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Abstract. The paper discuss the computation of the worst case uncertainty (WCU) in common measurement problems. The usefulness of computing the WCU besides the standard uncertainty is illustrated. A set of equations to compute the WCU in almost all practical situations is presented. The application of the equations to real-world cases is shown.

1. Introduction

The Guide to the Expression of Uncertainty in Measurements (GUM) [1] does not consider explicitly the "worst case uncertainty" (WCU), that is, the uncertainty with a 100% coverage probability. The GUM Uncertainty Framework (GUF) is focused on standard uncertainties, and on the computation of expanded uncertainties under the hypothesis of Gaussian distribution. Therefore, while in the GUF an expanded uncertainty with coverage probability of 95% or 99% is easily computed, the expanded uncertainty with 100% coverage probability is not. Indeed, either it is infinite (in the Gaussian approximation), or it needs separate propagation rules [2].

In practice, WCUs are widely used and, in general, worst-case analysis is a well-known and regarded tool in engineering [3], [4]. First of all, accuracy characteristics of almost *all* commercial instruments are given in terms of WCUs [5]. As a consequence, in common engineering and laboratory practice, simple functions of few measurements are assessed using the WCU, using simple and well-known propagation rules [6]. (For example, the absolute WCU of $x_1 - x_2$ is the sum of the absolute WCUs of x_1 and x_2; the relative WCU of x_1 / x_2 is the sum of the relative WCUs; and so on.) WCUs, in general, are the most convenient way of defining a credible interval whenever the involved distributions are bounded and without long tails, which is very common, e.g. when the measurement model involves few input quantities with uniform distribution. Therefore, the lack of an adequate theoretical recognition and sistematization of the WCU concept is not a desirable situation for the measurement science community.

In this paper, the propagation of the WCU is discussed. Firstly, it is pointed out that the information sufficient to compute the standard uncertainty (SU) is, in general, not sufficient to compute the WCU, but, for the same reason, computing *only* the SU may give insufficient information to state correctly the expanded uncertainty. Afterwards, the problem of computing the WCU without a full Monte Carlo analysis is considered. It is demonstrated that simple propagation formulae can be given in a very common situation, i.e. when measurement errors are a linear function of a set of statistically independent measurement errors. For this case, also the propagation of the SU is computed, avoiding explicit computation of correlation coefficients. The propagation formulae are shown to be congruent

with accuracy characteristics of instrument manufacturers, characteristics that cannot be written nor understood in the GUF.

Throughout the paper, the concept of measurement error is used. It is the difference $E = \hat{X} - X$, where $\hat{X}$ is the measurement and X is the unknown measurand. According to the Bayesian approach of Supplement 1 of GUM [7], $\hat{X}$ is deterministic, X is a random variable, and therefore E is a random variable. Computing the propagation using E instead of X does not change the final result (GUM, E.5) and has some advantages; in particular, it is necessary to clarify the accuracy characteristics of many instruments, which have an underlying "error model". The WCU is the maximum absolute value that the random variable E can assume: in the following, it is denoted by $U = \max|E|$.

2. Worst-case uncertainty for a general measurement model and general input quantities

2.1. Computation of WCU using joint distributions

Let $y = g(x_1,...,x_N)$ be the measurement model, E_i the measurement errors on the input quantities, $U(x_i) = \max|E_i| = U_i$ the WCUs on the input quantities ($i = 1,...,N$), E_y the error on the output quantity, and $U(y) = \max|E_y| = U_y$ the WCU on the output quantity.

Computing the WCU on the output quantity requires, in the general case, complete knowledge of the multivariate joint probability density function (pdf) of the errors $E_1,...,E_N$, and a cumbersome computation of the entire distribution of the error E. In particular, unlike SU, *correlation coefficients are useless* when computing the WCU. This is illustrated by the following example.

Let us consider two measurements x_1, x_2, affected by uncorrelated errors E_1, E_2 with identical (marginal) distributions $f_1(e_1) = f_2(e_2)$, supposed to be symmetric triangular in $[-1,1]$ (and therefore with WCU $U(x_1) = U(x_2) = U_x = 1$). The information on E_1, E_2 is sufficient to evaluate the SU, but not the WCU, of the sum $y = x_1 + x_2$. The inspection of the joint pdf in two different cases of uncorrelated errors clarifies the issue.

Fig. **1** shows the joint distribution of statistically independent triangular E_1, E_2, while Fig. **2** shows the distribution in a case when they are uncorrelated but not independent. The resulting distribution $f(e)$ of $E = E_1 + E_2$ in the two cases is depicted in Figs. **3** and **4**. Uncorrelated errors with the same distribution yield completely different output distributions and WCU ($U(y) = 2$ in the first case, $U(y) = 1$ in the second one).

