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RESEARCH PAPER

ANALYSIS AND SHAPING OF THE SELF-SUSTAINED OSCILLATIONS IN RELAY CONTROLLED FRACTIONAL-ORDER SYSTEMS

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Abstract

This work deals with Single-Input-Single-Output (SISO) fractional order systems with a discontinuous relay control element in the feedback loop. Stable self-sustained oscillations often occur in the closed loop relay system, and this work takes advantage of Describing Function (DF) analysis and of another more accurate approach, called Locus of a Perturbed Relay System (LPRS) method, for analyzing in the frequency domain the characteristics of the limit cycle oscillations. The use of fractional lead compensator is also suggested for the purpose of shaping the characteristics of the limit cycle. The proposed analysis and design procedures will be supported by thoroughly discussed simulation examples.

MSC 2010: Primary 34A08; Secondary 34C25, 93C80, 34A36

Key Words and Phrases: fractional-order systems, periodic solutions, relay feedback control systems, describing function, locus of a perturbed relay system

1. Introduction

Fractional–order systems (FOSs), i.e. dynamical systems described using fractional (or, more precisely, non-integer) order derivative and integral operators, are studied with rapidly growing interest in recent years (cf. [19, 13, 12, 14], and references therein).

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Discontinuous control approaches were recently suggested in the context of fractional order systems to exploit the underlying useful properties of accuracy and robustness possessed by the discontinuous control methodology (see [11, 9, 20, 10, 17]).

The focus of the present paper is on relay control systems with a fractional order plant in the loop. Particularly, we intend to study systems where the invariance condition of the sliding manifold is not fulfilled due to neglected dynamics (not necessarily fast) of plant, actuator and sensor, that cause an increase of the relative degree. When the difference between the overall transfer function's denominator and numerator degrees (fractional relative degree) is greater than one, or when there is a finite delay, the relay system is known to exhibit locally stable self-sustained oscillations.

The Describing Function (DF) methodology and the associated harmonic balance equation (see [1]) allow for a clear understanding of the phenomenon, and, in some cases, an accurate prediction of the oscillation frequency and amplitude. The Describing Function (DF) methodology was previously applied in the context of Fractional Order Systems (see [2, 21]) to different problems than that considered in this paper. The DF method has the main weakness that it is only an approximate procedure whose accuracy strictly relies on the level of satisfaction of the filtering hypothesis.

Recently suggested, the "Locus of a Perturbed Relay System" approach (in short LPRS, see [5, 3]) offers a simple alternative way to compute the exact frequency value of the self excited oscillation (the same computed by means of the Tsypkin method) while giving additional informations abour the equivalent gain which are not offered by the Tzypkin locus.

The first problem tackled in the paper is the oscillation analysis for the feedback relay FOS. DF and LPRS methods of analysis are developed for computing the oscillation parameters. The second problem tackled is oscillation shaping. Particularly, it is studied the problem of assigning an arbitrary frequency to the self-excited oscillations, and the task is achieved by means of a suitably tuned fractional lead/lag compensator (see ([15])). The frequency characteristics of the fractional lead/lag compensator were investigated in ([15]), and an auto-tuning methodology for such a controller, based on a modified relay test, was discussed in ([16]). It is worth to stress that the problem considered here is radically different, as the compensator is sought to be used along with the relay in order to adjust the oscillatory characteristics of the closed loop system.

The paper is structured as follows: in Section **2** the problem is formulated, and the DF and LPRS approaches to oscillation analysis are outlined. In Section **3**, the design of fractional order lead/lag compensators, tuned to assign the closed loop relay system a desired frequency of oscillations along with certain additional desirable properties, is addressed. Section **4** presents a thoroughly discussed analysis and design example, supported by computer simulation results, and Section **5** gives some concluding remarks.

2. Problem formulation

We shall study the feedback control system in Figure 1, where $P(s)$ is a strictly proper fractional order transfer function possibly including a delay as follows

$$P(s) = \frac{\sum_{i=1}^{m} b_i s^{\alpha_i}}{s^{\gamma_n} + \sum_{i=1}^{n-1} a_i s^{\gamma_i}} e^{-\delta s}, \tag{2.1}$$

along with a relay controller, possibly with hysteresis, in the negative feedback loop.

