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The concept of a profile decomposition formalizes concentration compactness arguments on the functional-analytic level, providing a powerful refinement of the Banach–Alaoglu weak-star compactness theorem. We prove existence of profile decompositions for general bounded sequences in uniformly convex Banach spaces equipped with a group of bijective isometries, thus generalizing analogous results previously obtained for Sobolev spaces and for Hilbert spaces. Profile decompositions in uniformly convex Banach spaces are based on the notion of Δ-convergence by Lim [Remarks on some fixed point theorems, *Proc. Amer. Math. Soc.* **60** (1976) 179–182] instead of weak convergence, and the two modes coincide if and only if the norm satisfies the well-known Opial condition, in particular, in Hilbert spaces and ℓ^p-spaces, but not in $L^p(\mathbb{R}^N)$, $p \neq 2$. Δ-convergence appears naturally in the context of fixed point theory for non-expansive maps. The paper also studies connection of Δ-convergence with Brezis–Lieb lemma and gives a version of the latter without an assumption of convergence a.e.

Keywords: Weak topology; Δ-convergence; Banach spaces; concentration compactness; cocompact imbeddings; profile decompositions; Brezis–Lieb lemma.

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1. Introduction

Finding solutions of equations in functional spaces, in particular of differential equations, typically involves the question of convergence of functional sequences, which in turn often relies on compactness properties of the problem. At the same time, infinite-dimensional Banach spaces have no local compactness. Lack of compactness in a sequence can be qualified in a variety of ways. For example, one can look for coarser topologies in which sequences of particular type, bounded in norm, become relatively compact. Banach–Alaoglu theorem assures that a closed ball in any Banach space is compact in the weak* topology. Concentration compactness principle (put in the terms of Willem and Chabrowski — see the presentation in [7]) addresses the situation when the norm in a functional space is expressed by means of integration of some measure-valued map, which we may call a Lagrangian, and when the Lagrangian, evaluated on a given sequence, has a weak measure limit, its singular support is called a concentration set. For specific sequences it is then possible to show that the singular part of the limit measure is zero, which typically yields convergence of the sequence in norm. As an illustration we sketch an argument for existence of minimizers (see [22, 31]) in the Sobolev inequality:

$$0 < S_{N,p} = \inf_{u\in \dot{W}^{1,p}(\mathbb{R}^N):\|u\|_{p^*}=1} \int |\nabla u|^p \mathrm{d}x, \quad N > p \geq 1, \quad p^* = \frac{pN}{N-p}.$$

The Sobolev imbedding $\dot{W}^{1,p}(\mathbb{R}^N) \hookrightarrow L^{p^*}(\mathbb{R}^N)$ is not compact. It is invariant, however, with respect to transformations $g[j,y](x) = 2^{j\frac{N-p}{p}}u(2^j(x-y))$, $j \in \mathbb{Z}$, $y \in \mathbb{R}^N$, and furthermore, if for any sequence $(j_k, y_k) \subset \mathbb{Z}\times\mathbb{R}^N$ one has $g[j_k, y_k]u_k \rightharpoonup 0$, then $u_k \to 0$ in L^{p^*} (see [21]). Therefore, if $\|u_k\|^p_{\dot{W}^{1,p}} \to S_{N,p}$ while $\|u_k\|_{p^*} = 1$, then, necessarily, there exist $(j_k, y_k)_{k\in\mathbb{N}} \subset \mathbb{Z} \times \mathbb{R}^N$ such that a renamed subsequence of $g[j_k, y_k]u_k$, which we denote v_k (and which, by invariance, is a minimizing sequence as well) converges weakly to some $w \neq 0$. A further reasoning that involves convexity may be then employed to show that $\limsup \|u_k\|^p_{\dot{W}^{1,p}} < S_{N,p}$ unless $\|w\|_{p^*} = 1$, and thus w is a minimizer.

Concentration arguments in application to variational analysis of PDE were developed and applied in the 1980s in works of Uhlenbeck, Brezis, Coron, Nirenberg, Aubin, Lieb, Struwe, and Lions, with perhaps the most notable application being the Yamabe problem of prescribed mean curvature [4, 37, 28]. This classical concentration compactness stimulated development of a more detailed analysis of loss of compactness in terms of *profile decompositions*, starting with the notions of *global compactness* (for bounded domains and critical nonlinearities) of Struwe [30] and of translational compactness in $\mathbb{R}^N$ (for subcritical nonlinearities) of Lions (1986). We do not aim here to provide a survey of concentration compactness and its applications over the decades, and refer the reader instead to the monographs of Chabrowski [7] and Tintarev and Fieseler [35].

AQ: "Lions (1986)" (or) "Lions [21]". Please check.

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A systematic *concentration analysis* extends the concentration compactness approach, from particular types of sequences in functional spaces to *general* sequences in functional spaces, and, further, to general sequences in Banach spaces, studied in relation to general concentration mechanisms modeled as actions of non-compact operator groups. Concentration analysis can thus avail itself to the methods of wavelet analysis: when the group, responsible for the concentration mechanism, generates a wavelet basis, concentration may be described in terms of sequence spaces of the wavelet coefficients. The counterpart of Fourier expansion in the concentration analysis is *profile decomposition.* A profile decomposition represents a given bounded sequence as a sum of its weak limit, decoupled elementary concentrations, and a remainder convergent to zero in a way appropriate for some significant application. An *elementary concentration* is a sequence $g_k w \rightharpoonup 0$, where g_k is a sequence of transformations (*dislocations*) involved in the loss of compactness and the function w is called a *concentration profile.* The 1995 paper of Solimini [29] has shown that in case of Sobolev spaces any bounded sequence admits a profile decomposition, with the only difference from the Palais–Smale sequences for semilinear elliptic functionals (for which such profile decompositions were previously known) being that it might contain countably many, rather than finitely many, decoupled elementary concentrations. The work of Solimini was independently reproduced in 1998–1999 by Gérard [12] and Jaffard [14], who, on the one hand, provided profile decompositions for fractional Sobolev spaces as well, but, on the other hand, gave a weaker form of remainder. It was subsequently realized by Schindler and Tintarev [27], that the notion of a profile decomposition can be given a functional-analytic formulation, in the setting of a general Hilbert space and a general group of isometries (see the alternative proof by Tao via non-standard analysis in [33, p. 168 ff.]) This, in turn stimulated the search for new concentration mechanisms, which included inhomogeneous dilations $j^{-1/2}u(z^j)$, $j \in \mathbb{N}$, with z^j denoting an integer power of a complex number, for problems in the Sobolev space $H_0^{1,2}(B)$ of the unit disk, related to the Trudinger–Moser functionals [2]; and the action of the Galilean invariance, together with shifts and rescalings, involved in the loss of compactness in Strichartz imbeddings for the nonlinear Schrdinger equation (see [32, 15]). The existence of profile decompositions involving the usual rescalings and shifts was established by Kyriasis [17] Koch [16], Bahouri, Cohen and Koch [5] for imbeddings involving Besov, Triebel–Lizorkin and BMO spaces, although, like all similar work based on the use of wavelet bases, it provided only a weak form of remainder. Related results involving Morrey spaces were obtained recently in [26], using, like here, more classical decompositions of spaces instead of wavelets. We refer the reader for details to the recent survey of profile decompositions, [34].

What is obviously missing in all the prior literatures are results about the existence of profile decompositions in the general setting of abstract Banach spaces. This paper introduces a general theory of concentration analysis in Banach space, as a sequel to an earlier Hilbert space version [27] and similar results for Sobolev spaces [29, 2] as well as their wavelet-based counterparts for Besov and Triebel–Lizorkin spaces [5]. The difference between the Hilbert space and the Banach space case is essential and is rooted in the hitherto absence, in a general Banach space, of a simple energy inequality that controls the total bulk of profiles. Our approach to convergence in this paper is based in finding clusters of concentrations with prescribed energy bounds, as there is no transparent relation between the total concentration energy and energies of elementary concentrations. Energy estimates that we obtain are not optimal and are based on the modulus of convexity.

In order to obtain such inequality we have had to abandon weak convergence in favor of Δ-*convergence* introduced by Lim [19], see Definition 3.1 below. The definition applies to metric spaces as well, and are considered in this more general setting in [9]. The notion of Δ-limit is connected to the notion of asymptotic center of a sequence (see [10, Appendix B]), namely, a sequence is Δ-convergent to x if x is an asymptotic center for each of its subsequences. In Hilbert spaces, Δ-convergence and weak convergence coincide (this can be observed following the calculations in a related statement of Opial, [25, Lemma 1]). More generally, the classical Opial's condition (Condition 2 of [25], see Definition 3.17) that has been in use for decades in the fixed point theory, is equivalent, for uniformly convex and uniformly smooth spaces, to the condition that Δ-convergence and weak convergence coincide. Opial's condition, however, does not hold in $L^p(\mathbb{R}^N)$-spaces unless $p \neq 2$, as shown in [25].

Similarly to the Banach–Alaoglu theorem, every bounded sequence in the uniformly convex Banach space has a Δ-convergent subsequence, which follows from the Δ-compactness theorem of Lim (see [19, Theorem 3]).

Role of Δ-convergence in profile decompositions. It is shown in [27] that any bounded sequence in a Hilbert space H, equipped with an appropriate group D of isometries (called *dislocations* or *gauges*), has a subsequence consisting of a sum of asymptotically orthogonal ***elementary concentrations*** and a remainder that ***converges to zero D-weakly***. These two terms mean the following. *An elementary concentration* (sometimes called a *bubble* or a *blow-up*) is an expression of the form $g_k w$, $k \in \mathbb{N}$, where $w \in H$ (called *concentration profile*) and $(g_k) \subset D$ is a sequence weakly convergent to zero in the operator sense, such that $g_k^{-1} u_k \rightharpoonup w$. A sequence $(u_k) \subset H$ is *convergent D-weakly to zero* if for any sequence $(g) \subset D$, $g_k u_k$ converges to zero weakly. D-weak convergence is generally stronger than weak convergence, and, in important applications, it implies convergence in the norm of some space X for which the imbedding $H \hookrightarrow X$ is not compact. Profile decompositions with a remainder vanishing despite the non-compactness of an imbedding express *defect of compactness* for a sequence, in form of a rigidly structured sum of elementary blow-ups. Of course, vanishing of the remainder in a useful norm depends on an appropriate choice of group D. It is easy to check that if D consists of all unitary operators, D-weak convergence becomes norm convergence, and if D is compact, D-weak convergence coincides with weak convergence. A useful group D lies somewhere between these extremes. A continuous imbedding $H \hookrightarrow X$ is called *cocompact relative to the group D* if any D-weakly convergent sequence in H is convergent in the norm of X. The notion of cocompactness extends naturally to Banach spaces, but the proof for the profile decomposition in Hilbert spaces cannot be generalized to the case of Banach spaces, as the summary bulk of concentration profiles of a sequence u_k is controlled by the inequality

$$\sum_{n\in\mathbb{N}} \|w^{(n)}\|^2 \leq \liminf \|u_k\|^2.$$

This inequality is, in turn, a consequence of the elementary relation

$$u_k \rightharpoonup u \Rightarrow \|u_k\|^2 = \|u_k - u\|^2 + \|u\|^2 + o(1). \tag{1.1}$$

(For convenience of presentation, in equalities and inequalities between terms of real-valued sequences, we use, as long as it does not cause ambiguity, a Bachmann–Landau notation $o(1)$ to denote a sequence of real numbers convergent to zero. In other words, $a_k = b_k + o(1)$ stays for $\lim_{k\to\infty}(a_k - b_k) = 0$, and $a_k \le b_k + o(1)$, $a_k, b_k \in \mathbb{R}$, $k \in \mathbb{N}$, stays for $\limsup_{k\to\infty}(a_k - b_k) \le 0$.) A plausible conjecture for the general uniformly convex Banach space (see Appendix A for definitions, in particular of the modulus of convexity, denoted as δ) would be, assuming for simplicity that $\liminf \|u_k\| \le 1$, that

$$\sum_{n\in\mathbb{N}} \delta(\|w^{(n)}\|) \le \liminf \|u_k\|, \tag{1.2}$$

where δ is the modulus of convexity for X. We have however, as the closest Banach space version of (1.1), the inequality

$$u_k \rightharpoonup u \Rightarrow \|u_k\| \ge \|u\| + \delta(\|u_k - u\|) + o(1), \tag{1.3}$$

proving which is an easy exercise using the definition of uniform convexity and weak lower semicontinuity of the norm that we leave to the reader. On the other hand, a desired inequality that leads to (1.2) is rather

$$\|u_k\| \ge \|u_k - u\| + \delta(\|u\|) + o(1), \tag{1.4}$$

and it is generally false when $u_k \rightharpoonup u$. It is true, however, if u is a Δ-limit, rather than a weak limit of u_k (see Lemma 3.7 below). In other words, concentration profiles for sequences in Banach spaces emerge not as weak limits of "deflation" sequences $g_k^{-1}u_k$, but as their Δ-limits.

