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REGULAR OBLIQUE DERIVATIVE PROBLEM IN MORREY SPACES

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Abstract. This article presents a study of the regular oblique derivative problem

$$\sum_{i,j=1}^{n} a^{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} = f(x)$$
$$\frac{\partial u}{\partial \ell(x)} + \sigma(x)u = \varphi(x)\,.$$

Assuming that the coefficients a^{ij} belong to the Sarason's class of functions with vanishing mean oscillation, we show existence and global regularity of strong solutions in Morrey spaces.

1. Introduction

The goal of the present paper is to study the global regularity in Morrey spaces for strong solutions to the *non-degenerate* oblique derivative problem

$$\begin{aligned} &\sum_{i,j=1}^{n} a^{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} = f(x) \quad \text{for almost all } x\in\Omega,\\ &\frac{\partial u}{\partial \ell(x)} + \sigma(x)u = \varphi(x) \quad \text{in the trace sense on } \partial\Omega\,. \end{aligned} \tag{1.1}$$

Here the coefficients of the uniformly elliptic operator may be discontinuous and the first order boundary operator, prescribed in terms of directional derivative with respect to a unit vector field $\ell(x)$, may be *nowhere* tangential to the boundary of Ω. More precisely, we assume that a^{ij}'s belong to the Sarason class, VMO, of functions with vanishing mean oscillation [22].

The interests in the study of boundary-value problems for elliptic operators with principal coefficients in VMO increased significantly in the last ten years. This is mainly due to the fact that VMO contains as a proper subspace $C^0(\overline{\Omega})$ that ensures the extension of the L^p-theory of operators with *continuous* coefficients to *discontinuous* coefficients [13, Chapter 9], [15]. On the other hand, the Sobolev spaces $W^{1,n}(\Omega)$ and $W^{\theta,\theta/n}(\Omega)$, $0<\theta<1$, are also contained in VMO, whence the discontinuities of a^{ij}'s expressed in terms of belonging to VMO become more general

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than those studied before (cf.[18], [6]). We refer the reader to the survey [5], where an excellent presentation of the state-of-the-art and relations with another similar results can be found concerning the regularizing properties of these operators in the framework of Sobolev spaces. The Dirichlet problem for such kind of equations has been well studied both in the linear ([6], [7]) and in the quasilinear ([20]) cases. Concerning the regular oblique derivative problems for elliptic operators with VMO principal coefficients, we should mention the articles [8] in the linear and [9] in the quasilinear case, respectively. The results of [8] have been extended also to elliptic operators with lower order terms and general boundary operators ([16]). Recently, the $W^{2,p}$-theory developed in [16] has been applied in the study of *degenerate* oblique derivative problem in Sobolev spaces (see [17]). The degeneracy means that the field ℓ can be tangential to the boundary of Ω at the points of some non-empty subset.

In the present paper we derive global regularizing property in Morrey spaces of elliptic operators with VMO coefficients. Precisely, it is proved that any strong solution ($u \in W^{2,p}(\Omega)$) of (1.1) with $f \in L^{p,\lambda}(\Omega)$ and $\varphi \in W^{(p,\lambda)}(\partial\Omega)$, $1 < p < +\infty$, $0 < \lambda < n$, admits second derivatives lying in the Morrey space $L^{p,\lambda}(\Omega)$ (Theorem 2.1). As consequence of that regularizing property we derive also strong solvability in $W^{2,p,\lambda}(\Omega)$ of (1.1) (Theorem 2.2) for any $f \in L^{p,\lambda}(\Omega)$ and $\varphi \in W^{(p,\lambda)}(\partial\Omega)$. (See the next Section for the definition of the spaces used.) Finally, the known relations between the Morrey and the Hölder spaces permit us to obtain *finer* Hölder continuity of the gradient Du of the strong solutions to (1.1) for suitable values of p and λ.

The crucial point of our investigations is the local boundary Morrey regularity of the strong solutions to (1.1) (Lemma 4.1). The approach is based on an explicit representation of solution's second derivatives (near the boundary) in terms of singular integral operators with Calderón–Zygmund kernels and their commutators and operators with positive kernels. This method has been already used in the study of Dirichlet problem ([7], see also [8]). Since the representation formula derived in [8] concerns constant coefficients elliptic and boundary operators, we apply here, in contrast to [8], a new approach in order to deal with non-homogeneous boundary conditions described by variable oblique derivative operator. This is reached by introducing a special auxiliary function which, roughly speaking, absorbs the right-hand side of the boundary condition. Thus, a new representation formula for the second derivatives occurs, which involves densities depending on the same second derivatives, *but also* on the strong solution and its gradient. To estimate effectively the Morrey norms of the second derivatives, we make use of a special non-dimensional norms. Indeed, that approach seems to be more natural when one studies the oblique derivative problem and this is due to the first order operator defined on the boundary $\partial\Omega$.

The rest of the paper is organized as follows. In Section 2 we state the problem, the assumptions on the data and the main results. Section 3 is devoted to some auxiliary results. Special emphasize is given on a construction and properties of the auxiliary function (Lemma 3.1) mentioned above, by the aid of which we are able to represent (locally near the boundary) the solution of (1.1). In Section 4 the local boundary Morrey regularity of the strong solutions to (1.1) is derived. Finally, a combination of that result with the interior regularity enables us to prove the main results of the paper (Section 5).

Results similar to the present here were derived for Dirichlet problem in [10] and [11].

2. The Problem and Assumptions

Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be a bounded domain with sufficiently smooth boundary $\partial\Omega$. Consider the unit vector field $\ell(x) = (\ell_1(x), \dots, \ell_n(x))$ prescribed on $\partial\Omega$ and the first-order boundary operator

$$\mathcal{B} \equiv \frac{\partial}{\partial \ell(x)} + \sigma(x) \quad x \in \partial\Omega.$$

In Ω we will consider the second order uniformly elliptic operator

$$\mathcal{L} \equiv a^{ij}(x) D_{ij}$$

where the usual summation convention on repeated indices is accepted and $D_{ij} \equiv \frac{\partial^2}{\partial x_i \partial x_j}$.

Our goal will be to study global regularity and strong solvability in the framework of Morrey spaces of the next oblique derivative problem

$$\begin{aligned} \mathcal{L}u &= f(x) \quad \text{for almost all } x \in \Omega, \\ \mathcal{B}u &= \varphi(x) \quad \text{in the trace sense on } \partial\Omega\,. \end{aligned} \tag{2.1}$$

Before giving the list of assumptions concerning the data of (2.1), let us recall some definitions and state useful notations. As usual, the classical Sobolev space of functions having weak derivatives up to order k which belong to $L^p(\Omega)$ will be denoted by $W^{k,p}(\Omega)$.

Let $p \in (1, +\infty)$ and $\lambda \in (0, n)$. The function $f \in L^1_{\text{loc}}(\Omega)$ is said to belong to the Morrey space $L^{p,\lambda}(\Omega)$ if

$$\|f\|_{L^{p,\lambda}(\Omega)} \equiv \Big(\sup_{\substack{\rho>0 \\ x\in\Omega}} \rho^{-\lambda} \int_{B_\rho(x)\cap\Omega} |f(y)|^p dy \Big)^{1/p} < +\infty\,,$$

where, hereafter $B_\rho(x)$ denotes an n-dimensional ball of radius ρ and centered at the point x.