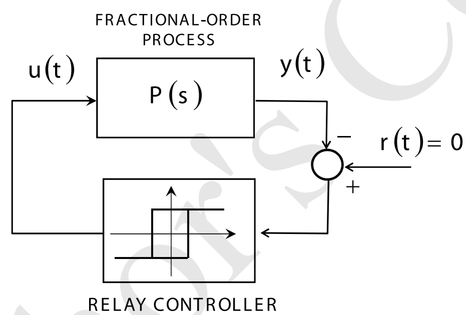

FIGURE 1. The feedback relay control system

Let c and b denote the amplitude and the hysteresis parameter of the relay, and a_y be the unknown-in-advance amplitude of the self-excited oscillatory motion of the output variable $y(t)$. For the hysteretic relay nonlinearity under the condition that $b < a_y$, the formula of the DF has been derived analytically (see [1]) as:

$$N(a_y) = \frac{4c}{\pi a_y}\sqrt{1 - \left(\frac{b}{a_y}\right)^2} - j\frac{4cb}{\pi a_y^2} \tag{2.2}$$

The periodic solution can be found from the equation of harmonic balance (see [1])

$$P(j\Omega) = -\frac{1}{N(a_y)} \tag{2.3}$$

which is a complex equation with two unknown values: the oscillation frequency Ω and amplitude a_y. Equation (2.3) has a convenient graphical interpretation. Note that the value on the left-hand side of the equation is considered a function of the frequency, and in fact is the Nyquist locus of the process. The value on the right-hand side is the negative reciprocal of the DF, which takes the form

$$-N^{-1}(a_y) = -\frac{\pi a_y}{4c}\sqrt{1-\left(\frac{b}{a_y}\right)^2} - j\frac{\pi b}{4c} \tag{2.4}$$

We can see from (2.4) that the imaginary part does not depend on the amplitude a_y, and the plot of $-N^{-1}(a_y)$ on the complex plane is a horizontal line (Fig.2), which lies on the left half-plane. The point corresponding to zero amplitude is located on the imaginary axis, and the real part of $-N^{-1}(a_y)$ tends to minus infinity as the amplitude a_y grows.

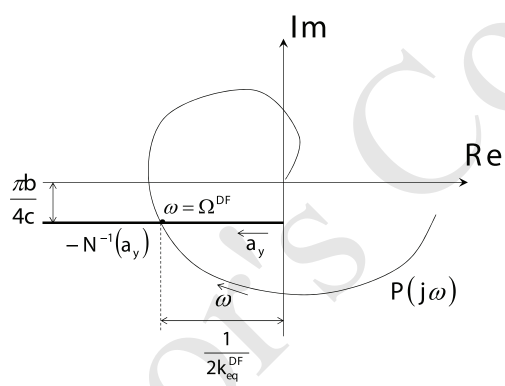

FIGURE 2. DF-analysis in the complex plane

The periodic solution of the equations of the relay servo system corresponds to the point of intersection of the Nyquist locus of the process (being a function of the frequency) and of the negative reciprocal of the DF of the hysteretic relay (being a function of the amplitude a_y), given by formula (2.4), on the complex plane.

The frequency of the periodic solution is estimated by DF method as the value Ω^{DF} such that

$$\mathrm{Im}\, P(j\Omega^{DF}) = -\frac{\pi b}{4c}\,. \tag{2.5}$$

The corresponding amplitude a_y is estimated as

$$a_y^{DF} = \frac{4c}{\pi}\left\|P(j\Omega^{DF})\right\|\,. \tag{2.6}$$

This periodic solution is approximate because of the approximate nature of the DF method itself, which is based upon the assumption about the harmonic shape of the input signal to the relay. However, if the linear part of the system has the property of the low-pass filter, so that the higher harmonics of the control signal are attenuated to a higher degree, the DF approximations (2.5)-(2.6) may give relatively precise results.

