

# Heat transport in bubbling turbulent convection

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Boiling is an extremely effective way to promote heat transfer from a hot surface to a liquid due to numerous mechanisms, many of which are not understood in quantitative detail. An important component of the overall process is that the buoyancy of the bubble compounds with that of the liquid to give rise to a much-enhanced natural convection. In this article, we focus specifically on this enhancement and present a numerical study of the resulting two-phase Rayleigh–Bénard convection process in a cylindrical cell with a diameter equal to its height. We make no attempt to model other aspects of the boiling process such as bubble nucleation and detachment. The cell base and top are held at temperatures above and below the boiling point of the liquid, respectively. By keeping this difference constant, we study the effect of the liquid superheat in a Rayleigh number range that, in the absence of boiling, would be between  $2 \times 10^6$  and  $5 \times 10^9$ . We find a considerable enhancement of the heat transfer and study its dependence on the number of bubbles, the degree of superheat of the hot cell bottom, and the Rayleigh number. The increased buoyancy provided by the bubbles leads to more energetic hot plumes detaching from the cell bottom, and the strength of the circulation in the cell is significantly increased. Our results are in general agreement with recent experiments on boiling Rayleigh–Bénard convection.

two-phase convection | latent heat | boundary layers | point bubble model | simulations

The greatly enhanced heat transfer brought about by the boiling process is believed to be due to several interacting components (1–3). With their growth the bubbles cause a microconvective motion on the heating surface, and as they detach by buoyancy, the volume they vacate tends to be replaced by cooler liquid. Especially in subcooled conditions, the liquid in the relatively stagnant microlayer under the bubbles can evaporate and condense on the cooler bubble top. This process provides for the direct transport of latent heat, which is thus able to bypass the low-velocity liquid region adjacent to the heated surface due to the no-slip condition. The bubble growth process itself requires latent heat and, therefore, also removes heat from the heated surface and the neighboring hot liquid. Finally, with their buoyancy, the bubbles enhance the convective motion in the liquid beyond the level caused by the well-known single-phase Rayleigh–Bénard (RB) convection mechanisms (4, 5). This last process is the aspect on which we focus in the present article.

In classical single-phase RB convection, the dimensionless heat transport,  $Nu$ , the Nusselt number, is defined as the ratio of the total heat transported through the cell to the heat that would be transported by pure conduction with a quiescent fluid. This ratio increases well above 1 as the Rayleigh number  $Ra = \frac{g\beta\Delta L^3}{\nu\kappa}$  is increased due to the onset of convective motion in the cell. Here  $g$  is the acceleration of gravity,  $\beta$  the isobaric thermal expansion coefficient,  $\Delta = T_h - T_c$  the difference between the temperature  $T_h$  of the hot bottom plate and the temperature  $T_c$  of the cold top plate,  $L$  the height of the cell,  $\nu$  the kinematic viscosity, and  $\kappa$  the thermal diffusivity. Further,  $Nu$  depends on the shape of the cell, its aspect ratio (defined for a cylindrical cell of diameter  $D$  as  $\Gamma = D/L$ ), and the Prandtl number  $Pr = \nu/\kappa$  of the liquid. For  $Ra$  in

the range  $10^7 - 10^{10}$  and  $Pr$  in the range  $0.7 - 7$ , the heat transport satisfies an approximate scaling relation  $Nu \propto Ra^{0.29-0.32}$  (4, 5).

How is this scaling modified if the hot plate temperature  $T_h$  is above the fluid saturation temperature  $T_{sat}$ , so that phase change can occur? The present article addresses this question by focusing on the enhanced convection caused by the bubble buoyancy, rather than attempting a comprehensive modeling of the actual boiling process in all its complexity. We carry out numerical simulations in the range  $2 \times 10^6 \leq Ra \leq 5 \times 10^9$  for a cylindrical cell with aspect ratio  $\Gamma = 1$  for  $Pr = 1.75$ , which is appropriate for water at 100 °C under normal conditions.

This work differs in two major respects from our earlier studies of the problem. In the first place, we are now able to reach a much higher Rayleigh number,  $5 \times 10^9$  as opposed to  $2 \times 10^5$  as in ref 6, and to include three times as many bubbles. Secondly, we now study the effect of the liquid superheat, which was held fixed before.

