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VARIATIONAL PROBLEMS FOR FÖPPL–VON KÁRMÁN PLATES*

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Abstract. Some variational problems for a Föppl–von Kármán plate subject to general equilibrated loads are studied. The existence of global minimizers is proved under the assumption that the out-of-plane displacement fulfils homogeneous Dirichlet condition on the whole boundary while the in-plane displacement fulfils nonhomogeneous Neumann condition. If the Dirichlet condition is prescribed only on a subset of the boundary, then the energy may be unbounded from below over the set of admissible configurations, as shown by several explicit conterexamples: in these cases the analysis of critical points is addressed through an asymptotic development of the energy functional in a neighborhood of the flat configuration. By a Γ-convergence approach we show that critical points of the Föppl–von Kármán energy can be strongly approximated by uniform Palais–Smale sequences of suitable functionals: this property leads to identifying relevant features for critical points of approximating functionals, e.g., buckled configurations of the plate. The analysis for rescaled thickness is performed by assuming that the plate-like structure is initially prestressed, so that the energy functional depends only on the out-of-plane displacement and exhibits asymptotic oscillating minimizers as a mechanism to relax compressive states.

Key words. Föppl–von Kármán, calculus of variations, elasticity, nonlinear Neumann problems, Monge–Ampère equation, critical points, Γ-convergence, asymptotic analysis, singular perturbations, mechanical instabilities

AMS subject classifications. 49J45, 74K30, 74K35, 74R10

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1. Introduction.

The Föppl–von Kármán model is widely used as a relevant theoretical tool in the study of the mechanical behavior of thin elastic plates, for its ability to capture the interplay between membrane and bending effects (see [1], [3] [18], [19], [20]). This interplay constitutes the source of a rich phenomenology affecting not only the macroscopic behavior but also the occurrence of local microinstabilities which are crucial also in the behavior of soft solids, biological tissues, and gels [30]. A relevant problem consists in detecting a precise geometric description of such creased equilibrium configurations in dependance of the geometric and constitutive properties of the plate.

Despite its long and controversial history, a rigorous analysis of the well posedness for variational problems associated to the Föppl–von Kármán functional under general boundary conditions is still far from complete. In particular, the minimization problem under general load conditions is quite subtle. The rigorous derivation of the Föppl–von Kármán plate model from three-dimensional (3D) nonlinear elasticity was proved by Friesecke, James, and Müller in the seminal paper [23] under the assumption of normal forces, while in [29] the authors carefully analyze the validity of such a theory under in-plane compressive forces and study in detail the instability issue under suitable coercivity hypotheses [29, Theorem 4].

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In this paper we study the existence of minimizers for the Föppl–von Kármán energy, under general load conditions. In particular, we deal with Dirichlet and Neumann conditions for the out-of-plane displacement on the whole boundary while the in-plane displacement fulfils nonhomogeneous Neumann condition, corresponding to general assumptions on the forces acting on the plate. The existence of minimizers is achieved in several cases by improving the techniques introduced in [4], [15] to circumvent the lack of coerciveness appearing in related nonconvex minimization problems and by taking advantage of some properties of the Monge–Ampère equation (see [25], [46]): in addition to the abstract setting of [4], [15], which leads to the existence of at least one compact sequence after suitable manipulations, in the present context we show that every minimizing sequence has a compact subsequence; moreover here the framework is not limited to a fixed functional but also to sequences of functionals and we skip the technical task of computing explicitly the recession functional.

We exhibit also examples where the energy of admissible configurations is not bounded from below, so that existence of minimizers fails and we turn our attention to the critical points by performing singular perturbation analysis of the functional in a neighborhood of a flat configuration. This analysis leads to detecting critical points of the Föppl–von Kármán energy by suitable approximations of Palais–Smale sequences associated to approximating functionals. Our procedure allows us to single out global buckling configurations, in cases when the plate has a rectangular shape. As is well known, wrinkling type phenomena and other microinstabilities (see [17], [21], [22], [24], [26], [45]) manifest themselves in sheets with very small thickness; therefore we focus our analysis on the behavior as thickness tends to 0 and highlight the energetic competition of oscillating configurations versus flat equilibrium configurations.