We restrict consideration of Banach spaces to the class of uniformly convex spaces, as the natural next step after having studied matters of weak convergence and profile decomposition in Hilbert spaces. Uniformly convex spaces have many common properties with Hilbert spaces, in particular, reflexivity, Kadec property ($u_k \rightharpoonup u$, $\|u_k\| \to \|u\| \Rightarrow \|u_k - u\| \to 0$), uniqueness of Δ-limits and sequential Δ-compactness of balls, that general Banach spaces do not necessarily possess.

It appears that sharper than (1.3) or (1.4) lower bounds for the norms of sequences in Banach spaces require the use of both the weak limit and the Δ-limit. Among these cases there is the important Brezis–Lieb lemma (see [6]), which states that if (Ω, μ) is a general measure space and $u_k \rightharpoonup u$ in $L^p(\Omega, \mu)$, $1 \le p < \infty$, and $u_k \to u$ μ-a.e. in Ω, then

$$\|u_k\|_{L^p}^p = \|u_k - u\|_{L^p}^p + \|u\|_{L^p}^p + o(1). \tag{1.5}$$

Remarkably, no a.e. convergence is required for (1.5) to hold when μ is a counting measure or when $p = 2$ (when it follows from (1.1)). If, however, one does not assume convergence a.e., one has the following analog of Brezis–Lieb lemma, proved in Sec. 5 for $p \geq 3$, namely an expression for a lower bound for the norm of the sequence

$$\|u_k\|_{L^p}^p \geq \|u_k - u\|_{L^p}^p + \|u\|_{L^p}^p + o(1), \tag{1.6}$$

where u is assumed to be both the weak limit and the Δ-limit of the sequence (but no a.e. convergence is assumed). It is shown in [3] that condition $p \geq 3$ is necessary, in particular, when (Ω, μ) is an interval with the Lebesgue measure.

For earlier discussions of a.e. convergence in Brezis-Lieb lemma see [23,24]

The paper is organized as follows. In Sec. 2 we give the precise definitions of the concepts arising in concentration analysis and formulate our main results. Section 3 studies basic properties of Δ-convergence in uniformly convex Banach spaces. In Sec. 4 we prove the inequality (1.6). In Sec. 5 we prove the existence of an abstract profile decomposition in terms of Δ-convergence, for every bounded sequence in a uniformly convex and uniformly smooth Banach space, whenever the relevant collection of bijective isometries on X satisfies appropriate hypotheses. It is important to note that the argument for existence of profile decomposition in Banach spaces is different both from the Sobolev space case (see [29]), where the norms show a natural asymptotic decoupling behavior with regard to distinct rescalings and from the general Hilbert spaces case (see [27]), where decoupling of distinct concentrations is expressed by their asymptotic orthogonality. In Sec. 6 we give a general discussion of cocompactness and related properties. In Sec. 7 we prove Theorem 2.6, discuss the remainder of the profile decomposition in the context of cocompact imbeddings, and give examples of the latter. In Appendix A we list definitions and elementary properties of uniformly convex and uniformly smooth Banach spaces, and in Appendix B we present the notion of asymptotic center and its connection to Δ-convergence. Appendix C discusses an equivalent form of the main condition for the groups involved in profile decompositions.

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The main results of the paper are:

- Profile decompositions: Theorem 5.5, its simplified version Theorem 2.6, and profile decomposition in the dual space: Proposition 6.10 and Theorem 2.10;
- Equivalence of the classical Opial's condition in uniformly convex and uniformly smooth spaces to the property that weak convergence and Δ-convergence coincide, Theorem 3.19;
- An analog of the Brezis–Lieb lemma, where the assumption of pointwise convergence replaced by the assumption of equal weak and Δ-limits, Theorem 4.2.

2. Basic Notions of Concentration Analysis and Statement of Results

The key element required for obtaining a cocompact imbedding of a Banach space X into a Banach space Y is a collection D of operators which act isometrically and surjectively (and thus bijectively) on X and which are chosen in such a way that any bounded sequence of elements in X which convergence weakly to zero under action of any sequence from D (see Definition 2.1 below) must converge to zero in the norm of Y. The operators of D are often referred to as “blow-up” or “rescaling” isometries since a frequently occurring example of D is the set of typical concentration actions $u \mapsto t^r u(t\,\cdot)$, $t > 0$. It seems better, however, to use some more general terminology, such as *gauges*, or *dislocations*, to refer to these operators, since D can be quite different in other important cases. For example, it may consist of actions of anisotropic or inhomogeneous dilations, of isometries on Riemannian manifolds, or of shifts in the Fourier variable. An elementary example is provided by a set of index shifts $u \mapsto u_{\cdot+j}$ on a sequence space.

Let D be a set of bijective isometries on a Banach space Z. We will use the following notation: $D^{-1} = \{h^{-1}\}_{h\in D}$.

Definition 2.1 (Gauged weak convergence). Let Z be a Banach space, and let $D \ni I$ be a bounded set of bijective isometries on Z such that D^{-1} is also a bounded set. One says that a sequence $(u_k)_{k\in\mathbb{N}}$ of elements in Z converges to zero D-weakly if $g_k^{-1}u_k \rightharpoonup 0$ for every choice of the sequence $(g_k)_{k\in\mathbb{N}} \subset D$. We use the notation $u_k \overset{D}{\rightharpoonup} 0$ to denote D-weak convergence and the notation $u_k \overset{D}{\rightharpoonup} u$ to mean that $u_k - u \overset{D}{\rightharpoonup} 0$.

We remark that in analogues of this definition appearing in earlier papers on this subject, the roles of D and D^{-1} are interchanged. This makes no difference when D is a group.

The definition below will be adapted to the different mode of convergence introduced in the course of argument, but remains relevant for the class of norms satisfying the Opial's condition which arise in most known applications.

Definition 2.2 (Cocompact subsets). Let Z be a Banach space, and let $D \ni I$ be a set of bijective isometries on Z. A set $B \subset Z$ is called D-cocompact if every D-weakly convergent sequence in B converges in norm in Z.

Clearly the limit in norm of such a sequence must be the same element as its D-weak limit. It is also clear that every precompact subset of X is also D-cocompact.

Definition 2.3 (Cocompact imbeddings). Let X be a Banach space continuously embedded into a Banach space Y. Let $D \ni I$ be a set of bijective isometries on X. Suppose that every sequence $(u_k)_{k\in\mathbb{N}}$ satisfying $u_k \overset{D}{\rightharpoonup} 0$ in X also satisfies $\|u_k\|_Y \to 0$. Then we say that the imbedding $X \hookrightarrow Y$ is *cocompact* relative to the set D, and we denote this by writing $X \overset{D}{\hookrightarrow} Y$.

It is easy to see that the following definition, under the additional assumptions it makes, is equivalent to Definition 2.3.

Definition 2.4. Let X be a Banach space continuously and embedded into a Banach space Y and assume that X is dense in Y and Y^* is dense in X^*. Let $D \ni I$ be a set of bijective isometries on Y, and assume that the set D_X of restrictions of operators in D to X defines a set of bijective isometries on X. One says that the imbedding $X \hookrightarrow Y$ is cocompact relative to the set D, if all bounded subsets of X are D-cocompact in Y.

In what follows, weak convergence of a sequence of operators $(A_k)_{k\in\mathbb{N}}$ on a Banach space X to an operator A, i.e. $A_k x \rightharpoonup Ax$ for each $x \in X$, will be denoted by $A_k \rightharpoonup A$. The following question arises immediately when one knows which set of bijective isometries D is responsible for concentration, or, in other words, when an imbedding $X \hookrightarrow Y$ is cocompact relative to a given set D: Is it possible, for any bounded sequence in X, to produce a subsequence which is norm convergent in Y by subtraction of *elementary concentrations*? We recall that by an elementary concentration for a sequence $(u_k)_{k\in\mathbb{N}} \subset X$ we mean a sequence of the special form $(g_k w)_{k\in\mathbb{N}}$, where $(g_k)_{k\in\mathbb{N}} \subset D$, $g_k \rightharpoonup 0$, and $g_k^{-1} u_k \rightharpoonup w \neq 0$ in X on some renamed subsequence. The use of word *concentration* originates in the case when the set D consists of dilation operators on a functional space, so that, as k tends to ∞, the graphs of the functions $g_k w$ become taller and narrower peaks clustering around some point of the underlying set. Such concentrations occur in scale-invariant PDE, such as semilinear elliptic equations with critical nonlinearities.

Definition 2.5. One says that a bounded sequence $(u_k)_{k\in\mathbb{N}}$ in a Banach space X admits a *profile decomposition* with respect to the set of bijective linear isometries $D \ni I$, if there exists a sequence $r_k \overset{D}{\rightharpoonup} 0$ and, for each $n \in \mathbb{N}$, there exists an element $w^{(n)} \in X$ and a sequence $(g_k^{(n)})_{k\in\mathbb{N}} \subset D$ such that $g_k^{(1)} = I$ and

$$(g_k^{(n)})^{-1} g_k^{(m)} \rightharpoonup 0 \quad \text{whenever } m \neq n \text{ (asymptotic decoupling of gauges)}, \tag{2.1}$$

and such that a renamed subsequence of $(u_k)_{k\in\mathbb{N}}$ can be represented in the form

$$u_k = \sum_{n=1}^{\infty} g_k^{(n)} w^{(n)} + r_k \quad \text{for each } k, \tag{2.2}$$

where the series $\sum_{n=1}^{\infty} g_k^{(n)} w^{(n)}$ is convergent in X unconditionally and uniformly in k. (It follows immediately then that $(g_k^{(n)})^{-1} u_k \rightharpoonup w^{(n)}, n \in \mathbb{N}$.)

Note that in general, any subset of *profiles* $w^{(n)}$ may consist of zero elements. In particular, the sum in (2.2) may be finite.

In the Banach space setting (restricted in the present study to uniformly convex spaces) we will establish the existence of a variant of this profile decomposition, based on Δ-convergence, studied in the next section. Δ-convergence, as we show in Theorem 3.19 below, coincides with weak convergence if and only if the Opial's condition (see, e.g., Definition 3.17) holds.

Our main result follows below. It uses a technical condition (2.3) that extends to the Banach space case the condition of dislocation group used in [27] for Hilbert space case, and it is verified in a great number of applications. We refer the reader for details to the book [35] and to the recent survey [34]. Our principle example of the class of spaces that satisfy conditions of two theorems below is Besov spaces $\dot{B}^{s,p,q}(\mathbb{R}^N)$ and Triebel–Lizorkin spaces $\dot{F}^{s,p,q}(\mathbb{R}^N)$ with $s \in \mathbb{R}$ and $p, q \in (1, \infty)$

when supplied with equivalent norms, based on Littlewood–Paley decomposition (see, e.g., [36, 1]).

Theorem 2.6. *Let X be a uniformly convex and uniformly smooth Banach space that satisfies the Opial's condition. Let D_0 be a group of linear isometries satisfying the property*

$$(g_k) \subset D_0,\ g_k \not\rightharpoonup 0 \Rightarrow \exists\,(k_j) \subset \mathbb{N} : (g_{k_j}^{-1}), (g_{k_j}) \text{ converge strongly (i.e. pointwise)} \tag{2.3}$$

and let $D \ni I$ be a subset of D_0. Then every bounded sequence $(u_k) \subset X$ admits a profile decomposition with respect to D. Moreover, if $\|u_k\| \le 1$ for all k, and δ is the modulus of convexity of X, then

$$\limsup \|r_k\| + \sum_n \delta(\|w^{(n)}\|) \le 1, \tag{2.4}$$

where r_k and $w^{(n)}$ are the elements arising in the profile decomposition as defined in (2.2).