The extensive literature on boiling leads to the expectation that the appearance of bubbles would cause a substantial increase in  $Nu$  with respect to single-phase convection (1). For RB convection, the effect of phase change has recently been studied in ref. 7 for the case of ethane near the critical point, and indeed a major increase of the heat transport has been found.

## Model

The present article is based on the same mathematical model and numerical method that we have used in ref. 6, and several other recent papers (8, 9). Briefly, under the Boussinesq approximation, conservation of mass, momentum, and thermal energy equations for the liquid are:

$$\nabla \cdot \mathbf{u} = 0, \quad [1]$$

$$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{u} + \beta(T - T_{sat})\mathbf{g} + \sum_{i=1}^{N_b} \mathbf{f}_i\delta(\mathbf{x} - \mathbf{x}_i), \quad [2]$$

and

$$\frac{DT}{Dt} = \kappa\nabla^2 T + \frac{1}{\rho c_p} \sum_{i=1}^N \mathbf{Q}_i\delta(\mathbf{x} - \mathbf{x}_i). \quad [3]$$

Here  $\mathbf{u}$ ,  $p$ , and  $T$  are the liquid velocity, pressure, and temperature, and  $\rho$  and  $c_p$  are the liquid density and specific heat, while  $N_b$  is the total number of bubbles. The bubbles are modeled as point sources of momentum and heat for the liquid. The  $i$ -th

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bubble offers a mechanical forcing  $\mathbf{f}_i = (4/3)\pi r_{b_i}^3 ([D\mathbf{u}/Dt]_{\mathbf{x}_i} - \mathbf{g})$  and a thermal forcing  $Q_i = h_{b_i}(4\pi r_{b_i}^2)(T_{sat} - T_i)$ . Here  $r_{b_i}$  is the radius of the  $i$ -th bubble,  $h_{b_i}$  the bubble heat transfer coefficient, and the liquid temperature  $T_i$  and acceleration  $[D\mathbf{u}/Dt]_{\mathbf{x}_i}$  are evaluated at the location  $\mathbf{x}_i$  of the bubble;  $T_{sat}$  is the bubble surface temperature assumed to be at saturation with respect to the static pressure.

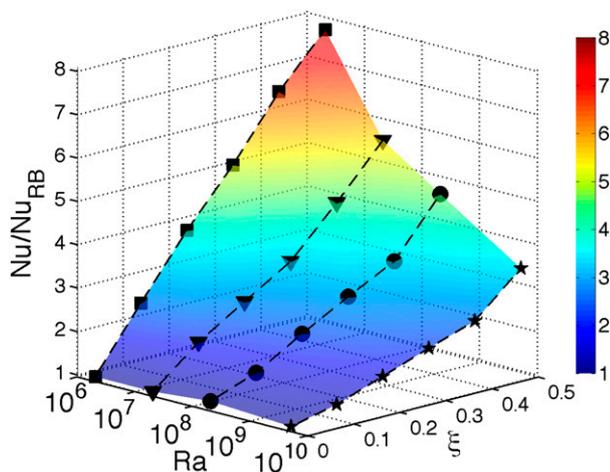
The motion of each bubble, envisaged as a sphere, is followed in a Lagrangian way by means of an equation that, in addition to buoyancy, includes drag, added mass, and lift:

$$\begin{aligned}
 & C_A \left[ \frac{4}{3}\pi r_{b_i}^3 \left( \frac{D\mathbf{u}}{Dt} - \frac{d\mathbf{V}_{b_i}}{dt} \right) + (\mathbf{u} - \mathbf{V}_{b_i}) \frac{d}{dt} \left( \frac{4}{3}\pi r_{b_i}^3 \right) \right] \\
 & - C_D \left[ \frac{1}{2}\pi r_{b_i}^2 |\mathbf{V}_{b_i} - \mathbf{u}| (\mathbf{V}_{b_i} - \mathbf{u}) \right] + \frac{4}{3}\pi r_{b_i}^3 \frac{D\mathbf{u}}{Dt} \\
 & + C_L \left[ \frac{4}{3}\pi r_{b_i}^3 (\nabla \times \mathbf{u}) \times (\mathbf{V}_{b_i} - \mathbf{u}) \right] - \frac{4}{3}\pi r_{b_i}^3 \mathbf{g} = 0, \quad [4]
 \end{aligned}$$

