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A note on "Double-diffusive Soret convection phenomenon in porous media: effect of Vadasz inertia term"

F. Capone[1] · R. De Luca[1] · M. Vitiello[2]

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Abstract
In this short paper, some analytical results found in "Double-diffusive Soret convection phenomenon in porous media: effect of Vadasz inertia term" by F. Capone, R. De Luca, M. Vitiello, Ricerche Mat. 68, 581–595 (2019), are recalled in order to better explore the dynamic of thermosolutal convection in a horizontal porous layer with the influence of Vadasz and Soret terms.

Keywords Steady convection · Hopf convection · Vadasz · Soret · Stability · Porous media

Mathematics Subject Classification 76E06 · 76S05 · 35B35

1 Introduction

In this note we recall the results on the onset of thermosolutal convection in a horizontal porous layer, uniformly heated and salted from below, with Soret and inertia effects, investigated in [1], with particular regard to analyze the kind of instability arising when the rest state is no longer observable. Denoting by $\mathtt{I}_{1n}$, $\mathtt{I}_{2n}$, $\mathtt{I}_{3n}$ the principal invariants of the linear operator governing the evolution of the nth component of perturbation,

✉ F. Capone
fcapone@unina.it

R. De Luca
roberta.deluca@unina.it

M. Vitiello
maria.vitiello@poliba.it

1 Department of Mathematics and Applications R. Caccioppoli, University of Naples Federico II, via Cintia, 80126 Naples, Italy

2 Department of Mechanics Mathematics and Management, Politecnico di Bari, via Orabona 4, 70125 Bari, Italy

the spectral equation is found to be

$$\lambda^3 - \mathtt{I}_{1n}\lambda^2 + \mathtt{I}_{2n}\lambda - \mathtt{I}_{3n} = 0. \tag{1}$$

The Routh–Hurwitz conditions, necessary and sufficient to guarantee that all the roots of (1) have negative real part, are

$$\mathtt{I}_{3n} < 0, \quad \mathtt{I}_{1n}\mathtt{I}_{2n} - \mathtt{I}_{3n} < 0 \tag{2}$$

since, for the problem under consideration, $\mathtt{I}_{1n} < 0$. These conditions are necessary and sufficient to guarantee that the thermal conduction solution is stable. Convection arises via a steady or oscillatory state if (1) admits the null root or a pure imaginary root: the lowest positive Rayleigh thermal number vanishing $\mathtt{I}_{3n}$ is the Rayleigh thermal number R_S for marginal steady state while the lowest positive Rayleigh thermal number such that $\mathtt{I}_{1n}\mathtt{I}_{2n} - \mathtt{I}_{3n} = 0$, is the Rayleigh thermal number R_O for marginal oscillatory state. The instability threshold is given by $R_{insta} = \min\{R_S, R_O\}$. On comparing these two numbers, in [1], sufficient conditions guaranteeing the onset of steady and Hopf convection have been found. In this short paper we reconsider the problem in [1] and perform the linear instability analysis of the thermal conduction solution of model (1) of [1] by using a different methodology. In this way, we are able to better state the results found in Theorem 1. Precisely, we are able to find conditions guaranteeing that oscillatory convection can not occur because R_O does not exist or because R_O exists and $R_O > R_S$.

2 Linear instability analysis

The linearized version of (4) in [1] is

$$\begin{cases} \dfrac{1}{\tilde{V}_a}\dfrac{\partial \hat{\mathbf{u}}}{\partial t} + \hat{\mathbf{u}} = -\nabla \hat{\Pi} + R_T \hat{\theta}\mathbf{k} - \dfrac{R_C}{Le}\hat{\Gamma}\mathbf{k} \\ \nabla \cdot \hat{\mathbf{u}} = 0 \\ \dfrac{\partial \hat{\theta}}{\partial t} = \hat{w} + \Delta\hat{\theta} \\ \omega\dfrac{\partial \hat{\Gamma}}{\partial t} = \hat{w} + \dfrac{1}{Le}\Delta\hat{\Gamma} + S_r\Delta\hat{\theta} \end{cases} \tag{3}$$

The third component of the double curl of $(3)_1$ gives

$$\frac{1}{\tilde{V}_a}\frac{\partial \Delta\hat{w}}{\partial t} + \Delta\hat{w} = R_T\Delta_1\hat{\theta} - \frac{R_C}{Le}\Delta_1\hat{\Gamma}, \tag{4}$$

where $\hat{\mathbf{u}} = (\hat{u}, \hat{v}, \hat{w})$ and Δ_1 is the two-dimensional Laplacian operator in the horizontal plane.

