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GLOBAL STRONG SOLVABILITY OF DIRICHLET PROBLEM FOR A CLASS OF NONLINEAR ELLIPTIC EQUATIONS IN THE PLANE

DIAN K. PALAGACHEV

Global solvability and uniqueness results are established for Dirichlet's problem for a class of nonlinear differential equations on a convex domain in the plane, where the nonlinear operator is elliptic in sense of Campanato. We prove existence by means of the Leray-Schauder fixed point theorem, using Aleksandrov-Pucci maximum principle in order to find a priori estimate for the solution.

1. Introduction.

In the present paper we deal with the global solvability and uniqueness of the Dirichlet problem on two-dimensional convex domain for a class of nonlinear second order differential equations represented by Carathéodory's functions. The principal part of the operator satisfies an ellipticity condition introduced by S. Campanato (see [2], [3]) who proved a local existence result for this kind of equations, assuming the measure of the domain to be sufficiently small. The global strong solvability result was proved by Bers and Nirenberg ([1]) for uniformly elliptic operators under assumption that functions representing the nonlinear operator are differentiable with respect to all their variables.

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The basic tool in our investigations is the Leray-Schauder fixed point theorem that reduces solvability of the boundary value problem under consideration to the establishment of *a priori* estimates for the solutions of an appropriate family of problems. Using Campanato's condition on ellipticity (see (A) below) we show that the nonlinear equation can be linearized in a suitable way that leads to the possibility to apply a semilinear variant of Aleksandrov-Pucci maximum principle in order to find L^∞ a priori bound for the solutions. On the other hand, the linearized operator has bad regularization properties (its principal coefficients are L^∞ only) that restricts us to the two-dimensional case.

The uniqueness result is a simple consequence of the Aleksandrov–Pucci maximum principle, as well as of further structural conditions on the nonlinear operator.

2. Setting of the problem.

Assume Ω to be C^2, bounded and convex domain in the plane $\mathbb{R}^2$, and let $a(x, z, p, \xi)$, $f(x, z, p)$ be real-valued functions satisfying *Carathéodory's* condition, i.e. they are measurable in x for all $(z, p, \xi) \in \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^4$ and continuous in the other variables for almost all $x \in \Omega$. Here $\mathbb{R}^4$ denotes the 4-dimensional space of real 2×2 matrices $\xi = \{\xi_{ij}\}_{i,j=1}^2$ with the norm $\|\xi\| = \left(\sum_{i,j=1}^2 \xi_{ij}^2\right)^{1/2}$. For a real-valued function $u \colon \Omega \to \mathbb{R}$ we denote by ∇u and $H(u)$ its gradient and Hessian matrix, respectively.

To fix our ideas we aimed at the study of global solvability of the next homogeneous Dirichlet problem for second order differential equation

$$\begin{cases} a(x, u, \nabla u, H(u)) = f(x, u, \nabla u) & \text{a.e. in } \Omega \\ u = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{1}$$

As usual, by strong solution of (1) we mean a twice weakly differentiable function ($u \in W^{2,q}(\Omega)$) satisfying equation in (1) almost everywhere in Ω, and that achieves boundary values in sense of $W^{1,q}(\Omega)$, i.e., $u \in W_0^{1,q}(\Omega)$, with an appropriate $q \geq 1$. Here $W^{2,q}(\Omega)$ denotes the usual Sobolev space equipped with the norm $\|\cdot\|_{W^{2,q}(\Omega)}$. The nonhomogeneous boundary value problem can be considered in the same way as below. The reason to dealing with zero boundary condition is to avoid some unessential complications of technical character.

In order to prove the solvability of (1) we need some structural conditions on the equation. The next "ellipticity" condition is introduced by S.Campanato (see [2], [3]):

(A) There exist positive constants α, γ and δ, $\gamma + \delta < 1$, and

$$|\mathrm{Tr}\,(\xi) - \alpha\,(a(x, z, p, \xi + \tau) - a(x, z, p, \tau))| \leq \gamma \|\xi\| + \delta |\mathrm{Tr}\,(\xi)|$$

for a.a. $x \in \Omega$; $\forall z \in \mathbb{R}$; $p \in \mathbb{R}^2$; $\xi, \tau \in \mathbb{R}^4$, and $a(x, z, p, 0) = 0$.

