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A TIME-DEPENDENT OPTIMAL HARVESTING PROBLEM WITH MEASURE-VALUED SOLUTIONS*

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Abstract. The paper is concerned with the optimal harvesting of a marine park, which is described by a parabolic heat equation with Neumann boundary conditions and a nonlinear source term. We consider a cost functional, which is linear with respect to the control; hence the optimal solution can belong to the class of measure-valued control strategies. For each control function, we prove existence and stability estimates for solutions of the parabolic equation. Moreover, we prove the existence of an optimal solution. Finally, some numerical simulations conclude the paper.

Key words. optimal control, differential games, measure-valued solutions, fish harvest

AMS subject classifications. 35Q93, 35K61, 49J20, 49N25, 49N90

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1. Introduction.

In this paper we consider an optimal control problem associated to a model for the evolution of fishes or of a group of individuals in a multidimensional domain $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$. Denote by $\varphi = \varphi(t,x)$ the density of fish at time t at the point $x \in \Omega$. In absence of fishing activity, we assume that the fish population evolves according to the parabolic heat equation with source term

$$\partial_t \varphi \;=\; \Delta\varphi + g(t,x,\varphi), \qquad t > 0,\; x \in \Omega\,,$$

with Neumann boundary conditions

$$\partial_\nu \varphi \;=\; 0, \qquad t > 0,\; x \in \partial\Omega, \tag{1.1}$$

and with the initial condition

$$\varphi(0,x) = \varphi_0(x), \qquad x \in \Omega. \tag{1.2}$$

Here $\nu = \nu(x)$ denotes the unit outer normal vector to the set Ω at the point $x \in \partial\Omega$. A typical choice for the source term is the logistic type function

$$g(t,x,\varphi) \;=\; \alpha(t,x)\big(h(t,x) - \varphi\big)\,\varphi,$$

where $h(t,x)$ denotes the maximum fish population supported by the habitat at x a time t, while α is a reproduction speed.

Denoting by $u = u(t,x)$ the intensity of harvesting conducted by a fishing company, we assume that, in the presence of this harvesting activity, the density φ evolves according to the partial differential equation

$$\partial_t \varphi \;=\; \Delta\varphi + g(t,x,\varphi) - \varphi u, \qquad t > 0,\; x \in \Omega, \tag{1.3}$$

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together with the Neumann boundary conditions (1.1) and the initial condition (1.2). The function u can be considered as a control function. Define also the cost functional

$$J(u) \;=\; \int_0^T \int_\Omega \varphi(t,x)u(t,x)dtdx - \Psi\left(\int_0^T \int_\Omega c(t,x)\,u(t,x)\,dtdx\right), \tag{1.4}$$

where $\varphi = \varphi(t,x)$ is the solution of (1.3) in correspondence of the harvesting strategy $u = u(t,x)$, $c(t,x)$ is the cost for a unit of fishing effort at time t and at the location $x \in \Omega$, and Ψ is a suitable function. The functional J is the net gain associated to the control u, being composed of two pieces representing respectively the income and the cost of the strategy. Various choices for the function c are meaningful. The simple one is c constant. Another possibility is to have a function c, which increases with respect to the distance from a point $\bar{x}$, representing the base of the fishing company. Note also that, setting $c = \infty$ on a set $\Omega_0 \subset \Omega$, one can consider regions where fishing is not permitted. As regards the control function u, it is reasonable to assume that it satisfies constraints of the form

$$u(t,x) \;\geq\; 0, \qquad\qquad \int_0^T \int_\Omega b(t,x)\,u(t,x)\,dtdx \;\leq\; 1,$$

for some nonnegative function b. The first constraint imposes that there is indeed a fishing activity and not a process of a population's increment, while the second one determines the maximum amount of harvesting power within the capabilities of the company. In practice, this may depend on the number of fishermen and on the size of fishing boats available.

Our interest is both the existence of a solution for problem (1.3) and the existence of optimal solutions with respect to the cost (1.4). Since the first part of the cost functional (1.4) has only linear growth w.r.t. u, there is no guarantee that the optimal strategy u will lie in the space $L^1((0,T) \times \Omega)$. Indeed, existence of optimal solutions will be proved within the larger space of functions with values in the space of the bounded Radon measures supported on the closure $\overline{\Omega}$ of the domain. An example where the optimal control is indeed a measure was constructed in [6] for the stationary case. We remark that a quadratic harvesting cost such as

$$\int_0^T \int_\Omega c(t,x)u^2(t,x)\,dtdx,$$

entirely natural from a mathematical point of view, guarantees that the optimal strategy is indeed a function. However, the linear cost provides a more realistic model. We also remark that most of the theory of partial differential equations with measure-valued right-hand side is concerned with elliptic equations with Dirichlet boundary conditions. In our fishery model, the Neumann boundary conditions (1.1) yield a more appropriate model.

Problems of optimal harvesting of a marine park, governed by a semilinear elliptic or parabolic equation, have been the subject of several investigations; see, for example, [1, 2, 7, 16, 17, 22]. The use of measure-valued strategies was first considered in the paper [5]. In [5] the authors study the well-posedness of the stationary elliptic problem on a real interval and prove the existence and uniqueness of the optimal control. Moreover, they considered a differential game, in which several fishing companies want to maximize a cost functional, and proved the existence of Nash equilibria. Those results have been extended to the time-dependent case in [12]. In [6, 12] necessary conditions on the Nash equilibria of a differential game were addressed. Finally, in [4] the elliptic problem in a multidimensional domain is considered. Control and optimization problems in measure spaces have been considered also in [8, 9, 10, 11, 20, 25].

The paper is organized as follows. Section 2 deals with a parabolic problem with smooth coefficients. In particular we prove existence and uniqueness of a solution for such a problem, which is the preliminary step for considering measure-valued controls. We use here the classical techniques of sub- and supersolutions. In section 3 we introduce a definition of weak solutions to (1.3), when the control u is a measure-valued function, and we prove existence and stability estimates. The proof is based on approximating the measure-valued control by smooth functions and proving that the approximate sequence of solutions has a limit point, which is the solution of the original problem. In section 4 we establish the existence of an optimal measure-valued control, by using compactness arguments in the space of control functions and the results of section 3. Finally, in section 5 we present some numerical simulations.

2. A parabolic problem with smooth coefficients. This section is dedicated to various technical results concerning the initial-boundary value problem

$$\begin{cases} \partial_t\varphi = \Delta\varphi - a(t,x)\varphi + g(t,x,\varphi), & 0<t<T,\ x\in\Omega,\\ \partial_\nu\varphi = 0, & 0<t<T,\ x\in\partial\Omega,\\ \varphi(0,x)=\varphi_0(x), & x\in\Omega. \end{cases} \tag{2.1}$$

Let us introduce the following hypotheses on Ω, g, and φ_0:

(H.1) The domain $\Omega \subset \mathbb{R}^N$ is open, bounded, connected with smooth boundary, and denoted by $\partial\Omega$.

(H.2) The nonlinear source term g can be written as $g(t,x,\varphi)=f(t,x,\varphi)\varphi$, where $f:[0,T]\times\overline{\Omega}\times\mathbb{R}\to\mathbb{R}$ is a smooth function satisfying

$$f(\cdot,\cdot,0)\in L^\infty((0,T)\times\Omega), \tag{2.2}$$

$$-\alpha_0 \le \partial_\varphi f(t,x,\varphi) < 0 \qquad \text{for all } (t,x,\varphi)\in[0,T]\times\overline{\Omega}\times\mathbb{R}, \tag{2.3}$$

$$f(t,x,\varphi) > 0 \qquad \text{if and only if} \quad \varphi < h(t,x), \tag{2.4}$$

where $h:[0,T]\times\overline{\Omega}\mapsto\mathbb{R}$ is a smooth bounded function such that $h\ge h_*$ and α_0, $h_*>0$ are constants.

(H.3) The initial datum φ_0 satisfies

$$\varphi_0\in L^\infty(\Omega), \qquad 0<\varphi_*\le\varphi_0(x) \qquad \text{for all } x\in\overline{\Omega},$$

for some constant φ_*.

For $\varepsilon>0$, define the sets

$$\Omega_\varepsilon = \left\{x\in\mathbb{R}^N : d(x,\Omega)<\varepsilon\right\}, \qquad \Omega_{-\varepsilon}=\{x\in\Omega : d(x,\partial\Omega)>\varepsilon\}.$$

LEMMA 2.1. *Assume* (H.1), (H.2), *and* (H.3). *Fix $T>0$ and a function $a\in C^\infty\left([0,T]\times\overline{\Omega}\right)$ such that $a(t,x)\ge 0$ for every $(t,x)\in[0,T]\times\Omega$. The initial-boundary value problem* (2.1) *admits a unique classical solution φ such that*

$$0\le\varphi(t,x)\le M := \max\left\{\|h\|_{L^\infty((0,T)\times\Omega)}, \|\varphi_0\|_{L^\infty(\Omega)}\right\} \tag{2.5}$$

for every $(t,x)\in[0,T]\times\Omega$.

Proof. Clearly 0 and M are respectively a subsolution and a supersolution to (2.1). By the comparison principle (see [24, Theorem 9.7] and [19, Theorem 2.16] for the case of Dirichlet boundary conditions), we deduce that (2.5) holds for every classical solution φ.