The detailed outline of the paper is as follows.

In section 2 we prove existence of minimizers for the Föppl–von Kármán energy (2.11) corresponding to a plate of prescribed thickness $h > 0$ under the action of balanced loads in three relevant cases:

(i) the plate is free at the boundary of a generic Lipschitz open set, while in plane uniform normal traction or mild uniform normal compression is prescribed on the whole boundary (Theorems 2.1, 2.3);

(ii) the plate is simply supported on the whole boundary of a convex set (Theorem 2.8);

(iii) the plate is clamped on the whole boundary of a generic Lipschitz open set (Theorem 2.11).

Moreover we focus the analysis on the cases when these conditions at the boundary are loosened, by showing explicit counterexamples where the energy is not bounded from below and minimizers do not exist, even for balanced loads and fixed thickness $h > 0$.

Section 3 is devoted to study asymptotic behavior of the energy near a flat configuration; this is achieved by scaling the out-of-plane displacements: in Theorem 3.3 we show that every critical point of a simpler limit functional (in the sense of Γ-convergence) can be approximated by a *uniform Palais–Smale sequence* (see Definition 3.2) whose construction is detailed in the statement of the same theorem. Despite the results proved in the present paper, the existence of nonminimizing critical points for the Föppl–von Kármán functional remains an open problem in the general case, at least at our knowledge; nevertheless uniform Palais–Smale sequences can be considered a surrogate of critical points in a small neighborhood of the flat configuration and Theorem 3.3 allows us to recover them starting from critical points of the limit functional. Moreover the analysis of the related Euler–Lagrange equations highlights buckled configurations, whose shape can be detailed in some explicit examples (see Examples 3.8, 3.9).

In section 4, given a plate with thickness $s =: hs_0$ ($h > 0$ is an adimensional parameter), we study the limit as $h \to 0$ of scaled Föppl–von Kármán energy $\mathcal{F}_h$ when in-plane forces scale as $\mathbf{f}_h = h^\alpha \mathbf{f}$ in functional (2.11).

If $\alpha \geq 2$ and $(\mathbf{u}_h^*, w_h^*)$ are minimizers, then $(h^{-\alpha}\mathbf{u}_h^*, h^{-\alpha/2} w_h^*)$ are weakly compact in $H^1 \times H^2$ and the corresponding energies, rescaled by $h^{-1-2\alpha}$, converge to a limit energy (see Theorem 4.1 and formula (4.2) therein). It is not surprising that for $\alpha = 2$ the limit energy is again the Föppl–von Kármán energy of a plate of thickness s_0, since h^{-5} is exactly the scaling factor in the hierarchy of [23] in order to obtain the Föppl–von Kármán plate model.

If $\alpha \in [0, 2)$, then the rescaled energies may be unbounded from below as $h \to 0_+$ for all cases: free, simply supported, and clamped plate (see Counterexample 4.4 and Remark 4.5).

The results obtained in sections 2–4 lead us to examine also the case $\alpha \in [0, 2)$, by studying the equilibrium configurations of the plate as $h \to 0$ through relaxation arguments applied to an energetic functional which takes into account a prestressed state of the plate. Precisely, in section 5 we perform the analysis of corresponding asymptotic sequence of minimizers; we show a competition between oscillating and flat equilibria and highlight how this competition is ruled by the mechanical and geometrical parameters: oscillating equilibria act as a mechanism to release compression states in the limit.

Eventually we exhibit a list of creased and noncreased equilibrium configurations of an annular plate (Examples 5.5–5.8), together with a general strategy (Remark 5.9) to build these examples: if both eigenvalues in the stress tensor of the prestressed state are strictly positive almost everywhere, then we can expect only the flat minimizer, whereas possible occurrence of oscillating configurations requires the presence of a compressive state on a region of positive measure (Proposition 5.3, Remark 5.4).