The main step in our considerations is ensured by the fact that condition (A) allows to rewrite the equation in (1) as an equation with linear structure, having L^∞ principal coefficients that depend on u, ∇u, $H(u)$.

On the function $f(x, z, p)$ we impose the next assumptions:

$$|f(x, z, p)| \leq f_1(|z|)\left(f_2(x) + |p|^q\right), \tag{2}$$

where $f_1 \in C^0(\mathbb{R}^+)$, $f_2 \in L^2(\Omega)$, and $q < 2$ is an arbitrary number;

$$-\operatorname{sign} z \cdot f(x, z, p) \leq \mu_1(x)|p| + \mu_2(x), \tag{3}$$

with μ_1 and μ_2 nonnegative $L^2(\Omega)$ functions.

The last condition is natural in the treatment of classical solvability of quasilinear elliptic equations, and it ensures L^∞ a priori estimate for solutions of (1).

We are in position now to formulate our existence result.

Theorem 1. *Assume $\Omega \subset \mathbb{R}^2$ to be C^2, bounded and convex domain, and let* (A), (2) *and* (3) *hold. Then the Dirichlet problem* (1) *possesses solution $u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$.*

Remarks. By means of Sobolev's imbedding theorem the solution of (1) is Hölder continuous function $u \in C^{0,\theta}(\overline{\Omega})$ for all $\theta < 1$, and therefore it attains continuously the boundary values on $\partial\Omega$.

It will be evident from the proof given below that if we impose the stronger condition $f_2 \in L^{q_0}(\Omega)$ in (2) with some $q_0 > 2$ and sufficiently close to 2 (see Theorem 3 in [3]), then the solution of (1) belongs to $W^{2,q_0}(\Omega) \cap W_0^{1,q_0}(\Omega)$. In particular, $u \in C^{1,1-2/q_0}(\overline{\Omega})$.

Moreover, let $f_2 \in L^{q_0}(\Omega)$ with *arbitrary* $q_0 > 2$. Then, using the linearized equation (11) below, and a result due to G. Talenti ([10]), it follows: $u \in C^{1,\beta}(\Omega) \cap C^{0,\theta}(\overline{\Omega})$ for each $\theta < 1$, and for all

$$0 < \beta < \min\left\{1 - \frac{2}{q_0}, \frac{1 - (\gamma + \delta)^2}{2(\gamma + \sqrt{2}(1 + \delta))}\right\}$$

(see the inequality (10)).

Unicity cannot be assured for the considered problem without further hypotheses on the nonlinear operator. It is possible to prove uniqueness for (1) in the wider functional class $C^0(\overline{\Omega}) \cap W^{2,2}_{\text{loc}}(\Omega)$ under additional structure conditions on the functions $a(x, z, p, \xi)$ and $f(x, z, p)$, as shows the next assertion.

Theorem 2. *Let the function a satisfies* (A), *and let it be independent of z and p. Assume $f(x, z, p)$ to be nondecreasing in z for almost all $x \in \Omega$, $\forall p \in \mathbb{R}^2$, and*

$$(4) \qquad \left|f(x, z, p) - f(x, z, p')\right| \leq f_3(x, z)|p - p'|$$

for a.a. $x \in \Omega$, $\forall z \in \mathbb{R}$, $\forall p, p' \in \mathbb{R}^2$, where $\sup_{|z| \leq M} f_3(\cdot, z) \in L^2(\Omega)$ for each fixed constant M.

Then, if $u, v \in C^0(\overline{\Omega}) \cap W^{2,2}_{\text{loc}}(\Omega)$ solve Dirichlet's problem (1) *we have $u = v$.*

3. Proofs of the results.

As was mentioned above we shall study the solvability of (1) using the Leray-Schauder fixed point theorem (Theorem 11.6 in [5]).

Without loss of generality we may assume $q > 1$ for the number q in (2). Thus, if $v \in W^{1,2q}(\Omega)$ Sobolev's imbedding theorem implies $v \in C^0(\overline{\Omega})$ and therefore

$$(5) \qquad f(x, v(x), \nabla v(x)) \in L^2(\Omega)$$

by means of (2).