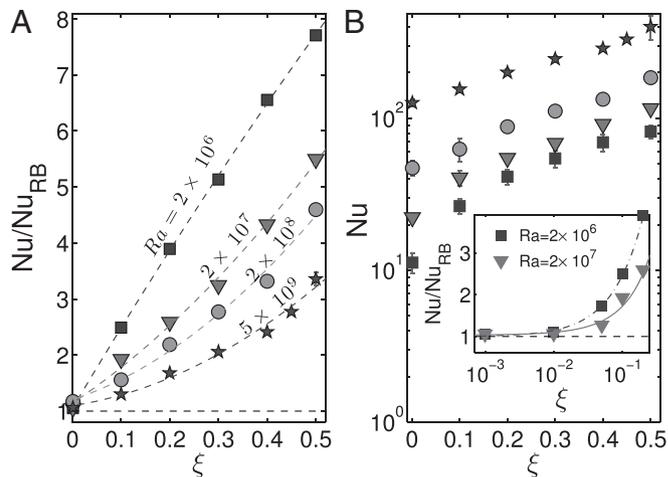
where  $C_A = 1/2$ ,  $C_L = 1/2$  are the added mass and lift coefficients, and  $\mathbf{V}_{b_i}$  is the bubble velocity.  $C_D$  is the drag coefficient (see ref. 6).

In its mechanical aspects, therefore, the model is similar to existing ones, which have been extensively used in the literature to simulate dilute disperse flows with bubbles and particles (10, 11). The novelty of our model lies in the addition of the thermal component. The heat exchange between the bubble and the liquid in its vicinity is modeled by means of a heat transfer coefficient dependent on the Péclet number of the bubble-liquid relative motion and on the Prandtl number of the liquid. The radial motion of the bubbles is slow enough that the vapor pressure remains essentially equal to the ambient pressure, which implies that the bubble surface temperature can be assumed to remain at the saturation value. The bubble volume, on which the enhanced buoyancy effect depends, is calculated by assuming that the entire heat absorbed by a bubble is used to generate vapor at the saturation density and pressure (for complete details, see ref. 6).

The calculation is carried out on a finite-difference grid based on cylindrical coordinates. The standard staggered-grid arrangement is used for the flow variables and the projection method for the calculation of the pressure and time stepping (12). No-slip conditions are applied on the bottom and top of the



**Fig. 1.**  $Nu(Ra, \xi)$  for boiling convection normalized by the corresponding single-phase value  $Nu_{RB}$  for  $N_b = 50,000$  bubbles. Here  $\xi$  is the normalized superheat,  $\xi \equiv (T_h - T_{sat})/\Delta$ . The symbols correspond to  $Ra = 2 \times 10^6$  (square),  $2 \times 10^7$  (triangles),  $2 \times 10^8$  (circles), and  $5 \times 10^9$  (stars).



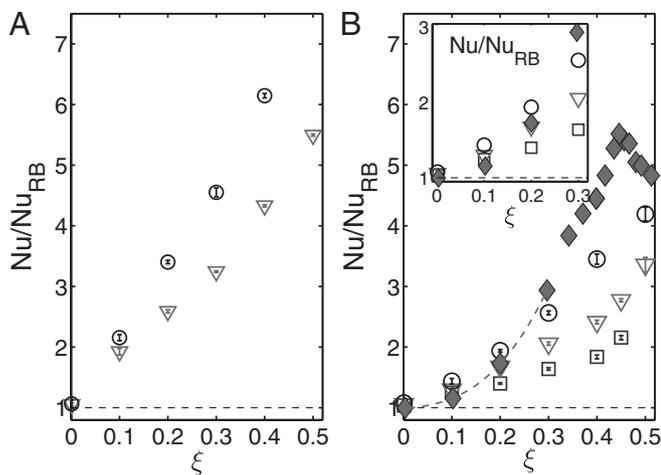
**Fig. 2.**  $Nu/Nu_{RB}$  (A) and  $Nu$  (B) as functions of the normalized superheat  $\xi$  for 50,000 bubbles. The symbols correspond to  $Ra = 2 \times 10^6$  (squares),  $Ra = 2 \times 10^7$  (triangles),  $Ra = 2 \times 10^8$  (circles), and  $Ra = 5 \times 10^9$  (stars). The inset shows a detail for small superheat  $\xi$  for  $Ra = 2 \times 10^6$  (squares) and  $Ra = 2 \times 10^7$  (triangles) with quadratic fits to the data.