Remark 2.7. The restriction $\|u_k\| \le 1$ is inessential. Unless (x_k) has a subsequence convergent to zero in X (in which case Theorem 5.5 holds with $w^{(n)} = 0$ for all n), one can apply Theorem 5.5 to a subsequence of $x_k/\|x_k\|$ with $\|x_k\| \to \nu > 0$. Then the assertion of Theorem 2.6 (and analogous statements further in this paper) will hold with the only modification being δ replaced by $\nu\delta(\frac{\cdot}{\nu})$.

Remark 2.8. The assumption of uniform convexity cannot be removed, as we can see from the example of $X = L^\infty(\mathbb{R})$ with D being a group of integer shifts. Let x_k be a characteristic function of a disjoint union of all intervals of the length $j/2^k$, $j = 1, \ldots, 2^k$, translated in such a manner that the distance between any two intervals exceeds k. Then the distinct profiles of x_k will be characteristic functions of all intervals $(0,t)$, $t \in (0,1]$, and thus form an uncountable set.

Corollary 2.9. *If, in addition to the assumptions of Theorem* 2.6, *the space X is D-cocompactly imbedded into another Banach space Y, then the remainder r_k converges to zero in the norm of Y.*

In the main body of the paper we first prove a more general statement, Theorem 5.5, similar to Theorem 2.6, that does not assume the Opial's condition, and then derive Theorem 2.6 from it as an elementary corollary. In absence of the Opial's condition, the argument is based on Δ- and D-Δ-convergence instead of, respectively, weak and D-weak convergence.

We also prove a conjecture by Cwikel (personal communication) that when $X \overset{D}{\hookrightarrow} Y$, the existence of profile decompositions in X implies the existence of "dual" profile decompositions in $Y^* \overset{D^\#}{\hookrightarrow} X^*$, where

$$D^\# = \{g^{*-1}, g \in D\}.$$

Theorem 2.10. *Let Y be a uniformly convex and uniformly smooth Banach space that satisfies the Opial's condition. Let $I \in D \subset D_0$ where D_0 is a group of linear isometries in X and Y satisfying* (2.3). *If $X \overset{D}{\hookrightarrow} Y$ and X is dense in Y, then $Y^* \overset{D^\#}{\hookrightarrow} X^*$ and any bounded sequence in Y^* has a profile decomposition relative to $D^\#$ with the remainder sequence $(r_k)_{k\in\mathbb{N}}$ converging in norm to 0 in X^*.*

3. Δ-Convergence in Uniformly Convex Spaces

3.1. *Definition and basic properties*

Definition 3.1. Let $(x_k)_{k\in\mathbb{N}}$ be a sequence in a Banach space X. One says that x is a Δ-limit of (x_k) if

$$\forall\, y \in X \quad \|x_k - x\| \le \|x_k - y\| + o(1). \tag{3.1}$$

We will use the notation $x_k \rightharpoondown x$ as well as $x = \overrightarrow{\lim}\, x_k$ to denote Δ-convergence. Obviously if x_k converges to x in norm, then x is a unique Δ-limit of (x_k).

Proposition 3.2. *Suppose that $(x_k)_{k\in\mathbb{N}}$ is a bounded sequence in a uniformly convex Banach space X and let $x \in X$. If $x_k \rightharpoondown x$, then for each element $z \in X$ with $z \ne x$ there exist a positive constant k_0 and a positive constant c depending on $\|x - z\|$ and $\sup_{k\in\mathbb{N}} \|x_k\|$ continuously in $(0,\infty) \times [0,\infty]$, such that*

$$\|x_k - x\| \le \|x_k - z\| - c \quad \textit{for all } k \ge k_0. \tag{3.2}$$

Proof. Given an element $z \ne x$ we first observe that $\liminf_{k\to\infty} \|x_k - z\|$ must be strictly positive, since otherwise there would be a subsequence of $\{x_k\}$ converging in norm to z.

Without loss of generality we may assume that $\|x_k - x\| < 1$ and note that it suffices to prove (3.2) for $\|x - z\| < 2$. By uniform convexity, and taking into account (3.1), we have

$$\begin{aligned}\|x_k - x\| &\le \left\| x_k + \frac{1}{2}(x+z) \right\| + o(1)\\ &= \left\| \frac{1}{2}[(x_k - x) + (x_k - z)] \right\| + o(1) \le \|x_k - z\| - \delta(\|x - z\|) + o(1),\end{aligned}$$

from which (3.2) is immediate. □

Corollary 3.3. *The Δ-limit in a uniformly convex Banach space is unique.*

It is shown in [10] that uniformly convex Banach spaces are asymptotically complete (a metric space is called asymptotically complete if every bounded sequence in it has an asymptotic center, see Appendix B). Since every bounded sequence in an asymptotically complete metric space has a Δ-convergent subsequence by [19, Theorem 3], we have the following analog of Banach–Alaoglu theorem.

Theorem 3.4. *Let X be a uniformly convex Banach space and let $(x_k) \subset X$ be a bounded sequence. Then (x_k) has a Δ-convergent subsequence.*

3.2. *Uniform boundedness theorem*

It is well known that for every $x \in X\backslash\{0\}$ there exists an element $x^* \in X$, called a conjugate of x, such that $\|x^*\| = 1$ and $\langle x^*, x\rangle = \|x\|$.

If X^* is strictly convex, namely, if

$$\xi, \eta \in X^*, \quad \xi \neq \eta, \quad \|\xi\| = \|\eta\| = 1, \Rightarrow \|t\xi + (1-t)\eta\| < 1 \quad \text{for all } t \in (0,1)$$

(in particular, when X^* is uniformly convex or, equivalently, when X is uniformly smooth, see Appendix A), then the element x^*, as one can immediately verify by contradiction, is unique.

Theorem 3.5. *Let X be a uniformly smooth and uniformly convex Banach space, and let $(x_k) \subset X$ be a Δ-convergent sequence. Then the sequence (x_k) is bounded.*

Proof. It suffices to prove the theorem for the case $x_k \rightharpoondown 0$, since, once we prove that, from $x_k \rightharpoondown x$ follows $x_k - x \rightharpoondown 0$ and thus $x_k - x$ is bounded. Assume that $\|x_k\| \to \infty$. Since X is uniformly smooth, there exists a function $\eta : [0,1] \to [0,\infty)$, $\lim_{t\to 0}\eta(t)/t = 0$, such that (see [20, p. 61])

$$|\|x+y\| - \|x\| - \langle x^*, y\rangle| \leq \eta(\|y\|), \text{ whenever } \|x\| = 1 \text{ and } \|y\| \leq 1.$$

Then, using the notation $\omega(x,y) = \|x+y\| - \|x\| - \langle x^*, y\rangle$, we have

$$\begin{aligned}\|x+y\|^2 - \|x\|^2 &= (\|x+y\| - \|x\|)(\|x+y\| - \|x\| + 2\|x\|)\\ &= (\omega(x,y) + \langle x^*, y\rangle)^2 + 2(\omega(x,y) + \langle x^*, y\rangle).\end{aligned}$$

Substitute now $x = \frac{x_k}{\|x_k\|}$ and $y = \frac{z}{\|x_k\|}$ with an arbitrary vector z. Then, by Proposition 3.2, we have

$$0 \leq \|x_k + z\|^2 - \|x_k\|^2 = \alpha_k^2 + 2\|x_k\|\alpha_k$$

for all k sufficiently large, where

$$\alpha_k = \|x_k\|\omega\left(\frac{x_k}{\|x_k\|}, \frac{z}{\|x_k\|}\right) + \langle x_k^*, z\rangle.$$

Consequently, either $\alpha_k \geq 0$ or $\alpha_k \leq -2\|x_k\| \to -\infty$. The latter case can be easily ruled out, since $\|x_k^*\| = 1$, $\langle x_k^*, z\rangle$ is bounded, $\|x_k\||\omega(\frac{x_k}{\|x_k\|}, \frac{z}{\|x_k\|})| \to 0$ as $\|x_k\| \to \infty$, and so α_k is bounded. Therefore we have necessarily, for large k,

$$\|x_k\|\omega\left(\frac{x_k}{\|x_k\|}, \frac{z}{\|x_k\|}\right) + \langle x_k^*, z\rangle \geq 0,$$

and, thus,

$$\langle x_k^*, z\rangle \geq -\eta(t_k)/t_k,$$

where $t_k = 1/\|x_k\|$. In other words, we have $|\langle \psi(\|x_k\|)x_k^*, z\rangle| \leq 1$, for k sufficiently large, where $\psi(t) = \frac{t^{-1}}{\eta(t^{-1})}$ satisfies $\psi(t) \to \infty$ when $t \to \infty$. By the uniform boundedness principle the sequence $\psi(\|x_k\|)$ is bounded, but this contradicts to the assumption $\|x_k\| \to \infty$, which proves the theorem. □

Note that without the condition of uniform smoothness, Δ-convergent sequences are not necessarily bounded (see [9, Example 3.1]).

3.3. *Characterization of* $\boldsymbol{\Delta}$*-convergence in terms of duality map*

Lemma 3.6. *Let X be a Banach space. If $(x_k)_{k\in\mathbb{N}}$ is a bounded sequence, $x_k \rightharpoondown x$ and $y_k \to y$, then $x_k + y_k \rightharpoondown x + y$.*

Proof. It suffices to prove the assertion for $x = y = 0$. Let $z \in X$. Then

$$\|x_k + y_k\| = \|x_k\| + o(1) \leq \|x_k - z\| + o(1) = \|x_k + y_k - z\| + o(1),$$

which proves the lemma. □

Lemma 3.7. *Let X be a uniformly convex Banach space with the modulus of convexity δ. If $u_k \rightharpoondown u$ in X and $\|u_k\| \leq 1$ for all $k \in \mathbb{N}$, then $\|u\| < 2$ and, for all sufficiently large k,*

$$\|u_k\| \geq \|u_k - u\| + \delta(\|u\|). \tag{3.3}$$

Proof. We can suppose that $u \neq 0$ since the result is a triviality for $u = 0$. Note that for k sufficiently large, $\|u_k - u\| < \|u_k\|$. This inequality implies that $\|u\| < 2\|u_k\| \leq 2$ and it also implies that $u_k \neq 0$ for these values of k. Thus we may apply (A.3) with $C_1 = \|u_k\|$ and $C_2 = 1$ to the elements u_k and $u_k - u$ to obtain that

$$\left\|u_k - \frac{1}{2}u\right\| = \left\|\frac{u_k + (u_k - u)}{2}\right\| \leq \|u_k\| - \delta(\|u\|).$$

Finally, since $u_k \rightharpoondown u$, one also has $\|u_k - u\| \leq \|u_k - \frac{1}{2}u\|$ for sufficiently large k and (3.3) follows. □

We have the following characterization of Δ-convergence by means of the duality map $x \mapsto x^*$.

Theorem 3.8. *Let X be a uniformly convex and uniformly smooth Banach space. Let $x \in X$ and let $(x_k)_{k\in\mathbb{N}} \subset X$ be a bounded sequence such that* $\liminf \|x_k - x\| > 0$. *Then $x_k \rightharpoondown x$ if and only if $(x_k - x)^* \rightharpoonup 0$.*

Proof. Without loss of generality we need only to consider the case $x = 0$.

Sufficiency. Suppose that $x_k^* \rightharpoonup 0$. Then for any $y \in X$, $\langle x_k^*, y\rangle \to 0$ and so

$$\|x_k\| = \langle x_k^*, x_k\rangle = |\langle x_k^*, x_k - y\rangle + \langle x_k^*, y\rangle| \leq \|x_k - y\| + o(1),$$

i.e. $x_k \rightharpoondown 0$.