The issues involved in the present article are closely related with a large class of instabilities, according to recent studies [5], [6], [7], [8], [9], [11], [12], [13], [16], [17], [28], [31], [32], [33], [35], [36], [37], [38], [39], [40], [42], [43], [44], [45].

Notation. $\mathrm{Sym}_{2,2}(\mathbb{R})$ denotes 2×2 real symmetric matrices; $\mathbf{a} \otimes \mathbf{b}$ denotes the matrix with entries $a_i b_j$, $\mathbf{a} \odot \mathbf{b} = \frac{1}{2}(\mathbf{a} \otimes \mathbf{b} + \mathbf{a} \otimes \mathbf{b})$ and $|\mathbf{a}|^2 = \sum_i a_i^2$ for every $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$; moreover $|\mathbb{A}|^2 = \sum_{i,j} A_{ij}^2$ and $\mathbb{A} : \mathbb{B} = \sum_{i,j} A_{ij} B_{ij}$, for every $\mathbb{A}, \mathbb{B} \in \mathrm{Sym}_{2,2}(\mathbb{R})$ with entries, respectively, A_{ij}, B_{ij}.

$H^k(\Omega)$ denotes the Sobolev space of functions in the open set $\Omega \subset \mathbb{R}^2$ whose distributional derivatives up to the order k belong to $L^2(\Omega)$; $H_0^k(\Omega)$ denotes the completion of compactly supported functions in the Sobolev H^k norm; and $H^1(\Omega, \mathbb{R}^2)$ denotes the vector fields with components in $H^1(\Omega)$. While notation H^1 refers to the Hilbertian case, $W^{1,p}(\Omega)$ denotes the Sobolev space of functions with first derivatives in $L^p(\Omega)$ with $p > 1$.

$⨍_A v \, d\mathbf{x} = |A|^{-1} \int_A v \, d\mathbf{x}$ for all measurable set A and every integrable function v defined on A. $\mathbf{1}_A(\mathbf{x}) = 1$ if $\mathbf{x} \in A$, $\mathbf{1}_A(\mathbf{x}) = 0$ if $\mathbf{x} \notin A$, $\chi_U(v) = 0$ if $v \in U$, and $\chi_U(v) = +\infty$ if $v \notin U$.

2. Minimization of Föppl–von Kármán functional. Let $\Omega \subset \mathbb{R}^2$ be a bounded open connected set with Lipschitz boundary $\partial\Omega$; $\mathbf{x} = (x_1, x_2)$ denotes the coordinates of points in Ω referring to the canonical reference frame in $\mathbb{R}^2$, and $s > 0$ is the thickness of a thin plate-like region whose reference configuration is $\Omega \times (-\frac{s}{2}, \frac{s}{2})$.

Moreover set $s := hs_0$, where h is an nondimensional scale factor which remains fixed throughout this section.

Let $\mathbf{u} : \Omega \to \mathbb{R}^2$ and $w : \Omega \to \mathbb{R}$ be, respectively, the in-plane and out-of-plane displacements. In the geometrical linear setting the *stretching tensor* $\mathbb{D}$ is given by

$$\mathbb{D}(\mathbf{u}, w) = \mathbb{E}(\mathbf{u}) + \frac{1}{2}\, Dw \otimes Dw, \tag{2.1}$$

where

$$\mathbb{E}(\mathbf{u}) = \frac{1}{2}\left(D\mathbf{u} + D\mathbf{u}^T\right) \tag{2.2}$$

denotes the linearized strain tensor.