cell, and also on the lateral boundary. The Lagrangian treatment of the bubbles proceeds by means of a third-order Runge-Kutta method. The energy and force imparted by each bubble to the liquid are interpolated to the grid points of the cell containing the bubble in such a way as to preserve the total energy and the resultant and moment of the force.

Simulations are carried out on computational grids with the angular, radial, and axial directions discretized by means of  $193 \times 49 \times 129$ ,  $385 \times 129 \times 257$ ,  $385 \times 129 \times 257$ , and  $769 \times 193 \times 385$  nodes for  $Ra = 2 \times 10^6$ ,  $2 \times 10^7$ ,  $2 \times 10^8$ , and  $5 \times 10^9$ . The simulations are therefore well resolved according to the requirements specified in refs. 13 and 14. We have also checked the global balances of appendix B in ref. 6, finding that they were satisfied to within 0.1%.

When a bubble reaches the top cold plate, it is removed from the calculation to model condensation and a new bubble is introduced at a random position on the bottom hot plate so that the total number of bubbles in the calculation remains constant. We do not attempt to model the nucleation process, which, on the basis of first principles and which would require addressing extremely complex multiscale issues. For our limited purpose of studying the bubble-induced increased buoyancy, it is sufficient to simply generate a new bubble at the hot plate. We do not model the process by which the bubble detaches from the plate but assume that it is free to rise immediately as it is introduced. The initial bubble radius is arbitrarily set at  $38 \mu\text{m}$ . As shown in ref. 9, the initial bubble size is immaterial provided it is in the range of a few tens of microns. In view of their smallness, the latent heat necessary for their generation is very small and is neglected. We show results for three values of the total number of bubbles  $N_b$ , namely  $N_b = 10,000$ ,  $50,000$ , and  $150,000$ . Another parameter we vary is the degree of superheat,  $T_h - T_{sat}$ , which we express in the dimensionless form  $\xi = (T_h - T_{sat})/\Delta$ .

The Nusselt number shown in the following is defined as  $Nu = q''_h L / (k\Delta)$ , where  $k$  is the liquid thermal conductivity and  $q''_h$  is the heat flux into the bottom plate. This quantity differs from  $q''_c$ , the heat flux at the upper plate, due to the heat stored in the bubbles. [The Nusselt number shown in our previous papers (6, 8, 9) are based on the average between  $q''_h$  and  $q''_c$ .] An important parameter introduced by the bubbles is the Jakob number  $Ja = \frac{\rho c_p (T_h - T_{sat})}{\rho_v h_{fg}} = \xi \frac{\rho c_p \Delta}{\rho_v h_{fg}}$ , where  $\rho$  and  $\rho_v$  are the densities of liquid



**Fig. 3.**  $Nu/Nu_{RB}$  versus  $\xi$  for three different bubble numbers,  $N_b = 10,000$  (squares, B), 50,000 (triangles), and 150,000 (circles); A is for  $Ra = 2 \times 10^7$  and B for  $Ra = 5 \times 10^9$ . The curved dash line is a fit to the experimental data of Zhong et al. (7) shown by the filled symbols. The inset is a blow-up for the range  $0 \leq \xi \leq 0.30$ . Error bars are shown inside the hollow symbols.

and vapor and  $h_{fg}$  is the latent heat for vaporization. Physically,  $Ja$  expresses the balance between the available thermal energy and the energy required for vaporization. With  $\Delta = 1$  °C,  $Ja$  varies between 0 and 1.68 as  $\xi$  varies between 0 and 1/2. For  $\xi = 0$ , the bubbles introduced at the hot plate can only encounter liquid at saturation temperature or colder, and therefore they cannot grow but will mostly collapse. On the other hand, for  $\xi = 1/2$ , they have significant potential for growth.