Moreover, local in time existence of solutions for (2.1) can be deduced by [24, Theorem 8.2] or by [19, Theorem 7.8]. The a priori estimates (2.5) imply the global existence of a solution. Finally, again by the comparison principle, uniqueness of solution is granted. □

LEMMA 2.2. *Assume* (H.1), (H.2), *and* (H.3). *Fix $T > 0$ and a function $a \in C^\infty\left([0,T]\times\overline{\Omega}\right)$ such that $a(t,x) \geq 0$ for every $(t,x) \in [0,T]\times\Omega$. The unique classical solution φ to* (2.1) *satisfies, for every $0 \le s \le t \le T$, the estimate*

$$\|\varphi(t,\cdot)\|^2_{L^2(\Omega)} + 2\int_s^t \|\nabla\varphi(\tau,\cdot)\|^2_{L^2(\Omega)}\,d\tau \le \|\varphi(s,\cdot)\|^2_{L^2(\Omega)} + C(t-s), \tag{2.6}$$

where

$$C = M^2\,\|f(\cdot,\cdot,0)\|_{L^\infty((0,T)\times\Omega)}\,|\Omega|. \tag{2.7}$$

Proof. Using (2.1) and Lemma 2.1, we obtain that

$$\begin{aligned}
\frac{d}{dt}\int_\Omega \frac{\varphi^2}{2}dx &= \int_\Omega \varphi\partial_t\varphi dx\\
&= \int_\Omega \varphi\Delta\varphi dx \underbrace{- \int_\Omega \varphi^2 a(t,x)dx}_{\le 0} + \int_\Omega \varphi g(t,x,\varphi)dx\\
&\le -\int_\Omega |\nabla\varphi|^2 dx + \underbrace{\int_\Omega \varphi^2 f(t,x,\varphi)dx}_{\text{(see (H.2))}}\\
&\le -\int_\Omega |\nabla\varphi|^2 dx + \underbrace{\int_\Omega \varphi^2 f(t,x,0)dx}_{\text{(see (2.2), (2.3), (2.5))}}\\
&\le -\int_\Omega |\nabla\varphi|^2 dx + M^2\,\|f(\cdot,\cdot,0)\|_{L^\infty((0,T)\times\Omega)}\,|\Omega|.
\end{aligned}$$

Integrating over (s,t) we obtain (2.6). □

3. A parabolic problem with measure-valued coefficients. This section is dedicated to the well-posedness of the initial-boundary value problem with the time-dependent measure-valued coefficient μ_t

$$\begin{cases}\partial_t\varphi = \Delta\varphi - \varphi\mu_t + g(t,x,\varphi), & 0<t<T,\ x\in\Omega,\\ \partial_\nu\varphi = 0, & 0<t<T,\ x\in\partial\Omega,\\ \varphi(0,x) = \varphi_0(x), & x\in\Omega.\end{cases} \tag{3.1}$$

We introduce the following hypothesis on the coefficient μ:

(H.4) There exist $\Phi_0 \in L^\infty\left(0,T;L^1\left(\mathbb{R}^N\right)\right)$ and $\Phi_1,\dots,\Phi_N \in L^\infty\left(0,T;L^2\left(\mathbb{R}^N\right)\right)$ such that the following conditions hold:

(1) $\mu_t = \Phi_0(t) - \operatorname{div}(\Phi_1(t), \dots, \Phi_N(t))$ in the sense of distributions for a.e. $t \in [0,T]$. More precisely, for every $v \in C_c^\infty(\mathbb{R}^N)$ and for a.e. $t \in [0,T]$

$$\int_{\mathbb{R}^N} v(x) d\mu_t(x) = \int_{\mathbb{R}^N} v(x)\Phi_0(t,x)dx + \sum_{j=1}^N \int_{\mathbb{R}^N} \Phi_j(t,x) D_{x_j} v(x) dx.$$

(2) $\Phi_0(t) - \operatorname{div}(\Phi_1(t), \dots, \Phi_N(t)) \geq 0$ in the sense of distributions for a.e. $t \in [0,T]$, i.e., for every $v \in C_c^\infty(\mathbb{R}^N)$ with $v \geq 0$ and for a.e. $t \in [0,T]$

$$\int_{\mathbb{R}^N} v(x)\Phi_0(t,x)dx + \sum_{j=1}^N \int_{\mathbb{R}^N} \Phi_j(t,x) D_{x_j} v(x) dx \geq 0.$$

(3) There exists $\bar{\varepsilon} > 0$ such that $\Phi_j(t,x) = 0$, for every $t \in [0,T]$, $x \in \mathbb{R}^N \setminus \Omega_{-\bar{\varepsilon}}$, $j \in \{0, \dots, N\}$.

Remark 3.1. Note that (H.4) implies that $\mu_t \in \mathcal{M}(\Omega)$ for a.e. $t \in [0,T]$, where $\mathcal{M}(\Omega)$ denotes the set of the bounded Radon measures on Ω. Let $v \in C_c^\infty(\mathbb{R}^N)$ and consider a function $\tilde{v} \in C_c^\infty(\Omega)$ such that $\tilde{v}(x) = v(x)$ for every $x \in \Omega_{-\bar{\varepsilon}}$. By (H.4) we have that

$$\begin{aligned}
\int_{\mathbb{R}^N} v(x) d\mu_t(x) &= \int_{\mathbb{R}^N} v(x)\Phi_0(t,x)dx + \sum_{j=1}^N \int_{\mathbb{R}^N} \Phi_j(t,x) D_{x_j} v(x) dx \\
&= \int_{\Omega_{-\bar{\varepsilon}}} v(x)\Phi_0(t,x)dx + \sum_{j=1}^N \int_{\Omega_{-\bar{\varepsilon}}} \Phi_j(t,x) D_{x_j} v(x) dx \\
&= \int_{\Omega_{-\bar{\varepsilon}}} \tilde{v}(x)\Phi_0(t,x)dx + \sum_{j=1}^N \int_{\Omega_{-\bar{\varepsilon}}} \Phi_j(t,x) D_{x_j} \tilde{v}(x) dx \\
&= \int_{\Omega} \tilde{v}(x) d\mu_t(x).
\end{aligned}$$

In particular, for every $v \in H^1(\Omega) \cap L^\infty(\Omega)$ and for a.e. $t \in [0,T]$, the integral

$$\int_{\Omega} v(x) d\mu_t(x) = \int_{\Omega} \Phi_0(t,x) v(x) dx + \sum_{i=1}^N \int_{\Omega} \Phi_i(t,x) D_{x_i} v(x) dx$$

is well defined.

Remark 3.2. Similarly to [4], one can also consider a function $\mu \in L^\infty(0,T; \mathcal{M}(\Omega))$ such that $\mu_t(A) = 0$ for a.e. $t \in [0,T]$ and for every Borel subset A of Ω with zero capacity. In this case, for a.e. $t \in [0,T]$, it is possible to write μ_t in a nonunique way as a sum $\widetilde{\mu}_t + \overline{\mu}_t$ with $\widetilde{\mu}_t \in L^1(\Omega)$ and $\overline{\mu}_t \in H^{-1}(\Omega)$; see [15, Proposition 2.5]. The nonuniqueness of the decomposition introduces the additional difficulty about the choice of $\widetilde{\mu}_t$ and of $\overline{\mu}_t$. Moreover, it is not clear whether the two functions $t \mapsto \widetilde{\mu}_t$ and $t \mapsto \overline{\mu}_t$ are strongly measurable.

In the following we shall use the following definition of solution.

DEFINITION 3.3. *A function* $\varphi : [0,T] \times \Omega \to \mathbb{R}$ *is a solution to* (3.1) *if*

$$\varphi \in L^\infty((0,T) \times \Omega) \cap L^2(0,T; H^1(\Omega))$$

and for every test function $v \in H^1\left(0,T;H^1(\Omega)\right) \cap L^\infty\left((0,T)\times\Omega\right)$ *such that* $v(T)=0$ *we have*

$$\begin{aligned}\int_0^T\int_\Omega \left(\partial_t v\varphi - \nabla v\cdot\nabla\varphi\right)dxdt - \int_0^T\int_\Omega v\varphi d\mu_t(x)dt \\ + \int_0^T\int_\Omega vg(t,x,\varphi)dxdt + \int_\Omega v(0,x)\varphi_0(x)dx = 0.\end{aligned} \tag{3.2}$$

The main result of this section is the following.

Theorem 3.4. *Assume that assumptions* (H.1), (H.2), (H.3), *and* (H.4) *hold. The initial-boundary value problem* (3.1) *admits a semigroup of solutions* S, *in the sense of Definition* 3.3, *such that if* $\varphi(t,x) = S_t(\varphi_0)(x)$, *then*

$$\begin{aligned}&0 \le \varphi(t,x) \le M, \qquad (t,x)\in[0,T]\times\overline{\Omega},\\ &\|\varphi(t,\cdot)\|^2_{L^2(\Omega)} + 2\int_0^t \|\nabla\varphi(s,\cdot)\|^2_{L^2(\Omega)}\,ds \le \|\varphi_0\|^2_{L^2(\Omega)} + Ct, \quad 0\le t\le T,\end{aligned} \tag{3.3}$$

where

$$C = M^2\,\|f(\cdot,\cdot,0)\|_{L^\infty((0,T)\times\Omega)}\,|\Omega|,$$

and M *is the constant defined in* (2.5). *Moreover, if* $\widehat{\varphi}$ *is the solution of* (3.1) *obtained in correspondence of the initial condition* $\widehat{\varphi}_0$, *i.e.,* $\widehat{\varphi} = S(\widehat{\varphi}_0)$, *the following stability estimate holds:*

$$\|\varphi(t,\cdot)-\widehat{\varphi}(t,\cdot)\|^2_{L^2(\Omega)} + 2e^{Kt}\int_0^t e^{-Ks}\,\|\nabla(\varphi(s,\cdot)-\widehat{\varphi}(s,\cdot))\|^2_{L^2(\Omega)}\,ds \le \|\varphi_0-\widehat{\varphi}_0\|^2_{L^2(\Omega)}\,e^{Kt} \tag{3.4}$$

for every $0\le t\le T$, *where*

$$K = \alpha_0 M + \|f(\cdot,\cdot,0)\|_{L^\infty((0,T)\times\Omega)}\,. \tag{3.5}$$

The strategy of the proof is the following:

- We approximate the measure μ_t by a sequence of smooth functions and we solve the approximated problems.
- Using some a priori estimates, we are able to pass to the limit obtaining a solution to (3.1).
- We improve the previous compactness argument showing that all the approximated sequence converges. In this way we are able to define a semigroup of solutions.

Note that in the statement of the theorem, there is not a uniqueness result for (3.1). Indeed the uniqueness issue is very delicate and we only prove the uniqueness of the solutions obtained as limit of approximations; see [13]. In general, for parabolic equations with measure-valued coefficients, a fine regularization technique such as Landes regularization [21] is necessary when dealing with nonsmooth test functions; see [14, 18, 26].