DF analysis also carries additional information regarding the propagation of the slow (as compared to the self-excited oscillation) external inputs through the relay control system. This problem is sometimes referred to in the literature as the "transfer properties" analysis of the relay control system (see [5, 3]). The "equivalent gain k_n^{DF} (see Fig.2)

$$k_n^{DF} = -\frac{1}{2\text{Re } P(j\Omega^{DF})} \tag{2.7}$$

comes into play as a fundamental concept. Basically, the behaviour of the relay control system in Figure 1 subject to a nonzero external input signal $r(t)$ slowly varying as compared to the self-excited oscillation is well approximated by the linearized system in which the relay is replaced by the associated equivalent gain. From this viewpoint, dynamic properties such as bandwidth and frequency response can be naturally associated to the transfer properties of the relay control system. The approximation is valid as long as the external inputs will be sufficiently slow as compared to the self excited oscillations.

An alternative approach, theoretically exact as it includes the higher harmonics up to a desired order, is going to be described. The "Locus of a Perturbed Relay System" (LPRS) $J(\omega)$ associated to the plant transfer function $P(s)$ takes the form (see [5, 3])

$$\begin{aligned} J(\omega) &= \sum_{k=1}^{\infty}(-1)^{k+1}\mathtt{Re}\ P(jk\omega) \\ &+ j\sum_{k=1}^{\infty}\frac{\mathtt{Im}\ P(j(2k-1)\omega)}{2k-1}, \quad i=1,2\,. \end{aligned} \tag{2.8}$$

The LPRS is employed graphically in a totally analogous way as it was done for the plant harmonic response function $P(j\omega)$ in the DF-analysis. Hence, the same graphical construction has to be made, with the Nyquist plot of the complex locus $J(\omega)$ to be used in place of that of $P(j\omega)$.

Thereby, the real part of $J(\omega)$ contains information about the equivalent gain and the imaginary part of $J(\omega)$ comprises the condition of the switching of the relay and, consequently, contains information about the frequency of the oscillations. The frequency Ω^{LPRS} of the oscillations can be computed by solving the equation:

$$\text{Im } J\left(\Omega^{LPRS}\right) = -\frac{\pi b}{4c}\,, \tag{2.9}$$

and LPRS gives the same frequency value as that computed by means of the Tsypkin method. The equivalent gain of the relay is computed according to

$$k_n^{LPRS} = \frac{1}{-2\,\mathtt{Re}\,\left[J(\Omega^{LPRS})\right]}\,. \tag{2.10}$$

The amplitude of the first and third harmonic of the oscillation are derived via the LPRS method as

$$a_{y,1}^{LPRS} = \frac{4c}{\pi}\left\|P(j\Omega^{LPRS})\right\|\,, \tag{2.11}$$

$$a_{y,3}^{LPRS} = \frac{4c}{3\pi}\left\|P(j3\Omega^{LPRS})\right\|\,. \tag{2.12}$$

3. Shaping of the self excited oscillation via lead/lag compensator

For simplicity, the non hysteretic relay (i.e., $b = 0$) will be studied. It is worth noting that the fundamental frequency of the oscillation is independent of the relay amplitude c. The deliberate introduction of (possibly additional) delays is a well known artifice to adjust the frequency of the limit cycle (see [16]). Dynamic compensators were suggested to this end (see [4]) along with a particularly effective tuning philosophy where the parameters of the compensator are selected by properly shaping the LPRS of the compensated system, rather then the Nyquist plot of its harmonic response. This offers high accuracy in setting the desired frequency of oscillation, since the LPRS analysis is more accurate than the DF-based one. This approach, however, requires that the LPRS of the process and that of the compensator are available in analytical form, or at least that the variation of the LPRS in response to changes in the system parameters is well understood, which is not the case, at the moment, in the fractional order systems setting. Thus, this research line will be explored in next activities. Other interesting directions for next research entail the use of more sophisticated controllers rather than the standard relay, in particular the "generalized" relay arising within the context of the so-called "high-order" sliding mode control theory (see [8, 6, 7, 18]).