To give an impression of the physical situation corresponding to our parameter choices, we may mention that 100 °C water in a 15-cm-high cylinder with an imposed temperature difference  $\Delta = 1$  °C would correspond to  $Ra \approx 5 \times 10^8$ . The Kolmogorov length scale based on the volume- and time-averaged kinetic energy dissipation in single-phase RB convection is 3 mm for  $Ra \sim 10^6$  and 0.5 mm for  $Ra \sim 10^{10}$  (5) and is therefore always much larger than the initial size of the bubbles (e.g., 13 times larger for the highest Rayleigh number). In our simulations bubbles grow at most to a diameter of  $130 d_{inj}$  (see Fig. 5). The bubble volume fractions are less than 0.01% and hence use of the point bubble model is justified.

### Observations on Heat Transport and Flow Organization

In Fig. 1, the dependence of  $Nu$  on the Rayleigh number  $Ra$  and the dimensionless superheat  $\xi$  is shown for  $N_b = 50,000$  bubbles. Here  $Nu$  is normalized by  $Nu_{RB}$ , the single-phase Nusselt number corresponding to the same value of  $Ra$ . Each symbol shows the result of a separate simulation carried out for the corresponding values of  $Ra$  and  $\xi$ . A colored surface is interpolated through the computed results with the color red corresponding to  $Nu/Nu_{RB} = 8$  and the color blue to  $Nu/Nu_{RB} = 1$ .

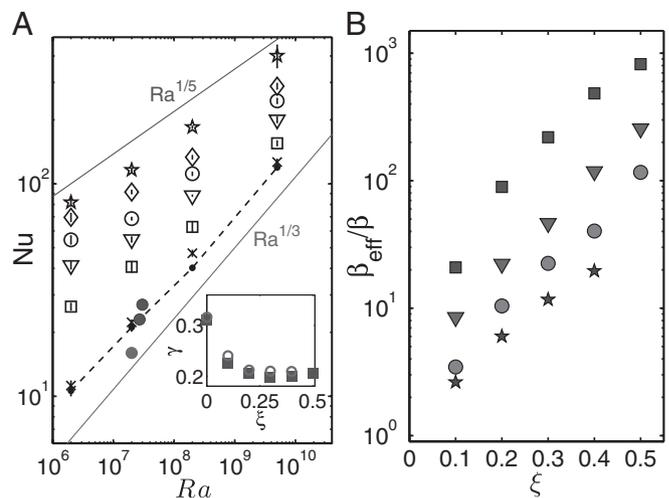
The same data are shown on a 2D plot of  $Nu/Nu_{RB}$  versus  $\xi$  in Fig. 2A for four different Rayleigh numbers in descending order; here the dashed lines are drawn as guides to the eye. It is evident that the relative enhancement of the heat transport is a decreasing function  $Ra$ . This statement, however, does not apply to the absolute heat transport shown in Fig. 2B, where  $Nu$  is not normalized by the single-phase value. Here  $Ra$  increases in ascending order, which shows that the bubbles always have a beneficial effect on the heat transport. For very small superheat, the heat transport approaches the single-phase value as shown in the inset of Fig. 2B.

Figs. 1 and 2 show results calculated keeping the bubble number fixed. This procedure, therefore, does not faithfully

reflect physical reality, as it is well known that the number of bubbles is an increasing function of superheat. The dependence is actually quite strong, with the number of bubbles proportional to  $T_h - T_{sat}$  raised to a power between 3 and 4 (1). However, varying independently  $N_b$  and  $\xi$  permits us to investigate separately the effect of these quantities.

The effect of changing the bubble number from 50,000 to 150,000 at the same  $\xi$  is shown in Fig. 3A and B for  $Ra = 2 \times 10^7$  and  $5 \times 10^9$ , respectively. In the latter case, we also include results for  $N_b = 10,000$ . For small  $\xi$ , the heat transfer enhancement is small as the bubbles will mostly encounter colder liquid, condense, and add very little to the system buoyancy. As the superheat  $\xi$  increases, however, the effect of the bubbles becomes stronger and stronger, and larger the larger their number.