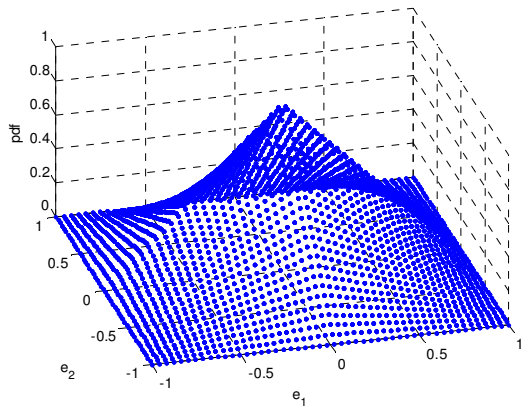

Figure 1. Joint pdf of independent errors with identical marginal pdf (triangular with $U_x = 1$).

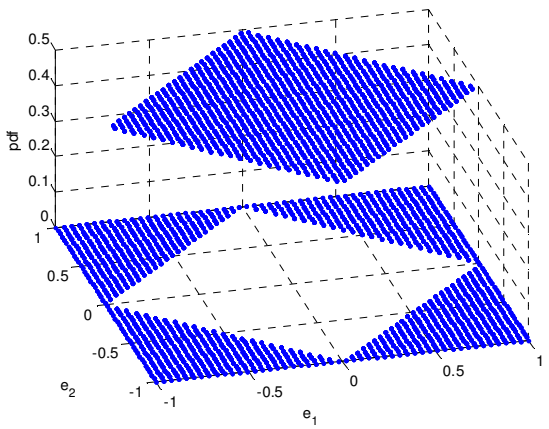

Figure 2. Joint pdf of uncorrelated, but not independent, errors, with the same marginal pdf.

For a generic joint pdf of the input errors E_i, computing the distribution of the output error E can be accomplished only via Monte Carlo simulations, in the same way described in GUM Supplement 1.

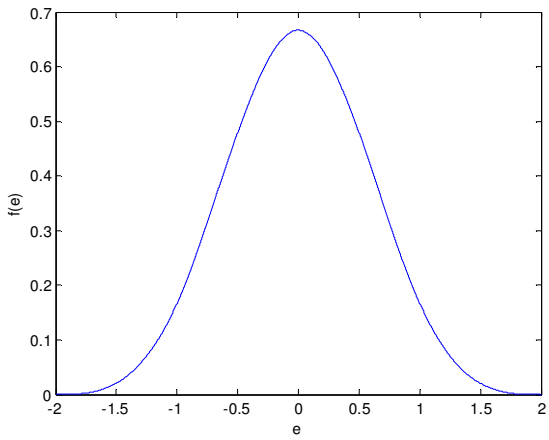

Figure 3. Distribution of $E = E_1 + E_2$ for the joint distribution depicted in Fig. **1**.

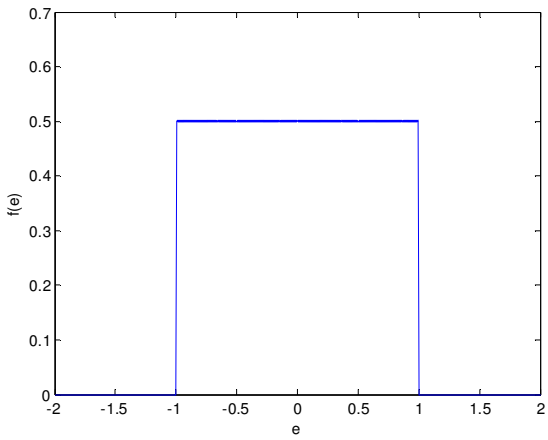

Figure 4. Distribution of $E = E_1 + E_2$ for the joint distribution depicted in Fig. **2**.

2.2. Reasons for computing the WCU

In the two cases above, let us assume knowledge of the WCU *only*, i.e. $U(y) = 2$ in the first case, $U(y) = 1$ in the second. The SU, due to the hypothesis of uncorrelated errors, is $u(y) = \sqrt{u^2(x_1) + u^2(x_2)} = U_x / \sqrt{3} = 0.577$ in both cases. Therefore, the ratio (WCU/SU) is $U(y)/u(y) = 3.464$ in the first case, and $U(y)/u(y) = \sqrt{3} = 1.732$ in the second case.