Proof of Theorem 3.4. Given $\bar{\varepsilon} > 0$ of hypothesis (H.4), by using convolution, there exist sequences of functions $\Phi_{0,n}$ and $\Phi_{i,n}$ ($i\in\{1,\dots,N\}$) in $C^\infty([0,T]\times\Omega)$ such that

(1) $\Phi_{0,n}(t,x) - \sum_{i=1}^{N} D_{x_i}\Phi_{i,n}(t,x) \geq 0$ for every $n \in \mathbb{N}$, $t \in [0,T]$ and $x \in \Omega$;
(2) $\Phi_{i,n}(t,x) = 0$ for every $t \in [0,T]$, $i \in \{0,\dots,N\}$, and $x \in \Omega \setminus \Omega_{-\frac{\bar{\varepsilon}}{2}}$;
(3) $\Phi_{0,n} \to \Phi_0$ as $n \to \infty$ in $L^p\left(0,T;L^1(\Omega)\right)$ for every $1 \leq p < \infty$;
(4) for every $i \in \{1,\dots,N\}$ and for every $1 \leq p < \infty$, $\Phi_{i,n} \to \Phi_i$ in $L^p\big(0,T;L^2(\Omega)\big)$ as $n \to \infty$;
(5) there exists $C > 0$ such that $\|\Phi_{0,n}\|_{L^\infty(0,T;L^1(\Omega_{\bar{\varepsilon}}))} + \sum_{i=1}^{N} ||D_{x_i}\Phi_{i,n}||_{L^\infty(0,T;L^1(\Omega_{\bar{\varepsilon}}))} \leq C$.

For every $n \in \mathbb{N}$, we consider the sequence of smooth and positive functions

$$a_n(t,x) = \Phi_{0,n}(t,x) - \sum_{i=1}^{N} D_{x_i}\Phi_{i,n}(t,x)$$

that is bounded in $L^\infty(0,T;L^1(\Omega_{\bar{\varepsilon}}))$. Consider, by Lemma 2.1, the classical solution φ_n of the initial-boundary value problem

$$\begin{cases} \partial_t\varphi_n = \Delta\varphi_n - a_n(t,x)\varphi_n + g(t,x,\varphi_n), & 0<t<T,\ x \in \Omega, \\ \partial_\nu\varphi_n = 0, & 0<t<T,\ x \in \partial\Omega, \\ \varphi_n(0,x) = \varphi_0(x), & x \in \Omega. \end{cases} \tag{3.6}$$

By Lemma 2.1, we obtain that, for every $t \in [0,T]$, $x \in \Omega$, and $n \in \mathbb{N}$,

$$0 \leq \varphi_n(t,x) \leq M, \tag{3.7}$$

proving that the sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ is bounded in $L^\infty((0,T)\times\Omega)$. Moreover, by Lemma 2.2 we deduce that, for every $t \in [0,T]$,

$$\|\varphi_n(t)\|^2_{L^2(\Omega)} + 2\int_0^t \|\nabla\varphi_n(s)\|^2_{L^2(\Omega)}\,ds \leq \|\varphi_0\|^2_{L^2(\Omega)} + Ct, \tag{3.8}$$

where C is defined in (2.7); hence the sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ is bounded in $L^\infty(0,T;L^2(\Omega))$ and in $L^2(0,T;H^1(\Omega))$. Therefore there exists a function φ and a subsequence of φ_n (again denoted by φ_n) such that

$$\varphi \in L^\infty((0,T)\times\Omega) \cap L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)), \tag{3.9}$$

$$\varphi_n \rightharpoonup \varphi \text{ weakly in } L^p((0,T)\times\Omega),\ L^2(0,T;H^1(\Omega)) \text{ for every } 1 \leq p < \infty, \tag{3.10}$$

$$\varphi_n \overset{\star}{\rightharpoonup} \varphi \quad \text{weakly-}* \text{ in } L^\infty((0,T)\times\Omega). \tag{3.11}$$

Since

$$\begin{aligned} \partial_t\varphi_n =& \Delta\varphi_n - a_n(t,x)\varphi_n + g(t,x,\varphi_n), \\ \{\Delta\varphi_n\}_n \quad & \text{is bounded in } L^2\left(0,T;H^{-1}(\Omega)\right) \text{ (see (3.10))}, \\ \{a_n\varphi_n\}_n \quad & \text{is bounded in } L^\infty\left(0,T;L^1(\Omega)\right) \text{ (see (3.7))}, \\ \{g(\cdot,\cdot,\varphi_n)\}_n \quad & \text{is bounded in } L^\infty\left((0,T)\times\Omega\right) \text{ (see (3.7) and (H.2))}, \end{aligned}$$

we have that

$$\{\partial_t\varphi_n\}_{n\in\mathbb{N}} \text{ is bounded in } L^1(0,T;H^{-s}(\Omega)) \text{ for every } s > \frac{N}{2}.$$

In light of [27, Corollary 4], we have that

$$\varphi_n \longrightarrow \varphi \text{ strongly in } L^2((0,T)\times\Omega) \tag{3.12}$$

and then, passing to a subsequence, we get that

$$\varphi_n \longrightarrow \varphi \text{ a.e. in } (0,T)\times\Omega. \tag{3.13}$$

Since φ_n is a classical solution to (3.6), then, for every $w \in H^1\left(0,T;H^1(\Omega)\right)$ such that $w(T,\cdot)=0$, it holds that

$$\begin{aligned}\int_0^T\int_\Omega\left(\partial_t w\varphi_n - \nabla w\cdot\nabla\varphi_n\right)dxdt - \int_0^T\int_\Omega a_n(t,x)w\varphi_n dxdt\\ + \int_0^T\int_\Omega wg(t,x,\varphi_n)dxdt + \int_\Omega w(0,x)\varphi_0(x)dx = 0.\end{aligned} \tag{3.14}$$

By using Lebesgue's dominated convergence theorem and by (3.10) and (3.13), passing to the limit as $n\to\infty$ in (3.14), one obtains that φ satisfies (3.2); hence it provides a solution to (3.1). Estimate (3.3) follows passing to the limit as $n\to\infty$ in (3.8).

We now have to prove the stability estimate (3.4). Consider two initial conditions φ_0 and $\widehat{\varphi}_0$ and the sequences

$$\Phi_{0,n},\Phi_{1,n},\ldots,\Phi_{N,n}\in C^\infty\left([0,T]\times\Omega\right), \qquad \widehat{\Phi}_{0,n},\widehat{\Phi}_{1,n},\ldots,\widehat{\Phi}_{N,n}\in C^\infty\left([0,T]\times\Omega\right)$$

such that

(1) $\Phi_{0,n}(t,x)-\sum_{i=1}^N D_{x_i}\Phi_{i,n}(t,x)\geq 0$ and $\widehat{\Phi}_{0,n}(t,x)-\sum_{i=1}^N D_{x_i}\widehat{\Phi}_{i,n}(t,x)\geq 0$ for every $n\in\mathbb{N}$, $t\in[0,T]$ and $x\in\Omega$;
(2) $\Phi_{j,n}(t,x)=0$ and $\widehat{\Phi}_{j,n}(t,x)=0$ for every $t\in[0,T]$ and $x\in\Omega\setminus\Omega_{-\frac{\bar\varepsilon}{2}}$;
(3) $\Phi_{0,n}\to\Phi_0$ and $\widehat{\Phi}_{0,n}\to\Phi_0$ as $n\to\infty$ in $L^p(0,T;L^1(\Omega))$ for every $1\leq p<\infty$;
(4) for every $i\in\{1,\ldots,N\}$ and for every $1\leq p<\infty$, $\Phi_{i,n}\to\Phi_i$ and $\widehat{\Phi}_{i,n}\to\Phi_i$ in $L^p(0,T;L^2(\Omega))$ as $n\to\infty$;
(5) there exists $C>0$ such that $\|\Phi_{0,n}\|_{L^\infty(0,T;L^1(\Omega_{\bar\varepsilon}))}+\sum_{i=1}^N||D_{x_i}\Phi_{i,n}||_{L^\infty(0,T;L^1(\Omega_{\bar\varepsilon}))}\leq C$ and $\|\widehat{\Phi}_{0,n}\|_{L^\infty(0,T;L^1(\Omega_{\bar\varepsilon}))}+\sum_{i=1}^N||D_{x_i}\Phi_{i,n}||_{L^\infty(0,T;L^1(\Omega_{\bar\varepsilon}))}\leq C$.

For every $n\in\mathbb{N}$, we consider the smooth and positive functions

$$a_n(t,x)=\Phi_{0,n}(t,x)-\sum_{i=1}^N D_{x_i}\Phi_{i,n}(t,x), \qquad \widehat{a}_n(t,x)=\widehat{\Phi}_{0,n}(t,x)-\sum_{i=1}^N D_{x_i}\widehat{\Phi}_{i,n}(t,x)$$

and consider, by Lemma 2.1, the classical solutions φ_n and $\widehat{\varphi}_n$, respectively, of

$$\begin{cases}\partial_t\varphi_n=\Delta\varphi_n-a_n(t,x)\varphi_n+g(t,x,\varphi_n), & 0<t<T,\ x\in\Omega,\\ \partial_\nu\varphi_n=0, & 0<t<T,\ x\in\partial\Omega,\\ \varphi_n(0,x)=\varphi_0(x), & x\in\Omega,\end{cases}$$

and of

$$\begin{cases}\partial_t\widehat{\varphi}_n=\Delta\widehat{\varphi}_n-\widehat{a}_n(t,x)\widehat{\varphi}_n+g(t,x,\widehat{\varphi}_n), & 0<t<T,\ x\in\Omega,\\ \partial_\nu\widehat{\varphi}_n=0, & 0<t<T,\ x\in\partial\Omega,\\ \widehat{\varphi}_n(0,x)=\widehat{\varphi}_0(x), & x\in\Omega.\end{cases}$$