Now, for fixed $\sigma \in [0, 1]$ and $v \in W^{1,2q}(\Omega)$ we consider

$$(6) \qquad \begin{cases} a(x, v, \nabla v, H(u)) = \sigma f(x, v, \nabla v) & \text{a.e. in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

The Dirichlet problem (6) is uniquely solvable in the space $W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega)$ as it was proved by S. Campanato (Theorem 3 in [3], Theorem 4.4 in [2]) and because of the convexity of Ω, (A) and (5). Moreover, the solution of (6) satisfies

$$(7) \qquad \left\|H(u)\right\|_{L^2(\Omega)} \leq \frac{\alpha}{1 - (\gamma + \delta)} \left\|\sigma f(x, v, \nabla v)\right\|_{L^2(\Omega)}.$$

So, we have defined a mapping

$$\mathcal{U} : \quad [0, 1] \times W^{1,2q}(\Omega) \longrightarrow W^{1,2q}(\Omega)$$

where the image $u = \mathcal{U}(\sigma, v)$ is the unique solution of (6). In such a way, the solvability of (1) is reduced to the existence of a fixed point of the mapping $\mathcal{U}(1, \cdot)$.

It is clear that $\mathscr{U}(0, v) = 0$ for each $v \in W^{1,2q}(\Omega)$ ($a(x, z, p, 0) = 0$ and Theorem 3 in [3]), and $\mathscr{U}$ is a compact operator from $[0, 1] \times W^{1,2q}(\Omega)$ into $W^{1,2q}(\Omega)$ ($\mathscr{U}(\sigma, v) \in W^{2,2}(\Omega) \hookrightarrow W^{1,r}(\Omega)$ compactly for each $r \geq 1$).

To prove continuity of $\mathscr{U}$ we get sequences $v_h \in W^{1,2q}(\Omega)$, $\sigma_h \in [0, 1]$ converging in the corresponding topologies to v and σ, respectively. Let $u_h = \mathscr{U}(\sigma_h, v_h) \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$, $u = \mathscr{U}(\sigma, v) \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$, i.e.,

$$a(x, v_h, \nabla v_h, H(u_h)) = \sigma_h f(x, v_h, \nabla v_h) \quad \text{a.e. in } \Omega$$

and

$$a(x, v, \nabla v, H(u)) = \sigma f(x, v, \nabla v) \quad \text{a.e. in } \Omega.$$

Hence

$$\begin{aligned}\Delta(u_h - u) =& \Delta(u_h - u) - \alpha\,(a(x, v_h, \nabla v_h, H(u_h)) - a(x, v_h, \nabla v_h, H(u))) \\ &+ \alpha\,(a(x, v, \nabla v, H(u)) - a(x, v_h, \nabla v_h, H(u))) \\ &+ \alpha\,(\sigma_h f(x, v_h, \nabla v_h) - \sigma f(x, v, \nabla v)).\end{aligned}$$

Since $\operatorname{Tr}(H(u_h - u)) = \Delta(u_h - u)$, condition (A) and the Young inequality lead to

$$\begin{aligned}|\Delta(u_h - u)|^2 \leq& (1 + \varepsilon)\left(\delta(\delta + \gamma)|\Delta(u_h - u)|^2 + \gamma(\delta + \gamma)\|H(u_h - u)\|^2\right) \\ &+ C(\varepsilon, \alpha)\Big(\big|a(x, v_h, \nabla v_h, H(u)) - a(x, v, \nabla v, H(u))\big|^2 \\ &+ \big|\sigma_h f(x, v_h, \nabla v_h) - \sigma f(x, v, \nabla v)\big|^2\Big)\end{aligned}$$

with arbitrary $\varepsilon > 0$. On the other hand $u_h - u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ and therefore

$$\int_\Omega \|H(u_h - u)\|^2\,dx \leq \int_\Omega |\Delta(u_h - u)|^2\,dx$$

because of the estimate of Miranda-Talenti (see [6], [9]), and the convexity of Ω.

It follows

$$\begin{aligned}&\int_\Omega \|H(u_h - u)\|^2\,dx \leq \\ &\leq C(\alpha)\Big(\int_\Omega \big|a(x, v_h, \nabla v_h, H(u)) - a(x, v, \nabla v, H(u))\big|^2\,dx \\ &\qquad + \int_\Omega \big|\sigma_h f(x, v_h, \nabla v_h) - \sigma f(x, v, \nabla v)\big|^2\,dx\Big)\end{aligned}$$

after choosing $\varepsilon > 0$ so small that $(1+\varepsilon)(\gamma+\delta)^2 < 1$.