We will consider also subspaces of $W^{k,p}(\Omega)$ formed by functions having their k-th order derivatives in $L^{p,\lambda}(\Omega)$. The symbol $W^{k,p,\lambda}(\Omega)$ stands for these subspaces. Precisely,

$$W^{k,p,\lambda}(\Omega) = \big\{ u \in W^{k,p}(\Omega) \colon \quad D^\alpha u \in L^{p,\lambda}(\Omega), \quad |\alpha| = k \big\}.$$

The norm in that space is naturally defined by

$$\|u\|_{W^{k,p,\lambda}(\Omega)} = \|u\|_{L^{p,\lambda}(\Omega)} + \|D^k u\|_{L^{p,\lambda}(\Omega)}.$$

By means of the interpolation inequality, it is clear that also the lower-order derivatives $D^\alpha u \in L^{p,\lambda}(\Omega)$ for $0 < |\alpha| < k$. We shall make use also of the non-dimensional norms

$$\|u\|^*_{W^{k,p,\lambda}(\Omega)} = \|u\|_{L^{p,\lambda}(\Omega)} + d^{k/2} \|D^k u\|_{L^{p,\lambda}(\Omega)}, \quad d = \operatorname{diam} \Omega.$$

To interpret the boundary condition in (2.1) in the trace sense on $\partial\Omega$, we will use the space of functions defined on $\partial\Omega$ which are traces of functions lying in $W^{1,p,\lambda}(\Omega)$. That functional class is well studied by Campanato (cf. [3]). Thus, define $W^{(p,\lambda)}(\partial\Omega)$ to be the Banach space formed by functions φ defined on $\partial\Omega$ and having the finite norm

$$\|\varphi\|_{W^{(p,\lambda)}(\partial\Omega)} = \Big(\sup_{\substack{\rho>0 \\ z'\in\partial\Omega}} \rho^{-\bar\lambda} \int\limits_{B_\rho(z')\cap\partial\Omega} |\varphi(x')|^p d\sigma_{x'} \Big)^{1/p} + \Big(\sup_{\substack{\rho>0 \\ z',\bar z'\in\partial\Omega}} \rho^{-\lambda} \int\limits_{B_\rho(z')\cap\partial\Omega} \int\limits_{B_\rho(\bar z')\cap\partial\Omega} \frac{|\varphi(x')-\varphi(\bar x')|^p}{|x'-\bar x'|^{p+n-2}} d\sigma_{x'} d\sigma_{\bar x'} \Big)^{1/p},$$

with $\bar\lambda = \max\{\lambda-1, 0\}$.

In order to formulate the regularity assumptions on the coefficients of the operator $\mathcal{L}$, we need also to recall the definitions of the John–Nirenberg space ([14]) of functions with bounded mean oscillation (BMO) and the Sarason class VMO of the functions with vanishing mean oscillation ([22]). A locally integrable function $f(x)$ is said to belong to BMO if

$$\|f\|_* \equiv \sup_{B\subset\mathbb{R}^n} \frac{1}{|B|} \int\limits_B |f(x) - f_B| dx < +\infty$$

with f_B being the integral average $\frac{1}{|B|}\int_B f(x)dx$ of the function $f(x)$ over the set B, and B ranges in the class of balls of $\mathbb{R}^n$. If $f(x) \in BMO$ denote

$$\gamma(r) = \sup_{\rho\le r,\ x\in\mathbb{R}^n} \frac{1}{|B_\rho|} \int\limits_{B_\rho} |f(x) - f_{B_\rho}| dx.$$

Then, $f(x) \in VMO$ if $\gamma(r) = o(1)$ as $r \to 0^+$ and refer to $\gamma(r)$ as the VMO-modulus of $f(x)$.

It should be noted that replacing the ball B above by the intersection $B \cap \Omega$, one obtains the definitions of $BMO(\Omega)$ and $VMO(\Omega)$. Later on, having a function defined on Ω that belongs to $BMO(\Omega)$ ($VMO(\Omega)$) it is possible to extend it to all $\mathbb{R}^n$ preserving its BMO (VMO) character (see [2, Proposition 1.3]).

We are in a position now to list our assumptions. Concerning the operator $\mathcal{L}$, we suppose that it is uniformly elliptic one with VMO coefficients. That is,

$$\begin{gathered} \exists\, \kappa > 0 : \ \kappa^{-1}|\xi|^2 \le a^{ij}(x)\xi_i\xi_j \le \kappa|\xi|^2 \quad \forall \xi \in \mathbb{R}^n, \text{ a.a. } x \in \Omega, \\ a^{ij}(x) \in VMO(\Omega), \quad a^{ij}(x) = a^{ji}(x). \end{gathered} \tag{2.2}$$

We set also $\gamma_{ij}(r)$ for the VMO-modulus of the function $a^{ij}(x)$ and let $\gamma(r) = \Big(\sum_{i,j=1}^n \gamma_{ij}^2(r)\Big)^{1/2}$. An immediate consequence of (2.2) is the essential boundedness of a^{ij}'s.

As it concerns the boundary operator $\mathcal{B}$, we assume

$$\begin{gathered} \ell_i(x),\ \sigma(x) \in C^{0,1}(\partial\Omega), \quad \partial\Omega \in C^{1,1}, \\ \ell(x)\cdot\nu(x) = \ell_i(x)\nu_i(x) > 0, \quad \sigma(x) < 0 \quad \text{for each } x \in \partial\Omega, \end{gathered} \tag{2.3}$$

with $\nu(x) = (\nu_1(x), \dots, \nu_n(x))$ being the unit *inward* normal to $\partial\Omega$. The simple geometric meaning of (2.3) is that the field $\ell(x)$ is nowhere tangential to $\partial\Omega$, that is, (2.1) is a *regular oblique derivative problem* (see [21]).

The main results of the paper are contained in the following theorems.

Theorem 2.1. *Let* (2.2) *and* (2.3) *be true,* $p \in (1, +\infty)$ *and* $\lambda \in (0, n)$*. Assume further that* $u \in W^{2,p}(\Omega)$ *solves the problem* (2.1) *with* $f \in L^{p,\lambda}(\Omega)$ *and* $\varphi \in W^{(p,\lambda)}(\partial\Omega)$*.*

Then $D_{ij}u \in L^{p,\lambda}(\Omega)$ *and there is a constant* $C = C(n, p, \lambda, \kappa, \gamma, \ell, \sigma, \partial\Omega)$ *such that*

$$\|u\|_{W^{2,p,\lambda}(\Omega)} \le C \left(\|u\|_{L^{p,\lambda}(\Omega)} + \|f\|_{L^{p,\lambda}(\Omega)} + \|\varphi\|_{W^{(p,\lambda)}(\partial\Omega)} \right). \tag{2.4}$$

The regularizing property of the couple $(\mathcal{L}, \mathcal{B})$ implies well-posedness of the oblique derivative problem (2.1) in the Morrey space $W^{2,p,\lambda}(\Omega)$.

Theorem 2.2. *Let* (2.2) *and* (2.3) *be satisfied,* $p \in (1, +\infty)$ *and* $\lambda \in (0, n)$*.*

Then, for every $f \in L^{p,\lambda}(\Omega)$ *and* $\varphi \in W^{(p,\lambda)}(\partial\Omega)$ *there exists a unique solution of the oblique derivative problem* (2.1)*. Moreover,*

$$\|u\|_{W^{2,p,\lambda}(\Omega)} \le C \left(\|f\|_{L^{p,\lambda}(\Omega)} + \|\varphi\|_{W^{(p,\lambda)}(\partial\Omega)} \right) \tag{2.5}$$

with a constant $C = C(n, p, \lambda, \kappa, \gamma, \ell, \sigma, \partial\Omega)$*.*

An immediate consequence of Theorem 2.1 and the imbedding properties of the Morrey spaces for suitable values of p and λ (cf. [4]) is the next global Hölder regularity result for the gradient Du of the strong solutions to (2.1).