The kernel of $\mathbb{E}$, which is the set of infinitesimal rigid displacements in Ω, is denoted by

$$\mathcal{R} := \{\mathbf{u} : \mathbb{E}(\mathbf{u}) = \mathbf{0}\} \tag{2.3}$$

and $\mathcal{R}(\mathbf{u})$ denotes the projection of $\mathbf{u} \in H^1(\Omega, \mathbb{R}^2)$ on $\mathcal{R}$. The elastic energy of a plate of thickness $hs_0 > 0$ is the sum of a membrane energy

$$F_h^m(\mathbf{u}, w) = hs_0 \int_\Omega J(\mathbb{D}(\mathbf{u}, w))\, d\mathbf{x} \tag{2.4}$$

and a bending energy

$$F_h^b(w) = \frac{h^3 s_0^3}{12} \int_\Omega J(D^2 w)\, d\mathbf{x}\,. \tag{2.5}$$

We assume that for every $\mathbb{A} \in \mathrm{Sym}_{2,2}(\mathbb{R})$ the energy density J is given by

$$J(\mathbb{A}) = \frac{E}{2(1-\nu^2)}\left(|\,\mathrm{Tr}(\mathbb{A})|^2 - 2(1-\nu)\det \mathbb{A}\right) = \frac{E}{2(1+\nu)}\,|\mathbb{A}|^2 + \frac{E\,\nu}{2(1-\nu^2)}\,|\mathrm{Tr}\mathbb{A}|^2, \tag{2.6}$$

where $E > 0$ is the Young modulus and ν is the Poisson ratio, $-1 < \nu < 1/2$.

A straightforward consequence of (2.6) which will be exploited in subsequent computations is

$$c_\nu\, \frac{E}{2}\, |\mathbb{A}|^2 \le J(\mathbb{A}) \le C_\nu\, \frac{E}{2}\, |\mathbb{A}|^2, \tag{2.7}$$

where $0 < c_\nu := \min\{(1-\nu)^{-1}, (1+\nu)^{-1}\} \le C_\nu := \max\{(1-\nu)^{-1}, (1+\nu)^{-1}\} < +\infty$. By denoting the unit outer normal to $\partial\Omega$ by $\mathbf{n}$, we define

$$\begin{aligned} &\mathcal{A}^0 := \left\{ w \in H^2(\Omega) | w = \frac{\partial w}{\partial \mathbf{n}} = 0 \text{ on } \Gamma \right\}, \\ &\mathcal{A}^1 := \left\{ w \in H^2(\Omega) | w = 0 \text{ on } \Gamma \right\}, \\ &\mathcal{A}^2 := H^2(\Omega), \end{aligned} \tag{2.8}$$

where spaces $\mathcal{A}^0 = \mathcal{A}^0(\Gamma)$, $\mathcal{A}^1 = \mathcal{A}^1(\Gamma)$ do depend on the set Γ. We assume in general that

$$\Gamma \subset \partial\Omega \quad \text{is a Borel set s.t.} \quad \mathcal{H}^1(\Gamma) > 0\,. \tag{2.9}$$

Let

$$\mathbf{f}_h \in\ L^2(\partial\Omega, \mathbb{R}^2)\,, \quad g_h \in L^2(\Omega) \tag{2.10}$$

be, respectively, the densities of a given in-plane load distribution and of a given out-of-plane load distribution.

By taking into account the work of external loads and different types of boundary conditions, we define the *Föppl–von Kármán functional* (FvK) in what follows,

$$\begin{aligned}\mathcal{F}_h(\mathbf{u},w) = hs_0\int_\Omega J(\mathbb{D}(\mathbf{u},w))\,d\mathbf{x}\\ + \frac{h^3s_0^3}{12}\int_\Omega J(D^2w)\,d\mathbf{x} - hs_0\int_\Omega g_h\,w\;d\mathbf{x} - hs_0\int_{\partial\Omega}\mathbf{f}_h\cdot\mathbf{u}\,d\mathcal{H}^1.\end{aligned} \tag{2.11}$$

Throughout the paper we choose units of measurement such that $s_0 = 1$.