We have that

$$\begin{aligned}
\frac{d}{dt}\int_\Omega \frac{(\varphi_n-\widehat{\varphi}_n)^2}{2}dx &= \int_\Omega \partial_t(\varphi_n-\widehat{\varphi}_n)(\varphi_n-\widehat{\varphi}_n)dx\\
&= \int_\Omega (\varphi_n-\widehat{\varphi}_n)\Delta(\varphi_n-\widehat{\varphi}_n)dx - \int_\Omega (\varphi_n-\widehat{\varphi}_n)\left(a_n(t,x)\varphi_n - \widehat{a}_n(t,x)\widehat{\varphi}_n\right)dx\\
&\quad + \int_\Omega (\varphi_n-\widehat{\varphi}_n)(g(t,x,\varphi_n)-g(t,x,\widehat{\varphi}_n))dx\\
&= -\int_\Omega |\nabla(\varphi_n-\widehat{\varphi}_n)|^2\,dx \underbrace{-\int_\Omega a_n(t,x)\,(\varphi_n-\widehat{\varphi}_n)^2\,dx}_{\leq 0}\\
&\quad - \int_\Omega (\varphi_n-\widehat{\varphi}_n)\,\widehat{\varphi}_n\,(a_n(t,x)-\widehat{a}_n(t,x))\,dx + \int_\Omega\int_{\widehat{\varphi}_n}^{\varphi_n}(\varphi_n-\widehat{\varphi}_n)\partial_\varphi g(t,x,\xi)d\xi dx\\
&\leq -\int_\Omega |\nabla(\varphi_n-\widehat{\varphi}_n)|^2\,dx - \int_\Omega (\varphi_n-\widehat{\varphi}_n)\,\widehat{\varphi}_n\left(\Phi_{0,n}(t,x)-\widehat{\Phi}_{0,n}(t,x)\right)dx\\
&\quad + \sum_{i=1}^N \int_\Omega (\varphi_n-\widehat{\varphi}_n)\,\widehat{\varphi}_n D_{x_i}\left(\Phi_{i,n}(t,x)-\widehat{\Phi}_{i,n}(t,x)\right)dx\\
&\quad + \int_\Omega\int_{\widehat{\varphi}_n}^{\varphi_n}(\varphi_n-\widehat{\varphi}_n)f(t,x,\xi)d\xi dx + \int_\Omega\int_{\widehat{\varphi}_n}^{\varphi_n}(\varphi_n-\widehat{\varphi}_n)\xi\partial_\varphi f(t,x,\xi)d\xi dx.
\end{aligned}$$

By using the divergence theorem on the term

$$\sum_{i=1}^N \int_\Omega (\varphi_n-\widehat{\varphi}_n)\,\widehat{\varphi}_n D_{x_i}\left(\Phi_{i,n}(t,x)-\widehat{\Phi}_{i,n}(t,x)\right)dx$$

we deduce that

$$\begin{aligned}
\frac{d}{dt}\int_\Omega \frac{(\varphi_n-\widehat{\varphi}_n)^2}{2}dx \leq &-\int_\Omega |\nabla(\varphi_n-\widehat{\varphi}_n)|^2\,dx + K\int_\Omega (\varphi_n-\widehat{\varphi}_n)^2\,dx\\
&- \int_\Omega (\varphi_n-\widehat{\varphi}_n)\,\widehat{\varphi}_n\left(\Phi_{0,n}(t,x)-\widehat{\Phi}_{0,n}(t,x)\right)dx\\
&- \sum_{i=1}^N \int_\Omega D_{x_i}\left[(\varphi_n-\widehat{\varphi}_n)\,\widehat{\varphi}_n\right]\left(\Phi_{i,n}(t,x)-\widehat{\Phi}_{i,n}(t,x)\right)dx,
\end{aligned}$$

where K is defined in (3.5). Denoting

$$\begin{aligned}
\gamma_n(t) = &-\int_\Omega (\varphi_n-\widehat{\varphi}_n)\,\widehat{\varphi}_n\left(\Phi_{0,n}(t,x)-\widehat{\Phi}_{0,n}(t,x)\right)dx\\
&- \sum_{i=1}^N \int_\Omega D_{x_i}\left[(\varphi_n-\widehat{\varphi}_n)\,\widehat{\varphi}_n\right]\left(\Phi_{i,n}(t,x)-\widehat{\Phi}_{i,n}(t,x)\right)dx
\end{aligned}$$

and $z_n(t) = \int_\Omega \frac{(\varphi_n-\widehat{\varphi}_n)^2}{2}dx$, we get that

$$\frac{d}{dt}z_n(t) \leq -\int_\Omega |\nabla(\varphi_n-\widehat{\varphi}_n)|^2\,dx + Kz_n(t) + \gamma_n(t).$$

By Gronwall's lemma, we deduce that

$$z_n(t) \leq e^{kt}z_n(0) - \int_0^t e^{K(t-s)}\int_\Omega |\nabla(\varphi_n-\widehat{\varphi}_n)|^2\,dxds + \int_0^t e^{K(t-s)}\gamma_n(s)ds;$$

hence

$$\int_\Omega \frac{(\varphi_n-\widehat{\varphi}_n)^2}{2}dx \le e^{kt}\int_\Omega \frac{(\varphi_0-\widehat{\varphi}_0)^2}{2}dx - \int_0^t e^{K(t-s)}\int_\Omega \left|\nabla\left(\varphi_n-\widehat{\varphi}_n\right)\right|^2 dxds + \int_0^t e^{K(t-s)}\gamma_n(s)ds. \tag{3.15}$$

Now

$$\begin{aligned}
|\gamma_n(t)| &\le 2M^2\left\|\Phi_{0,n}(t,\cdot)-\widehat{\Phi}_{0,n}(t,\cdot)\right\|_{L^1(\Omega)} \\
&\quad + \sum_{i=1}^N \left\|D_{x_i}\left[\left(\varphi_n-\widehat{\varphi}_n\right)\widehat{\varphi}_n\right]\right\|_{L^2(\Omega)}\left\|\Phi_{i,n}(t,\cdot)-\widehat{\Phi}_{i,n}(t,\cdot)\right\|_{L^2(\Omega)} \\
&\le 2M^2\left\|\Phi_{0,n}(t,\cdot)-\widehat{\Phi}_{0,n}(t,\cdot)\right\|_{L^1(\Omega)} \\
&\quad + M\sum_{i=1}^N \left\|D_{x_i}\left(\varphi_n-\widehat{\varphi}_n\right)\right\|_{L^2(\Omega)}\left\|\Phi_{i,n}(t,\cdot)-\widehat{\Phi}_{i,n}(t,\cdot)\right\|_{L^2(\Omega)} \\
&\quad + 2M\sum_{i=1}^N \left\|D_{x_i}\widehat{\varphi}_n\right\|_{L^2(\Omega)}\left\|\Phi_{i,n}(t,\cdot)-\widehat{\Phi}_{i,n}(t,\cdot)\right\|_{L^2(\Omega)} \\
&\le 2M^2\left\|\Phi_{0,n}(t,\cdot)-\widehat{\Phi}_{0,n}(t,\cdot)\right\|_{L^1(\Omega)} \\
&\quad + M\left\|\varphi_n-\widehat{\varphi}_n\right\|_{H^1(\Omega)}\sum_{i=1}^N \left\|\Phi_{i,n}(t,\cdot)-\widehat{\Phi}_{i,n}(t,\cdot)\right\|_{L^2(\Omega)} \\
&\quad + 2M\left\|\widehat{\varphi}_n\right\|_{H^1(\Omega)}\sum_{i=1}^N \left\|\Phi_{i,n}(t,\cdot)-\widehat{\Phi}_{i,n}(t,\cdot)\right\|_{L^2(\Omega)}.
\end{aligned}$$

Therefore, for every $t\in[0,T]$,

$$\begin{aligned}
\left|\int_0^t e^{K(t-s)}\gamma_n(s)ds\right| &\le 2M^2\int_0^t \left\|\Phi_{0,n}(s,\cdot)-\widehat{\Phi}_{0,n}(s,\cdot)\right\|_{L^1(\Omega)}ds \\
&\quad + M\sum_{i=1}^N\int_0^t \left\|\varphi_n-\widehat{\varphi}_n\right\|_{H^1(\Omega)}\left\|\Phi_{i,n}(s,\cdot)-\widehat{\Phi}_{i,n}(s,\cdot)\right\|_{L^2(\Omega)}ds \\
&\quad + 2M\sum_{i=1}^N\int_0^t \left\|\widehat{\varphi}_n\right\|_{H^1(\Omega)}\left\|\Phi_{i,n}(s,\cdot)-\widehat{\Phi}_{i,n}(s,\cdot)\right\|_{L^2(\Omega)}ds \\
&\le 2M^2\left\|\Phi_{0,n}-\widehat{\Phi}_{0,n}\right\|_{L^1(0,T;L^1(\Omega))} \\
&\quad + M\left\|\varphi_n-\widehat{\varphi}_n\right\|_{L^2(0,T;H^1(\Omega))}\sum_{i=1}^N\left\|\Phi_{i,n}-\widehat{\Phi}_{i,n}\right\|_{L^2(0,T;L^2(\Omega))} \\
&\quad + 2M\left\|\widehat{\varphi}_n\right\|_{L^2(0,T;H^1(\Omega))}\sum_{i=1}^N\left\|\Phi_{i,n}-\widehat{\Phi}_{i,n}\right\|_{L^2(0,T;L^2(\Omega))}.
\end{aligned}$$

By the previous analysis, the terms $\|\varphi_n-\widehat{\varphi}_n\|_{L^2(0,T;H^1(\Omega))}$ and $\|\widehat{\varphi}_n\|_{L^2(0,T;H^1(\Omega))}$ are uniformly bounded and so we deduce that, for every $t\in[0,T]$,

$$\left|\int_0^t e^{K(t-s)}\gamma_n(s)ds\right| \le K_1\left[\left\|\Phi_{0,n}-\widehat{\Phi}_{0,n}\right\|_{L^1(0,T;L^1(\Omega))}+\sum_{i=1}^N \left\|\Phi_{i,n}-\widehat{\Phi}_{i,n}\right\|_{L^2(0,T;L^2(\Omega))}\right],$$

where K_1 is a suitable positive constant. Therefore, for every $t \in [0,T]$,

$$\lim_n \left|\int_0^t e^{K(t-s)}\gamma_n(s)ds\right| = 0.$$

Passing to the limit in (3.15) as $n \to \infty$, we have that, for every $t \in [0,T]$,

$$\int_\Omega \frac{(\varphi(t,x)-\widehat{\varphi}(t,x))^2}{2}dx + \int_0^t e^{K(t-s)}\int_\Omega |\nabla(\varphi(s,x)-\widehat{\varphi}(s,x))|^2\,dxds$$
$$\le e^{kt}\int_\Omega \frac{(\varphi_0-\widehat{\varphi}_0)^2}{2}dx,$$

proving (3.4). □

4. Existence of an optimal measure-valued harvesting strategy. In this section we study the existence of an optimal strategy μ^* for the problem

$$\text{(4.1)}\quad \text{maximize:}\qquad J(\mu) := \int_0^T\int_{\overline{\Omega}}\varphi(t,x)d\mu_t(x)dt - \Psi\left(\int_0^T\int_{\overline{\Omega}}c(t,x)d\mu_t(x)dt\right),$$

where φ is the solution, associated to the semigroup constructed in Theorem 3.4, of (3.1) and μ is a nonnegative measure-valued function on $\overline{\Omega}$ which satisfies the constraint

$$\text{(4.2)}\qquad \int_{\overline{\Omega}} b(t,x)d\mu_t(x) \le 1 \qquad \text{a.e. } t \in (0,T).$$

We perform the maximization stated in (4.1) on the set

$$\mathcal{N} = \left\{\mu \in L^\infty(0,T;L^1(\Omega));\ \mu \text{ satisfies (H.4) and (4.2)}\right\}.$$

In addition to the hypotheses (H.1)–(H.4), made in sections 2 and 3, we now assume on b, c, Ψ the following:

(H.5) The functions $b, c \in L^2\left(0,T;H^1(\Omega)\cap C\left(\overline{\Omega}\right)\right)$ satisfy

$$b(t,x) \ge 0, \qquad c(t,x) \ge c_* > 0, \qquad \text{for a.e. } (t,x) \in (0,T)\times\Omega,$$

for some positive constant c_*.