Looking for *normal type* solutions, in view of the periodicity in the horizontal directions, we set

$$\begin{pmatrix} \hat{w} \\ \hat{\theta} \\ \hat{\Gamma} \end{pmatrix} = e^{\sigma t + i a_x x + a_y y} \begin{pmatrix} \bar{w}(z) \\ \bar{\theta}(z) \\ \bar{\Gamma}(z) \end{pmatrix}, \tag{5}$$

with $\sigma \in \mathbb{C}$. Hence, in view of the boundary conditions

$$\hat{w} = \hat{\theta} = \hat{\Gamma} = 0, \quad \text{on} \quad z = 0, 1, \tag{6}$$

one can choose

$$\begin{pmatrix} \bar{w} \\ \bar{\theta} \\ \bar{\Gamma} \end{pmatrix} = \begin{pmatrix} w_0 \\ \theta_0 \\ \Gamma_0 \end{pmatrix} \sin n\pi z, \quad n \in \mathbb{N} \tag{7}$$

with w_0, θ_0, Γ_0 constants and (4), $(3)_3$–$(3)_4$ become

$$\begin{cases} \dfrac{\xi_n \sigma}{\tilde{V}_a} w_0 + \xi_n w_0 = R_T a^2 \theta_0 - \dfrac{R_C}{Le} a^2 \Gamma_0, \\ \sigma \theta_0 = w_0 - \xi_n \theta_0 \\ \omega \sigma \Gamma_0 = w_0 - \frac{\xi_n}{Le} \Gamma_0 - S_r \xi_n \theta_0 \end{cases} \tag{8}$$

being $a^2 = a_x^2 + a_y^2$, $\xi_n = a^2 + n^2\pi^2$. Requiring the vanishing of the determinant of the coefficient matrix in (8), one has

$$R_T = \frac{\xi_n + \sigma - S_r \xi_n}{\xi_n + \sigma \omega Le} R_C + \frac{\xi_n (\xi_n + \sigma)(\tilde{V}_a + \sigma)}{a^2 \tilde{V}_a}. \tag{9}$$

2.1 Rayleigh number for a steady marginal state

The Rayleigh number for a steady marginal state, is obtained by (9) on substituting $\sigma = 0$ and looking for the minimum with respect to $(n, a^2) \in \mathbb{N} \times \mathbb{R}^+$. Then, it is given by

$$R_S = (1 - S_r) R_C + \min_{(n,a^2) \in \mathbb{N} \times \mathbb{R}^+} \frac{\xi_n^2}{a^2}, \tag{10}$$

i.e.

$$R_S = (1 - S_r) R_C + 4\pi^2. \tag{11}$$

2.2 Rayleigh number for an oscillatory marginal state

In order to determine R_O, let us set $\sigma = i\sigma_1,\ \sigma_1 \in \mathbb{R}\backslash\{0\}$ (i being the imaginary unit) in (9). Then

$$R_T = Re(R_T) + i\sigma_1 Im(R_T), \tag{12}$$

with

$$Re(R_T) = \frac{\xi_n^2(1-S_r) + \sigma_1^2\omega Le}{\xi_n^2 + \sigma_1^2\omega^2 Le^2} R_C + \frac{\xi_n(\xi_n\tilde{V}_a - \sigma_1^2)}{a^2\tilde{V}_a}, \tag{13}$$

and

$$Im(R_T) = \frac{\xi_n - \omega Le\xi_n(1-S_r)}{\xi_n^2 + \sigma_1^2\omega^2 Le^2} R_C + \frac{\xi_n(\tilde{V}_a + \xi_n)}{a^2\tilde{V}_a}. \tag{14}$$

Requiring $Im(R_T) = 0$, it follows that σ_1^2 has to verify

$$\omega^2 Le^2(\tilde{V}_a + \xi_n)\sigma_1^2 + [1 - \omega Le(1-S_r)]\, a^2\tilde{V}_a R_C + \xi_n^2(\tilde{V}_a + \xi_n) = 0. \tag{15}$$

Remark 1 Equation (15) has no real solution if

$$S_r \geq \frac{\omega Le - 1}{\omega Le}, \tag{16}$$

or if

$$\begin{cases} S_r < \dfrac{\omega Le - 1}{\omega Le} \\ R_C \leq \dfrac{\xi_n^2(\tilde{V}_a + \xi_n)}{a^2\tilde{V}_a\,[\omega Le(1-S_r) - 1]}. \end{cases} \tag{17}$$

Hence, in the cases (16) or (17), R_O does not exist and only steady convection can occur.