It is the main task of this work that of studying the possibility of adjusting the frequency of the limit cycle by cascading the relay controller with a properly designed fractional lead/lag compensator. We take advantage of DF analysis be seeking for a compensator that shapes the frequency response in such a way that the DF-frequency of oscillation Ω_c^{DF} of the compensated system is equal to some desired value Ω_{des}. The compensator

$$C(s) = k_c\left(\frac{s + 1/\lambda}{s + 1/x\lambda}\right)^{\nu}, \quad 0 < x < 1,\ \lambda > 0, \tag{3.1}$$

is under study, where ν is the real valued fractional order of the compensator, with $\nu > 0$ for lead compensation, and $\nu < 0$ for lag compensation.

Assume that $\Omega_{des} > \Omega_u^{DF}$, where Ω_u^{DF} is the oscillation frequency of the uncompensated system. Then a positive phase lead has to be applied at frequency Ω_{des} to shift the oscillation frequency towards the desired value, and a lead-type compensation is needed.

The magnitude and phase diagrams of compensator (3.1) have a qualitative shape as that in Fig. 3, with parameters (see ([15]))

$$M_0 = 20 \log \left(k_c x^\nu\right), \qquad M_\infty = 20 \log k_c, \tag{3.2}$$

$$\omega_z = \frac{1}{\lambda}, \qquad \bar{\omega} = \frac{1}{\lambda\sqrt{x}}, \qquad \omega_p = \frac{1}{\lambda x}, \tag{3.3}$$

$$\begin{aligned} \phi_m &= \nu\left(\arctan\left(\frac{1}{\sqrt{x}}\right) - \arctan\left(\sqrt{x}\right)\right) \\ &= \nu \arcsin\left(\frac{1-x}{1+x}\right) \end{aligned} \tag{3.4}$$

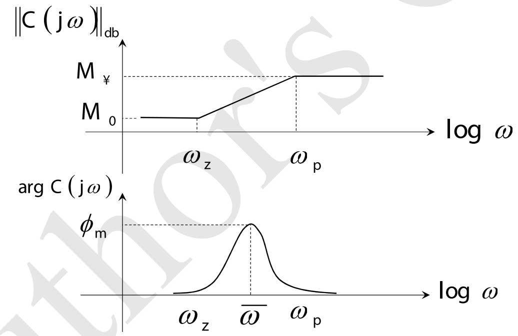

FIGURE 3. Bode diagrams of compensator (3.1) with $\nu > 0$

Interestingly, the next relation holds

$$\|C(j\omega)\|_{dB} = \nu\, \|C_1(j\omega)\|_{dB}, \tag{3.5}$$

$$\arg C(j\omega) = \nu\, \arg C_1(j\omega), \tag{3.6}$$

where $C_1(s)$ is the integer order compensator

$$C_1(s) = k_c \frac{s + 1/\lambda}{s + 1/x\lambda} = k_c x \frac{1 + \lambda s}{1 + x\lambda s}\,. \tag{3.7}$$

Therefore, the chart depicting the normalized magnitude and phase diagrams of the compensator (3.7), widely adopted in loop shaping design for integer order systems, can be used for tuning the parameters of (3.1), by taking into account, while exploiting the chart, the scaling properties (3.5) and (3.6). The normalized phase diagram is shown in Figure 4.

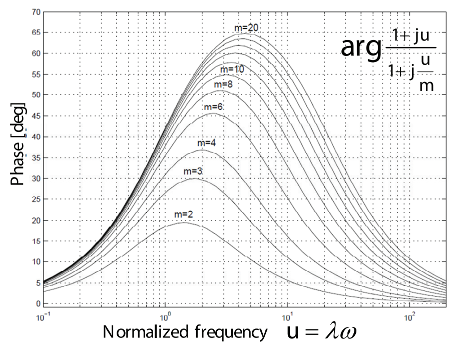

FIGURE 4. Normalized phase diagram of the integer order lead compensator

The m parameter of the normalized chart corresponds to the reciprocal of the x parameter, $m = 1/x$. Let us now address more explicitly the tuning of the compensator.