The right-hand side above tends to 0 as $h \to 0$ since the nonlinear operators $v \to a(x, v, \nabla v, \xi)$ and $v \to f(x, v, \nabla v)$ are continuous mappings from $W^{1,2q}(\Omega)$ into $L^2(\Omega)$. This fact is a simple consequence of the growth condition (2), condition (A) that implies

$$|a(x, z, p, \xi)| \le C\|\xi\|,$$

and Theorem 16.11 in [4]. Therefore $u_h \to u$ in $W^{2,2}(\Omega)$ (and moreover in $W^{1,2q}(\Omega)$) as $h \to 0$ that shows continuity of the mapping $\mathscr{U}$.

In order to apply Leray-Schauder's theorem to the operator $\mathscr{U}$, we must prove the a priori estimate

$$\|u\|_{W^{1,2q}(\Omega)} \le C \tag{8}$$

for any solution $u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ of the problem

$$\begin{cases} a(x, u, \nabla u, H(u)) = \sigma f(x, u, \nabla u) & \text{a.e. in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{9}$$

where the constant C does not depend on σ and u.

The first step in establishment of (8) is an a priori bound for $\|u\|_{L^\infty(\Omega)}$. The next result shows that (A) really implies uniform ellipticity of our nonlinear operator, and ensures us the possibility to apply Aleksandrov-Pucci maximum principle (see [5], [8]).

Lemma. *Assume $a(x, z, p, \xi)$ satisfies condition* (A). *Then the function $\xi \to a(x, z, p, \xi)$ is differentiable almost everywhere in $\mathbb{R}^4$. If $a^{ij}(x, z, p, \xi) = \frac{\partial a}{\partial \xi_{ij}}(x, z, p, \xi)$, $i, j = 1, 2$, then $a^{ij} \in L^\infty(\Omega \times \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^4)$ and*

$$\sum_{i,j=1}^{2} a^{ij}(x, z, p, \xi)\lambda_i\lambda_j \ge \frac{1-(\gamma+\delta)^2}{2\alpha}|\lambda|^2 \tag{10}$$

for all $(x, z, p, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^4$, $\forall \lambda \in \mathbb{R}^2$.

Proof. Employing the evident inequality $|\mathrm{Tr}(\xi)| \le \sqrt{2}\|\xi\|$ it follows from (A),

$$\left|a(x, z, p, \xi+\tau) - a(x, z, p, \tau)\right| \le \frac{\gamma + \sqrt{2}(1+\delta)}{\alpha}\|\xi\|,$$

i.e., the function $\xi \to a(x, z, p, \xi)$ is Lipschitz continuous function on $\mathbb{R}^4$, uniformly with respect to (x, z, p). The Rademacher theorem implies now that the partial derivatives $\frac{\partial a(x,z,p,\xi)}{\partial \xi_{ij}}$ exist for almost all $\xi \in \mathbb{R}^4$, and they are bounded by $\frac{\gamma+\sqrt{2}(1+\delta)}{\alpha}$. Without loss of generality we may define $a^{ij} = \frac{1-(\gamma+\delta)^2}{2\alpha}\delta^{ij}$ (with Cronecker's δ^{ij}) at the set in $\mathbb{R}^4$ with measure 0, where the above derivatives do not exist.

The ellipticity condition (10) is proved in [2] (see p.6) in the case $a \in C^1$. We will proceed in the same manner.

For, it follows from (A)

$$\left|\mathrm{Tr}\,(\tau) - \alpha\Big(a(x,z,p,\xi+\tau) - a(x,z,p,\xi)\Big)\right|^2 \leq$$
$$\leq \gamma(\gamma+\delta)\|\tau\|^2 + \delta(\gamma+\delta)|\mathrm{Tr}\,(\tau)|^2$$

and therefore

$$2\alpha\mathrm{Tr}\,(\tau)\big(a(x,z,p,\xi+\tau) - a(x,z,p,\xi)\big) \geq$$
$$\geq \big(1-\delta(\gamma+\delta)\big)|\mathrm{Tr}\,(\tau)|^2 - \gamma(\gamma+\delta)\|\tau\|^2.$$