The ratio U/u is the maximum sensible coverage factor and therefore indicates, even without knowing the actual distributions $f(e)$, the applicability of the normal approximation. In the first case the ratio $U/u = 3.5$ indicates that coverage factors like $k = 2$ or $k = 2.58$ (for probabilities of 95% or 99%, respectively) are applicable and accurate, while, in the second case, the ratio $U/u = 1.732$ demonstrates that these coverage factors do not make sense. In the first case, for a coverage factor $k = 2$, the actual coverage probability p is $p = 95.75\%$, with a negligible difference from the Gaussian value 95.45%. In the second case, instead, the ratio $U/u = 1.732$ immediately suggests that the error must have a uniform distribution, and the measurement uncertainty is effectively quantified by the WCU. Summing up, a simple procedure to compute the WCU, *without* computing the entire distribution, is desirable. The reasons are essentially the following:

- the ratio WCU/SU shows the applicability of the normal approximation (i.e., of the GUF);
- the WCU is the maximum expanded uncertainty that may be attached to the measurement.

3. Practical computation of worst-case uncertainty

3.1. Worst-case uncertainty propagation for statistically independent errors

Under the hypothesis of statistically independent errors, computing the WCU is very simple and even more straightforward then computing the SU. This well-known fact is recalled here to introduce concepts and notation used later. In particular, a compact matrix notation is used. We introduce the symbols $\mathbf{x} = [x_1,...,x_N]^T$, $y = g(\mathbf{x})$, $\mathbf{E}_x = [E_1,...E_N]^T$, $\mathbf{U}_x = [U(x_1),...,U(x_N)]^T$. We denote with $\mathbf{C} = \partial y / \partial \mathbf{x} = [\partial y / \partial x_1,...,\partial y / \partial x_N]$ the Jacobian matrix of the measurement model (for a scalar output it is a row vector). This is the vector of the sensitivity coefficients. The well-known first-order approximation of the error propagation law is

$$E_y \cong \mathbf{C} \cdot \mathbf{E}_x . \tag{1}$$

If the input errors are statistically independent, the maximum absolute value of the output error, $\max|E_y| \models \max|\mathbf{C} \cdot \mathbf{E}_x|$, is simply factorized in $|\mathbf{C}| \cdot \max|\mathbf{E}_x|$, where $|\mathbf{C}|$ denotes the matrix of the absolute values of the elements of **C**. Therefore the WCU is given by:

$$U_y \cong |\mathbf{C}| \cdot \mathbf{U}_x . \tag{2}$$

With the same notation, the SU is simply expressed by

$$u_y^2 = \mathbf{C} \cdot \mathbf{\Sigma}_x \cdot \mathbf{C}^T \tag{3}$$

where $\mathbf{\Sigma}_x$, the covariance matrix, in this case is diagonal $\mathbf{\Sigma}_x = \mathrm{diag}(u^2(x_1),...u^2(x_N))$.

3.2. Extending the propagation formulae to non-independent errors

The simple formulae above can be extended to the case of non-independent errors in a case that is very common in practice:

- the errors, although not independent, can be expressed as a linear function of statistically independent errors.

A typical example of this kind of situation is the static error introduced by an A/D converter, which is usually expressed in the form [8]

$$e(x) = \Delta G \cdot x + O + inl(x) + e_q(x) \tag{4}$$

where ΔG is the gain error, O is the offset error, $inl(x)$ is the integral nonlinearity error, and $e_q(x)$ is the quantization error. The experimental results about the variability of the systematic errors in ADC-based instrument are reported in [9], [10].

If many measurements $x_1,...,x_N$ are taken by the A/D converter in repeatability conditions (but with different measurand), gain and offset errors are essentially the same in all the measurements, while the integral nonlinearity and the quantization errors are not. Therefore, the N measurement errors are a function of $2N+2$ independent errors: ΔG, O, $inl(x_1)$, $e_q(x_1)$, …, $inl(x_N)$, $e_q(x_N)$. Many other examples of this situation are possible, including the common case of "cascaded" measurement models.

In all such cases, the propagation formulae for the WCU are a simple extension of those for independent errors. Let $\mathbf{E}' = [E'_1,...,E'_M]^T$ be the vector of the statistically independent errors. The input errors $\mathbf{E}_x = [E_1,...E_N]^T$ are given by

$$\mathbf{E}_x = \mathbf{T} \cdot \mathbf{E}' \tag{5}$$

where **T** is a transformation matrix. Equations (1), (2), (3) becomes

$$E_y = \mathbf{C} \cdot \mathbf{T} \cdot \mathbf{E}',\ U_y = |\mathbf{C} \cdot \mathbf{T}| \cdot \mathbf{U}',\ u_y^2 = \mathbf{C} \cdot \mathbf{T} \cdot \mathbf{\Sigma}_x \cdot \mathbf{T}^T \cdot \mathbf{C}^T \tag{6)-(7)-(8)}$$

Equation (7) is the propagation law for the WCU. An example of its application is easily done, considering errors of the kind (4).