Corollary 2.3. *Let* $u \in W^{2,p}(\Omega)$ *be a strong solution to* (2.1) *with* $f \in L^{p,\lambda}(\Omega)$ *and* $\varphi \in W^{(p,\lambda)}(\partial\Omega)$*.*

Then, if $n - p < \lambda < n$*, the gradient* Du *is a Hölder continuous function on* $\overline{\Omega}$ *with exponent* $\alpha = 1 - (n - \lambda)/p$ *and*

$$\|Du\|_{C^{0,\alpha}(\overline{\Omega})} = \sup_{x,y \in \Omega} \frac{|Du(x) - Du(y)|}{|x - y|^{\alpha}} \le C \left(\|f\|_{L^{p,\lambda}(\Omega)} + \|\varphi\|_{W^{(p,\lambda)}(\partial\Omega)} \right).$$

Let us point out that the solely assumptions $f \in L^p(\Omega)$ and $\varphi \in W^{1-1/p,p}(\partial\Omega)$ imply $u \in W^{2,p}(\Omega)$ (see [8]). Thus, if $p > n$ the Sobolev imbedding theorem yields $Du \in C^{\beta}(\overline{\Omega})$ with $\beta = 1 - n/p$. On the other hand, Corollary 2.3 ensures Hölder continuity of the gradient *also* for $p \in (1, n]$, assuming *finer* regularity of the data expressed in terms of their belonging to the Morrey space $L^{p,\lambda}(\Omega)$ with $\lambda \in (n - p, n)$.

Remark 2.4. The results presented here can be applied in studying Morrey regularity of the strong solutions to (2.1) for general elliptic operators

$$\mathcal{L} \equiv a^{ij}(x)D_{ij} + b^i(x)D_i + c(x)$$

with $a^{ij} \in VMO(\Omega)$ and the lower order coefficients $b^i(x)$ and $c(x)$ owning suitable Lebesgue integrability. We refer the reader to [16] for details concerning the case of Sobolev spaces.

3. Auxiliary Results

Let $\tilde{\Gamma}$ be a portion of the hyperplane $\{x_n = 0\}$, $x = (x_1, \dots, x_{n-1}, x_n) \equiv (x', x_n)$, and let $\tilde{\varphi}(x')$ be a function defined on $\tilde{\Gamma}$ which belongs to $W^{(p,\lambda)}(\tilde{\Gamma})$. The Banach space $W^{(p,\lambda)}(\tilde\Gamma)$ is equipped now with the non-dimensional norm

$$\|\tilde\varphi\|^*_{W^{(p,\lambda)}(\tilde\Gamma)} = \Big(\sup_{\substack{\rho\in(0,d]\\ z'\in\tilde\Gamma}} \rho^{-\bar\lambda} \int\limits_{B'_\rho(z')\cap\tilde\Gamma} |\tilde\varphi(x')|^p dx' \Big)^{1/p} + d^{1/2}\Big(\sup_{\substack{\rho\in(0,d]\\ z',\,\bar z'\in\tilde\Gamma}} \rho^{-\lambda} \int\limits_{B'_\rho(z')\cap\tilde\Gamma} \int\limits_{B'_\rho(\bar z')\cap\tilde\Gamma} \frac{|\tilde\varphi(x')-\tilde\varphi(\bar x')|^p}{|x'-\bar x'|^{p+n-2}} dx' d\bar x' \Big)^{1/p},$$

and $B'_\rho(z')$ is an $(n-1)$-dimensional ball of radius ρ and centered at $z'\in\{x_n=0\}$, $\bar\lambda=\max\{\lambda-1,0\}$, $d=\operatorname{diam}\tilde\Gamma$.

Now, following [13] (see the proof of Theorem 6.26 therein), we take a function $\eta(y')\in C^2_0(\mathbb{R}^{n-1})$ such that $\int_{\mathbb{R}^{n-1}}\eta(y')dy'=1$. Fixing arbitrary $x_0=(x'_0,0)$ and $R>0$, and denoting $B^+_R=B_R(x_0)\cap\{x_n>0\}$, $\Gamma_R=B_R(x_0)\cap\{x_n=0\}$, without loss of generality we may take Γ_R instead of $\tilde\Gamma$ at the above definition of the norm $\|\tilde\varphi\|^*_{W^{(p,\lambda)}(\tilde\Gamma)}$ and set $d=R$. Later, having $\tilde\varphi\in W^{(p,\lambda)}(\Gamma_R)$ we suppose that $\tilde\varphi$ is extended to all $\mathbb{R}^{n-1}$ as a function with a compact support, preserving its $W^{(p,\lambda)}$-norm.

Supposing that the boundary $\partial\Omega$ is locally flatten near the point x_0 such that $\Omega\subset\{x_n>0\}$, we recall that the *regular obliqueness condition* (2.3) ensures $\ell_n(x_0)\neq 0$. Consider now the function

$$\phi(x)=\phi(x',x_n)=\frac{x_n}{\ell_n(x_0)}\int\limits_{\mathbb{R}^{n-1}}\tilde\varphi(x'-x_ny')\eta(y')dy'. \tag{3.1}$$

Essential step in our further considerations is ensured by the next

Lemma 3.1. *The function $\phi(x)$ belongs to $W^{2,p,\lambda}(B^+_R)$ and satisfies*

$$\phi(x',0)=0,\quad \frac{\partial\phi}{\partial x_n}(x',0)=\frac{\tilde\varphi(x')}{\ell_n(x_0)}\quad \text{for}\quad x'\in\Gamma_R. \tag{3.2}$$

Moreover,

$$\|\phi\|^*_{W^{2,p,\lambda}(B^+_R)}=\|\phi\|_{L^{p,\lambda}(B^+_R)}+R\|D^2\phi\|_{L^{p,\lambda}(B^+_R)}\le CR^{1/2}\|\tilde\varphi\|^*_{W^{(p,\lambda)}(\Gamma_R)} \tag{3.3}$$

with $C=C(n,p,\lambda,\ell,\eta)$.

Proof. We will prove Lemma 3.1 in two steps.

Step 1: A bound of $\|\phi\|_{L^{p,\lambda}(B^+_R)}$. Let $\rho\in(0,R]$, $\bar x\in B^+_R$ and $B^+_\rho(\bar x)=B_\rho(\bar x)\cap\{x_n>0\}$. Then, making use of the Jensen integral inequality as well as of Fubini's theorem, we obtain

$$\rho^{-\lambda}\int\limits_{B^+_\rho(\bar x)\cap B^+_R}|\phi(x)|^pdx=\frac{1}{[\ell_n(x_0)]^p}\rho^{-\lambda}\int\limits_{B^+_\rho(\bar x)\cap B^+_R}\Big|x_n\int\limits_{\mathbb{R}^{n-1}}\tilde\varphi(x'-x_ny')\eta(y')dy'\Big|^pdx$$
$$\le C(n,p,\ell,\operatorname{supp}\eta)\rho^{-\lambda}\int\limits_{\operatorname{supp}\eta}|\eta(y')|^p\Big(\int\limits_{B^+_\rho(\bar x)\cap B^+_R}x_n^p|\tilde\varphi(x'-x_ny')|^pdx\Big)dy'.$$

Now, setting $I_{B^+_\rho(\bar x)\cap B^+_R}(y')=\rho^{-\lambda}\int_{B^+_\rho(\bar x)\cap B^+_R}x_n^p|\tilde\varphi(x'-x_ny')|^pdx$ and $Q_\rho(\bar x)$ for the cube $\big\{x\in\mathbb{R}^n:\ |x_i-\bar x_i|\le\rho \text{ for } i\le n-1;\ \max\{0,-\rho+\bar x_n\}\le x_n\le\rho+\bar x_n\big\}$, we have

$$I_{B^+_\rho(\bar{x})\cap B^+_R}(y') \le I_{Q_\rho(\bar{x})}(y') = \rho^{-\lambda} \int\limits_{Q_\rho(\bar{x})} x_n^p |\tilde{\varphi}(x'-x_ny')|^p dx'dx_n$$

$$\le \rho^{-\lambda} \int\limits_{\max\{0,-\rho+\bar{x}_n\}}^{\rho+\bar{x}_n} x_n^p \int\limits_{Q'_\rho(\bar{x})} |\tilde{\varphi}(z')|^p dz'dx_n$$

with $Q'_\rho(\bar{x}) = \big\{z' \in \mathbb{R}^{n-1}\colon\ -\rho+\bar{x}_i - x_ny_i \le z_i \le \rho + \bar{x}_i - x_ny_i,\quad i \le n-1\big\}$.