According to DF method, in order to rise up the frequency of oscillation to the desired value Ω_{des} the required phase lead, to be provided by the compensator at the desired oscillation frequency, is

$$\Delta\phi = -180^\circ - arg\left(P(j\Omega_{des})\right) . \tag{3.8}$$

Note that $arg\left(P(j\Omega_{des})\right)$ can be estimated by a simple identification test on the plant. Additionally, one wants to limit the magnitude increase provided by the compensator at the frequency Ω_{des}. It is therefore appropriate to select small values of the normalized frequency u. For any given pair (u, m), the compensator order is to be chosen according to

$$\nu = \frac{\Delta\phi}{\phi^*(u,m)} , \tag{3.9}$$

where $\phi^*(u, m)$ is the phase lead read in the normalized phase chart in correspondence of the (u, m) pair. The compensator parameters tuning formulas are

$$\lambda = \frac{u}{\Omega_{des}}, \quad x = \frac{1}{m}, \quad k_c = \frac{1}{x^\nu} . \tag{3.10}$$

The gain k_c is selected to assign the unit value to the compensator low frequency gain.

In some cases the higher magnitude attenuation provided by the plant at the desired frequency can effectively compensate for the magnitude amplification caused by the controller. In this case one can select higher values of u. A particularly simple and systematic tuning procedure can be devised if one selects to work at the point at which the compensator gives the maximal phase lead ϕ_m. In this case, the design can be done analytically without employing the normalized charts. Given $x \in (0, 1)$, the parameter λ and the fractional order ν are set according to the relations

$$\lambda = \frac{1}{\Omega_{des}\sqrt{x}}, \qquad \nu = \frac{\Delta\phi}{arcsin\left(\frac{1-x}{1+x}\right))}. \tag{3.11}$$

Values of x maximizing the equivalent gain

$$k_n^{DF} = -\frac{1}{2\mathrm{Re}\,(\,P(j\Omega_{des})C(j\Omega_{des}))} \tag{3.12}$$

should be preferred. The effect of x on k_n^{DF} is not immediate to capture, however, in most cases, one observes that the equivalent gain increases with increasing x. On the other hand, values of x too close to 1 cause an excessive increase in the order of the compensator and a peak-wise phase response with poor robustness properties. So, a compromise has to be found.

4. Design example and simulation results

Consider the delayed fractional-order transfer function

$$P(s) = \frac{1.01}{s^{3\alpha} + 1.2s^{2\alpha} + 1.21s^{\alpha} + 1.01}e^{-0.1s} \tag{4.1}$$

with commensurate order $\alpha = 0.8$, and with the coefficients chosen to assign the s^α poles of the system the locations $\{p_1, p_2, p_3\} = \{-1, -0.1 \pm i, \}$ in order to make system (4.1) asymptotically stable in the open loop according to Matignon stability condition. Let a standard relay with unit magnitude and without hysteresis be used.

4.1. Oscillation analysis

The harmonic response of the plant is shown in the Bode plots of Figure 5. The phase plot puts into evidence that the frequency of self excited oscillations predicted by the DF method, the frequency at which the phase crosses -180 degrees, is near $2rad/sec$. A precise evaluation of frequency Ω^{DF} according to (2.5) (specialized to the current value of $b = 0$) and of the corresponding predicted oscillation amplitude (2.6) gives

$$\Omega^{DF} = 1.97\ rad/sec, \qquad a_y^{DF} = 0.2359. \tag{4.2}$$

The LPRS approach is exploited to perform a more accurate analysis. The imaginary part of the LPRS function (2.8) versus frequency is shown in the Figure 6. The frequency value Ω^{LPRS} has to be evaluated in accordance with the relation (2.9), which gives