Remark 2 Simple calculation shows that $(17)_2$ is satisfied $\forall (n, a^2) \in \mathbb{N} \times \mathbb{R}^+$ iff

$$R_C \leq \min_{(n,a^2)\in\mathbb{N}\times\mathbb{R}^+} \frac{\xi_n^2(\tilde{V}_a + \xi_n)}{a^2\tilde{V}_a\,[\omega Le(1-S_r) - 1]}, \tag{18}$$

i.e. iff

$$R_C \leq \bar{R}_C := \frac{(\bar{a}^2 + \pi^2)^2(\tilde{V}_a + \bar{a}^2 + \pi^2)}{\bar{a}^2\tilde{V}_a\,[\omega Le(1-S_r) - 1]}, \tag{19}$$

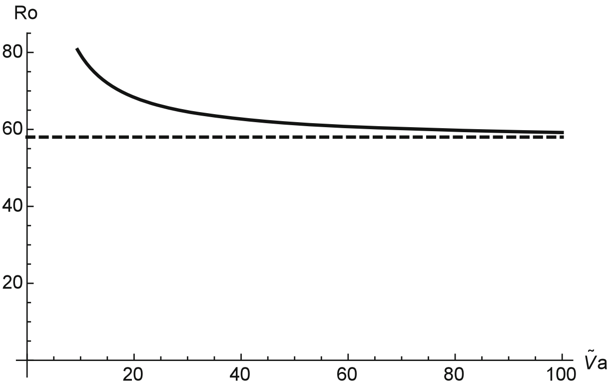

Fig. 1 Asymptotic behaviour of R_O for $Le = 20$; $\omega = 0.2$; $S_r = 0.5$; $R_C = 30$ (continuous line); Dotted line denotes R_O for $\tilde{V}_a \to \infty$

with $\bar{a}^2$ unique positive solution of

$$2x^3 + (\tilde{V}_a + 2\pi^2)x^2 - \pi^4(\tilde{V}_a + \pi^2) = 0. \tag{20}$$

In view of (16)–(17) and Remark 1, the condition

$$S_r < \frac{\omega Le - 1}{\omega Le}, \quad R_C > \bar{R}_C, \tag{21}$$

is *necessary* for the occurrence of Hopf convection. When (21) holds, (15) admits the solution

$$\sigma_1^2 = \frac{[\omega Le(1 - S_r) - 1]\, a^2 \tilde{V}_a R_C - \xi_n^2(\tilde{V}_a + \xi_n)}{\omega^2 Le^2(\tilde{V}_a + \xi_n)}. \tag{22}$$

Substituting (22) into (13) and looking for the minimum, with respect to $(n, a^2) \in \mathbb{N} \times \mathbb{R}^+$, one has

$$R_O = \min_{(n,a^2)\in\mathbb{N}\times\mathbb{R}^+} g(n, a^2), \tag{23}$$

with

$$g(n, a^2) = \frac{\xi_n(1 + S_r\omega Le) + \tilde{V}_a\omega Le}{\omega^2 Le^2(\tilde{V}_a + \xi_n)} R_C + \frac{\xi_n^2}{a^2}\frac{\omega Le + 1}{\omega Le} + \frac{\xi_n^3}{a^2}\frac{\omega Le + 1}{\tilde{V}_a\omega^2 Le^2}. \tag{24}$$

The instability threshold is given by

$$R_{insta} = \min\{R_S, R_O\} \tag{25}$$

and coincides with that one found in [1].

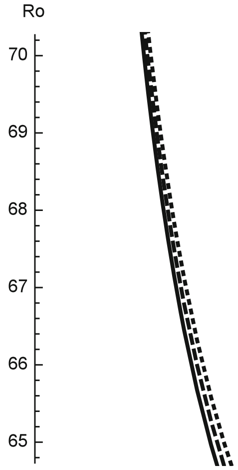

Fig. 2 Asymptotic behaviour of R_O versus $\tilde{V}_a$ for $Le = 20$; $\omega = 0.2$; $R_C = 30$ and $S_r = 0.5$ (continuous line); $S_r = 0.6$ (dashed line); $S_r = 0.7$ (dotted line)