Since, $\int_{Q'_\rho(\bar{x})} |\tilde{\varphi}(z')|^p dz' \le \rho^{\bar{\lambda}} \left(\|\tilde{\varphi}\|^*_{W^{(p,\lambda)}(\Gamma_R)}\right)^p$, $\bar{\lambda} = \max\{\lambda-1, 0\}$, using $\bar{x}_n \le R$, $\rho \le R$, one has

$$I_{B^+_\rho(\bar{x})\cap B^+_R}(y') \le \rho^{\bar{\lambda}-\lambda} \left(\|\tilde{\varphi}\|^*_{W^{(p,\lambda)}(\Gamma_R)}\right)^p \int\limits_{\max\{0,-\rho+\bar{x}_n\}}^{\rho+\bar{x}_n} x_n^p dx_n$$

$$\le C(n,p,\ell) R^{p+\max\{1-\lambda,0\}} \left(\|\tilde{\varphi}\|^*_{W^{(p,\lambda)}(\Gamma_R)}\right)^p.$$

The last bound and the fact that $y' \in \operatorname{supp}\eta$ show that

$$\|\phi\|_{L^{p,\lambda}(B^+_R)} \le C(n,p,\ell,\operatorname{supp}\eta) R^{1+\max\{1-\lambda,0\}/p} \|\tilde{\varphi}\|^*_{W^{(p,\lambda)}(\Gamma_R)}. \tag{3.4}$$

Step 2: An estimate for $\|D^2\phi\|_{L^{p,\lambda}(B^+_R)}$. We will calculate now the first and second derivatives of the function ϕ given by (3.1). For, after the change $z' = x' - x_ny'$ of the variables in (3.1), one has

$$\phi(x',x_n) = \frac{x_n^{2-n}}{\ell_n(x_0)} \int\limits_{\mathbb{R}^{n-1}} \tilde{\varphi}(z')\eta\left(\frac{x'-z'}{x_n}\right) dz',$$

whence

$$\frac{\partial\phi}{\partial x_i}(x',x_n) = \frac{x_n^{1-n}}{\ell_n(x_0)} \int\limits_{\mathbb{R}^{n-1}} \tilde{\varphi}(z')\frac{\partial\eta}{\partial x_i}\left(\frac{x'-z'}{x_n}\right) dz' \quad \text{for} \quad i<n,$$

$$\frac{\partial\phi}{\partial x_n}(x',x_n) = \frac{(2-n)x_n^{1-n}}{\ell_n(x_0)} \int\limits_{\mathbb{R}^{n-1}} \tilde{\varphi}(z')\eta\left(\frac{x'-z'}{x_n}\right) dz'$$

$$- \frac{x_n^{-n}}{\ell_n(x_0)} \int\limits_{\mathbb{R}^{n-1}} \tilde{\varphi}(z')D'\eta\left(\frac{x'-z'}{x_n}\right)\cdot(x'-z')dz'$$

with $D' = (\partial/\partial y_1, \dots, \partial/\partial y_{n-1})$, $x'\cdot y' = \sum_{j=1}^{n-1} x_jy_j$.

Returning to the original variables, we obtain

$$\frac{\partial\phi}{\partial x_i}(x',x_n) = \frac{1}{\ell_n(x_0)} \int\limits_{\mathbb{R}^{n-1}} \tilde{\varphi}(x'-x_ny')D_i\eta(y')dy' \quad \text{for} \quad i<n, \tag{3.5}$$

$$\frac{\partial\phi}{\partial x_n}(x',x_n) = \frac{2-n}{\ell_n(x_0)} \int\limits_{\mathbb{R}^{n-1}} \tilde{\varphi}(x'-x_ny')\eta(y')dy'$$

$$- \frac{1}{\ell_n(x_0)} \int\limits_{\mathbb{R}^{n-1}} \tilde{\varphi}(x'-x_ny')D'\eta(y')\cdot y'dy'. \tag{3.6}$$

Now, remembering $\eta \in C_0^2(\mathbb{R}^{n-1})$ and $\int_{\mathbb{R}^{n-1}} \eta(y')dy' = 1$, the divergence theorem implies

$$
\begin{aligned}
&\int_{\mathbb{R}^{n-1}} D_i\eta(y')dy' = \int_{\mathbb{R}^{n-1}} D_{ij}\eta(y')dy' = \int_{\mathbb{R}^{n-1}} D'(D_i\eta)(y')\cdot y'dy' = 0 \ \forall i,j \le n-1;\\
&\int_{\mathbb{R}^{n-1}} \eta(y')dy' = 1, \quad \int_{\mathbb{R}^{n-1}} D'\eta(y')\cdot y'dy' = 1-n, \quad \int_{\mathbb{R}^{n-1}} D'(D'\eta\cdot y')\cdot y'dy' = (1-n)^2.
\end{aligned}
\tag{3.7}
$$

Therefore, (3.2) follows from (3.1) and (3.6) putting $x_n = 0$ therein.

Since $\eta \in C_0^2(\mathbb{R}^{n-1})$, we can differentiate (3.5) and (3.6) once again. Thus, straightforward calculations yield

$$
\begin{aligned}
D_{ij}\phi(x',x_n) &= \frac{1}{\ell_n(x_0)}\frac{1}{x_n}\int_{\mathbb{R}^{n-1}} \tilde{\varphi}(x'-x_ny')D_{ij}\eta(y')dy' \quad i,j \le n-1,\\
D_{in}\phi(x',x_n) &= \frac{1-n}{\ell_n(x_0)}\frac{1}{x_n}\int_{\mathbb{R}^{n-1}} \tilde{\varphi}(x'-x_ny')D_i\eta(y')dy'\\
&\quad - \frac{1}{\ell_n(x_0)}\frac{1}{x_n}\int_{\mathbb{R}^{n-1}} \tilde{\varphi}(x'-x_ny')D'(D_i\eta)(y')\cdot y'dy' \quad i \le n-1,\\
D_{nn}\phi(x',x_n) &= \frac{(2-n)(1-n)}{\ell_n(x_0)}\frac{1}{x_n}\int_{\mathbb{R}^{n-1}} \tilde{\varphi}(x'-x_ny')\eta(y')dy'\\
&\quad + \frac{2n-3}{\ell_n(x_0)}\frac{1}{x_n}\int_{\mathbb{R}^{n-1}} \tilde{\varphi}(x'-x_ny')D'\eta(y')\cdot y'dy'\\
&\quad + \frac{1}{\ell_n(x_0)}\frac{1}{x_n}\int_{\mathbb{R}^{n-1}} \tilde{\varphi}(x'-x_ny')D'(D'\eta\cdot y')\cdot y'dy'.
\end{aligned}
$$

These formulae and (3.7) lead to

$$
\begin{aligned}
D_{ij}\phi(x',x_n) &= \frac{1}{\ell_n(x_0)}\frac{1}{x_n}\int_{\mathbb{R}^{n-1}} [\tilde{\varphi}(x'-x_ny')-\tilde{\varphi}(x')]D_{ij}\eta(y')dy', \quad i,j \le n-1,\\
D_{in}\phi(x',x_n) &= \frac{1-n}{\ell_n(x_0)}\frac{1}{x_n}\int_{\mathbb{R}^{n-1}} [\tilde{\varphi}(x'-x_ny')-\tilde{\varphi}(x')]D_i\eta(y')dy'\\
&\quad - \frac{1}{\ell_n(x_0)}\frac{1}{x_n}\int_{\mathbb{R}^{n-1}} [\tilde{\varphi}(x'-x_ny')-\tilde{\varphi}(x')]D'(D_i\eta)(y')\cdot y'dy', \quad i \le n-1,\\
D_{nn}\phi(x',x_n) &= \frac{(2-n)(1-n)}{\ell_n(x_0)}\frac{1}{x_n}\int_{\mathbb{R}^{n-1}} [\tilde{\varphi}(x'-x_ny')-\tilde{\varphi}(x')]\eta(y')dy'\\
&\quad + \frac{2n-3}{\ell_n(x_0)}\frac{1}{x_n}\int_{\mathbb{R}^{n-1}} [\tilde{\varphi}(x'-x_ny')-\tilde{\varphi}(x')]D'\eta(y')\cdot y'dy'\\
&\quad + \frac{1}{\ell_n(x_0)}\frac{1}{x_n}\int_{\mathbb{R}^{n-1}} [\tilde{\varphi}(x'-x_ny')-\tilde{\varphi}(x')]D'(D'\eta\cdot y')\cdot y'dy'.
\end{aligned}
\tag{3.8}
$$

Now, getting a look on the formulae (3.8), it is clear that the integrals appearing there are all of the type (modulo a constant multiplier)

$$\psi(x) = \psi(x', x_n) = \frac{1}{x_n} \int\limits_{\mathbb{R}^{n-1}} [\tilde{\varphi}(x' - x_n y') - \tilde{\varphi}(x')]\mu(y')dy'$$

with $\mu(y')$ being $\eta(y')$, $D_i\eta(y')$, $y' \cdot D'\eta(y')$, $D_{ij}\eta(y')$ or $D'(D'\eta \cdot y') \cdot y'$.