Now, if $\xi \in \mathbb{R}^4$ is a point at which there exists the differential $d_\xi a$, getting $\tau = t\{\lambda_i\lambda_j\}_{i,j=1}^2$, $t \in \mathbb{R}\backslash\{0\}$, $\lambda \in \mathbb{R}^2 \setminus \{0\}$, we have

$$a(x,z,p,\xi + t\{\lambda_i\lambda_j\}) - a(x,z,p,\xi) = t\sum_{i,j=1}^{2} \frac{\partial a(x,z,p,\xi)}{\partial \xi_{ij}}\lambda_i\lambda_j + o(|\,t\,|),$$

and therefore

$$2\alpha t|\lambda|^2\left(t\sum_{i,j=1}^{2}\frac{\partial a(x,z,p,\xi)}{\partial \xi_{ij}}\lambda_i\lambda_j + o(|t|)\right) \geq \big(1-(\gamma+\delta)^2\big)t^2|\lambda|^4.$$

The inequality (10) follows after dividing the both sides by $t^2|\lambda|^2$ and letting $t \to 0$. Of course, (10) holds in the points $\xi \in \mathbb{R}^4$ where the differential $d_\xi a$ does not exists, because of our definition $a^{ij} = \frac{1-(\gamma+\delta)^2}{2\alpha}\delta^{ij}$ there. $\square$

Returning to our problem (9) and using the already proved result, we may rewrite the equation in the next manner:

$$A^{ij}(x)D_{ij}u = \sigma f(x,u,\nabla u) \tag{11}$$

where

$$A^{ij}(x) = \int_0^1 \frac{\partial a}{\partial \xi_{ij}}(x,u(x),\nabla u(x),tH(u)(x))\,dt \in L^\infty(\Omega)$$

(recall $a(x, z, p, 0) = 0$) and $\{A^{ij}(x)\}_{i,j=1}^{2}$ is positively definite matrix with eigenvalues $\geq \frac{1-(\gamma+\delta)^2}{2\alpha} > 0$. Taking into account (3), we have on the set $\Omega^+ = \{x \in \Omega : u(x) > 0\}$:

$$\sigma f(x, u(x), \nabla u(x)) \geq -\sigma\mu_1(x) \mid \nabla u(x) \mid -\sigma\mu_2(x) \geq$$
$$\geq -\sigma\mu_1(x)\sum_{i=1}^{2} \operatorname{sign}\left(\frac{\partial u}{\partial x_i}(x)\right)\cdot\frac{\partial u}{\partial x_i}(x) - \sigma\mu_2(x) \qquad \text{a.e. in } \Omega^+,$$

and it follows from (11):

$$A^{ij}(x)D_{ij}u + \sigma\mu_1(x)\sum_{i=1}^{2} \operatorname{sign}\left(\frac{\partial u}{\partial x_i}(x)\right)\cdot\frac{\partial u}{\partial x_i}(x) \geq -\sigma\mu_2(x) \quad \text{a.e. in } \Omega^+,$$

with $A^{ij}(x) \in L^\infty(\Omega)$, $\sigma\mu_1(x)\operatorname{sign}\left(\frac{\partial u}{\partial x_i}(x)\right) \in L^2(\Omega)$, $\sigma\mu_2 \in L^2(\Omega)$.

Now, the Aleksandrov-Pucci maximum principle ($u \in C^0(\overline{\Omega}) \cap W^{2,2}(\Omega)$) asserts:

$$\sup_\Omega u \leq C\left(\alpha, \gamma, \delta, \operatorname{diam}\Omega, \left\|\mu_1\right\|_{L^2(\Omega)}\right)\left\|\mu_2\right\|_{L^2(\Omega)}. \tag{12}$$

To estimate u from below, we consider the function $w = -u$ that satisfies the equation

$$A^{ij}(x)D_{ij}w = -\sigma f(x, -w, -\nabla w) \quad \text{a.e. in } \Omega$$

and

$$-\operatorname{sign} z\big(-f(x, -z, p)\big) = -\operatorname{sign}(-z)\, f(x, -z, p) \leq \mu_1(x)|p| + \mu_2(x).$$

Repeating the same arguments as above we arrive at:

$$-\inf_\Omega u = \sup_\Omega w \leq C\left\|\mu_2\right\|_{L^2(\Omega)}.$$

The last inequality combined with (12) gives us

$$\sup_\Omega |u| \leq C. \tag{13}$$

Now, inequalities (2) and (13) lead to:

$$\int_\Omega \left|\sigma f(x, u, \nabla u)\right|^2 dx \leq C\left(\left\|f_2\right\|^2_{L^2(\Omega)} + \int_\Omega |\nabla u|^{2q}\, dx\right)$$

whence

$$\int_\Omega |H(u)|^2\,dx \le C\left(1 + \|\nabla u\|^{2q}_{L^{2q}(\Omega)}\right) \tag{14}$$

by means of (7).