$$\Omega^{LPRS} = 1.93\ rad/sec. \tag{4.3}$$

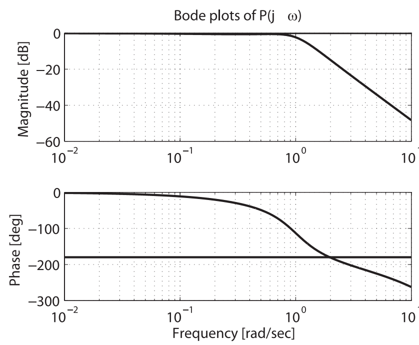

FIGURE 5. Harmonic response of the plant

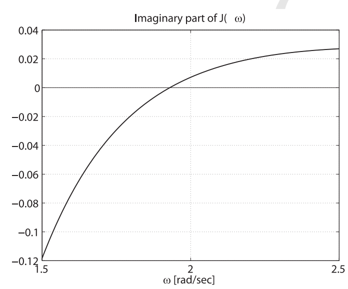

FIGURE 6. Imaginary part of the LPRS function

The predicted amplitudes of the first and third harmonics of oscillation are derived in accordance with (2.11)-(2.12) as

$$a_{y,1}^{LPRS} = \frac{4}{\pi} \left\| P(j\Omega^{LPRS}) \right\| = 0.2478\,, \tag{4.4}$$

$$a_{y,3}^{LPRS} = \frac{4}{3\pi} \left\| P(j3\Omega^{LPRS}) \right\| = 0.00598\,. \tag{4.5}$$

We now present the simulation results of the closed loop system. An integration method of the forward-Euler type was used to solve the next state-space formulation of the closed loop system under investigation:

$$\begin{array}{rcl} {}^{RL}D^{\alpha}x_1 & = & x_2 \\ {}^{RL}D^{\alpha}x_2 & = & x_3 \\ {}^{RL}D^{\alpha}x_3 & = & -1.01x_1 - 1.21x_2 - 1.2x_3 \\ & & \quad -1.01 sign(x_1(t-\delta))\,. \end{array} \tag{4.6}$$

The sampling step of the forward-Euler algorithm is set to $\tau = 0.002s$ to achieve an accurate solution. The time evolution of the output $y(t) = x_1(t)$ is displayed in the next Figure 7.

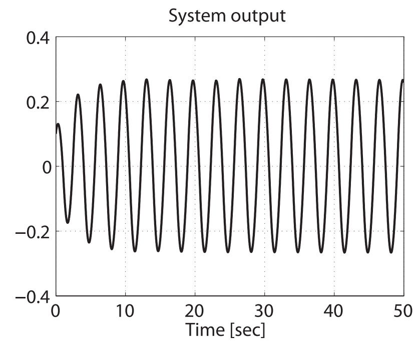

FIGURE 7. Iime evolution of the output $y(t) = x_1(t)$

The oscillation frequency of the output was continuously inspected by an ad-hoc mechanism that detects the distance between adjacent zero crossings and computes the frequency on the basis of the time elapsed between them. The plot in Figure 8 shows that the frequency estimate (4.3) provided by the LPRS method is more accurate that that in (4.2) provided by the DF approach.

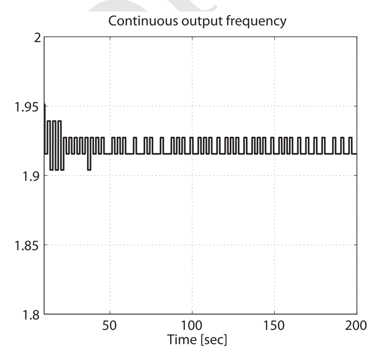

FIGURE 8. Continuous measure of the output frequency.

In order to carefully investigate the spectral content of the output, thereby checking the accuracy of the predicted oscillation magnitudes, an accurate FFT analysis was made, whose result is displayed in the Figures 9. The amplitude of the third harmonics is much smaller than that of the first, that is why the upper plot does not allow for appreciating it. However, the lower zoomed plots showing the peak values of the first and third harmonics show that the oscillation amplitude estimates (4.4)-(4.5) provided by the LPRS method are both very accurate.

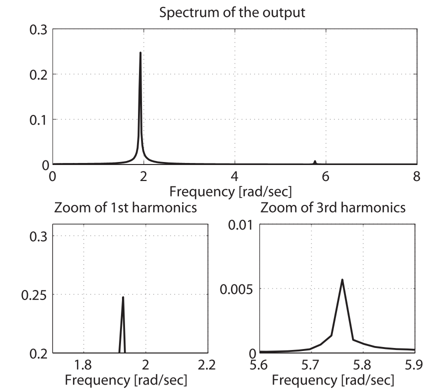

FIGURE 9. Continuous measure of the output frequency.