(H.6) $\Psi \in C^1(\mathbb{R};\mathbb{R})$ is a nondecreasing and convex function such that

$$\Psi(0) = 0, \qquad \Psi'(0) = 1.$$

The main result of this section is the following.

THEOREM 4.1. *Assume* (H.1), (H.2), (H.3), (H.5), *and* (H.6). *There exist*

$$\mu^* \in L^\infty_w\left(0,T;\mathcal{M}_+\left(\overline{\Omega}\right)\right), \qquad \varphi \in L^\infty((0,T)\times\Omega)\cap L^2(0,T;H^1(\Omega)),$$

such that

(1) μ^* *satisfies* (4.2)*;*

(2) φ *solves* (3.1) *in correspondence of* μ^**, in the sense of Definition* 3.3*;*

(3) *the following inequality holds:*

$$\sup_{\mu\in\mathcal{N}} J(\mu) \leq \int_0^T \int_{\overline{\Omega}} \varphi(t,x) d\mu_t^*(x) dt - \Psi\left(\int_0^T \int_{\overline{\Omega}} c(t,x) d\mu_t^*(x) dt\right).$$

Remark 4.2. We underline the fact that in Theorem 4.1 we solve a relaxed problem, in the sense that we are able to find a solution that may not belong to the set $\mathcal{N}$. In general, the *optimal solution* μ^* does not belong to $\mathcal{N}$, because the decomposition stated in (H.4) does not hold and because the space $L^\infty(0,T;L^1(\Omega))$ is not closed with respect to the weak$-*$ convergence. As a consequence, we are not able to define the semigroup S, introduced in Theorem 3.4, and the functional J on μ^*. Anyway, we obtain a weak solution φ to (3.1) in correspondence of μ^*.

LEMMA 4.3. *Fix* $\mu \in \mathcal{N}$*. There exists a sequence of measures* $\{\mu_n\}$ *satisfying* (H.4) *such that* $\mu_n \in C^\infty\left([0,T]\times\overline{\Omega}\right)$ *for every* $n\in\mathbb{N}$ *and*

$$\lim_n |J(\mu) - J(\mu_n)| = 0 \tag{4.3}$$

and, for a.e. $t \in (0,T)$*,*

$$\limsup_n \int_{\overline{\Omega}} b(t,x)\, d\mu_n(t,x) \leq 1. \tag{4.4}$$

Proof. By (H.4), there exist $\Phi_0 \in L^\infty\left(0,T;L^1\left(\mathbb{R}^N\right)\right)$ and $\Phi_1,\dots,\Phi_N \in L^\infty\left(0,T;L^2\left(\mathbb{R}^N\right)\right)$ such that $\mu_t = \Phi_0(t) - \operatorname{div}\left(\Phi_1(t),\dots,\Phi_N(t)\right)$ in the sense of measures. Arguing as in the proof of Theorem 3.4, we can construct a sequence of functions $\Phi_{0,n}$ and $\Phi_{i,n}$ ($i\in\{1,\dots,N\}$, $n\in\mathbb{N}$) in $C^\infty([0,T]\times\overline{\Omega})$ such that

(1) $\Phi_{0,n}(t,x) - \sum_{i=1}^N D_{x_i}\Phi_{i,n}(t,x) \geq 0$ for every $t\in[0,T]$, $n\in\mathbb{N}$, and $x\in\Omega$;

(2) $\Phi_{i,n}(t,x) = 0$ for every $t\in[0,T]$, $i\in\{0,\dots,N\}$, $n\in\mathbb{N}$, and $x\in\Omega\setminus\Omega_{-\frac{\varepsilon}{2}}$;

(3) $\lim_{n\to\infty}\|\Phi_{0,n}-\Phi_0\|_{L^p(0,T;L^1(\Omega))} = 0$ for every $1\leq p<\infty$;

(4) for every $i\in\{1,\dots,N\}$ and for every $1\leq p<\infty$, $\lim_{n\to\infty}||\Phi_{i,n}-\Phi_i||_{L^p(0,T;L^2(\Omega))} = 0$;

(5) there exists $C>0$ such that $\|\Phi_{0,n}\|_{L^\infty(0,T;L^1(\Omega))} + \sum_{i=1}^N \|D_{x_i}\Phi_{i,n}\|_{L^\infty(0,T;L^1(\Omega))} \leq C$ for every $n\in\mathbb{N}$.

Define

$$\mu_{n,t}(x) = \Phi_{0,n}(t,x) - \sum_{i=1}^N D_{x_i}\Phi_{i,n}(t,x)$$

which is a smooth and positive function.

Let φ and φ_n be the solutions of (3.1) in correspondence of μ and μ_n (see Theorem 3.4). We have

$$\begin{aligned}|J(\mu)-J(\mu_n)| \leq &\left|\int_0^T\int_{\overline{\Omega}}\varphi(t,x)\,d\mu_t(x)\,dt - \int_0^T\int_{\overline{\Omega}}\varphi_n(t,x)\,d\mu_{n,t}(x)\,dt\right| \\ &+\left|\Psi\left(\int_0^T\int_{\overline{\Omega}}c(t,x)\,d\mu_t(x)\,dx\right) - \Psi\left(\int_0^T\int_{\overline{\Omega}}c(t,x)\,d\mu_{n,t}(x)\,dx\right)\right|\end{aligned}$$

$$\leq \left| \int_0^T \int_{\overline{\Omega}} [\varphi(t,x) - \varphi_n(t,x)] \, \Phi_0(t,x) \, dx \, dt \right| + \sum_{i=1}^N \left| \int_0^T \int_{\overline{\Omega}} D_{x_i} [\varphi(t,x) - \varphi_n(t,x)] \, \Phi_i(t,x) \, dx \, dt \right| + \left| \int_0^T \int_{\overline{\Omega}} \varphi_n(t,x) \left[\Phi_0(t,x) - \Phi_{0,n}(t,x) \right] \, dx \, dt \right| + \sum_{i=1}^N \left| \int_0^T \int_{\overline{\Omega}} D_{x_i} \varphi_n(t,x) \left[\Phi_i(t,x) - \Phi_{i,n}(t,x) \right] \, dx \, dt \right| + \left| \Psi \left(\int_0^T \int_{\overline{\Omega}} c(t,x) d\mu_t(x) dx \right) - \Psi \left(\int_0^T \int_{\overline{\Omega}} c(t,x) d\mu_{n,t}(x) dx \right) \right|. \tag{4.5}$$

Thanks to (3.3), (3.13), and (H.4), we have that

$$\lim_n \int_0^T \int_{\overline{\Omega}} [\varphi(t,x) - \varphi_n(t,x)] \, \Phi_0(t,x) \, dx \, dt = 0 \tag{4.6}$$

by the Lebesgue theorem. By the fact that $\nabla \varphi_n \rightharpoonup \nabla \varphi$ in $L^2\left((0,T) \times \Omega\right)$ (see (3.10)) and (H.4), we deduce that

$$\lim_n \sum_{i=1}^N \left| \int_0^T \int_{\overline{\Omega}} D_{x_i} [\varphi(t,x) - \varphi_n(t,x)] \, \Phi_i(t,x) \, dx \, dt \right| = 0. \tag{4.7}$$

Moreover, by (3.3), (3.10), and (H.4), we have that

$$\left| \int_0^T \int_{\overline{\Omega}} \varphi_n(t,x) \left[\Phi_0(t,x) - \Phi_{0,n}(t,x) \right] \, dx \, dt \right| \leq M \left\| \Phi_0 - \Phi_{0,n} \right\|_{L^1((0,T) \times \Omega)} \longrightarrow 0 \tag{4.8}$$

and

$$\sum_{i=1}^N \left| \int_0^T \int_{\overline{\Omega}} D_{x_i} \varphi_n(t,x) \left[\Phi_i(t,x) - \Phi_{i,n}(t,x) \right] \, dx \, dt \right| \leq \sum_{i=1}^N \left\| D_{x_1} \varphi_n \right\|_{L^2((0,T) \times \Omega)} \left\| \Phi_i - \Phi_{i,n} \right\|_{L^2((0,T) \times \Omega)} \longrightarrow 0 \tag{4.9}$$

as $n \to \infty$. Thanks to (H.5) and (H.6), there exists a positive constant $\overline{K}$ such that

$$\left| \Psi \left(\int_0^T \int_{\overline{\Omega}} c(t,x) d\mu_t(x) dx \right) - \Psi \left(\int_0^T \int_{\overline{\Omega}} c(t,x) d\mu_{n,t}(x) dx \right) \right| \leq \overline{K} \left| \int_0^T \int_{\overline{\Omega}} c(t,x) \, d\left(\mu_t(x) - \mu_n(t,x) \right) \, dt \right|$$

and so

$$\lim_n \left| \Psi \left(\int_0^T \int_{\overline{\Omega}} c(t,x) d\mu_t(x) dx \right) - \Psi \left(\int_0^T \int_{\overline{\Omega}} c(t,x) d\mu_{n,t}(x) dx \right) \right| = 0. \tag{4.10}$$