About the various parameters issue we notice that FvK energies are derived explicitly in [23] only when the measure units are chosen such that $s := hs_0 = 1$; nevertheless, by a careful inspection of the proof, we claim that an analogous scaling argument leads to the same result for a generic s, say, to functional (2.11): indeed it is enough starting from a 3D cylinder of thickness $(-ths_0/2, ths_0/2)$ (where s_0 is a fixed thickness, $h, t > 0$ are adimensional parameters) and letting $t \to 0^+$ in the 3D energies scaled by t^{-5}. Concerning this issue we refer also to the analysis about energy bounds available in [10], [9].

Equilibrium configurations of the plate under prescribed loads $\mathbf{f_h}$ and g_h are obtained by minimizing the functional (2.11) over $H^1(\Omega,\mathbb{R}^2)\times\mathcal{A}^i$, $i = 0, 1, 2$, corresponding, respectively, to clamped, simply supported, and free plate. The present section focuses on issues related to existence and nonexistence of these minimizers: we study in detail the existence of such minimizers according to the various choices $i = 0, 1, 2$ of boundary conditions and loads and we exhibit some counterexamples in which the functional is unbounded from below, and hence global minimizers do not exist.

The main obstruction in applying the direct methods of the calculus of variations to this problem relies in the possible lack of coerciveness of the functional (2.11): indeed the kernel of the membrane energy density, which in general is a subset of the set of solutions of the Monge–Ampère equation in Ω (see Lemma 2.5 below), may be too large to allow balancing of the internal membrane energy versus the effect of external forces, in order to achieve an equilibrium configuration. Notwithstanding this difficulty, an existence theorem can be proved either assuming a sign condition on boundary forces, or an homogeneous Dirichlet condition on the transverse displacement. In the first case the work of the external forces is bounded away from zero on the kernel of the membrane energy density, thus allowing the global energy to be bounded from below; in the second one a uniqueness result in the theory of Monge–Ampère equation implies that the kernel of bending energy reduces to the null transverse displacement (see also [31], [32], [33]). These settings together with a tuning of some techniques introduced in [4] and [15] yield compactness of minimizing sequences, and hence the existence of minimizers via the direct method.

Assuming $\mathbf{f}_h = f_h\mathbf{n}$, we prove the existence of minimizers for $\mathcal{F}_h$ in $H^1(\Omega,\mathbb{R}^2)\times H^2(\Omega)$, first under the assumption that f_h is a nonnegative constant (Theorem 2.1), and second under the assumption that f_h is a small negative constant (Theorem 2.3).

THEOREM 2.1 (uniform boundary tension of a free plate). *Assume that $\Omega\subset\mathbb{R}^2$ is a bounded connected Lipschitz open set and*

$$\int_\Omega g_h\,d\mathbf{x} = \int_\Omega x_1 g_h\,d\mathbf{x} = \int_\Omega x_2 g_h\,d\mathbf{x} = 0\ , \tag{2.12}$$

$$\mathbf{f}_h = f_h\mathbf{n}\quad \text{on } \partial\Omega\,,\qquad f_h \ge 0 \text{ is a constant.} \tag{2.13}$$

Then, for every fixed $h > 0$, $\mathcal{F}_h$ achieves a minimum over $H^1(\Omega,\mathbb{R}^2)\times H^2(\Omega)$.

Proof. In order to achieve the proof it will be enough to show a minimizing sequence equibounded in $H^1(\Omega,\mathbb{R}^2)\times H^2(\Omega)$, since $\mathcal{F}_h$ is sequentially l.s.c. with respect to the weak convergence in such space. Due to $\inf_{H^1\times H^2}\mathcal{F}_h \le \mathcal{F}_h(\mathbf{0},0)\le 0$, if $\mathcal{F}_h(\mathbf{u}_n,w_n)\to\inf_{H^1\times H^2}\mathcal{F}_h$, we may suppose $\mathcal{F}_h(\mathbf{u}_n,w_n)\le 1$ so, by the divergence theorem, (2.13), and (2.7) we also get

$$c_\nu\frac{h^3 E}{24}\int_\Omega |D^2 w_n|^2 + c_\nu\frac{h E}{2}\int_\Omega |\mathbb{D}(\mathbf{u}_n,w_n)|^2 \le h f_h\int_\Omega \mathrm{div}\ \mathbf{u}_n + h\int_\Omega g_h w_n + 1. \tag{2.14}$$