Necessity. Suppose that $x_k \rightharpoondown 0$. By Proposition 3.2, for any $y \in X$, there exists an integer $k(y)$ such that $\|x_k\| \leq \|x_k - y\|$ for all $k \geq k(y)$. Then

$$\begin{aligned}\|x_k\| \leq \|x_k - y\| &= \langle (x_k - y)^*, x_k - y\rangle = \langle (x_k - y)^*, x_k\rangle - \langle (x_k - y)^*, y\rangle \\ &\leq \|x_k\| - \langle (x_k - y)^*, y\rangle.\end{aligned}$$

Consequently we have

$$\langle (x_k - y)^*, y\rangle \leq 0 \quad \text{for all } k \geq k(y).$$

Since $\liminf \|x_k\| > 0$, we may assume that $k(y)$ is large enough so that $\|x_k\| \geq 2\lambda$ for some positive constant λ whenever $k \geq k(y)$. So, if we consider only those y which satisfy $\|y\| \leq \lambda$ and those x_k for which $k \geq k(y)$, we can assert that x_k and $x_k - y$ are both contained in the set $E = \{x \in X : \|x\| \geq \lambda\}$ and therefore deduce from Lemma A.2, for each $\epsilon \in (0, 1/4)$, that $\|(x_k - y)^* - x_k^*\| \leq \epsilon$ whenever $0 < \|y\| \leq \min\{\frac{3\lambda}{2}\delta(\epsilon), \frac{\lambda}{2}\}$. For such choices of y we will therefore have

$$\left\langle x_k^*, \frac{y}{\|y\|}\right\rangle \leq \epsilon \quad \text{for all } k \geq k(y).$$

Applying the same reasoning to the element $-y$, we obtain that $|\langle x_k^*, \frac{y}{\|y\|}\rangle| \leq 2\epsilon$ whenever $0 < \|y\| \leq \min\{\frac{3\lambda}{2}\delta(\epsilon), \frac{\lambda}{2}\}$ and $k \geq k_0 = \max\{k(y), k(-y)\}$. In other words, given any $w \in X$ with $\|w\| = 1$, we know that $|\langle x_k^*, w\rangle| \leq 2\epsilon$ for all $k \geq k_0(w, \epsilon)$ for some sufficiently large $k_0(w, \epsilon)$. Consequently, $x_k^* \rightharpoonup 0$. □

Corollary 3.9. *Let X be either a Hilbert space or the ℓ^p-space with $1 < p < \infty$, and let (x_n) be a sequence in X. Then $x_n \rightharpoonup x$ if and only if $x_n \rightharpoondown x$.*

Proof. We may assume that $\liminf \|x_n - x\| > 0$, since for subsequences that converge to x in norm the result is trivial.

Let X be a Hilbert space and recall that we are using the definition of conjugate dual with the unit norm. If $x_n \rightharpoonup x$, then for any $y \in X$,

$$\begin{aligned}|((x_n - x)^*, y)| &= \left|\left(\frac{x_n - x}{\|x_n - x\|}, y\right)\right| \\ &\leq \frac{1}{\liminf \|x_n - x\|}\limsup |(x_n - x, y)| + o(1) \to 0.\end{aligned}$$

Conversely, if $x_n \rightharpoondown x$, then for any $y \in X$, taking into account that (x_n) is bounded by Theorem 3.5, we have

$$|(x_n - x, y)| = \|x_n - x\||((x_n - x)^*, y)| \leq (\sup \|x_n\| + \|x\|)|((x_n - x)^*, y)| \to 0.$$

Let now $X = \ell^p$. If $x_n \rightharpoonup x$, then the sequence (x_n) is bounded and converges to x by components. Then $(x_n - x)^* \rightharpoonup 0$ in $\ell^{p'}$, and by Theorem 3.8 it follows that $x_n \rightharpoondown x$.

Conversely, if $x_n \rightharpoondown x$, then by Theorem 3.8 $(x_n - x)^* \rightharpoonup 0$ in $\ell^{p'}$, and then x_n converges to x by components. Since by Theorem 3.5 Δ-convergent sequences are bounded, this implies that $x_n \rightharpoonup x$. □

Remark 3.10. Another proof that weak and Δ-convergence in Hilbert space coincide can be inferred from the definition of Δ-convergence, Proposition 3.2 and the elementary identity

$$\|x_n - x + y\|^2 = \|x_n - x\|^2 + \|y\|^2 + 2(x_n - x, y).$$

Remark 3.11. From Theorem 3.8 it follows that Δ-limit is not additive, that is, the relation $\overset{\rightharpoondown}{\lim}\,(x_n + y_n) = \overset{\rightharpoondown}{\lim}\, x_n + \overset{\rightharpoondown}{\lim}\, y_n$ is generally false. Consider, for example, $L^4((0,9))$. Set $x_0(t) = 2$ for $t \in (0,1]$ and $x_0(t) = -1$ for $t \in (1,9]$. Define $x_n(t) = x_0(nt)$ when $0 < t \leq \frac{9}{n}$ and extend it periodically to $(0,9)$. Set $y_0(t) = -1$ for $t \in (0, \frac{9}{2}]$ and $y_0(t) = 1$ for $t \in (\frac{9}{2}, 9]$ and define y_n similarly to x_n. Observe that $x_n^3 \rightharpoonup 0$, $y_n^3 \rightharpoonup 0$, but $(x_n + y_n)^3 \rightharpoonup \frac{1}{2}$.

Remark 3.12. Using Theorem 3.8 one can also show that norms are not necessarily lower semicontinuous with respect to Δ-convergence. Let (v_k) be a normalized sequence in $L^4([0,1])$, such that $v_k \rightharpoondown 0$ and $v_k \rightharpoonup a$ where a is a positive constant (one constructs such sequence by fixing a step function v_0 such that $\int v_0^3 = 0$ and $\int v_0 > 0$, rescaling it by the factor k and extending it periodically). By Theorem 3.8, $v_k^3 \rightharpoonup 0$ in $L^{4/3}$. Let $u_k = u - tv_k$, $t > 0$ with some positive function u. Then

$$\int u^4 - \int u_k^4 = 4t^3 \int uv_k^3 - 6t^2 \int u^2 v_k^2 + 4t \int u^3 v_k - t^4 \int v_k^4$$
$$\geq -6t^2 \int u^2 v_k^2 + 4ta \int u^3 - t^4 + to(1).$$

Taking into account that $\int u^2 v_k^2$ is bounded as $k \to \infty$, we have that for t sufficiently small the right-hand side is bounded away from zero for all k sufficiently large.

3.4. Δ-*convergence versus weak convergence*

As we have shown above, Δ-limits and weak limits coincide in Hilbert spaces in ℓ^p-spaces, $1 < p < \infty$ (Corollary 3.9). In general it can happen that the weak limit and the Δ-limit of a sequence both exist but are different.

Example 3.13. An example of Opial [25, Sec. 5] allows an immediate interpretation in terms of Δ-limit and then says that in the space $L^p((0, 2\pi))$, $p \neq 2$, $1 < p < \infty$, there exist sequences whose Δ-limit and weak limit are different functions. (Cwikel has brought the authors' attention to the fact that the number 3/4 which appears twice in the definition of function ϕ on p. 596 of [25] is a misprint and is to be read in both places as 4/3.)

Remark 3.14. Furthermore, if Ψ_n is the primitive function of ψ_n of Opial's counterexample, normalized in $W^{1,p}((0, 2\pi))$, the sequence $\{\Psi_n\}$ in $W^{1,p}((0, 2\pi))$ also has a Δ-limit and a weak limit with different values (note that because of the normalization coefficient the non-gradient portion of the Sobolev norm for this sequence is vanishing).

Remark 3.15. It is not clear at this point when Δ-convergence can be associated with a topology, except when Δ-convergence coincides with weak convergence. See a preliminary discussion in [9].

Remark 3.16. In general, weakly lower semicontinuous functionals are not lower semicontinuous with respect to Δ-convergence. From Example 3.13 it follows that this is the case already for continuous linear functionals acting on L^p, $p \neq 2$.

3.5. *The Opial's condition in uniformly convex spaces*

In this subsection we show that the Opial's condition (Condition (2) in [25]), which plays significant role in the fixed point theory, has, for uniformly convex and uniformly Banach spaces, two equivalent formulations. One is that weak and Δ-convergence coincide and the other is that the Frecht derivative of the norm is weak-to-weak continuous away from zero. The latter is similar to [25, Lemma 3] (which makes a weaker assertion under weaker conditions).

Definition 3.17. Let X be a Banach space. One says that a sequence $(x_n)_{n\in\mathbb{N}} \subset X$, which is weakly convergent to a point $x_0 \in X$, satisfies the *Opial's condition* if

$$\liminf \|x_n - x_0\| \leq \liminf \|x_n - x\| \quad \text{for every } x \in X. \tag{3.4}$$

One says that a Banach space X satisfies the Opial's condition if any weakly convergent sequence $(x_k)_{k\in\mathbb{N}}$ in X satisfies the Opial's condition.

Remark 3.18. It is immediate from respective definitions that if a sequence in a Banach space satisfies the Opial's condition and is both weakly convergent and Δ-convergent, then its Δ-limit equals its weak limit.

Theorem 3.19. *Let X be a uniformly convex and uniformly smooth Banach space. Then X satisfies the Opial's condition if and only if for any sequence $(x_n)_{n\in\mathbb{N}} \subset X$,*

$$x_n \rightharpoonup x \Leftrightarrow x_n \rightharpoondown x, \tag{3.5}$$

or, equivalently, if for any bounded sequence which does not have a strongly convergent subsequence,

$$x_n \rightharpoonup x \text{ in } X \Leftrightarrow (x_n - x)^* \rightharpoonup 0 \text{ in } X^*. \tag{3.6}$$

Proof. The Opial's condition follows immediately from (3.5) and the definition of Δ-convergence. Assume now that Opial's condition holds. By the Banach–Alaoglu theorem and Theorem 3.4 (once we take into account Theorem 3.5), it suffices to consider sequences that have both a weak and a Δ-limit. Then by (3.4) the weak limit of such sequence satisfies the definition of Δ-limit. The last assertion of the theorem follows from Theorem 3.8. □

Remark 3.20. It should also be noted that Δ-convergence, unlike weak convergence, depends on the choice of an equivalent norm. Theorem 1 of van Dulst [38], proves that in a separable Banach space one can always find an equivalent norm (that one may call a *van Dulst norm*) such that every weakly convergent sequence in the space satisfies Opial's condition (3.4), i.e. that Δ-convergence associated with a van Dulst norm is associated with the weak topology. In practice, however, renorming the space may change conditions of a problem where the Opial's condition is needed. In particular, since van Dulst's construction uses a basis in a Banach space Y which contains X isometrically, it is not clear if one can preserve the invariance of the equivalent norm with respect to a given group of operators without existence of a wavelet basis associated with this group. Theorem 2.6 requires that the new norm will remain uniformly convex and invariant with respect to a fixed group of isometries, which is not assured by the van Dulst's construction. For the purpose of applications to functional spaces, uniformly convex norms, satisfying strong Opial's condition and invariant with respect to Euclidean shifts and dyadic dilations, are known (see [8]) for Besov and Triebel–Lizorkin spaces $\dot{B}^{s,p,q}$ and $\dot{F}^{s,p,q}$ with $p,q \in (1,\infty)$, $s \in \mathbb{R}$ (which includes Sobolev spaces $\dot{H}^{s,p}$ for all $s \in \mathbb{R}$, $p \in (1,\infty)$)) for all Besov and Triebel–Lizorkin spaces $\dot{B}^{s,p,q}$ and $\dot{F}^{s,p,q}$ with $p,q \in (1,\infty)$, $s \in \mathbb{R}$ (which includes Sobolev spaces $\dot{H}^{s,p}(\mathbb{R}^N)$ for all $s \in \mathbb{R}$ and $p \in (1,\infty)$). Motivation for the choice of norm, based on the Littlewood–Paley decomposition, can be found by the proof of cocompactness of Sobolev imbeddings in [15, Chap. 4] (note that the authors call the property of cocompactness *inverse imbedding*). The argument of Cwikel is based on verifying (3.6) using the definition of the equivalent norm for Besov and Triebel–Lizorkin spaces from [36, Definition 2, p. 238], based on the Littlewood–Paley decomposition, and it reduces both weak and polar convergence, by straightforward calculations, to obvious pointwise convergence of the sequence $(2^{ns}F^{-1}\varphi_0(2^{-n}\cdot)F(u_k-u))_{k\in\mathbb{N}}$, where F denotes the Fourier transform, φ_0 is a smooth function supported in an annulus, $n \in \mathbb{Z}$ and $s \in \mathbb{R}$.