Proceeding as in Step 1 with $\bar{x} \in B_R^+$, $\rho \in (0, R]$, we obtain

$$\rho^{-\lambda} \int\limits_{B_\rho^+(\bar{x}) \cap B_R^+} |\psi(x)|^p dx \le C(n, p, \ell, \operatorname{supp}\mu) \int\limits_{\operatorname{supp}\mu} |\mu(y')|^p J(y')dy'$$

with

$$J(y') = \rho^{-\lambda} \int\limits_{B_\rho^+(\bar{x}) \cap B_R^+} \frac{1}{x_n^p} |\tilde{\varphi}(x' - x_n y') - \tilde{\varphi}(x')|^p dx' dx_n.$$

Since

$$J(y') \le C \sum_{j=1}^{n-1} \rho^{-\lambda} \int\limits_{B_\rho^+(\bar{x}) \cap B_R^+} \frac{1}{x_n^p} |\tilde{\varphi}(x_1, \dots, x_{j-1}, x_j - x_n y_j, x_{j+1}, \dots, x_{n-1}) - \tilde{\varphi}(x')|^p dx,$$

replacing the balls at the last integrals by sets of the type $T_j = \{x \in \mathbb{R}^n\colon\ -\rho + \bar{x}_i \le x_i \le \rho + \bar{x}_i\ (i \ne j),\ -\rho + \bar{x}_j \le x_j \le \bar{x}_j - x_n\}$, it is easily seen that

$$\begin{aligned} &\rho^{-\lambda} \int\limits_{T_j} \frac{1}{x_n^p} |\tilde{\varphi}(x_1, \dots, x_{j-1}, x_j - x_n y_j, x_{j+1}, \dots, x_{n-1}) - \tilde{\varphi}(x')|^p dx \\ &\quad \le C \sup_{\substack{\rho > 0 \\ z', \bar{z}' \in \Gamma_R}} \rho^{-\lambda} \int\limits_{B_\rho(z') \cap \Gamma_R} \int\limits_{B_\rho(\bar{z}') \cap \Gamma_R} \frac{|\tilde{\varphi}(x') - \tilde{\varphi}(\bar{x}')|^p}{|x' - \bar{x}'|^{p+n-2}} d\sigma_{x'} d\sigma_{\bar{x}'} \end{aligned}$$

(see [1], [3], [19] for details). This implies

$$\|D^2\phi\|_{L^{p,\lambda}(B_R^+)} \le C\|\psi\|_{L^{p,\lambda}(B_R^+)} \le C\|\tilde{\varphi}\|_{W^{(p,\lambda)}(\Gamma_R)}. \tag{3.9}$$

The estimates (3.4) and (3.9) yield (3.3). ♢

In our further considerations we will need some precise results on the boundedness in Morrey spaces of suitable integral operators. We refer the readers to the corresponding theorems and proofs given in [11] and [12].

Proposition 3.2. [12, Theorem 2.3] *Let U be an open subset of $\mathbb{R}^n$, $f \in L^{p,\lambda}(U)$, $p \in (1, +\infty)$, $\lambda \in (0, n)$, $a \in VMO \cap L^\infty(\mathbb{R}^n)$. Let $k(x, z)$ be a Calderón–Zygmund kernel (see* [7]*) in the z variable for almost all $x \in U$ such that*

$$\max_{|\alpha| \le 2n} \left\| \frac{\partial^\alpha}{\partial z^\alpha} k(x, z) \right\|_{L^\infty(D \times \Sigma)} = M < +\infty,$$

with $\Sigma = \{x \in \mathbb{R}^n :\ |x| = 1\}$. *For an arbitrary* $\varepsilon > 0$ *set*

$$K_\varepsilon f(x) = \int\limits_{\substack{|x-y|>\varepsilon \\ x\in U}} k(x, x-y) f(y) dy,$$

$$C_\varepsilon(a,f)(x) = \int\limits_{\substack{|x-y|>\varepsilon \\ x\in U}} k(x, x-y)(a(x) - a(y)) f(y) dy.$$

There exist Kf, $C(a,f) \in L^{p,\lambda}(U)$ *such that*

$$\lim_{\varepsilon\to 0} \|K_\varepsilon f - Kf\|_{L^{p,\lambda}(U)} = \lim_{\varepsilon\to 0} \|C_\varepsilon(a,f) - C(a,f)\|_{L^{p,\lambda}(U)} = 0.$$

Moreover,

$$\|Kf\|_{L^{p,\lambda}(U)} \le C\|f\|_{L^{p,\lambda}(U)}, \quad \|C(a,f)\|_{L^{p,\lambda}(U)} \le C\|a\|_* \|f\|_{L^{p,\lambda}(U)}$$

for some positive constant $C = C(n, p, \lambda, M)$.

Proposition 3.3. [11, Theorem 2.5] *Let* $x \in \mathbb{R}^n_+$ *and define*

$$\tilde{K} f(x) = \int\limits_{\mathbb{R}^n_+} \frac{f(y)}{|\tilde{x} - y|^n} dy, \quad \tilde{x} \equiv (x_1, \dots, x_{n-1}, -x_n).$$

There exists a constant C *independent of* $f(x)$, *such that*

$$\|\tilde{K} f\|_{L^{p,\lambda}(\mathbb{R}^n_+)} \le C\|f\|_{L^{p,\lambda}(\mathbb{R}^n_+)}.$$

Proposition 3.4. [11, Theorem 2.6] *Let* $f \in L^{p,\lambda}(\mathbb{R}^n_+)$, $p \in (1, +\infty)$, $\lambda \in (0, n)$, $a \in VMO \cap L^\infty(\mathbb{R}^n_+)$. *Then, for any* $x \in \mathbb{R}^n_+$ *the commutator*

$$\tilde{C}(a,f)(x) = \int\limits_{\mathbb{R}^n_+} \frac{|a(x) - a(y)|}{|\tilde{x} - y|^n} f(y) dy$$

is bounded from $L^{p,\lambda}(\mathbb{R}^n_+)$ *into itself. There exists a constant* C *independent of* $a(x)$ *and* $f(x)$ *such that*

$$\|\tilde{C}(a,f)\|_{L^{p,\lambda}(\mathbb{R}^n_+)} \le C\|a\|_* \|f\|_{L^{p,\lambda}(\mathbb{R}^n_+)}.$$

4. Boundary Morrey Regularity

As in the previous section, we suppose that the boundary $\partial\Omega$ is locally flatten near an arbitrary point $x_0 \in \partial\Omega$ such that $\Omega \subset \{x_n > 0\}$. The following result implies boundary regularizing property of the couple $(\mathcal{L}, \mathcal{B})$ in Morrey spaces:

Lemma 4.1. *Let* (2.2) *and* (2.3) *be satisfied and* $p \in (1, +\infty)$, $1 < q \le p < +\infty$, $\lambda \in (0, n)$. *Suppose* $r > 0$ *and let* $u \in W^{2,q}(B^+_r)$ *be a solution to the equation* $\mathcal{L}u = f \in L^{p,\lambda}(B^+_r)$ *such that* $\mathcal{B}u = \varphi$ *on* $B_r \cap \{x_n = 0\}$ *with* $\varphi \in W^{(p,\lambda)}(B_r \cap \{x_n = 0\})$.