4.2. Oscillation shaping

Let us now address the problem of oscillation shaping. We aim to rise up the principal frequency of oscillation to the desired value $\Omega_{des} = 6 rad/sec$. The previously outlined procedures are applied to derive the parameters involved in the tuning of the lead compensator. The required phase lead to be provided by the compensator at the desired oscillation frequency is

$$\Delta\phi = -180^\circ - arg\,(P(j\Omega_{des})) = 53.7^\circ\,. \tag{4.7}$$

By using the normalized phase and magnitude charts of the integer order compensator (3.7), and taking into account the previously given considerations about the convenience of working with sufficiently small values of the normalized frequency u, we select the (u, m) pair as $(u, m)= (0.8, 12)$, in such a way that $\phi^*(u, m) \approx 35^\circ$. This working point appears to be a reasonable compromise between the need of obtaining more than 50 degrees of phase lead without having an excessive magnitude increase. The resulting compensator order is to be chosen according to (3.9), as the ratio between the required phase lead and the value read on the normalized charts, i.e.

$$\nu = \frac{\Delta\phi}{\phi^*(u, m)} = 1.53\,. \tag{4.8}$$

The compensator parameters tuning formulas (3.10) specialize to

$$\lambda = \frac{u}{\Omega_{des}} = 0.133, \quad x = \frac{1}{m} = 0.08, \quad k_c = \frac{1}{x^\nu} = 46.28\,. \tag{4.9}$$

The Bode diagram of the compensated system (see Fig. 10) meets the design requirement of having 180 degrees of phase lag at the desired oscillation frequency. The simulated output response (see Fig. 11) shows an oscillation frequency $\Omega \approx 5 rad/sec$, and an amplitude of 0.032. LPRS analysis predicts the simulated values with higher accuracy. Estimated parameters by DF and LPRS method are given as follows

$$\Omega^{DF} = 5.99\ rad/sec, \qquad a_y^{DF} = 0.023. \tag{4.10}$$
$$\Omega^{LPRS} = 4.61\ rad/sec, \qquad a_{y,1}^{LPRS} = 0.036\,. \tag{4.11}$$

Again, the LPRS method outperforms the DF. Therefore, to compensate for the discrepancy between the imposed and observed frequency, a new design iteration should be adopted, rising up the desired phase lead of the compensator.

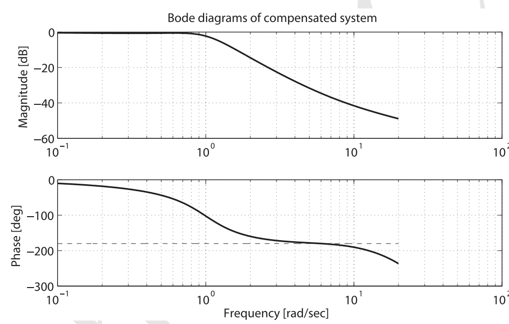

FIGURE 10. Bode diagrams of compensated system.

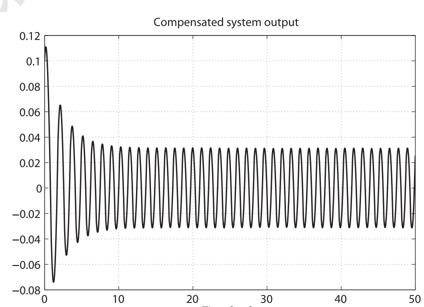

FIGURE 11. Output of compensated system.

5. Conclusions

The output oscillations occurring in relay system with linear fractional-order plants have been analyzed in the frequency domain, and a design method to adjust the corresponding frequency is described, based on a dynamical compensator. Among the most interesting directions for future improvements, different controllers than standard relays can be studied (e.g., the "generalized suboptimal" relay, cf. [5]). Additionally, analytical computation of the LPRS of some basic low-order fractional order processes would be useful to devise enhanced, more accurate, constructions of compensator design for oscillation shaping.

Acknowledgements

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