4. A Discussion Concerning the Brezis–Lieb Lemma

It is interesting to note that while weak convergence of $(x_k)_{k\in\mathbb{N}}$ to an element x in a Banach space implies that $\|x_k\| \geq \|x\| + o(1)$ (weak lower semicontinuity of the norm), Δ-convergence of such a sequence to x implies that $\|x_k\| \geq \|x_k - x\| + o(1)$, while in the case of sequences in a Hilbert space, both of these inequalities can also be deduced from the stronger condition

$$\|x_k\|^2 = \|x_k - x\|^2 + \|x\|^2 + o(1) \tag{4.1}$$

When the space X is uniformly convex, Lemma 3.7 gives a lower bound for the norm of the Δ-convergent sequence in the form $\|u_k\| \geq \|u_k - u\| + \delta(\|u\|) + o(1)$. Another relation that allows to estimate the norm of the sequence (u_k) by the norms of its weak limit u and of the remainder sequence $u_k - u$ when $X = L^p$, $1 \leq p < \infty$, is the important Brezis–Lieb lemma [6]. Remarkably, in the case $p = 2$ Brezis–Lieb lemma follows from (4.1), while for $p \neq 2$ it requires, in addition to the

[8]

/Delta

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Remarkably, Brezis-Lieb lemma in the case p=2 follows from (4.1) unded assumption of weak convergence, but when $p \neq 2$ a stronger assumption of a.e. convergence is required.

assumption of weak convergence, also convergence almost everywhere. One may, however, interpret convergence a.e. as a sufficient condition for Δ-convergence of the sequence to its weak limit, as one can see from the Brezis–Lieb lemma itself, or, alternatively, from the following argument.

Lemma 4.1. *Let (Ω, μ) be a measure space and let u_k be a bounded sequence in $L^p(\Omega, \mu), p \in (1, \infty)$. If $u_k \rightharpoonup u$ and $u_k \to u$ a.e. then $u_k \rightharpoondown u$.*

Proof. Without loss of generality we may assume that $u = 0$. Let $u_k^* \rightharpoondown w$ on a renamed subsequence. Then $w = 0$ on every set where a.e. convergence becomes uniform, and therefore, by Egoroff theorem, $w = 0$ outside of a set of arbitrarily small measure, and thus a.e. Thus u_k^* has no subsequence with a non-zero Δ-limit, i.e. $u_k \rightharpoondown 0$. □

It is natural to pose the question, what may remain of the assertion of the Brezis–Lieb lemma if we replace its conditions with a weaker requirement that both Δ-limit and weak limit exist and are equal.

Theorem 4.2. *Let (Ω, μ) be a measure space. Assume that $u_k \rightharpoonup u$ and $u_k \rightharpoondown u$ in $L^p(\Omega, \mu)$. If $p \geq 3$ then*

$$\int_\Omega |u_k|^p d\mu \geq \int_\Omega |u|^p d\mu + \int_\Omega |u_k - u|^p d\mu + o(1). \tag{4.2}$$

Proof. In order to prove the assertion it suffices to verify the elementary inequality

$$(1+t)^p \geq 1 + |t|^p + p|t|^{p-2}t + pt, \tag{4.3}$$

since it implies $|u_k|^p \geq |u_k - u|^p + |u|^p + p|u|^{p-2}u(u_k - u) + p|u_k - u|^{p-2}(u_k - u)u$, with the integrals of the last two terms vanishing by assumption. The elementary inequality is equivalent to the inequalities

$$f_+(t) = (1+t)^p - 1 - t^p - pt^{p-1} - pt \geq 0, \quad t \geq 0$$

and, assuming without any restriction (in view of the symmetry of the formula) that $|t| \leq 1$

$$f_-(t) = (1-t)^p - 1 - t^p + pt^{p-1} + pt \geq 0, \quad t \in [0,1].$$

To prove them, note that both functions vanish at zero, so it suffices to show that their derivatives are non-negative. We have

$$\frac{1}{p} f_+'(t) = (1+t)^{p-1} - t^{p-1} - 1 - (p-1)t^{p-2},$$

which is also a function vanishing at zero, so it suffices to show that its derivative is non-negative, i.e.

$$\frac{1}{p(p-1)} f_+''(t) = (1+t)^{p-2} - t^{p-2} - (p-2)t^{p-3} \geq 0.$$

Let $s = t^{-1}$ and $q = p - 2$. Then

$$\frac{s^q}{p(p-1)} f_+''(s^{-1}) = (1+s)^q - 1 - qs \geq 0, \quad s \geq 1,$$

which is true by convexity of the first term, since $q \geq 1$ (i.e. $p \geq 3$).

Consider now the derivative of f_-:

$$\frac{1}{p} f_-'(t) = -(1-t)^{p-1} - t^{p-1} + 1 + (p-1)t^{p-2}.$$

It remains to notice that $(1-t)^{p-1} + t^{p-1} \leq 1$. □

Remark 4.3. Easy calculations show that inequality (4.3) used in the proof of Theorem 4.2 does not hold unless $p \geq 3$, and the argument of homogenization type is used in [3] to show that condition $p \geq 3$ is indeed necessary for (4.2), unless $p = 2$. For $p = 2$, as we already mentioned, inequality (4.2) holds, and, moreover, becomes an equality, which can be easily verified.

Remark 4.4. The inequality in (4.2) can be strict. Indeed, one can easily calculate by binomial expansion for $p = 4$ that if $u_k \rightharpoonup u$ and $u_k \rightharpoondown u$ (i.e. $(u_k - u)^3 \rightharpoonup 0$ in $L^{4/3}$), then

$$\int_\Omega |u_k|^4 d\mu = \int_\Omega |u|^4 d\mu + \int_\Omega |u_k - u|^4 d\mu + 6 \int u^2 (u_k - u)^2 d\mu + o(1).$$

Let $\Omega = (0, 3)$ equipped with Lebesgue measure and consider three sequences of disjoint sets $A_{1;k}, \ldots, A_{3;k}$, $k \in \mathbb{N}$, such that $(\frac{m-1}{k}, \frac{m}{k}) \subset A_{\mathrm{rem}(m,3);k}$ where $\mathrm{rem}(m, 3)$ is the remainder of division of m by 3 and $m = 1, \ldots, 3k$. Set $u_k = \sum_{i=1}^3 a_i \chi_{A_{i;k}}$ where $a_1 = 1$, $a_2 = 2$ and $a_3 = 0$. Then $u_k \rightharpoonup 1$ and $(u_k - u)^3 \rightharpoonup \frac{1}{3} \sum_i (a_i - 1)^3 = 0$, while $\int u^2 (u_k - u)^2 d\mu \to 2 > 0$.

Remark 4.5. Δ-convergence is *necessary* for the assertion of Brezis–Lieb lemma, and even a weaker statement (4.2), to hold. More accurately, if a sequence $(u_k) \subset L^p(\Omega, \mu)$, $p \in [1, \infty)$, and a function $u \in L^p(\Omega, \mu)$ are such that for any $v \in L^p(\Omega, \mu)$,

$$\int_\Omega |u_k - v|^p d\mu \geq \int_\Omega |u - v|^p d\mu + \int_\Omega |u_k - u|^p d\mu + o(1), \tag{4.4}$$

then $u_k \rightharpoondown u$ by the definition of Δ-limit.

5. Profile Decomposition in Terms of Δ-Convergence

Throughout this section we assume that X is a uniformly convex and uniformly smooth Banach space. We also assume that D is a subset, containing the identity operator, of a group D_0 of isometries on X. In this section we prove that every bounded sequence in X has a subsequence with a profile decomposition based on

Δ-convergence. The reason that motivates us to define concentration profiles as Δ-limits, rather than weak limits, is that Δ-convergence yields estimates of the energy type (3.3) which are not readily available when usual weak convergence is used.

We need to modify some of the definitions of previous sections, which are based on weak convergence, by changing the mode of convergence involved to Δ-convergence.

Definition 5.1. One says that a sequence $(u_k) \subset X$ has a D-Δ-limit u (to be denoted $u_k \overset{D}{\rightharpoondown} u$), if for every sequence $(g_k) \subset D$, $g_k^{-1}(u_k - u) \rightharpoondown 0$.

Equivalently, if we take into account the supremum of the norms of the Δ-profiles of a given sequence by setting

$$p((u_k)_{k\in\mathbb{N}}) = \sup\{\|w\| : \exists \text{ subsequences } (u_{n_k}) \subset (u_k) \text{ and } (g_k) \subset D, \\ \text{such that } g_k^{-1}(u_{n_k}) \rightharpoondown w\},$$

we can say that $u_k \overset{D}{\rightharpoondown} u$ if and only if $p((u_k - u)_{k\in\mathbb{N}}) = 0$.

Definition 5.2. One says that a bounded sequence (u_k) in a Banach space X *admits a Δ-profile decomposition relative to the set of isometries $D \subset D_0$*, if there exist sequences $(g_k^{(n)})_k \subset D$ with $g_k^{(1)} = \mathrm{Id}$, elements $w^{(n)} \in X$, $n \in \mathbb{N}$, and a sequence $r_k \overset{D}{\rightharpoondown} 0$ such that

$$(g_k^{(n)})^{-1} g_k^{(m)} \rightharpoonup 0 \quad \text{whenever } m \neq n \text{ (asymptotic decoupling of gauges)}, \tag{5.1}$$

and a renamed subsequence of u_k can be represented in the form

$$u_k = \sum_{j=1}^{\infty} g_k^{(j)} w^{(j)} + r_k, \tag{5.2}$$

where the series $\sum_{j=1}^{\infty} g_k^{(j)} w^{(j)}$ is convergent in X ~~absolutely~~ and uniformly with respect to k. In this case we also have unconditionally

$$(g_k^{(n)})^{-1} u_k \rightharpoondown w^{(n)}, \quad n \in \mathbb{N}.$$

Definition 5.3. We shall say that the group D_0 of isometries on a Banach space X is a *dislocation group* (to be denoted $D_0 \in \mathcal{I}_X$) if it satisfies

$$(g_k) \subset D_0, \quad g_k \not\rightharpoonup 0 \\ \Rightarrow \exists\, (k_j) \subset \mathbb{N} : (g_{k_j}^{-1}) \text{ and } (g_{k_j}) \text{ converge operator-strongly (i.e. pointwise)} \tag{5.3}$$

and

$$u_k \rightharpoondown 0, \quad w \in X, \quad (g_k) \subset D_0, \quad g_k \rightharpoonup 0 \Rightarrow u_k + g_k w \rightharpoondown 0. \tag{5.4}$$

Remark 5.4. Note also that condition (5.4) is trivially satisfied if Opial's condition holds, and in particular, in a Hilbert space, so this definition agrees with the definition of the dislocation group used previously in [35]. It is easy to prove that when D_0 is a dislocation group, the profiles $w^{(n)}$ in Definition 5.2 are unique, up to the choice of subsequence and up to multiplication by an operator $g \in D_0$. The argument is repetitive of that in [35, Proposition 3.4], which considers the case of Hilbert space.

Theorem 5.5. *Let X be a uniformly convex and uniformly smooth Banach space and let $D \ni \mathrm{Id}$ be subset of a dislocation group D_0. Then every bounded sequence $(x_k) \subset X$ admits a Δ-profile decomposition relative to D. Moreover, if $\|x_k\| \le 1$, and δ is the modulus of convexity of X, then $\|w^{(n)}\| \le 2$ for all $n \in \mathbb{N}$ and*

$$\limsup \|r_k\| + \sum_n \delta(\|w^{(n)}\|) \le 1. \tag{5.5}$$

We prove the theorem via a sequence of lemmas.

Lemma 5.6. *Let $(g_k) \subset D_0$. If $g_k \rightharpoonup 0$ then $g_k^{-1} \rightharpoonup 0$.*

Proof. Assume that $g_k^{-1} \not\rightharpoonup 0$. Then by (5.3) the sequence (g_k) has a strongly convergent subsequence, whose limit is an isometry, and thus it cannot be zero. □

Lemma 5.7. *Let $(g_k) \subset D$ be such that g_k^{-1} is operator-strongly convergent. If $x_k \rightharpoondown 0$, then $g_k x_k \rightharpoondown 0$.*

Proof. It is immediate from the assumption that there is a linear isometry h, such that $g_k^{-1} y \to hy$ for every $y \in X$. Then

$$\langle (g_k x_k)^*, y\rangle = \langle x_k^*, g_k^{-1} y\rangle = \langle x_k^*, hy\rangle + o(1) \to 0. \quad □$$

Our next lemma assures that dislocation sequences (g_k) that provide distinct profiles are asymptotically decoupled.

