable for the time steps in the range of analysis (from 0.005 to 0.045 sec) both for covariance and linearised coefficients. An interesting result is the evaluation of computational cost under different time

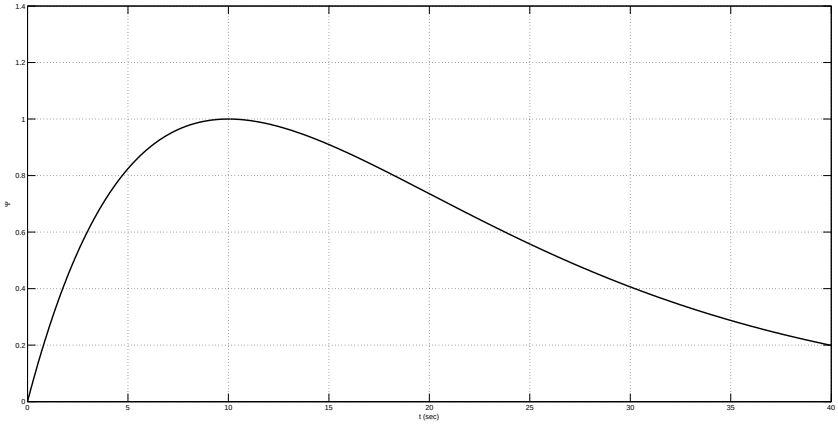

Figure 4: Modulation function.

steps that has been evaluated as the number of iterations j_{tot} necessary to reache convergence in the inner loop of predictor-corrector scheme. This value is multiplied by the number of time steps necessary to simulate the whole process in the time window $[0, T_{\text{tot}} = 40(\text{sec})]$. Figure 8 shows that, for a given accuracy, computational costs are strongly influenced by time steps. Actually when the time step is too small the method requires less iterations to satisfy the stop criterion, but a greater number of integration points is required. On the contrary, increasing the time step size, the required integration points decrease, but to reach the given accuracy a greater number of iterations is required at each time step. It is evident that there is an equilibrium, corresponding to a minimum of total evaluation, between number of integration points and number of iterations useful for each time step, and it is independent from the stiffness ration used. This is important to decide the most appropriate time step but it depends on the problem and its parameters.

6. Conclusions and future work

In this work a numerical approach to integrate nonstationary Lyapunov matrix equation has been presented to improve stochastic linearization technique adopted for nonlinear systems representation. The proposed algorithm is implemented using a predictor/corrector scheme, where the linearised co-

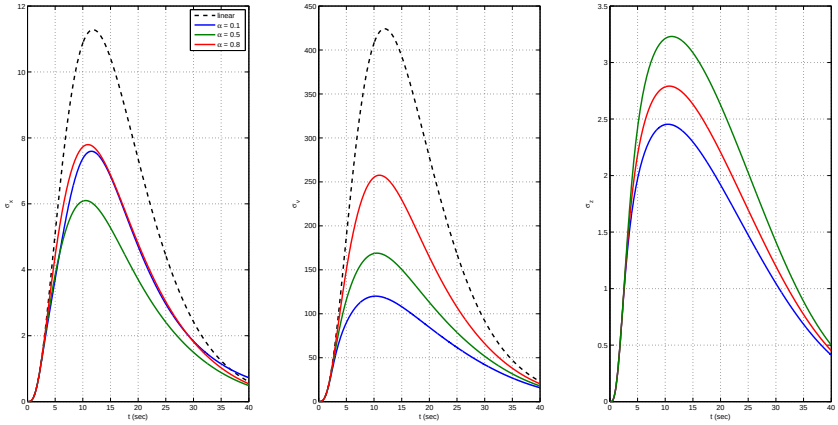

Figure 5: Evolutive structural covariance $(\sigma_x, \sigma_{\dot{x}}, \sigma_z)$.

efficients update is performed by an iterative scheme at each step. The numerical integration algorithm has been used for an important case of mechanical nonlinearity, the hysteretic Bouc-Wen model. A single degree of freedom system modelled and subject to a nonstationary filtered white noise has been evaluated under different mechanical conditions and time step sizes. Results show that the proposed algorithm is stable and accurate, while time step size plays a central role in total computational costs, so that an optimum time step should be deducted comparing total computational costs for different steps Δt.