Using (4.5), (4.6), (4.7), (4.8), (4.9), and (4.10) we deduce that (4.3) holds. Thanks to (H.4) and (H.5) we deduce that

$$\int_0^T \left| \int_{\overline{\Omega}} b(t,x)\, d\left(\mu_t(x) - \mu_n(t,x)\right) \right| dt \le \int_0^T \left| \int_{\overline{\Omega}} b(t,x) \left[\Phi_0(t,x) - \Phi_{0,n}(t,x)\right] dx \right| dt$$
$$+ \sum_{i=1}^N \int_0^T \left| \int_{\overline{\Omega}} D_{x_i} b(t,x) \left[\Phi_i(t,x) - \Phi_{i,n}(t,x)\right] dx \right| dt$$
$$\le \|b\|_{L^2(0,T;L^\infty(\Omega))} \|\Phi_0 - \Phi_{0,n}\|_{L^2(0,T;L^1(\Omega))}$$
$$+ \sum_{i=1}^N \|D_{x_i} b\|_{L^2((0,T)\times\Omega)} \|\Phi_i - \Phi_{i,n}\|_{L^2((0,T)\times\Omega)}$$

and so

$$\lim_n \int_0^T \left| \int_{\overline{\Omega}} b(t,x)\, d\left(\mu_t(x) - \mu_n(t,x)\right) \right| dt = 0.$$

Hence, passing to a subsequence, for a.e. $t \in (0,T)$,

$$\lim_n \left| \int_{\overline{\Omega}} b(t,x)\, d\left(\mu_t(x) - \mu_n(t,x)\right) \right| = 0;$$

therefore, for a.e. $t \in (0,T)$,

$$\limsup_n \int_{\overline{\Omega}} b(t,x)\, d\mu_n(t,x) \le \int_{\overline{\Omega}} b(t,x)\, d\mu_t(x) + \limsup_n \left| \int_{\overline{\Omega}} b(t,x)\, d\left(\mu_t(x) - \mu_n(t,x)\right) \right| \le 1$$

proving (4.4). ∎

Proof of Theorem 4.1. Let $\{\mu_n\}_{n\in\mathbb{N}} \subset \mathcal{N}$ be a maximizing sequence of J, i.e., $\lim_n J(\mu_n) = \sup_{\mathcal{N}} J$, and let $\{\varphi_n\}_{n\in\mathbb{N}}$ be the sequence of the corresponding solutions of (3.1), according to Theorem 3.4.

For every $n \in \mathbb{N}$, thanks to Lemma 4.3, there exists $\widetilde{\mu}_n$ satisfying (H.4) such that $\widetilde{\mu}_n \in C^\infty\left([0,T] \times \overline{\Omega}\right)$ and

$$|J(\mu_n) - J(\widetilde{\mu}_n)| < \frac{1}{n}, \qquad 1 \le p < \infty. \tag{4.11}$$

As a consequence, $J(\widetilde{\mu}_n) \to \sup_{\mathcal{N}} J$ as $n \to \infty$. Let $\{\widetilde{\varphi}_n\}_{n\in\mathbb{N}}$ be the sequence of the corresponding smooth solutions of (3.1), according to Lemmas 2.1 and 2.2.

By (H.5), there exists a constant $c_* > 0$ so that $c(t,x) > c_*$ for a.e. $(t,x) \in [0,T] \times \overline{\Omega}$. Consider the strategy μ'_n defined, for every $t \in [0,T]$, by

$$\mu'_{n,t} := \chi_{A_{n,t}}\, \widetilde{\mu}_{n,t}, \tag{4.12}$$

where

$$A_{n,t} := \left\{x \in \overline{\Omega};\ \widetilde{\varphi}_n(t,x) \ge c_*\right\}.$$

Since $\widetilde{\varphi}_n$ and M provide sub- and supersolutions to (3.1) in correspondence of μ'_n, we have the existence of a solution φ'_n to the same problem; see [19, 24]. Moreover, we have

$$0 \le \widetilde{\varphi}_n \le \varphi'_n \le M. \tag{4.13}$$

Using the definition of μ'_n, (4.13), (H.5), and (H.6), we get

$$\begin{aligned}
J(\widetilde{\mu}_n) - J(\mu'_n) &= \int_0^T \int_{\overline{\Omega}} \widetilde{\varphi}_n d\widetilde{\mu}_{n,t}(x)dt - \int_0^T \int_{\overline{\Omega}} \varphi'_n d\mu'_{n,t}(x)dt \\
&\quad - \Psi\left(\int_0^T \int_{\overline{\Omega}} cd\widetilde{\mu}_{n,t}(x)dt\right) + \Psi\left(\int_0^T \int_{\overline{\Omega}} cd\mu'_{n,t}(x)dt\right) \\
&= \int_0^T \int_{\overline{\Omega}} \widetilde{\varphi}_n d\widetilde{\mu}_{n,t}(x)dt - \int_0^T \int_{A_{n,t}} \varphi'_n d\widetilde{\mu}_{n,t}(x)dt \\
&\quad - \Psi\left(\int_0^T \int_{\overline{\Omega}} cd\widetilde{\mu}_{n,t}(x)dt\right) + \Psi\left(\int_0^T \int_{A_{n,t}} cd\widetilde{\mu}_{n,t}(x)dt\right) \\
&\le \int_0^T \int_{\overline{\Omega}\setminus A_{n,t}} \widetilde{\varphi}_n d\widetilde{\mu}_{n,t}(x)dt - \int_0^T \int_{\overline{\Omega}\setminus A_{n,t}} cd\widetilde{\mu}_{n,t}(x)dt \\
&\le \int_0^T \int_{\overline{\Omega}\setminus A_{n,t}} (\widetilde{\varphi}_n - c_*)d\widetilde{\mu}_{n,t}(x)dt \le 0,
\end{aligned}$$

proving that

$$\liminf_n J(\mu'_n) \ge \sup_{\mathcal{N}} J. \tag{4.14}$$

We claim that

$$\varphi'_n \ge \kappa = \min\{c_*, h_*, \varphi_*\}. \tag{4.15}$$

Thanks to (4.12) and (4.13), we know that

$$\varphi'_n \ge c_* \qquad \text{on} \bigcup_{0\le t\le T} \operatorname{supp}(\mu'_{n,t}). \tag{4.16}$$

Consider the set

$$Q = \{\varphi'_n < c_*\},$$

which is open, due the boundedness of μ'_n, and which guarantees the continuity of φ'_n; see [19, 24].

Due to (4.16), φ'_n solves the problem

$$\begin{cases}
\partial_t \varphi'_n = \Delta\varphi'_n + g(t,x,\varphi'_n), & (t,x) \in ((0,T)\times\Omega)\cap Q,\\
\partial_\nu \varphi'_n = 0, & (t,x) \in ((0,T)\times\partial\Omega)\setminus\partial Q,\\
\varphi'_n = c_*, & (t,x) \in \partial Q \setminus ((0,T)\times\partial\Omega),\\
\varphi'_n = \varphi_0, & (0,x) \in (\{0\}\times\Omega)\cap Q.
\end{cases} \tag{4.17}$$

Since κ is a subsolution of (4.17) we have (4.15).

Due to (4.2) and (H.5), the sequence $\{\mu_n\}_{n\in\mathbb{N}}$ is uniformly bounded in $L^\infty(0,T;\mathcal{M}(\overline{\Omega}))$. Moreover, by (H.4),

$$\begin{aligned}
\|\widetilde{\mu}_n\|_{L^\infty(0,T;\mathcal{M}(\overline{\Omega}))} &= \operatorname{ess\,sup}_{t\in(0,T)} |\widetilde{\mu}_{t,n}|\left(\overline{\Omega}\right) = \operatorname{ess\,sup}_{t\in(0,T)} \widetilde{\mu}_{t,n}\left(\overline{\Omega}\right) \\
&= \operatorname{ess\,sup}_{t\in(0,T)} \int_{\Omega} \Phi_{0,n}(t,x)\,dx \le C,
\end{aligned}$$

where $C > 0$ is the constant defined in the construction of $\widetilde{\mu}_n$. This implies that the sequence $\{\widetilde{\mu}_n\}_{n\in\mathbb{N}}$ is uniformly bounded in $L^\infty(0,T;\mathcal{M}(\overline{\Omega}))$. Finally, thanks to (4.12), the sequence $\{\mu'_n\}_{n\in\mathbb{N}}$ is uniformly bounded in $L^\infty(0,T;\mathcal{M}(\overline{\Omega}))$. Therefore, there exists function $\mu \in L^\infty_w(0,T;\mathcal{M}(\overline{\Omega}))$ such that, passing to a subsequence,

$$\mu'_n \overset{\star}{\rightharpoonup} \mu, \text{weakly-}* \text{ in } L^\infty_w(0,T;\mathcal{M}(\overline{\Omega})). \tag{4.18}$$

Since $L^\infty_w(0,T;\mathcal{M}(\overline{\Omega})) = (L^1(0,T;C(\overline{\Omega})))'$ (see, for example, [3]), (4.18) says that for every $v \in L^1(0,T;C(\overline{\Omega}))$

$$\int_0^T \int_{\overline{\Omega}} v(t,x)d\mu'_{n,t}(x)dt \longrightarrow \int_0^T \int_{\overline{\Omega}} v(t,x)d\mu_t(x)dt.$$

Moreover, if we define $\nu_n = \varphi'_n\mu'_n$ we have that also the sequence $\{\nu_n\}_{n\in\mathbb{N}}$ is uniformly bounded in $L^\infty(0,T;\mathcal{M}(\overline{\Omega}))$. Therefore, there exists a function $\nu \in L^\infty_w(0,T;\mathcal{M}(\overline{\Omega}))$ such that, passing to a subsequence,

$$\nu_n \overset{\star}{\rightharpoonup} \nu, \qquad \text{weakly-}* \text{ in } L^\infty_w(0,T;\mathcal{M}(\overline{\Omega})).$$

In addition, arguing in the same way as the proof of Theorem 3.4, we have that the sequence $\{\varphi'_n\}_{n\in\mathbb{N}}$ is bounded in $L^\infty((0,T)\times\Omega) \cap L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$ and the sequence $\{\partial_t\varphi'_n\}_{n\in\mathbb{N}}$ is bounded in $L^1(0,T;H^{-s}(\Omega))$ with $s > \frac{N}{2}$. Therefore there exists a function

$$\varphi \in L^\infty((0,T)\times\Omega) \cap L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$$

such that, passing to a subsequence,

$\varphi'_n \rightharpoonup \varphi$ weakly in $L^p((0,T)\times\Omega), L^p(0,T;L^2(\Omega)), L^2(0,T;H^1(\Omega))$ for every $1 \le p < \infty$,
$\varphi'_n \longrightarrow \varphi$ strongly in $L^1(0,T;L^2(\Omega))$ and a.e. in $(0,T)\times\Omega$.