For two measurements x_1, x_2 taken in repeatability conditions, affected by errors (4), the independent errors are the 6×1 (unknown) vector

$$\mathbf{E}' = [\Delta G, O, inl(x_1), inl(x_2), e_q(x_1), e_q(x_2)]^T . \tag{9}$$

with associated (known) WCUs given by

$$\mathbf{U}' = [U_G, U_O, U_{inl}, U_q, U_{inl}, U_q]^T . \tag{10}$$

On the basis of the error model (4), the transformation matrix **T** is:

$$\mathbf{T} = \begin{bmatrix} x_1 & 1 & 1 & 0 & 1 & 0 \\ x_2 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}. \tag{11}$$

It is now immediate to compute the worst-case uncertainty for any given measurement model $g(x_1, x_2)$. For example, if $y = x_1 - x_2$, the sensitivity coefficients are $\mathbf{C} = [1, -1]$, and therefore

$$\mathbf{C} \cdot \mathbf{T} = [x_1 - x_2, 0, 1, 1, 1, 1]. \tag{12}$$

The WCU is:

$$U_y = |\mathbf{C} \cdot \mathbf{T}| \cdot \mathbf{U}' = |x_1 - x_2| U_G + 2U_{inl} + 2U_q . \tag{13}$$

The elements of the vector $\mathbf{C} \cdot \mathbf{T}$ are modified sensitivity coefficients: they are sensitivity coefficients of the independent errors. In the example, $x_1 - x_2$ is the coefficient of the gain error, 0 is the coefficient of the offset error, etc.

4. Application to real-world instruments and to the GUM

The derived propagation formulae applies, of course, to real-world measurements. An example are the accuracy characteristics provided by Agilent Technologies [11], [12]. This manufacturer gives accuracy specification in terms of WCUs, like almost all instrument manufacturers, but usually gives also separate formulae for the uncertainty of the single measurement x, and for the difference of two measurements in repeatability conditions ("dual cursor measurements"), $y = x_2 - x_1$. Therefore, it is very simple to write down the vector $\mathbf{U}'$ and the matrix **T** for a pair of measurements (x_1, x_2).

Below, we report $\mathbf{U}'$ and **T** for different instruments, together with the formulae for the WCU of the single measurement x_1 and the difference $y = x_1 - x_2$. The formulae are computed according to the developed mathematics, and of course coincide with those reported in the technical sheets. In the formulae, V_{FS} is the selected full-scale range, and V_P is the selected "vertical position".

Agilent 54600B:

$$\mathbf{U}' = [1.9\% \;\; 1\%V_{FS} \;\; 0.5\%|V_P| \;\; 0.2\% \cdot V_{FS} \;\; 0.2\%V_{FS}]$$

$$\mathbf{T} = \begin{bmatrix} x_1 & 1 & 1 & 1 & 0 \\ x_2 & 1 & 1 & 0 & 1 \end{bmatrix}$$

$$U(x_1) = 1.9\%|x_1| + 1.2\%V_{FS} + 0.5\%|V_P|$$

$$U(x_1 - x_2) = 1.9\%|x_1 - x_2| + 0.4\%V_{FS}$$

Agilent U1610A:

$$\mathbf{U}' = [4\% \;\; 0.1 \text{ div} \;\; 2 \text{ mV} \;\; 1.6\%|V_P| \;\; 0.2\% \cdot V_{FS} \;\; 0.2\%V_{FS}]$$

$$\mathbf{T} = \begin{bmatrix} x_1 & 1 & 1 & 1 & 1 & 0 \\ x_2 & 1 & 1 & 1 & 0 & 1 \end{bmatrix}$$

$$U(x_1) = 4\%|x_1| + 0.1 \text{ div} + 2 \text{ mV} + 1.6\%V_P + 0.2\%V_{FS}$$

$$U(x_1 - x_2) = U(x_1) = 4\%|x_1 - x_2| + 0.4\%V_{FS}$$

For the U1610A, in the final declared characteristics the manufacturer substitutes $4\% \mid x_1 \mid$ and $4\% \mid x_1 - x_2 \mid$ with a common upper bound, $4\% V_{FS}$.

5. Conclusions

Worst case analysis is a precious and commonly used tool in engineering, but it is not included in the GUM, mainly because the WCU does not fit in the propagation equations for the SU. The paper shows briefly that WCU can be included in the GUM framework with a set of separate equations, which are, however, very simple and formally analogous to those for SU. WCU can be computed simply and automatically when measurement errors are independent, and in almost all the practical cases of non-independent errors. The WCU/SU ratio indicates when the Gaussian approximation of the GUF is applicable, and when not.

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