Set $\lambda_n := \|\mathbb{E}(\mathbf{u}_n)\|_{L^2}$ and suppose by contradiction that $\sup\lambda_n=+\infty$, and hence (up to subsequences without relabeling) $\lambda_n\to+\infty$. Let $\zeta_n := \lambda_n^{-1/2}w_n,\quad \mathbf{v}_n:=\lambda_n^{-1}\mathbf{u}_n$ and $\mathbf{x}_\Omega$ is the center of mass of Ω. Possibly different constants denoted by C actually depend only on Ω. Then by substituting in (2.14) and dividing times λ_n, we get via (2.12) and Poincaré inequality

$$\begin{aligned}
&c_\nu\frac{h^3 E}{24}\int_\Omega |D^2\zeta_n|^2 + \lambda_n c_\nu\frac{h E}{2}\int_\Omega |\mathbb{D}(\mathbf{v}_n,\zeta_n)|^2\\
&\le h f_h\int_\Omega \mathrm{div}\ \mathbf{v}_n + \lambda_n^{-1/2}\,h\int_\Omega g_h\zeta_n + \lambda_n^{-1}\\
&= h f_h\int_\Omega \mathrm{div}\ \mathbf{v}_n + \lambda_n^{-1/2}\,h\int_\Omega g_h\left(\zeta_n - \fint_\Omega\zeta_n - (\mathbf{x}-\mathbf{x}_\Omega)\fint_\Omega D\zeta_n\right) + \lambda_n^{-1}\\
&\le h f_h\int_\Omega \mathrm{div}\ \mathbf{v}_n + \lambda_n^{-1/2}\,h\,\|g_h\|^2_{L^2} + \lambda_n^{-1/2}\,C\int_\Omega |D^2\zeta_n|^2 + \lambda_n^{-1}.
\end{aligned}\tag{2.15}$$

The above inequality together with $\|\mathbb{E}(\mathbf{v}_n)\|_{L^2}=1$ entails

$$c_\nu\frac{h^3 E}{24}\int_\Omega |D^2\zeta_n|^2 + \lambda_n c_\nu\frac{h E}{2}\int_\Omega |\mathbb{D}(\mathbf{v}_n,\zeta_n)|^2 \le C \tag{2.16}$$

for large n. Exploiting $\|\mathbb{E}(\mathbf{v}_n)\|_{L^2}=1$, once more, we get that $D\zeta_n$ are then equibounded in $H^1(\Omega,\mathbb{R}^2)$, and, up to subsequences, $\zeta_n-\fint_\Omega\zeta_n\to\zeta$ weakly in $H^2(\Omega)$, $D\zeta_n\to D\zeta$ in $L^4(\Omega,\mathbb{R}^2)$ due to the Rellich theorem and $\mathbf{v}_n\to\mathbf{v}$ weakly in $H^1(\Omega,\mathbb{R}^2)$.

By taking into account (2.12) we get

$$h f_h\int_\Omega \mathrm{div}\ \mathbf{v}_n + \lambda_n^{-1/2}h\int_\Omega g_h\zeta_n = h f_h\int_\Omega \mathrm{div}\ \mathbf{v}_n + \lambda_n^{-1/2}h\int_\Omega g_h\left(\zeta_n - \fint_\Omega\zeta_n\right)\to h f_h\int_\Omega \mathrm{div}\ \mathbf{v}\,. \tag{2.17}$$