Due to (4.15) we have

$$\varphi \geq \kappa. \tag{4.19}$$

Therefore, it makes sense to define the strategy

$$\mu^* \doteq \frac{\nu}{\varphi}.$$

Clearly φ is a solution to

$$\begin{cases} \partial_t\varphi = \Delta\varphi - \varphi\mu^* + g(t,x,\varphi), & 0 < t < T,\ x \in \Omega, \\ \partial_\nu\varphi = 0, & 0 < t < T,\ x \in \partial\Omega, \\ \varphi(0,x) = \varphi_0(x), & x \in \Omega, \end{cases} \tag{4.20}$$

in the sense of Definition 3.3.

We now establish the key inequality

$$\mu^* \leq \mu. \tag{4.21}$$

To prove that (4.21) holds, it suffices to show that

$$\int_0^T \int_{\overline{\Omega}} v d\mu^*_t(x)dt \leq \int_0^T \int_{\overline{\Omega}} v d\mu_t(x)dt \qquad \text{for every } v \in C^\infty_c([0,T]\times\overline{\Omega}),\quad v \geq 0.$$

From the equation for φ_n', (4.15), (4.19), and (4.20), we get

$$\partial_t \log(\varphi_n') = \Delta \log(\varphi_n') + \frac{|\nabla \varphi_n'|^2}{(\varphi_n')^2} - \mu_n' + f(t,x,\varphi_n'),$$
$$\partial_t \log(\varphi) = \Delta \log(\varphi) + \frac{|\nabla \varphi|^2}{\varphi^2} - \mu^* + f(t,x,\varphi).$$

Let $v \in C_c^\infty([0,T] \times \overline{\Omega})$, $v \geq 0$. Since

$$\begin{aligned}\lim_n \int_0^T \int_\Omega & (\partial_t v \log(\varphi_n') + \Delta v \log(\varphi_n') + v f(t,x,\varphi_n'))\, dx\, dt \\ &= \int_0^T \int_\Omega (\partial_t v \log(\varphi) + \Delta v \log(\varphi) + v f(t,x,\varphi))\, dx\, dt, \\ \liminf_n \int_0^T \int_\Omega v \frac{|\nabla \varphi_n'|^2}{(\varphi_n')^2}\, dx\, dt &\geq \int_0^T \int_\Omega v \frac{|\nabla \varphi|^2}{\varphi^2}\, dx\, dt,\end{aligned}$$

we have

$$\begin{aligned}\int_0^T \int_{\overline{\Omega}} v d\mu_t(x) dt &= \lim_n \int_0^T \int_{\overline{\Omega}} v d\mu_{n,t}'(x) dt \\ &= \liminf_n \int_0^T \int_\Omega \left(\partial_t v \log(\varphi_n') + \Delta v \log(\varphi_n') + v \frac{|\nabla \varphi_n'|^2}{(\varphi_n')^2} + v f(t,x,\varphi_n') \right) dx\, dt \\ &\quad + \int_\Omega v(0,x)\varphi_0(x) dx \\ &\geq \int_0^T \int_\Omega \left(\partial_t v \log(\varphi) + \Delta v \log(\varphi) + v \frac{|\nabla \varphi|^2}{\varphi^2} + v f(t,x,\varphi) \right) dx\, dt + \int_\Omega v(0,x)\varphi_0(x) dx \\ &= \int_0^T \int_{\overline{\Omega}} v d\mu_t^*(x) dt.\end{aligned}$$

Thanks to (4.14), using the monotonicity of Ψ and (4.21), we obtain

$$\begin{aligned}\sup_{\mu \in \mathcal{N}} J(\mu) \leq \liminf_n J(\mu_n') &= \lim_n \int_0^T \int_{\overline{\Omega}} \varphi_n' d\mu_{n,t}'(x) dt - \lim_n \Psi \left(\int_0^T \int_{\overline{\Omega}} c d\mu_{n,t}'(x) dt \right) \\ &= \lim_n \int_0^T \int_{\overline{\Omega}} d\nu_{n,t}(x) dt - \Psi \left(\int_0^T \int_{\overline{\Omega}} c d\mu_t(x) dt \right) \\ &= \int_0^T \int_{\overline{\Omega}} d\nu_t(x) dt - \Psi \left(\int_0^T \int_{\overline{\Omega}} c d\mu_t(x) dt \right) \\ &= \int_0^T \int_{\overline{\Omega}} \varphi d\mu_t^*(x) dt - \Psi \left(\int_0^T \int_{\overline{\Omega}} c d\mu_t(x) dt \right) \\ &\leq \int_0^T \int_{\overline{\Omega}} \varphi d\mu_t^*(x) dt - \Psi \left(\int_0^T \int_{\overline{\Omega}} c d\mu_t^*(x) dt \right).\end{aligned}$$

Finally, hypothesis (H.5) and (4.4) yield, for a.e. $t \in (0,T)$,

$$\int_{\overline{\Omega}} b\, d\mu_t^*(x) \leq \int_{\overline{\Omega}} b\, d\mu_t(x) \leq \lim_n \int_{\overline{\Omega}} b\, d\mu_{n,t}'(x) \leq \limsup_n \int_{\overline{\Omega}} b\, d\widetilde{\mu}_{n,t}(x) \leq 1.$$

This completes the proof. ∎

5. Numerical simulations. In this section we present various numerical simulations on system (3.1) to show qualitative features of the solutions. The parabolic equation in (3.1) is solved by means of an explicit forward finite difference method of the first order, while the Neumann boundary condition was achieved by using ghost cells; see, for example, [23, Chapter 2.12]. Here we do not perform any numerical optimization algorithm. We simply compare the costs of solutions obtained in correspondence of some reasonable strategies.

In all the simulations, we consider a rectangular domain $\Omega = (0,30) \times (0,20)$, a fixed time interval $(0,T)$ with $T = 10$, and a fixed time step $dt = \max\left\{(dx)^2, (dy)^2\right\}/5$, where dx and dy are the sizes of the meshes, respectively, for x and y. In the following the function $\chi_B(x)$ denotes the characteristic function of the set B.

5.1. Marine park. We consider here the case of a marine park, in which fishing activity is forbidden. The conjecture is that the optimal measure is indeed singular along the boundary of the nonfishing zone. Numerical simulations seem to confirm this fact. We perform four different simulations using the following functions (see (3.1) and (4.1)):

$$g(t,x,y,u) = 0.3 \cdot u\,(2-u)\,, \qquad \varphi_0(x,y) \equiv 1,$$

$$c(t,x,y) = \begin{cases} \frac{1}{20} & \text{if } (x,y) \in (0,a) \times (0,20), \\ \infty & \text{if } (x,y) \in (a,30) \times (0,20), \end{cases} \qquad \Psi(\xi) = \xi,$$

with $a = 1$. Note that the region, where the cost is equal to ∞, corresponds to the marine park. The fishing strategies, denoted by μ^1, μ^2, μ^3, and μ^4, are

$$\mu_t^1 = 0.1 * \chi_{(0,a)\times(0,20)}(x,y)dx\,dy,$$
$$\mu_t^2 = 0.05 * \chi_{(0,a/2)\times(0,20)}(x,y)dx\,dy + 0.15 * \chi_{(a/2,a)\times(0,20)}(x,y)dx\,dy,$$
$$\mu_t^3 = 0.05 * \chi_{(0,3a/4)\times(0,20)}(x,y)dx\,dy + 0.25 * \chi_{(3a/4,a)\times(0,20)}(x,y)dx\,dy,$$
$$\mu_t^4 = 0.05 * \chi_{(0,7a/8)\times(0,20)}(x,y)dx\,dy + 0.45 * \chi_{(7a/8,a)\times(0,20)}(x,y)dx\,dy$$

so that $\mu_t^1(\Omega) = \mu_t^2(\Omega) = \mu_t^3(\Omega) = \mu_t^4(\Omega)$. All the measures are supported in the fishing zone $(0,a) \times (0,20)$. Figure 1 displays the qualitatively similar behaviors of the solutions at final time $t = 10$ respectively for the different fishing strategies. The cost for μ^1 is approximately 31.71, for μ^2 is 31.82, for μ^3 is 32.49, and for μ^4 is 33.70; see Figure 2. The cost slightly increases as the fishing strategy concentrates near the boundary of the marine park.

5.2. Different fishing locations. We consider here three different fishing strategies. More precisely the total strength of the three strategies is the same, while the positions where the fishing activities take place are different. In all the cases, we use the following functions (see (3.1) and (4.1)):

$$g(t,x,u) = 0.3 \cdot u\,(2-u)\,, \qquad \varphi_0(x,y) \equiv 1, \qquad c(t,x,y) \equiv \frac{1}{2}, \qquad \text{and} \qquad \Psi(\xi) = \xi.$$

The fishing strategies, denoted by μ^1, μ^2, and μ^3, are

$$\mu_t^1 = \chi_{B_1}(x,y)dx\,dy, \qquad \mu_t^2 = \chi_{B_2}(x,y)dx\,dy, \qquad \mu_t^3 = \chi_{B_3}(x,y)dx\,dy,$$

where $B_1 = (10,20) \times \left(\frac{20}{3}, \frac{40}{3}\right)$, B_2 is a disjoint union of 4 rectangles with edges of length 5 and $\frac{10}{3}$, while B_3 is a disjoint union of 16 rectangles with edges of length $\frac{5}{2}$ and $\frac{5}{3}$. The sizes of the mesh are $dx = dy = \frac{1}{50}$. Figure 3 shows the contour of the