Lemma 5.8. *Let $(u_k) \subset X$ be a bounded sequence. Assume that there exist two sequences $(g_k^{(1)})_k \subset D$ and $(g_k^{(2)})_k \subset D$, such that $(g_k^{(1)})^{-1} u_k \rightharpoondown w^{(1)}$ and $(g_k^{(2)})^{-1}(u_k - g_k^{(1)} w^{(1)}) \rightharpoondown w^{(2)} \neq 0$. Then $(g_k^{(1)})^{-1}(g_k^{(2)}) \rightharpoonup 0$.*

Proof. Assume that $(g_k^{(1)})^{-1}(g_k^{(2)})$ does not converge weakly to zero. Then by (5.3), on a renamed subsequence, $(g_k^{(1)})^{-1}(g_k^{(2)})$ converges operator-strongly to some isometry h. Then by Lemma 5.7,

$$(g_k^{(1)})^{-1}(g_k^{(2)})[(g_k^{(2)})^{-1}(u_k - g_k^{(1)} w^{(1)}) - w^{(2)}] \rightharpoondown 0,$$

which implies, taking into account (5.4),

$$(g_k^{(1)})^{-1} u_k - w^{(1)} - hw^{(2)} \rightharpoondown 0.$$

However, this contradicts the definition of $w^{(1)}$ and the assumption that $w^{(2)} \neq 0$. □

The next statement assures that one can find decoupled elementary concentrations by iteration.

Lemma 5.9. *Let u_k be a bounded sequence in X and let $(g_k^{(n)})_k \subset D, w^{(n)} \in X, n = 1, \ldots, M$, be such that $g_k^{(1)} = I, (g_k^{(n)})^{-1} u_k \rightharpoonup w^{(n)}, n = 1, \ldots M$, and $(g_k^{(n)})^{-1}(g_k^{(m)}) \rightharpoonup 0$ whenever $n < m$. Assume that there exists a sequence $(g_k^{(M+1)}) \subset D$ such that, on a renumbered subsequence, $(g_k^{(M+1)})^{-1}(u_k - w^{(1)} - g_k^{(2)} w^{(2)} - \cdots - g_k^{(M)} w^{(M)}) \rightharpoonup w^{(M+1)} \neq 0$. Then $(g_k^{(n)})^{-1}(g_k^{(M+1)}) \rightharpoonup 0$ for $n = 1, \ldots, M$.*

Proof. We can replace u_k by $u_k - \sum_{m \neq n} g_k^{(m)} w^{(m)}$ and then, thanks to (5.4), apply Lemma 5.8 with 1 replaced by n and 2 by $M + 1$. □

We may now start the construction needed for the proof of Theorem 5.5. As we have remarked before, we may without loss of generality assume that $\|x_k\| \leq 1$.

Let us introduce a partial strict order relation between sequences in X, to be denoted as $>$. First, given two sequences $(x_k) \subset X$ and $(y_k) \subset X$, we shall say that $(x_k) \succ (y_k)$ if there exists a sequence $(g_k) \subset D$, an element $w \in X \setminus \{0\}$, and a renumeration (n_k) such that $g_{n_k}^{-1} x_{n_k} \rightharpoonup w$ and $y_k = x_{n_k} - g_{n_k} w$. From Lemma 3.7 it follows that if $(x_k) \succ (y_k)$ and $\|x_k\| \leq 1$, then $\|y_k\| \leq 1$ for k sufficiently large, and therefore it follows from sequential Δ-compactness of bounded sequences that for every sequence $(x_k) \subset X$, $\|x_k\| \leq 1$, which is not D-Δ- convergent to 0, there is a sequence $(y_k) \subset X$, such that $\|y_k\| \leq 1$ and $(x_k) \succ (y_k)$.

Then we shall say that $(x_k) > (y_k)$ in one step, if $(x_k) \succ (y_k)$ and in m steps, $m \geq 2$, if there exist sequences $(x_k^1) \succ (x_k^2) \succ \ldots \succ (x_k^m)$, such that $(x_k^1) = (x_k)$ and $(x_k^m) = (y_k)$. Note that, for every sequence $(x_k) \subset X$, $\|x_k\| \leq 1$, either there exists a finite number of steps $m_0 \in \mathbb{N}$ such that $(x_k) > (y_k)$ in m_0 steps for some $(y_k) \subset X$, $\|y_k\| \leq 1$, and $p((y_k)) = 0$, or for every $m \in \mathbb{N}$ there exists a sequence $(y_k) \subset X$, $\|y_k\| \leq 1$, such that $(x_k) > (y_k)$ in m steps. We will say that $(x_k) \geq (y_k)$ if either $(x_k) > (y_k)$ or $(x_k) = (y_k)$.

Define now

$$\sigma((x_k)) = \inf_{(y_k) \geq (x_k)} \sup_k \|y_k\|$$

and observe that if $(x_k) \geq (z_k)$, then $\sigma((x_k)) \leq \sigma((z_k))$, since the set of sequences (y_k) dominating (z_k) is a subset of sequences dominating (x_k).

Lemma 5.10. *Let $(x_k) > (y_k)$ in m steps, $\|x_k\| \leq 1$ and $\eta > 0$. Then there exist elements $w^{(1)}, \ldots, w^{(m)}$, and sequences $(g_k^{(1)}), \ldots, (g_k^{(m)})$ in D, and a renumeration (n_k) such that*

$$y_k = x_{n_k} - \sum_{n=1}^{m} g_{n_k}^{(n)} w^{(n)},$$

$(g_{n_k}^{(p)})^{-1}g_{n_k}^{(q)} \rightharpoonup 0$ *for* $p \neq q$, *and for any set* $J \subset J_m = (1, \ldots, m)$,

$$\delta\left(\left\|\sum_{n\in J} g_{n_k}^{(n)} w^{(n)}\right\|\right) \leq \sup \|x_{n_k}\| - \sigma((x_{n_k})) + \eta, \quad \textit{for all } k \textit{ sufficiently large.} \tag{5.6}$$

Proof. The first assertion follows from Lemma 5.9. Let

$$\alpha_k = x_{n_k} - \sum_{n\in J_m\setminus J} g_k^{(n)} w^{(n)},$$

$$\beta_k = x_{n_k} - \sum_{n\in J_m\setminus J} g_k^{(n)} w^{(n)} - \frac{1}{2}\sum_{n\in J} g_k^{(n)} w^{(n)} = \frac{1}{2}(\alpha_k + y_k).$$

By Lemma 3.7, $\|y_k\| \leq \|\alpha_k\| \leq \|x_k\| \leq 1$ and $\beta_k \leq 1$ for all k large. Note that, as in the construction above, we can take k large enough so that $\sup \|\beta_k\| \leq \inf \|\beta_k\| + \eta$. By uniform convexity, for large k we have

$$\|\beta_k\| \leq \|\alpha_k\| - \delta(\alpha_k - y_k).$$

Therefore

$$\delta\left(\left\|\sum_{n\in J} g_k^{(n)} w^{(n)}\right\|\right) \leq \|\alpha_k\| - \|\beta_k\| \leq \sup \|x_k\| - \sigma((x_k)) + \eta. \qquad \square$$

Proof of Theorem 5.5. For every $j \in \mathbb{N}$ define $\epsilon_j = \delta(\frac{1}{2^j})$. Let $(x_k^{(1)}) \subset X$ be such that $(x_k) > (x_k^{(1)})$ and $\sup\|x_k^{(1)}\| < \sigma((x_k)) + \epsilon_1$. Consider the following iterations. Given $(x_k^{(j)})_k$, either $p((x_k^{(j)})_k) = 0$, in which case there is a profile decomposition with $r_k = x_k^{(j)}$, or there exists a sequence $(x_k^{(j+1)})_k < (x_k^{(j)})_k$, such that $\sup_k \|x_k^{(j+1)}\| < \sigma((x_k^{(j)})_k) + \frac{\epsilon_j}{2}$, $j \in \mathbb{N}$. Let us denote as n_k^j the cumulative enumeration of the original sequence that arises at the jth iterative step, and denote as m_{j+1} the number of elementary concentrations that are subtracted at the transition from $(x_k^{(j)})_k$ to $(x_k^{(j+1)})_k$ (using the convention $x_k^{(0)} := x_k$). Set $M_j = \sum_{i=1}^j m_i$, $M_0 = 0$. Then the sequence $(x_k^{(j)})_k$ admits the following representation:

$$x_k^{(j)} = x_{n_k^j} - \sum_{n=1}^{M_j} g_{n_k^j}^{(n)} w^{(n)}, \quad k \in \mathbb{N}.$$

By Lemma 5.10, under an appropriate renumeration such that (5.6) holds for all k,

$$\delta\left(\left\|\sum_{n=M_{j-1}+1}^{M_j} g_{n_k^j}^{(n)} w^{(n)}\right\|\right) \leq \sup \|x_k^{(j+1)}\| - \sigma((x_k^{(j)})) + \frac{\epsilon_j}{2} < \epsilon_j, \quad k \in \mathbb{N},$$

and thus

$$\left\|\sum_{n=M_{j-1}+1}^{M_j} g_{n_k^j}^{(n)} w^{(n)}\right\| \leq 2^{-j}, \quad j \in \mathbb{N}.$$

Let us now diagonalize the double sequence $x_k^{(j)}$ by considering

$$x_k^{(k)} = x_{n_k^k} - \sum_{n=1}^{M_k} g_{n_k^k}^{(n)} w^{(n)}.$$

Let us show that $x_k^{(k)} \overset{D}{\rightharpoonup} 0$. Indeed, by definition of functional p and Lemma 5.10, $\delta(p(x_k) \leq \sup \|x\_k\| - \sigma(x_k)$, and therefore, for any $j \in \mathbb{N}$ and all $k \geq j$,

$$p(x_k^{(k)}) \leq p(x_k^{(j)}) \leq \sup \|x_k^{(j)}\| - \sigma(x_k^{(j)}) \leq \epsilon_j.$$

Since j is arbitrary, this implies $p(x_k^{(k)}) = 0$. Furthermore, denoting as J_j an arbitrary subset, of $\{M_j + 1, \ldots, M_{j+1}\}$, $j \in \mathbb{N}$, we have

$$\left\| \sum_{n=M_k+1}^{\infty} g_{n_k^k}^{(n)} w^{(n)} \right\| \leq \sum_{j=k}^{\infty} \left\| \sum_{n \in J_j} g_{n_k^k}^{(n)} w^{(n)} \right\| \leq \frac{1}{2^{k-1}}.$$

We have therefore

$$x_{n_k^k} - \sum_{n=1}^{\infty} g_{n_k^k}^{(n)} w^{(n)} \overset{D}{\rightharpoonup} 0,$$

where the series is understood as the sum $S_k + S_k'$, where $S_k = \sum_{n=1}^{M_k} g_{n_k^k}^{(n)} w^{(n)}$ is a finite, not a priori bounded, sum, and a series $S_k' = \sum_{n=M_k+1}^{\infty} g_{n_k^k}^{(n)} w^{(n)}$ that converges unconditionally and uniformly in k.

Note, however, that S_k is a sum of a bounded sequence $x_{n_k^k}$, a D-Δ-vanishing (and thus bounded) sequence, and the convergent series S_k' bounded with respect to k. Therefore the sum S_k' is bounded with respect to k and, consequently, the series $S_k + S_k'$ is convergent in norm, unconditionally and uniformly in k. Note that the construction can be carried out without further modifications if one prescribes in the beginning $g_k^{(1)} = \mathrm{Id}$ whenever $w^{(1)} = \overset{\rightharpoonup}{\lim}\, x_{n_k} \neq 0$, while in the case $x_k \rightharpoonup 0$ one can add the zero term $g_k^{(1)} w^{(1)}$ to the sum. □

6. General Properties of Cocompactness and Profile Decompositions

In this section we discuss some general functional-analytic properties of sequences related to cocompactness, following the discussion for Sobolev spaces in [29]. The reader whose interest is focused on profile decompositions may skip to the next section after reading the definition below. We will assume throughout this section that X is a strictly convex Banach space, unless specifically stated otherwise, and that the set D will be a non-empty subset of a group D_0 of linear isometries on X.

Definition 6.1. A continuous imbedding of two Banach spaces $X \hookrightarrow Y$, given a set D of bijective linear isometries of both X and Y, is called D, X-*cocompact* (to be denoted $X \overset{D,X}{\hookrightarrow} Y$), if any D-Δ-convergent sequence in X is convergent in the norm of Y. It will be called D, Y-*cocompact* (to be denoted $X \overset{D,Y}{\hookrightarrow} Y$), if any sequence bounded in X and D-Δ-convergent in Y, is convergent in the norm of Y.

Note that when for weak and Δ-convergence in X (respectively, Y) coincide, D, X-(respectively, D, Y-) cocompacntess coincides with D-cocompactness.