Then there exists $R \in (0, r)$ *small enough such that* $D_{ij}u \in L^{p,\lambda}(B^+_R)$. *Moreover, there is a constant* $C = C(n, \kappa, p, \lambda, \ell, \sigma, \partial\Omega)$ *such that*

$$\|D_{ij}u\|_{L^{p,\lambda}(B^+_R)} \le C\left(\|u\|_{L^{p,\lambda}(B^+_R)} + \|f\|_{L^{p,\lambda}(B^+_R)} + \|\varphi\|_{W^{(p,\lambda)}(B_R\cap\{x_n=0\})}\right). \quad (4.1)$$

Proof. We will utilize the explicit representation formula of the second derivatives D^2u derived in [8, Lemma 4.2]. However, as that formula concerns oblique derivative problem for constant coefficients elliptic operator and homogeneous boundary condition with constant coefficients boundary operator, first of all we shall reduce the original problem to a homogeneous one.

Without loss of generality we may suppose that the ball B_r is centered at the origin. Let $x_0 = (x'_0, x_{0n})$, $x'_0 = (x_{01}, \dots, x_{0\,n-1})$. Obviously, we have

$$\begin{aligned} a^{ij}(x_0)D_{ij}u(x) &= \big[a^{ij}(x_0) - a^{ij}(x)\big]D_{ij}u(x) + f(x) \quad \text{a.e. in } B_r^+, \\ \ell_i(x'_0)D_iu(x') + \sigma(x'_0)u(x') &= \big[\ell_i(x'_0) - \ell_i(x')\big]D_iu(x') \\ &\quad + \big[\sigma(x'_0) - \sigma(x')\big]u(x') + \varphi(x') \quad x' \in B_r \cap \{x_n = 0\}. \end{aligned}$$

Consider now the right-hand side of the boundary condition above and denote it by $\tilde\varphi$. That is,

$$\tilde\varphi(x', u) = \big[\ell_i(x'_0) - \ell_i(x')\big]D_iu(x') + \big[\sigma(x'_0) - \sigma(x')\big]u(x') + \varphi(x'). \tag{4.2}$$

Define $\phi(x) = \phi(x, u)$ by (3.1) with $\tilde\varphi$ given by (4.2). Since $\tilde\varphi(x', u)$ depends *affinely* on u, it is clear that also the dependence of ϕ on u will be affine one. Later, remembering the properties of ϕ established in Lemma 3.1 (see (3.2)), it is obvious that

$$\frac{\partial\phi}{\partial\ell(x'_0)}(x) + \sigma(x'_0)\phi(x) = \tilde\varphi(x', u) \quad \text{for } x_n = 0.$$

That is why, the function $u(x) - \phi(x)$ satisfies

$$\begin{aligned} a^{ij}(x_0)D_{ij}(u(x) - \phi(x)) &= \big[a^{ij}(x_0) - a^{ij}(x)\big]D_{ij}u(x) \\ &\quad + f(x) - a^{ij}(x_0)D_{ij}\phi(x) \quad \text{a.e. in } B_r^+, \\ \frac{\partial(u-\phi)}{\partial\ell(x'_0)} + \sigma(x'_0)(u(x') - \phi(x')) &= 0 \quad x' \in B_r \cap \{x_n = 0\}. \end{aligned}$$

Therefore, [8, Lemma 3.1] implies

$$u(x) = \phi(x) + \int_{B_r^+} G(x_0, x, y)\left\{\big(a^{ij}(x_0) - a^{ij}(y)\big)D_{ij}u(y) + f(y) - a^{ij}(x_0)D_{ij}\phi(y)\right\}dy,$$

where

$$G(x_0, x, y) = \Gamma(x_0, x - y) - \Gamma(x_0, T(x, x_0) - y) + \theta(x_0, T(x, x_0) - y);$$

$\Gamma(x_0, \xi)$ is the normalized fundamental solution of the operator $a^{ij}(x_0)D_{ij}$:

$$\Gamma(x_0, \xi) = \frac{1}{n(2-n)\omega_n\sqrt{\det\{a^{ij}(x_0)\}}}\big(A^{ij}(x_0)\xi_i\xi_j\big)^{(2-n)/2}$$

with ω_n and $A^{ij}(x_0)$ being the measure of the unit ball in $\mathbb{R}^n$ and the inverse matrix of $\{a^{ij}(x_0)\}$, respectively;

$$T(x, y) = x - \frac{2x_n}{a^{nn}(y)}\boldsymbol{a}^n(y), \quad T(x) = T(x, x), \quad \boldsymbol{a}^n(y) = (a^{1n}(y), \dots, a^{nn}(y));$$

$$\theta(x_0,\xi)=\frac{2}{n\omega_n\sqrt{\det\{a^{ij}(x_0)\}}}\frac{\ell_n(x_0')}{a^{nn}(x_0)}$$
$$\times\int_0^\infty \frac{e^{\sigma(x_0')s}(\xi+sT(\ell(x_0')))_n}{\left(A^{ij}(x_0)(\xi+sT(\ell(x_0')))_i(\xi+sT(\ell(x_0')))_j\right)^{n/2}}ds$$

with $(\xi+sT(\ell(x_0')))_i$ being the i-th component of the vector $\xi+sT(\ell(x_0'))\in\mathbb{R}^n$.

Now, similar arguments as these used in the proof of [8, Lemma 4.2] lead to

$$\begin{aligned}D_{ij}u(x)&=D_{ij}\phi(x)\\&\quad+\text{P.V.}\int_{B_r^+}\Gamma_{ij}(x,x-y)\left\{\left(a^{ij}(x)-a^{ij}(y)\right)D_{ij}u(y)+f(y)-\mathcal{L}(x)\phi(y)\right\}dy\\&\quad+c_{ij}(x)\left(f(x)-\mathcal{L}(x)\phi(x)\right)+I_{ij}(x,x)+J_{ij}(x,x)\end{aligned}\tag{4.3}$$

for almost all $x\in B_r^+$, where $\mathcal{L}(x)\phi(y)=a^{ij}(x)D_{ij}\phi(y)$ and $\Gamma_i(x,\xi)=D_{\xi_i}\Gamma(x,\xi)$, $\Gamma_{ij}(x,\xi)=D_{\xi_i\xi_j}\Gamma(x,\xi)$, $\theta_i(x,\xi)=D_{\xi_i}\theta(x,\xi)$, $\theta_{ij}(x,\xi)=D_{\xi_i\xi_j}\theta(x,\xi)$,

$$c_{ij}(x)=\int_{|\xi|=1}\Gamma_i(x,\xi)\xi_j d\sigma_\xi;$$

$$I_{ij}(x,z)=\int_{B_r^+}\Gamma_{ij}(z,T(x,z)-y)\left\{\left(a^{hk}(z)-a^{hk}(y)\right)D_{hk}u(y)+f(y)-\mathcal{L}(x)\phi(y)\right\}dy$$

for $i,j<n$;

$$I_{in}(x,z)=\int_{B_r^+}\Gamma_{ij}(z,T(x,z)-y)\left\{\left(a^{hk}(z)-a^{hk}(y)\right)D_{hk}u(y)+f(y)-\mathcal{L}(x)\phi(y)\right\}B_j(z)dy$$

for $i<n$;

$$I_{nn}(x,z)=\int_{B_r^+}\Gamma_{ij}(z,T(x,z)-y)\left\{\left(a^{hk}(z)-a^{hk}(y)\right)D_{hk}u(y)+f(y)-\mathcal{L}(x)\phi(y)\right\}B_i(z)B_j(z)dy;$$

$$J_{ij}(x,z)=\int_{B_r^+}\theta_{ij}(z,T(x,z)-y)\left\{\left(a^{hk}(z)-a^{hk}(y)\right)D_{hk}u(y)+f(y)-\mathcal{L}(x)\phi(y)\right\}dy$$

for $i,j<n$;

$$J_{in}(x,z)=\int_{B_r^+}\theta_{ij}(z,T(x,z)-y)\left\{\left(a^{hk}(z)-a^{hk}(y)\right)D_{hk}u(y)+f(y)-\mathcal{L}(x)\phi(y)\right\}B_j(z)dy$$

for $i<n$;

$$J_{nn}(x,z)=\int_{B_r^+}\theta_{ij}(z,T(x,z)-y)\left\{\left(a^{hk}(z)-a^{hk}(y)\right)D_{hk}u(y)+f(y)-\mathcal{L}(x)\phi(y)\right\}B_i(z)B_j(z)dy.$$