Remark 3 We remark that, since

$$\min_{a^2} \frac{(a^2+\pi^2)^2(\tilde{V}_a + a^2 + \pi^2)}{a^2} = \frac{(\bar{a}^2+\pi^2)^2(\tilde{V}_a + \bar{a}^2 + \pi^2)}{\bar{a}^2}, \tag{26}$$

with $\bar{a}^2$ positive solution of (20), then

$$\bar{R}_C < R_{C_1}, \tag{27}$$

with R_{C_1} (determined in [1]) given by

$$R_{C_1} = \frac{(a_1^2+\pi^2)^2(\tilde{V}_a + a_1^2 + \pi^2)}{a_1^2 \tilde{V}_a \left[\omega Le(1-S_r) - 1\right]}. \tag{28}$$

In view of Remarks 1, 3 and Theorem 1 in [1], one can conclude that when $R_T < R_{C_1}$ convection can arise only via a steady state. In particular:

$$\begin{aligned} &R_T < \bar{R}_C \Rightarrow \nexists R_O, \\ &\bar{R}_C < R_T < R_{C_1} \Rightarrow \exists R_O > R_S. \end{aligned} \tag{29}$$

3 Discussion

In this short paper, we reconsider the onset of thermosolutal convection in a horizontal porous layer, uniformly heated and salted from below, with Soret and Vadasz inertia terms effect, investigated in [1]. Via a different methodology, some analytical results to perform the linear instability analysis of the rest state, found in [1], have been deeply examined. In particular, we have better explored the analytical results found in Theorem 1 of [1]. Precisely, we have been able to specify that, when $R_T < R_{C_1}$,

Fig. 3 Asymptotic behaviour of wave number a_c^2 versus $\tilde{V}_a$ for $Le = 20;\ \omega = 0.2;\ R_C = 30$ and $S_r = 0.5$ (continuous line); $S_r = 0.6$ (dashed line); $S_r = 0.7$ (dotted line)

Table 1 Oscillatory Rayleigh critical numbers in the presence and in the absence of Vadasz inertia term for $R_C = 50$

S_r	ω	Le	$\tilde{V}_a$	a_c^2	R_O^0	R_O
0.5	2.5	1	1.2	5.3779	75.2698	386.271
0.5	2.5	1	10	7.0557	75.2698	114.066
0.5	2.5	1	100	9.2539	75.2698	79.2383
0.7	2.3	1.5	10	7.4030	65.4142	92.4059
0.5	2.3	1.5	10	7.4480	65.4142	90.5693
0.5	2	1.5	10	7.2781	65.3046	99.5709
0.5	2.3	1.5	10	7.48758	64.5776	88.6800

thermosolutal convection can arise only via a steady state since or R_O does not exist or R_O exists but $R_O > R_S$. In view of the obtained results, the numerical simulations given in Section 6 of [1] have to be restated. We write in the following, the right version.

R_O reverts to the critical Rayleigh number for the onset of Hopf convection in the absence of Vadasz inertia term as illustrated in Fig. 1.

The behaviour of R_O versus $\tilde{V}_a$ is illustrated in Fig. 2 on fixing $Le = 20;\ \omega = 0.2$, $R_C = 30$ and letting S_r vary. The numerical simulations show that R_O decreases with respect to $\tilde{V}_a$ and increases with respect to S_r.

Figure 3 shows the wave number for Hopf convection behaviour versus $\tilde{V}_a$ on fixing $Le = 20;\ \omega = 0.2$, $R_C = 30$ and letting S_r vary. In Table 1 the critical Rayleigh numbers for the onset of oscillatory convection in the absence and in the presence of Vadasz inertia term denoted respectively by R_O^0 and R_O and the wave number a_c^2, are collected for $R_C = 50$ and for different values of Le, ω, S_r, $\tilde{V}_a$. R_O decreases with $\tilde{V}_a$ (lines 1–3) while increases with S_r (lines 4–5). Lines 5–6 show that, as one is expected, R_O decreases with ω. In fact, ω is proportional to the porosity of the medium and, the larger the pores, the more convection is inhibited (i.e. increasing in porosity has a stabilizing effect). Lines 5, 7 concern the behaviour of R_O versus Le and, in particular, one can remark that Le has a stabilizing effect on the onset of oscillatory convection.

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Declarations

Conflict of interest The authors declare that they have no conflicts of interest.

Reference

1. Capone, F., De Luca, R., Vitiello, M.: Double-diffusive Soret convection phenomenon in porous media: effect of Vadasz inertia term. Ricerche Mat. **68**, 581–595 (2019)