Analogously to the notion of D-cocompact set in Definition 2.2, we say that a set $B \subset X$, is D, X-*cocompact*, if every D-Δ-convergent sequence in X is strongly convergent.

Definition 6.2. A subset B of a Banach space X is called D-Δ-*bounded* if for every sequence $(g_k) \subset D$, $g_k \rightharpoonup 0$, and any sequence B, $x_k \rightharpoondown x$, one has $g_k^{-1}(x_k - x) \rightharpoondown 0$. It is called D-*bounded* if it possesses analogous property with Δ-convergence replaced by weak convergence.

Definition 6.3. A Banach space X is called locally D-Δ-cocompact if every bounded subset of X is D-Δ-cocompact. It is called locally D-cocompact if it possesses analogous property with Δ-convergence replaced by weak convergence.

We have two examples of locally cocompact spaces.

Example 6.4. (cf. [14, Remarks, p. 395]) The imbedding $\ell^p(\mathbb{Z}) \hookrightarrow \ell^\infty(\mathbb{Z})$, $1 \leq p \leq \infty$, is D-cocompact with $D = \{u \mapsto u(\cdot + y)\}_{y \in \mathbb{Z}}$. In particular, ℓ^∞ is locally cocompact. To see that observe that $u_k \overset{D}{\rightharpoonup} 0$ implies $u_k(y_k) \to 0$ for any y_k, in particular when y_k is a point such that $|u_k(y_k)| \geq \frac{1}{2}\|u_k\|_\infty$. As an immediate consequence we also have $\ell^p \overset{D}{\hookrightarrow} \ell^q$ whenever $q > p$.

Example 6.5. Another example of a locally cocompact space is $L^\infty(\mathbb{R})$, equipped with $D = \{u \mapsto u(2^j \cdot + y)\}_{j \in \mathbb{Z}, y \in \mathbb{R}}$. Indeed, assume, without loss of generality, that $A \geq \operatorname{ess\,sup} u_k(x) = \|u_k\|_\infty \geq \eta > 0$. Then for every k there exists a Lebesgue point x_k of the set $X_k = \{x : u_k(x_k) \geq \eta/2\}$. Therefore, for every k and for every $\alpha \in (0, 1)$ there exists $\delta_{\alpha,k} > 0$ such that $|X_k \cap [x_k - \delta_{\alpha,k}, x_k + \delta_{\alpha,k}]| \geq 2\alpha\delta_{\alpha,k}$.

Let $\tilde{u}_k(x) = u_k(\delta_{\alpha_k}^{-1}(x + x_k))$. Consider the set $\tilde{X}_k = \{x : \tilde{u}_k(x_k) \geq \eta/2\}$ and note that $|\tilde{X}_k \cap [-1, 1]| \geq 2\alpha$. Therefore, choosing any $\alpha \in (\frac{2A}{2A+\eta}, 1)$, we get

$$\int_{[-1,1]} \tilde{u}_k \geq \alpha\eta - A(2 - 2\alpha) = (\eta + 2A)\alpha - 2A > 0.$$

Consequently, $\tilde{u}_k \not\rightharpoonup 0$. It is easy to show that if $j(\alpha, k) \in \mathbb{N}$ is such that $2^{-j(\alpha,k)} \leq \delta_{\alpha,k} \leq 2^{1-j(\alpha,k)}$, then $u_k(2^{j(\alpha,k)}(x + x_k)) \not\rightharpoonup 0$ as well and D-cocompactness of bounded sets in $L^\infty(\mathbb{R})$ follows.

We have the following immediate criterion of local cocompactness.

Proposition 6.6. *A Banach space X is locally D-Δ-cocompact (respectively, D-cocompact) if and only if its every D-Δ-bounded (respectively, D-bounded) set is compact.*

Definition 6.7. A set B in a Banach space X is called profile-compact relative to a set D of bijective isometries on X if any sequence in B admits a strong profile decomposition, i.e. a profile decomposition whose remainder term vanishes in the norm of X.

Proposition 6.8. *Let B be a profile-compact subset of a Banach space X and let D be a non-empty subset of a dislocation group D_0. Then the profiles $w^{(n)}$ for a sequence $(u_k) \subset B$ are given by $w^{(n)} = \overset{\rightharpoonup}{\lim}\,(g_k^{(n)})^{-1}u_k = \overset{\rightarrow}{\lim}\,(g_k^{(n)})^{-1}u_k$.*

Proof. Without loss of generality we may consider profile decompositions with finitely many terms. Then from (5.4) by an elementary induction argument, we see that the weak and the Δ-limits of $(g_k^{(n)})^{-1}u_k$ coincide. □

Remark 6.9. Consider for simplicity a uniformly convex and uniformly smooth Banach space X with the Opial's condition. Conclusion of Theorem 5.5 is analogous to the conclusion of the Banach–Alaoglu theorem, in the sense that every bounded sequence is "profile-weakly-compact" (that is, has a subsequence that admits a profile decomposition). Similarly to compactness of imbeddings, an imbedding $X \hookrightarrow Y$ is cocompact relative to a set of bijective isometries $D \subset D_0 \in \mathcal{I}_X$, which extend to bijective isometries $D \subset D_0 \in \mathcal{I}_Y$, if and only if a "profile-weakly-compact" sequence in X becomes profile-compact (i.e. "profile-strongly-compact") in Y, that is, if it gets a strongly vanishing remainder.

Our next question is if a dual imbedding $Y^* \hookrightarrow X^*$ of a cocompact imbedding $X \hookrightarrow Y$ is cocompact. The answer is positive, but it involves the two different modes of cocompactness (or requires the Opial condition).

Proposition 6.10. *Let X be a reflexive Banach space equipped with a set D of linear bijective isometries on X and Y. Assume that $X \overset{D,X}{\hookrightarrow} Y$, and that every bounded sequence in X admits a Δ-profile decomposition. Then the dual imbedding $Y^* \hookrightarrow X^*$ is $D^\#$-cocompact, where*

$$D^\# = \{(g^*)^{-1},\ g \in D\}.$$

Proof. Consider (v_k), $v_k \overset{D^\#}{\rightharpoonup} 0$ in Y^*, as a sequence in X^*, and let $v_k^* \in X$ be a dual conjugate of v_k. Consider a Δ-profile decomposition for v_k^* in X. Then

$$\|v_k\|_{X^*} = \sum_n \langle v_k, g_k^{(n)} w^{(n)}\rangle_X + \langle v_k, r_k\rangle_X \le \sum_n \langle g_k^{(n)*} v_k, w^{(n)}\rangle_X + \|v_k\|_{Y^*}\|r_k\|_Y.$$

It remains to observe that the sum in the right-hand side is uniformly convergent relative to k, and each term vanishes by the assumption on v_k. The last term in the right-hand side vanishes, since v_k is bounded in Y^* and the remainder of profile decomposition vanishes in Y. □

We can now prove Theorem 2.10.

Proof of Theorem 2.10. Note first that condition (2.3) holds for $D_0^\#$ in Y^*. Indeed, if $(g_k^*)^{-1} \not\rightharpoonup 0$, then $\langle v, g_k^{-1}u\rangle \not\to 0$ for some $u, v \in Y$, and thus $g_k^{-1} \not\rightharpoonup 0$ in Y, and, by density, $g_k^{-1} \not\rightharpoonup 0$ in X. Then, by (2.3), on a renamed subsequence, $g_k^{-1} \to g^{-1}$ in the strong operator sense in X and, by imbedding, in Y. In particular, g^{-1} is an isometry and so also, by a simple duality argument, is $(g^*)^{-1}$. Then, for any $v \in Y^*$, $(g_k^*)^{-1}v \rightharpoonup (g^*)^{-1}v$, and $\|(g_k^*)^{-1}v\|_{Y^*} = \|(g^*)^{-1}v\|_{Y^*} = \|v\|_{Y^*}$. Since by assumption Y^* is uniformly convex, we have $(g_k^*)^{-1} \to (g^*)^{-1}$ in the strong sense.

It remains now to combine Proposition 6.10 with Theorem 5.5, taking into account that weak and Δ-convergence of bounded sequences coincide by the Opial's condition. □

7. Profile Decompositions: Convergence of Remainder

We start this section with the proof of Theorem 2.6.

Proof of Theorem 2.6. By the Opial's condition, Δ-convergence in the uniformly convex and uniformly smooth space X is equivalent to the weak convergence in X. Consequently, $D \in \mathcal{I}_X$. Moreover, D-Δ-convergence for bounded sequences coincides with D-weak convergence. Consequently, since every bounded sequence in X has a Δ-profile decomposition by Theorem 5.5, it has a profile decomposition in the sense of Definition 2.5. □

The rest of this section deals with general terms for interpretation of D, X-weak of D-weak convergence as convergence in some norm. In most cases this cannot be the norm of X, and verifying convergence in a suitable weaker norm typically involves some hard analytic proof. We give one example below where the group D is sufficiently robust to achieve convergence of D-weakly convergent sequences in the norm of X. Our main concern, however, is cocompactness of imbeddings of spaces of Sobolev type, which are discussed at the end of this section.

Theorem 7.1. *Let X be a uniformly convex and uniformly smooth Banach space and let D be a non-empty subset of a dislocation group D_0 on X. Then every bounded sequence $(u_k) \subset X$ admits a Δ-profile decomposition and* (5.5) *holds. Furthermore, if X is X, D -cocompactly imbedded into a Banach space Y, or if X satisfies the Opial's condition and X is D-cocompactly imbedded into Y, then the remainder r_k converges to zero in the norm of Y.*

We also have a partial analog of this theorem that imposes some of the assumptions on Y instead of X.

Theorem 7.2. *Let X be a Banach space densely imbedded into a uniformly convex and uniformly smooth Banach space Y, and let D be a non-empty subset of a dislocation group D_0 on Y such that with $D_0|_X$ is a dislocation group on X. Then every* *bounded sequence* $(u_k) \subset X$ *admits a* Δ*-profile decomposition in* Y *and* (5.5) *holds (in* Y*). Furthermore, if there are only finitely many profiles* $w^{(n)} \neq 0, Y$ *satisfies the Opial's condition, and* X *is* D*-cocompactly imbedded into* Y*, then the remainder* r_k *converges to zero in the norm of* Y*.*

Proof. Apply Theorem 5.5 in Y. Since Y satisfies the Opial's condition, all profiles are defined as weak limits in Y, and, since (x_k) is bounded in X, they are elements of X. Since there are only finitely many profiles, the remainder r_k is bounded in X. Since r_k is bounded in X and $r_k \overset{D}{\rightharpoonup} 0$ in Y, we have also $r_k \overset{D}{\rightharpoonup} 0$ in X, and therefore, by D-cocompactness of the imbedding, $r_k \to 0$ in Y. □

We now consider Besov and Triebel–Lizorkin spaces equipped with the group of rescalings D_r, $r \in \mathbb{R}$, defined as the product group of Euclidean shifts and, for some and dyadic dilations $g_r^j u(x) \mapsto 2^{rj} u(2^j x)$, $j \in \mathbb{Z}$. We refer to the definition in the book of Triebel [36, (Definition 2, p. 238] (see also a similar exposition in [1]), based on the Littlewood–Paley decomposition, of equivalent norm for Besov spaces $\dot{B}^{s,p,q}(\mathbb{R}^N)$ and Triebel–Lizorkin spaces $\dot{F}^{s,p,q}(\mathbb{R}^N)$. It is shown by Cwikel [8] that for all $s \in \mathbb{R}$, and $p, q \in (0, \infty)$ (i.e. when the corresponding spaces are uniformly convex and uniformly smooth), the equivalent norm, which remains scale-invariant, satisfies the Opial's condition. The latter work also gives direct proofs of cocompactness of some of the imbeddings of Besov and Triebel–Lizorkin spaces, which were implicitly proved, via wavelet argument, in [5]. We refer the reader to the survey [34] for explanations why Assumption 1, verified in [5] for Besov and Triebel–Lizorkin spaces, implies cocompactness. We summarize the imbeddings whose cocompactness is proved in [5] (another proof, based on Littlewood–Paley decomposition rather than on wavelet decomposition will be given in a forthcoming paper [8]).