By sequential lower semicontinuity together with (2.17), (2.15) we get

$$\begin{aligned}
c_\nu\frac{h^3 E}{24}\int_\Omega |D^2\zeta|^2 &\le \liminf c_\nu\frac{h^3 E}{24}\int_\Omega |D^2\zeta_n|^2\\
&\le \liminf\left\{h f_h\int_\Omega \mathrm{div}\ \mathbf{v}_n + \lambda_n^{-1/2}\,h\int_\Omega g_h\left(\zeta_n - \fint_\Omega\zeta_n\right) + \lambda_n^{-1}\right\}\\
&= h f_h\int_\Omega \mathrm{div}\ \mathbf{v}.
\end{aligned}\tag{2.18}$$

Moreover, by taking into account that $\lambda_n\to+\infty$,

$$\lambda_n c_\nu\frac{h E}{2}\int_\Omega |\mathbb{D}(\mathbf{v}_n,\zeta_n)|^2 \le h f_h\int_\Omega \mathrm{div}\ \mathbf{v}_n + \lambda_n^{-1} + \lambda_n^{-1/2}\,h\int_\Omega g_h\left(\zeta_n - \fint_\Omega\zeta_n\right)\le C \tag{2.19}$$

and by $D\zeta_n \to D\zeta$ in $L^4(\Omega,\mathbb{R}^2)$, we have also

$$c_\nu \frac{h\,E}{2}\int_\Omega |\mathbb{D}(\mathbf{v},\zeta)|^2 \le \liminf c_\nu \frac{h\,E}{2}\int_\Omega |\mathbb{D}(\mathbf{v}_n,\zeta_n)|^2 \le C \liminf \lambda_n^{-1} = 0. \tag{2.20}$$

Hence $\mathbb{D}(\mathbf{v}_n,\zeta_n) \to \mathbb{D}(\mathbf{v},\zeta) = \mathbb{O}$, $\mathbb{E}(\mathbf{v}_n) \to \mathbb{E}(\mathbf{v})$ both in $L^2(\Omega, \mathrm{Sym}_{2,2}(\mathbb{R}))$ and $2\,\mathrm{div}\,\mathbf{v} = -|D\zeta|^2$.

Therefore by (2.18)

$$c_\nu \frac{h^3\,E}{24}\int_\Omega |D^2\zeta|^2 + \frac{1}{2} h f_h \int_\Omega |D\zeta|^2 \le 0 \tag{2.21}$$

and by taking into account that $⨍_\Omega \zeta = 0$ we get $\zeta = 0$ and $\mathbb{E}(\mathbf{v}) = 0$, a contradiction since $\|\mathbb{E}(\mathbf{v}_n)\|_{L^2} = 1$ and $\mathbb{E}(\mathbf{v}_n) \to \mathbb{E}(\mathbf{v})$ in $L^2(\Omega,\mathrm{Sym}_{2,2}(\mathbb{R}))$. So $\lambda_n \le C$ for some $C > 0$ and $\mathbf{u}_n - \mathcal{R}(\mathbf{u}_n)$ are equibounded in $H^1(\Omega,\mathbb{R}^2)$ by the Korn inequality, while equiboundedness of $w_n - ⨍_\Omega w_n$ in $H^2(\Omega)$ follows from (2.14). Existence of minimizers is then straightforward via direct method. ∎

If $f < 0$, then the analogue of Theorem 2.1 for in-plane compression along the whole boundary cannot be true, as shown by the next particularly telling Counterexample 2.2. Anyway we can deal also with load corresponding to small negative f, as shown by Theorem 2.3 below.

COUNTEREXAMPLE 2.2 (uniform boundary compression). *Assume*

$$\Omega = (-2,2)\times(-1,1)\,, \qquad \Gamma = \{-2\}\times[-1,1], \quad g_h \equiv 0, \tag{2.22}$$

$$\mathbf{f}_h = f_h \mathbf{n} \;\; \textit{on } \partial\Omega, \qquad \textit{where } f_h \textit{ is a given constant s.t.} \quad f_h < -\tfrac{C_\nu E}{64} h^2. \tag{2.23}$$