The vector $B(z)=(B_1(z),\dots,B_n(z))$ above is given by the formula

$$B(z)=\frac{\partial}{\partial x_n}T(x,z),\quad\text{that is}\quad B(z)=\left(-2\frac{a^{1n}(z)}{a^{nn}(z)},\dots,-2\frac{a^{n-1,n}(z)}{a^{nn}(z)},-1\right).$$

Suppose now $q < p$ and let $s \in [q,p]$. Take an arbitrary $w \in W^{2,s,\lambda}(B_r^+)$ and define

$$\mathcal{S}w = \phi(x) + \int_{B_r^+} G(x_0,x,y)\left\{\left(a^{ij}(x_0) - a^{ij}(y)\right) D_{ij}w(y) + f(y) - a^{ij}(x_0) D_{ij}\phi(y)\right\} dy,$$

with $\tilde{\varphi} = \tilde{\varphi}(x', w)$ given by (4.2) and $\phi(x) = \phi(x,w)$ defined by (3.1).

The idea in proving Lemma 4.1 will be to show that $\mathcal{S}$ is a *contraction mapping* from $W^{2,s,\lambda}(B_R^+)$ into itself for small enough $R \in (0,r)$. (The fact that $\mathcal{S}$ maps $W^{2,s}(B_r^+)$ into itself will follow from the calculations below.) Then, having in mind that $u \in W^{2,q}(B_R^+)$ is a fixed point of the map $\mathcal{S}$ it will follow easily the statement of Lemma 4.1 and the estimate (4.1).

Take now two arbitrary functions $w_1,\ w_2 \in W^{2,s,\lambda}(B_r^+)$. Denoting $w = w_1 - w_2$, one has

$$\begin{aligned}\mathcal{S}w_1 - \mathcal{S}w_2 &= \phi(x,w)\\ &\quad + \int_{B_r^+} G(x_0,x,y)\left\{\left(a^{ij}(x_0) - a^{ij}(y)\right) D_{ij}w(y) - a^{ij}(x_0) D_{ij}\phi(y,w)\right\} dy\end{aligned}$$

with

$$\tilde{\varphi}(x', w) = \left[\ell_i(x_0') - \ell_i(x')\right] D_i w(x') + \left[\sigma(x_0') - \sigma(x')\right] w(x'), \tag{4.4}$$

and

$$\phi(x) = \phi(x', x_n) = \frac{x_n}{\ell_n(x_0)} \int_{\mathbb{R}^{n-1}} \tilde{\varphi}(x' - x_n y')\eta(y')dy' \quad \text{(see (3.1))}.$$

Taking into account the assumptions (2.2), we have

$$\begin{aligned}\|\mathcal{S}w_1 - \mathcal{S}w_2\|_{L^{s,\lambda}(B_r^+)} &\le C(n,\kappa)\Bigg(\|\phi(x,w)\|_{L^{s,\lambda}(B_r^+)}\\ &\quad + \Big\|\int_{B_r^+} G(x_0,x,y)\left((a^{ij}(x_0) - a^{ij}(y))D_{ij}w(y) + D_{ij}\phi(y,w)\right) dy\Big\|_{L^{s,\lambda}(B_r^+)}\Bigg).\end{aligned}$$

Since $G(x_0,x,y) = O(|x-y|^{2-n})$ as $|x-y| \to 0$ (see [8, Lemma 3.1, Remark 3.1]) and $a^{ij} \in L^\infty(\Omega)$, the integral $\int_{B_r^+} G(x_0,x,y)((a^{ij}(x_0) - a^{ij}(y))D_{ij}w(y) + D_{ij}\phi(y,w))dy$ is a Riesz potential. Thus, the classical theory (cf. [13, Lemma 7.12], [4, Lemma I.1]) implies

$$\begin{aligned}&\left\|\int_{B_r^+} G(x_0,x,y)\left((a^{ij}(x_0) - a^{ij}(y))D_{ij}w(y) + D_{ij}\phi(y,w)\right) dy\right\|_{L^{s,\lambda}(B_r^+)}\\ &\qquad \le C(n,s)r^2\left(\left\|D^2 w\right\|_{L^{s,\lambda}(B_r^+)} + \left\|D^2\phi(\cdot,w)\right\|_{L^{s,\lambda}(B_r^+)}\right).\end{aligned}$$

Further, according to (4.3) one has

$$\begin{aligned} D_{ij}(\mathcal{S}w_1 & -\mathcal{S}w_2)(x) = D_{ij}\phi(x,w) \\ & + \text{P.V.}\int_{B_r^+} \Gamma_{ij}(x,x-y)\left\{\left(a^{ij}(x)-a^{ij}(y)\right)D_{ij}w(y)-\mathcal{L}(x)\phi(y,w)\right\}dy \\ & - c_{ij}(x)\mathcal{L}(x)\phi(x,w)+I_{ij}(x,x,w)+J_{ij}(x,x,w) \quad \text{for a.a. } x\in B_r^+, \end{aligned}$$

with c_{ij}, $I_{ij}(x,x,w)$ and $J_{ij}(x,x,w)$ being as above with u replaced by w and missing term $f(y)$ at the integrands.

Since $\Gamma_{ij}(x,\xi)$ are Calderón–Zygmund kernels in the ξ variable, Proposition 3.2 implies

$$\begin{aligned} & \left\|\text{P.V.}\int_{B_r^+} \Gamma_{ij}(x,x-y)\left\{\left(a^{ij}(x)-a^{ij}(y)\right)D_{ij}w(y)-\mathcal{L}(x)\phi(y,w)\right\}dy\right\|_{L^{s,\lambda}(B_r^+)} \\ & \quad \le C(n,s,\kappa,\gamma_{ij},M,\partial\Omega)\left(\gamma(r)\|D^2w\|_{L^{s,\lambda}(B_r^+)}+\|D^2\phi(\cdot,w)\|_{L^{s,\lambda}(B_r^+)}\right) \end{aligned}$$

with $M=\max_{i,j=1,\dots,n}\max_{|\alpha|\le 2n}\left\|\frac{\partial^\alpha\Gamma_{ij}(x,\xi)}{\partial\xi^\alpha}\right\|_{L^\infty(\Omega\times\Sigma)}$.