The L^{2q}-norm of the right-hand side can be estimated with the help of Gagliargo-Nirenberg estimate ([7]) and (13):

$$\|\nabla u\|_{L^{2q}(\Omega)} \le C\|H(u)\|^{1/2}_{L^2(\Omega)} \cdot \|u\|^{1/2}_{L^{\frac{2q}{2-q}}(\Omega)} \le C\|H(u)\|^{1/2}_{L^2(\Omega)},$$

and applying Young's inequality ($q < 2$), we have

$$\|\nabla u\|^{2q}_{L^{2q}(\Omega)} \le C\|H(u)\|^{q}_{L^2(\Omega)} \le \varepsilon\|H(u)\|^{2}_{L^2(\Omega)} + C(\varepsilon)$$

with arbitrary $\varepsilon > 0$. In other words, (14) has the form

$$\int_\Omega |H(u)|^2\,dx \le \varepsilon\int_\Omega |H(u)|^2\,dx + C(\varepsilon),$$

and choosing ε to be sufficiently small we arrive at the desired estimate (8). Therefore, by virtue of the Leray-Schauder theorem the operator $\mathscr{U}(1,\cdot)$ has a fixed point, i.e., $u = \mathscr{U}(1,u) \in W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega)$ is a solution of (1), that proves Theorem 1. □

Proof of Theorem 2. Let us introduce the function $w = u - v \in C^0(\overline{\Omega}) \cap W^{2,2}_{\text{loc}}(\Omega)$. Then

$$\begin{aligned} a(x,H(u)) - a(x,H(v)) - f(x,u,\nabla u) + f(x,u,\nabla v)& \\ - f(x,u,\nabla v) + f(x,v,\nabla v) = 0 &\qquad \text{a.e. in } \Omega. \end{aligned} \tag{15}$$

Since $f(x,z,p)$ is nondecreasing in z, we have

$$-f(x,u,\nabla v) + f(x,v,\nabla v) \le 0 \qquad \text{a.e. in } \Omega^+,$$

where $\Omega^+ = \{x \in \Omega : \ w(x) = u(x) - v(x) > 0\}$.

On the other hand

$$a(x,H(u)) - a(x,H(v)) = \sum_{i,j=1}^{2} a^{ij}(x) D_{ij} w$$

with

$$a^{ij}(x) = \int_0^1 \frac{\partial a}{\partial \xi_{ij}}\big(x, tH(w)(x) + H(v)(x)\big)\,dt \in L^\infty(\Omega)$$

according to the above proved Lemma.

Moreover, condition (4) ensures Lipschitzity of f with respect to p, and therefore

$$f(x,u,\nabla v) - f(x,u,\nabla u) = \sum_{i=1}^{2} b^i(x) D_i w$$

where

$$b^i(x) = -\int_0^1 \frac{\partial f}{\partial p_i}\big(x, u(x), t\nabla w(x) + \nabla v(x)\big)\,dt \in L^2(\Omega).$$

It follows from (15):

$$\sum_{i,j=1}^{2} a^{ij}(x) D_{ij} w + \sum_{i=1}^{2} b^i(x) D_i w \geq 0 \qquad \text{a.e. in } \Omega^+$$

with $a^{ij} \in L^\infty(\Omega)$, $b^i \in L^2(\Omega)$. Applying the Aleksandrov-Pucci maximum principle again, we arrive at the estimate:

$$\sup_{\Omega^+} w \leq \sup_{\partial\Omega^+} w^+$$

that shows

$$w(x) \leq \sup_{\Omega} w \leq 0 \quad \text{on } \overline{\Omega},$$

since $\sup_{\partial\Omega^+} w^+ = 0$ (we recall $w = 0$ on $\partial\Omega$). Repeating the same arguments with $-w$ instead of w, we obtain $w \geq 0$ on $\overline{\Omega}$, that completes the proof of Theorem 2. □

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Technological University of Sofia,
Department of Mathematics,
8, "Kl. Ohridski" blv.,
Sofia - 1156, (Bulgaria)