The main future research is to use a variable step size technique for the predictor-corrector scheme, like the following, if

$$e_{n+1} = \frac{\left\| \mathbf{A}_{\text{eq}} \left(\mathbf{R}_{(h+1)}^{j+1} \right) - \mathbf{A}_{\text{eq}} \left(\mathbf{R}_{(h+1)}^{j} \right) \right\|}{\left\| \mathbf{A}_{\text{eq}} \left(\mathbf{R}_{(h+1)}^{j} \right) \right\|}$$

where the norm is the Frobenius or the infinity one, $\delta > \varepsilon_{\max}$, $a > 1$, $b < 1$, with δ, a and b chosen parameters (see [6, 7]), it is possible to define Δt according to the following criterion: if $e_{n+1} \leq \varepsilon_{\max}$ then $\Delta t = a\Delta t$ else if $e_{n+1} \geq \delta$ then $\Delta t = b\Delta t$. The step Δt unchanges if $\varepsilon_{\max} < e_{n+1} < \delta$. Some possible values could be $a = 1.2$, $b = 0.8$, $\varepsilon = 10^{-5}$ and $\delta = 10^{-4}$.

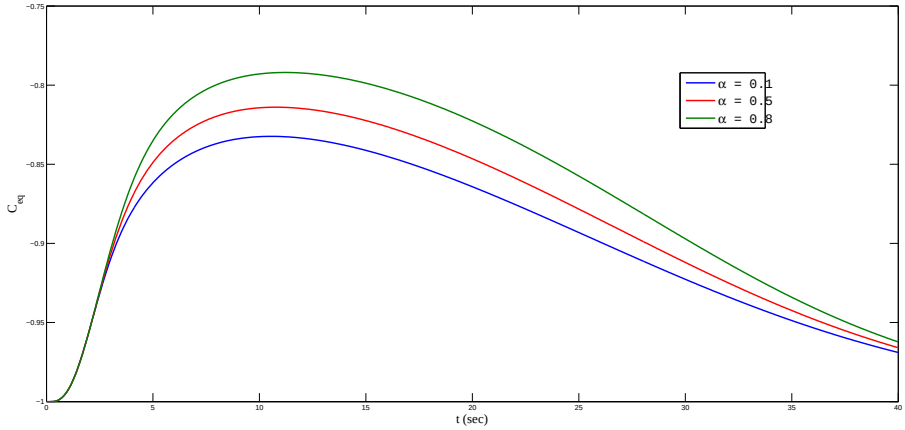

Figure 6: Evolution of linearized coefficients c_{eq}.

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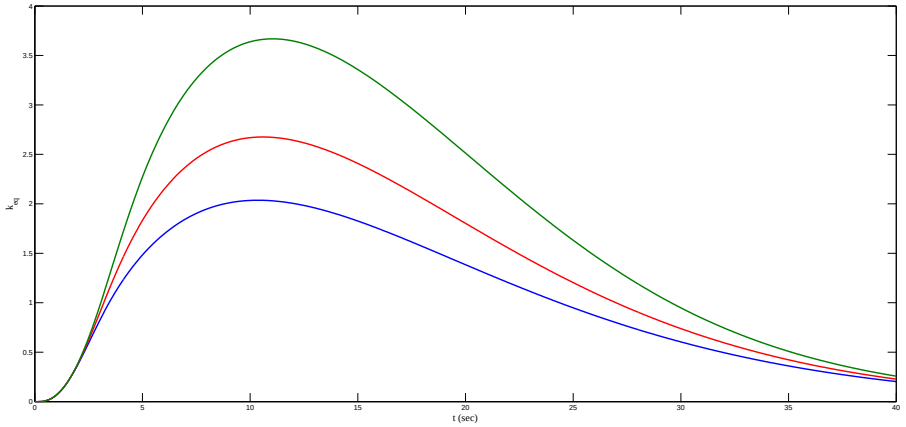

Figure 7: Evolution of linearized coefficients k_{eq}.

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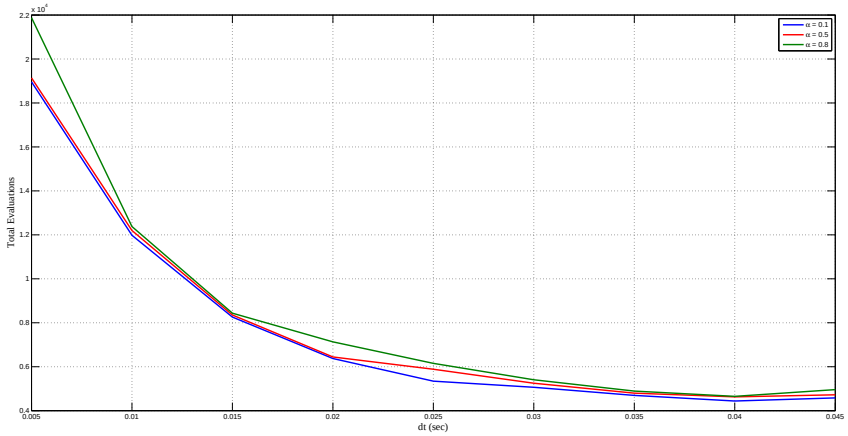

Figure 8: Iterations for different post-yielding stiffness.

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