In Fig. 3B, the solid symbols are the data of ref. 7 taken at a higher Rayleigh number,  $Ra \approx 3 \times 10^{10}$ . (This article reports data for both increasing and decreasing superheat. We show here only the latter data because, for increasing superheat, there is a threshold for fully developed boiling conditions that pushes the onset of bubble appearance beyond  $\xi = 0.35$ . For decreasing  $\xi$ , on the other hand, fully developed boiling conditions prevail all the way to small values of  $\xi$ .) The inset in the figure shows our computed results and the experimental data for  $\xi \leq 0.3$ . A major difference between our simulations and the experiment is that, in the latter, the number of bubbles increases with the superheat, while it remains constant with  $\xi$  in the simulations. We can nevertheless attempt a comparison as follows. Quadratic interpolation using our results for the three values of  $N_b$  suggests that, to match the experimental values, we would need  $N_b \approx 63,000$  for  $\xi = 0.2$  and  $N_b \approx 250,000$  for  $\xi = 0.3$ . If, as suggested by experiment, the actual physical process results in a relation of the form  $N_b \propto \xi^m$ , we find  $m \approx 3.4$ , which falls in the experimental range  $3 < m < 4$  mentioned before. With this value of  $m$ , we can estimate the number of bubbles necessary to account for the measured  $Nu$  at  $\xi = 0.1$ . Using  $N_b(\xi = 0.1) = (0.1/\xi)^m N_b(\xi)$ , we find  $N_b(0.1) \approx 5,968$  for  $\xi = 0.2$  and  $N_b(0.1) \approx 6,000$  for  $\xi = 0.3$ .



**Fig. 4.** (A)  $Nu$  versus  $Ra$  and (B)  $\beta_{eff}/\beta$  versus  $\xi$  for 50,000 bubbles. In A, the numerical results are shown as crosses ( $\xi = 10^{-3}$ ), squares ( $\xi = 0.1$ ), triangles ( $\xi = 0.2$ ), circles ( $\xi = 0.3$ ), diamonds ( $\xi = 0.4$ ), and stars ( $\xi = 0.5$ ). Simulations without bubbles are also shown for comparison as a dashed line joining small dots and data from the LB simulations of ref. 15 as filled circles, light gray for no-boiling, and dark gray for boiling. In the inset, the effective scaling exponent  $\gamma(\xi)$  obtained from power-law fits of the form  $Nu \propto Ra^\gamma$  is shown as a function of  $\xi$  for 50,000 (squares) and 150,000 (circles) bubbles. In B, the effective buoyancy has been computed from Eq. 5. The symbols are the same as in Fig. 2A. Error bars are shown inside the hollow symbols.





layers is provided in Fig. 8. The figure makes evident that the temperature distribution in the upper and lower layers is not symmetric. Furthermore, the layers become thinner as  $N_b$  increases, a clear manifestation of the enhanced convective circulation promoted by the bubbles.

For the hot plate, one can define the thermal boundary layer thickness as  $\lambda_\theta = (T_m - T_h) / [dT/dz]_{z=0}$ , where  $[dT/dz]_{z=0}$  is the mean temperature gradient at the hot plate. Replotting the data of the bottom panel in Fig. 8 as functions of  $z/\lambda_\theta$ , we find that the three sets of data collapse on a single line in the range  $0 \leq z/\lambda_\theta \leq 0.5$  (Fig. 9). The small differences farther away from the wall reflect differences in the shape factor of the boundary layers.

### Summary and Conclusions

In summary, our investigation of a simple model of RB convection with boiling has demonstrated the effect of the degree of superheat and of the bubble number on heat transport. Comparison with existing data suggests a basic conformity of our

results with some physical features of a real system. Vapor bubbles significantly enhance the heat transport primarily by increasing the strength of the circulatory motion in the cell. The velocity and thermal fluctuations of the boundary layers are increased and, by releasing their latent heat upon condensation in the bulk fluid, the bubbles also act as direct carriers of energy. We have shown that the heat transfer enhancement can be interpreted in terms of an enhanced buoyancy, which is shown in Eq. 1 and Fig. 4B. The relative effect of the bubbles diminishes as  $Ra$  increases.

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