Further, the geometric properties of the mapping T ensure $c_1|\tilde{x}-y|\le|T(x)-y|\le c_2|\tilde{x}-y|$ (cf. [7]) for some positive constants c_1 and c_2. Thus, Propositions 3.3 and 3.4 yield

$$\begin{aligned} & \|I_{ij}(\cdot,\cdot,w)\|_{L^{s,\lambda}(B_r^+)},\ \|I_{ij}(\cdot,\cdot,w)\|_{L^{s,\lambda}(B_r^+)} \\ & \quad \le C(n,s,\kappa,\gamma_{ij},M,\partial\Omega)\left\|\sum_{h,k=1}^n \tilde{C}(a^{hk},D_{hk}w)+\tilde{K}(\mathcal{L}\phi(\cdot,w))\right\|_{L^{s,\lambda}(B_r^+)} \\ & \quad \le C\left(\gamma(r)\|D^2w\|_{L^{s,\lambda}(B_r^+)}+\|D^2\phi(\cdot,w)\|_{L^{s,\lambda}(B_r^+)}\right). \end{aligned}$$

Finally,

$$\|c_{ij}(x)\mathcal{L}(x)\phi(x,w)\|_{L^{s,\lambda}(B_r^+)}\le C\|D^2\phi(\cdot,w)\|_{L^{s,\lambda}(B_r^+)}.$$

Therefore,

$$\begin{aligned} \|\mathcal{S}w_1-\mathcal{S}w_2\|^*_{W^{2,s,\lambda}(B_r^+)} \le C\Big(& r\gamma(r)\|D^2w\|_{L^{s,\lambda}(B_r^+)}+r^2\|D^2w\|_{L^{s,\lambda}(B_r^+)} \\ & +r\|D^2\phi(\cdot,w)\|_{L^{s,\lambda}(B_r^+)}+\|\phi(\cdot,w)\|_{L^{s,\lambda}(B_r^+)}\Big) \end{aligned} \tag{4.5}$$

with $C=C(n,s,\kappa,\gamma_{ij},M,\partial\Omega)$.

To express the last two norms above in terms of $\|w\|^*_{W^{2,s,\lambda}(B_r^+)}$, we use Lemma 3.1. Thus,

$$\begin{aligned} \|\phi\|^*_{W^{2,s,\lambda}(B_r^+)} &= \|\phi(\cdot,w)\|_{L^{s,\lambda}(B_r^+)}+r\|D^2\phi(\cdot,w)\|_{L^{s,\lambda}(B_r^+)} \\ &\le cr^{1/2}\|\tilde{\varphi}\|^*_{W^{(s,\lambda)}(B_r^+\cap\{x_n=0\})}. \end{aligned}$$

On the other hand, (4.4) implies

$$\begin{aligned} r^{1/2}\|\tilde{\varphi}(\cdot,w)\|^*_{W^{(s,\lambda)}(B_r^+\cap\{x_n=0\})} \le Cr^{1/2}\Big(& \|(\ell_i(x_0')-\ell_i(x'))D_iw\|^*_{W^{(s,\lambda)}(B_r^+\cap\{x_n=0\})} \\ & +\|(\sigma(x_0')-\sigma(x'))w\|^*_{W^{(s,\lambda)}(B_r^+\cap\{x_n=0\})}\Big). \end{aligned}$$

Remembering the Lipschitz regularity of the coefficients of the boundary operator (2.3), the Rademacher theorem and [3, Theorem 1.2] yield

$$\begin{aligned} r^{1/2}\|(\ell_i(x_0')-\ell_i(x'))D_iw\|^*_{W^{(s,\lambda)}(B_r^+\cap\{x_n=0\})} \\ +r^{1/2}\|(\sigma(x_0')-\sigma(x'))w\|^*_{W^{(s,\lambda)}(B_r^+\cap\{x_n=0\})} \leq Cr^{1/2}\|w\|^*_{W^{2,s,\lambda}(B_r^+)}. \end{aligned}$$

Therefore, (3.3) implies

$$\|\phi(\cdot,w)\|^*_{W^{2,s,\lambda}(B_r^+)} \leq Cr^{1/2}\|w\|^*_{W^{2,s,\lambda}(B_r^+)}.$$

Taking into account (2.2), (4.5) reads

$$\|\mathcal{S}w_1-\mathcal{S}w_2\|^*_{W^{2,s,\lambda}(B_r^+)} \leq C(r)\|w_1-w_2\|^*_{W^{2,s,\lambda}(B_r^+)}, \quad C(r)=o(1) \text{ as } r\to 0,$$

where $C(r)=C(\gamma(r)+r+r^{1/2})$. Taking $r=R$ to be sufficiently small above we have $C(R)<1$, that is, $\mathcal{S}$ is a contraction mapping from $W^{2,s,\lambda}(B_R^+)$ (equipped with the norm $\|\cdot\|^*_{W^{2,s,\lambda}(B_R^+)}$) into itself for each $s\in[q,p]$. Now, remembering that $u\in W^{2,q}(B_R^+)$ is a fixed point of $\mathcal{S}$, and using the imbedding $W^{2,p,\lambda}(B_R^+)\subset W^{2,q,\lambda}(B_R^+)\subset W^{2,q}(B_R^+)$, as well as the fact that the fixed point of $\mathcal{S}$ should be *unique* one, we obtain $D^2u\in L^{p,\lambda}(B_R^+)$.

To get the estimate (4.1), we have to take the L^p-norm of the both sides of (4.3). The calculations are similar to these already carried out in obtaining (4.5). Precisely, taking $w_1=u$ and $w_2=0$ we have

$$\begin{aligned} \|u\|^*_{W^{2,p,\lambda}(B_r^+)} &= \|\mathcal{S}u\|^*_{W^{2,p,\lambda}(B_r^+)} \\ &\leq \|\mathcal{S}u-\mathcal{S}0\|^*_{W^{2,p,\lambda}(B_r^+)}+\|\mathcal{S}0\|^*_{W^{2,p,\lambda}(B_r^+)}. \end{aligned}$$

The first norm above is estimated exactly as in (4.5), while the second one gives $\|f\|_{L^{p,\lambda}(B_R^+)}$ and $\|\varphi\|_{W^{(p,\lambda)}(B_R\cap\{x_n=0\})}$.

This completes the proof of Lemma 4.1. ♢

5. Global Morrey Regularity and Solvability of the Problem (2.1)

Proof of Theorem 2.1. Bearing in mind the *interior Morrey regularity* ([12, Theorem 3.3]), the statement of Theorem 2.1 and the bound (2.4) follow from Lemma 4.1 through a suitable partition of unity. ♢

Proof of Theorem 2.2. The functions $f\equiv 0$ and $\varphi\equiv 0$ lie in $L^{p,\lambda}(\Omega)$ and $W^{(p,\lambda)}(\partial\Omega)$, respectively, for each $p>1$ and each $\lambda\in(0,n)$. In particular, this holds true for $p>n$. Thus, bearing in mind the Aleksandrov–Bakelman–Pucci maximum principle ([23, Theorem 2.6.2]) it follows that $u(x)=0$ is the *unique solution* of the homogeneous oblique derivative problem (2.1) ($f\equiv 0$, $\varphi\equiv 0$). This proves uniqueness of the solution to (2.1).

Concerning the strong solvability in the space $W^{2,p,\lambda}(\Omega)$ of the problem (2.1), we note that $L^{p,\lambda}(\Omega)\subset L^p(\Omega)$. Therefore, in view of [8, Theorem 1.2], there exists a unique solution $u\in W^{2,p}(\Omega)$ of (2.1). Further, Theorem 2.1 asserts $u\in W^{2,p,\lambda}(\Omega)$.

To derive the estimate (2.5) we have for the linear operator

$$(\mathcal{L},\mathcal{B})\colon\ W^{2,p,\lambda}(\Omega)\to L^{p,\lambda}(\Omega)\times W^{(p,\lambda)}(\partial\Omega)$$

that

$$\begin{aligned}\|(\mathcal{L},\mathcal{B})u\|_{L^{p,\lambda}(\Omega)\times W^{(p,\lambda)}(\partial\Omega)} &= \|\mathcal{L}u\|_{L^{p,\lambda}(\Omega)} + \|\mathcal{B}u\|_{W^{(p,\lambda)}(\partial\Omega)} \\ &\le C\left(\|u\|_{L^{p,\lambda}(\Omega)} + \|Du\|_{L^{p,\lambda}(\Omega)} + \|D^2u\|_{L^{p,\lambda}(\Omega)}\right) \\ &\le C\|u\|_{W^{2,p,\lambda}(\Omega)}.\end{aligned}$$

This shows continuity of $(\mathcal{L},\mathcal{B})$. Further, $(\mathcal{L},\mathcal{B})$ is injective and surjective mapping as it was shown before. Thus, the Banach theorem on inverse mappings implies continuity of the operator $(\mathcal{L},\mathcal{B})^{-1}$, i.e., the bound (2.5). $\diamondsuit$

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