Theorem 7.3. *The following imbeddings are cocompact relative to rescalings group* $D_{N/p-s}$:

(i) $\dot{B}^{s,p,q} \hookrightarrow \dot{F}^{t,q,b}$, $\frac{1}{p} - \frac{1}{q} = \frac{s-t}{N} > 0$.
(ii) $\dot{B}^{s,p,a} \hookrightarrow \dot{B}^{t,q,b}$, $\frac{1}{p} - \frac{1}{q} = \frac{s-t}{N} \geq 0$, $a < b$.
(iii) $\dot{B}^{s,p,p} \hookrightarrow \mathrm{BMO}$, $s = \frac{N}{p} > 0$.
(iv) $\dot{B}^{s,p,a} \hookrightarrow L^{q,b}$, $\frac{1}{p} - \frac{1}{q} = \frac{s}{N} > 0$, $a < b$.
(v) $\dot{F}^{s,p,a} \hookrightarrow \dot{F}^{t,q,b}$, $\frac{1}{p} - \frac{1}{q} = \frac{s-t}{N} > 0$, $a, b > 1$.
(vi) $\dot{F}^{s,p,a} \hookrightarrow \dot{B}^{t,q,p}$, $\frac{1}{p} - \frac{1}{q} = \frac{s-t}{N} > 0$.

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Appendix A. Uniformly Convex and Uniformly Smooth Banach Spaces

Definition A.1. We recall that a normed vector space X is called uniformly convex if the following function on $[0, 2]$, called the *modulus of convexity of* X, is strictly positive for all $\epsilon > 0$:

$$\delta(\epsilon) = \inf_{x,y\in X,\|x\|=\|y\|=1,\|x-y\|=\epsilon} 1 - \left\|\frac{x+y}{2}\right\|.$$

As shown by Figiel [11, Proposition 3, p. 122] the function $\epsilon \mapsto \delta(\epsilon)/\epsilon$ is non-decreasing on $(0, 2]$, and thus $\epsilon \mapsto \delta(\epsilon)$ is strictly increasing if $\delta(\epsilon) > 0$. Uniform convexity can be equivalently defined by the property

$$x, y \in X, \quad \|x\| \le 1, \quad \|y\| \le 1 \Rightarrow \left\|\frac{x+y}{2}\right\| \le 1 - \delta(\|x-y\|) \tag{A.1}$$

(see [11, Lemma 4, p. 124]).

It is an obvious consequence of (A.1) that

$$\left\|\frac{u+v}{2}\right\| \le \|v\|\left(1 - \delta\left(\frac{\|u-v\|}{\|v\|}\right)\right) \tag{A.2}$$

for any two elements $u, v \in X$ which satisfy $\|u\| \le \|v\|$ and $v \ne 0$. This in turn implies that every two elements $u, v \in X$ which are not both zero satisfy

$$\left\|\frac{u+v}{2}\right\| \le C_1 - C_2\delta\left(\frac{\|u-v\|}{C_2}\right) \quad \text{for all } C_1 \text{ and } C_2 \text{ in } [\max\{\|u\|, \|v\|\}, \infty). \tag{A.3}$$

If $C_1 = C_2 = \max\{\|u\|, \|v\|\}$ then (A.3) is exactly (A.2), possibly with u and v interchanged. To extend this to larger values of C_1 and C_2 we simply use the fact that $t \mapsto t\delta(\frac{\|u-v\|}{t})$ is a non-increasing function.

A Banach space X is called uniformly smooth if for every $\epsilon > 0$ there exists $\delta > 0$ such that if $x, y \in X$ with $\|x\| = 1$ and $\|y\| \le \delta$ then $\|x+y\| + \|x-y\| \le 2 + \epsilon\|y\|$. It is known that X^* is uniformly convex if and only if X is uniformly smooth (see [20, Proposition 1.e.2]) and that if X is uniformly convex, then the norm of X, as a function $\phi(x) = \|x\|$, considered on the unit sphere $S_1 = \{x \in X, \|x\| = 1\}$, is uniformly Gateau differentiable, which immediately implies that ϕ' is a uniformly continuous function $S_1 \to S_1^*$ (see [20, p. 61]). Considering ϕ as a function on the whole X, one has by homogeneity $\phi'(x) = \phi'(x/\|x\|) \in S_1^*$ for all $x \ne 0$, and an elementary argument shows that $\phi'(x)$ coincides with the uniquely defined x^*. We summarize this characterization of the duality conjugate as the following statement.

Lemma A.2. *Let X be a uniformly convex and uniformly smooth Banach space. Then the map $x \mapsto x^*$ is a continuous map $X\setminus\{0\} \to X^*$ with respect to the norm* *topologies on X and X^* and is in fact uniformly continuous on all closed subsets of $X\backslash\{0\}$.*

Appendix B. Asymptotic Centers and Δ-Convergence

We follow the presentation of the Chebyshev and asymptotic centers from Edelstein [10], in restriction to a particular case: the objects in [10] are defined there relative to a subset C of a Banach space X, and here we consider only the case $C = X$. We follow the presentation of Δ-convergence from Lim [19].

A bounded set A in a Banach space X can be assigned a positive number

$$R_A = \inf_{y\in X} \sup_{x\in A} \|x - y\|,$$

called the Chebyshev radius of A. The Chebyshev radius is attained (and is therefore a minimum) by weak lower semicontinuity of the norm and the corresponding minimizer is called the Chebyshev center of A. When X is uniformly convex, the value R_A cannot be attained at two different points $y' \neq y''$, since from uniform convexity one immediately has $\sup_{x\in A} \|x - \frac{y'+y''}{2}\| < R_A$. Consequently, the Chebyshev center of any set in a uniformly convex space is unique. Theorem 1 in [10] gives the following.

Theorem B.1. *Let X be a uniformly convex Banach space and let (x_n) be a bounded sequence in X. Then the sequence of Chebyshev centers (y_N) of the sets $A_N = (x)_{k\geq N}$ converges in norm.*

By definition of the Chebyshev center of A_N, $N \in \mathbb{N}$, we have $\sup_{k\geq N} \|x_k - y_N\| \leq \sup_{k\geq N} \|x_k - y\|$ for all $y \in X$, and the asymptotic center y_0 of the sequence (x_n) satisfies

$$\limsup \|x_n - y_0\| \leq \limsup \|x_n - y\|. \tag{B.1}$$

From uniform convexity it follows immediately that

$$s\limsup \|x_n - y_0\| < \limsup \|x_n - y\|, \quad y \neq y_0, \tag{B.2}$$

so the asymptotic center in a uniformly convex space is unique. An equivalent definition of Δ-limit in [19, (2)] says that y_0 is a Δ-limit of (x_n) if relation (B.1) holds for every subsequence of (x_n). In particular, if a sequence is Δ-convergent, its Δ-limit is also its asymptotic center. On the other hand, an asymptotic center is not necessarily the Δ-limit. If, for example, (x_n) is an alternating sequence of two points a and b, its asymptotic center is $\frac{a+b}{2}$, which is not a Δ-limit limit of the sequence.

The property of a space that every bounded sequence has an asymptotic center is called in [19] Δ-*completeness*, so by [10], uniformly convex spaces are Δ-complete. Sequential Δ-compactness follows from existence of a *regular* subsequence, i.e. a subsequence whose any further subsequence has the same asymptotic radius. This is the content of [13, Lemma 15.2], whose proof we reproduce below. Note that, unlike the proof of Δ-compactness in [19], no use is made of the Axiom of Choice.

Proof of Theorem A.1. Let $(x_k)_{k\in\mathbb{N}} \subset X$ be a bounded sequence. We use the notation $(v_n) \prec (u_n)$ to indicate that (v_n) is a subsequence of (u_n) and denote asymptotic radius of a sequence (v_n) by $\mathrm{rad}_{n\to\infty}(v_n)$. Set

$$r_0 = \inf\{\mathrm{rad}_{n\to\infty}(v_n) : (v_n) \prec (x_n)\}.$$

Select $(v_n^1) \prec (x_n)$ such that

$$\mathrm{rad}_{n\to\infty}(v_n^1) < r_0 + 1$$

and let

$$r_1 = \inf\{\mathrm{rad}_{n\to\infty}(v_n) : (v_n) \prec (v_n^1)\}.$$

Continuing by induction, and having defined $(v_n^i) \prec (v_n^{i-1})$ set

$$r_i = \inf\{\mathrm{rad}_{n\to\infty}(v_n) : (v_n) \prec (v_n^i)\}$$

and select $(v_n^{i+1}) \prec (v_n^i)$ so that

$$\mathrm{rad}_{n\to\infty}(v_n^{i+1}) < r_i + 1/2^{i+1}. \tag{B.3}$$

Note that $r_0 \le r_1 \le \cdots$ so that $\lim_{i\to\infty} \mathrm{rad}_{n\to\infty}(v_n^i) = r := \lim r_i$.

Consider a diagonal sequence (v_k^k). Since $(v_k^k) \prec (v_k^{i+1})$, we have $\mathrm{rad}_{k\to\infty}(v_k^k) \ge r$, while from (B.3) it follows that $\mathrm{rad}_{k\to\infty}(v_k^k) \le r$. Then $\mathrm{rad}_{k\to\infty}(v_k^k) = r$, and since the same argument applies to every subsequence of (v_k^k), the sequence (v_k^k) is regular. □

Appendix C. An Equivalent Condition to (2.3)

Condition (2.3), while it is verified in a great number of applications, has a quite technical appearance. While we cannot remedy this, we would like in this appendix to give it an equivalent formulation. We will use the notation $\xrightarrow{s}$ for the strong operator convergence.

Proposition C.1. *Let X be a uniformly convex separable Banach space and let D_0 be a group of isometries on X. Then condition* (2.3) *is equivalent to the following condition*:

$$(g_k) \subset D_0,\ g_k \not\rightharpoonup 0,\ u_k \rightharpoonup 0 \Rightarrow g_k u_k \rightharpoonup 0 \quad \text{on a subsequence.} \tag{C.1}$$

Lemma C.2. *If $(g_k) \subset D_0, g_k \rightharpoonup g \ne 0$, is such that*

$$u_k \rightharpoonup 0 \Rightarrow g_k u_k \rightharpoonup 0$$

then $g_k^* \xrightarrow{s} g^*$.

Proof. Let $v \in X^*$. We will verify that $g_k^* \xrightarrow{s} g^*$ once we show that for every bounded sequence (u_k),

$$\langle g_k^* v - g^* v, u_k\rangle \to 0.$$

Without loss of generality assume that $u_k \rightharpoonup u$. Then

$$\langle g_k^* v - g^* v, u_k\rangle = \langle g_k^* v - g^* v, u\rangle + \langle v, g_k(u_k - u)\rangle + \langle g^* v, (u_k - u)\rangle \to 0,$$

with the middle term vanishing by assumption and the remaining two vanishing by the weak convergence. □

Lemma C.3. *If* $(g_k) \subset D_0, g_k \rightharpoonup g \neq 0$, *is such that*

$$u_k \rightharpoonup 0 \quad \textit{in } X \Rightarrow g_k u_k \rightharpoonup 0 \quad \textit{on a subsequence},$$

then

$$v_k \rightharpoonup 0 \textit{ in } X^* \Rightarrow g_k^* v_k \rightharpoonup 0 \quad \textit{on a subsequence.}$$

Proof. By Lemma C.2 we have $g_k^* \overset{s}{\rightharpoonup} g^*$ and g^* is necessarily a bijective isometry. This implies $g_k^* \rightharpoonup g^*$ and therefore $g_k \rightharpoonup g$, since $\langle v, (g_k - g)u\rangle = \langle (g_k^* - g^*)v, u\rangle$. At the same time, $\|g_k u\| = \|u\| = \|gu\|$, and thus, due to the uniform convexity, $g_k u \to gu$, i.e. $g_k \overset{s}{\to}$g. □

Combining Lemma C.2 and Lemma C.3, we have the following statement.

Lemma C.4. *If* $(g_k) \subset D_0, g_k \rightharpoonup g \neq 0$, *is such that*

$$u_k \rightharpoonup 0 \Rightarrow g_k u_k \rightharpoonup 0 \quad \textit{on a subsequence},$$

then $g_k \overset{s}{\to} g$.

We can now prove the proposition.

Proof. ***Necessity***. Relation (C.1) follows from (2.3) immediately.

Sufficiency. Let (e_n) be a normalized unconditional basis in X. Since $\|g_k e_n\| = 1$ for every k and n, $(g_k e_n)_k$ has a weakly convergent subsequence. By diagonalization, we easily conclude that (g_k) has a weakly convergent subsequence. By assumption the weak limit is non-zero. The sufficiency in the proposition follows now from Lemma C.4. □

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