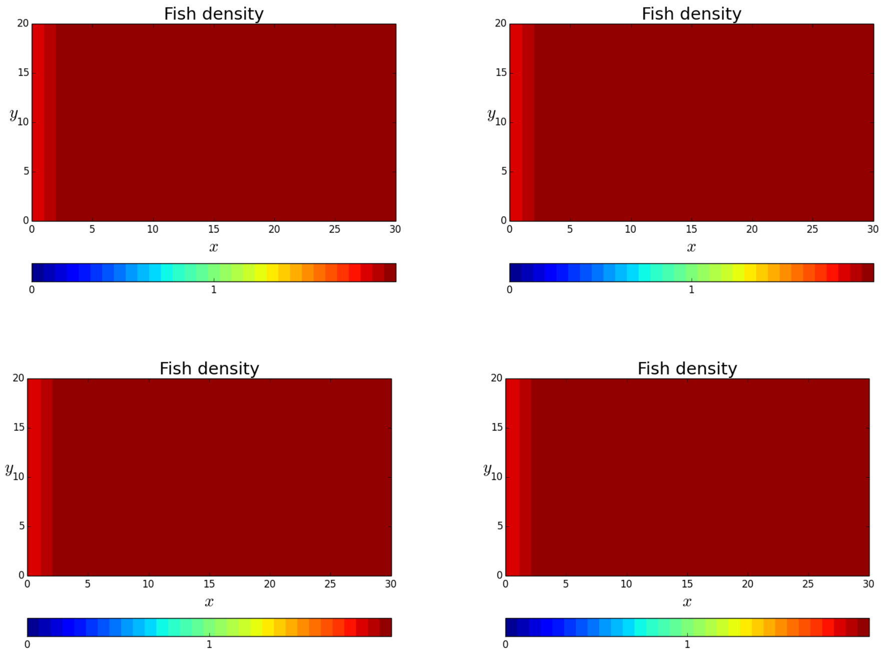

FIG. 1. *Contour plots of the solution to* (3.1). *The pictures represent the solutions at time* $t = 10$ *corresponding to the strategies of subsection* 5.1.

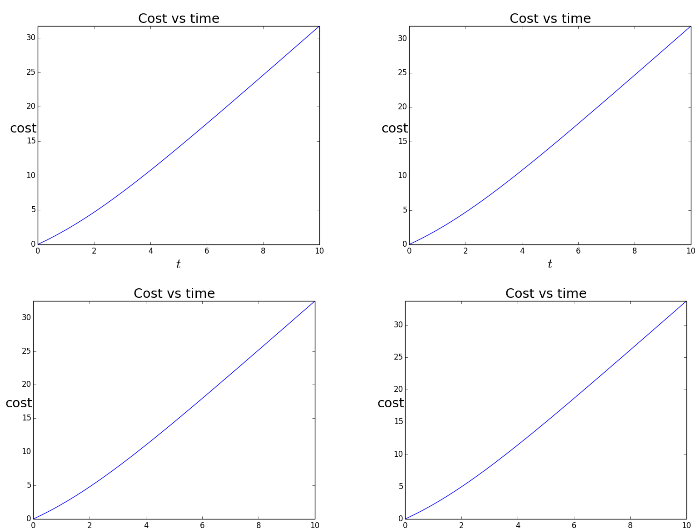

FIG. 2. *Graph of the cost* (4.1) *with respect to time for the four different strategies implemented in subsection* 5.1.

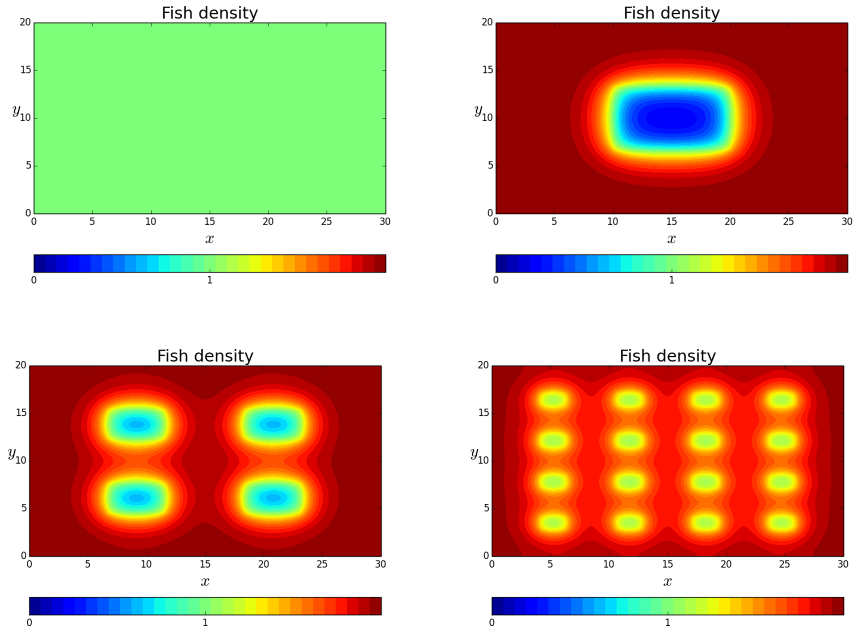

FIG. 3. *Contour plots of the solution to* (3.1). *The first picture displays the common initial condition. The other pictures represent the solutions at time* $t = 10$ *corresponding to the strategies of subsection* 5.2.

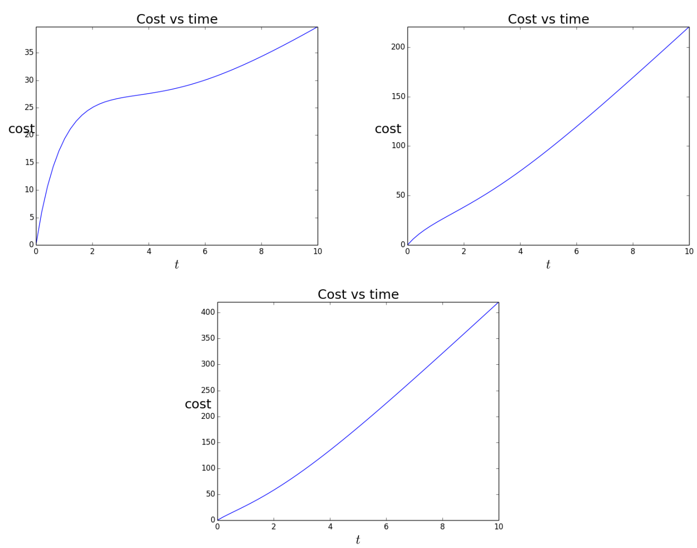

FIG. 4. *Graph of the cost* (4.1) *with respect to time for the three different strategies implemented in subsection* 5.2.

initial condition and of the solution at the final time $t = 0$ with respect to the fishing strategies μ^1, μ^2, and μ^3. The cost for μ^1 is approximately 39.73, for μ^2 is 220.64, and for μ^3 is 420.34; see Figure 4. These simulations seem to suggest that in order to

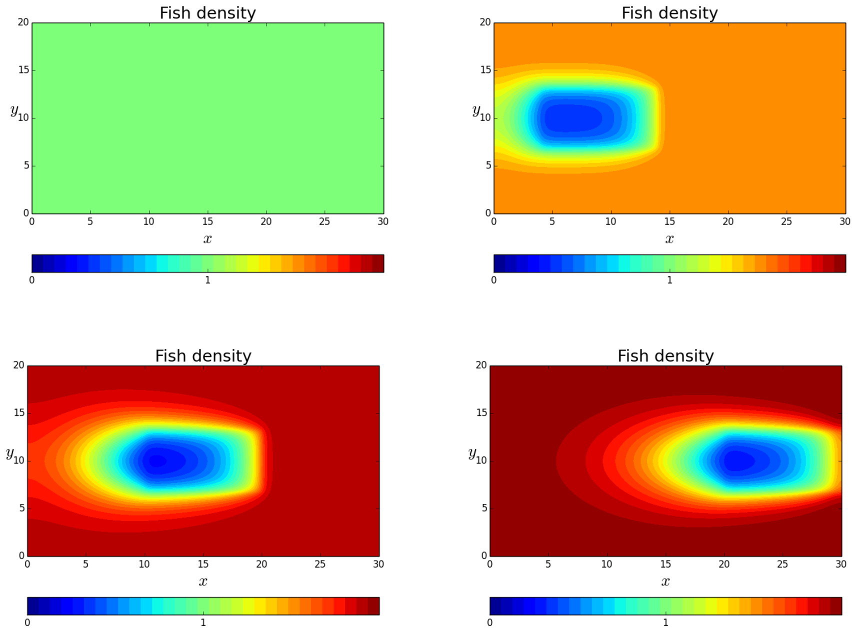

FIG. 5. *Contour plots of the solution to* (3.1) *at times* $t = 0$, $t = 2$, $t = 5$, *and* $t = 10$. *Here we use the functions and parameters of subsection* 5.3.

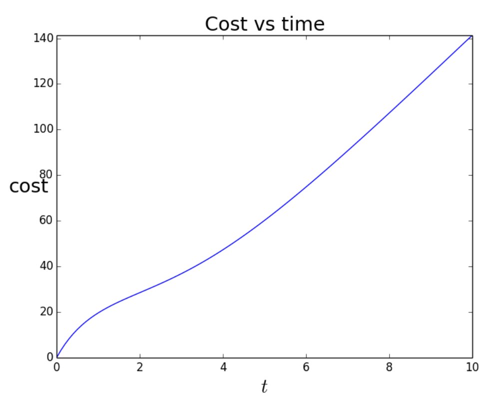

FIG. 6. *Graph of the cost* (4.1) *with respect to time for the solution of subsection* 5.3.

maximize (4.1) with the constraint (4.2), one possible strategy could be distributing the fishing activity in several different locations.

5.3. Time-dependent fishing activity. We consider here a simple situation in which the fishing activity happens in a rectangular region moving with constant speed from the left to the right. More precisely we consider here the system (3.1) with the following choices:

$$\mu_t = \chi_{B_t}(x, y) dx\, dy, \qquad g(t, x, y, u) = \frac{3}{10} u\,(2 - u)\,, \qquad \varphi_0(x, y) \equiv 1,$$

where $B_t = (2t, 2t + 10) \times \left(\frac{20}{3}, \frac{40}{3}\right)$. As regards the cost functional (4.1), we consider the following functions:

$$c(t, x, y) \equiv \frac{1}{2} \qquad \text{and} \qquad \Psi(\xi) = \xi.$$

Finally, we used $dx = dy = \frac{1}{50}$. In Figure 5 the plots of the solution to (3.1) at times $t = 0$, $t = 2$, $t = 5$, and $t = 10$ are shown. The overall cost (4.1) for such a problem is approximately 141.35; see Figure 6.

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