Then $\inf \mathcal{F}_h = -\infty$ *over both* $H^1(\Omega,\mathbb{R}^2)\times\mathcal{A}^1$ *and* $H^1(\Omega,\mathbb{R}^2)\times\mathcal{A}^2$. *Indeed, let*

$$\mathbf{u} = -\frac{(2+x_1)^3}{6}\mathbf{e}_1, \qquad \varphi = \frac{(2+x_1)^2}{2},$$

and $\mathbf{u}_n := n\mathbf{u}$, $\varphi_n := \sqrt{n}\varphi$; *then* $2\mathbb{E}(\mathbf{u}_n) = -D\varphi_n \otimes D\varphi_n$ *and by* (2.7)

$$\begin{aligned}\mathcal{F}_h(\mathbf{u}_n,\varphi_n) &\le \frac{h^3 C_\nu\, nE}{24}\int_\Omega |D^2\varphi|^2\,d\mathbf{x} - \,nh\,f_h \int_{\partial\Omega} \mathbf{n}\cdot\mathbf{u}\,d\mathcal{H}^1\\ &= \frac{h^3 C_\nu nE}{24}\int_\Omega |D^2\varphi|^2\,d\mathbf{x} - \,nh\,f_h\int_\Omega \mathrm{div}\,\mathbf{u}\,d\mathbf{x}\\ &= \frac{h^3 C_\nu nE}{24}\int_\Omega |D^2\varphi|^2\,d\mathbf{x} + \frac{nhf_h}{2}\int_\Omega |D\varphi|^2\,d\mathbf{x}\\ &= \frac{nh\,C_\nu}{3}\left(h^2E + 64\,f_h {C_\nu}^{-1}\right) \to -\infty\,.\end{aligned}$$

Referring to the bounded connected Lipschitz open set $\Omega \subset \mathbb{R}^2$, denote by $K(\Omega)$ the best constant such that

$$\int_\Omega \left|\mathbf{v} - ⨍_\Omega \mathbf{v}\right|^2 d\mathbf{x} \;\le\; K(\Omega)\int_\Omega |D\mathbf{v}|^2\;d\mathbf{x} \qquad \forall\, \mathbf{v}\in H^1\left(\Omega,\mathbb{R}^2\right)\,. \tag{2.24}$$

THEOREM 2.3 (mild uniform boundary compression of a simply supported plate). *Assume that* $\Omega \subset \mathbb{R}^2$ *is a bounded connected Lipschitz open set,* $g_h \in L^2(\Omega)$, *and*

$$\mathbf{f}_h = f_h \mathbf{n} \quad \textit{on } \partial\Omega, \tag{2.25}$$

where f_h *is a given constant such that*

$$f_h > -\frac{h^2 c_\nu E}{12\, K(\Omega)}. \tag{2.26}$$

Then, for every fixed $h>0$, $\mathcal{F}_h$ *achieves a minimum over* $H^1(\Omega,\mathbb{R}^2)\times H^2(\Omega)\cap H^1_0(\Omega)$.

Proof. Here, by setting $\Gamma=\partial\Omega$, we have $\mathcal{A}^1 = H^2(\Omega)\cap H^1_0(\Omega)$. Let $\mathcal{F}_h(\mathbf{u}_n,w_n)\to \inf_{H^1\times\mathcal{A}^1}\mathcal{F}_h$ and assume by contradiction that $\|\mathbb{E}(\mathbf{u}_n)\|\to+\infty$. By arguing as in the proof of Theorem 2.1 we can build a sequence $(\mathbf{v}_n,\zeta_n)\to(\mathbf{v},\zeta)$ weakly in $H^1(\Omega,\mathbb{R}^2)\times H^2(\Omega)$, $\|\mathbb{E}(\mathbf{v}_n)\|=1$, $\mathbb{D}(\mathbf{v}_n,\zeta_n)\to\mathbb{D}(\mathbf{v},\zeta)=\mathbb{O}$, $\mathbb{E}(\mathbf{v}_n)\to\mathbb{E}(\mathbf{v})$ both in $L^2(\Omega,\mathrm{Sym}_{2,2}(\mathbb{R}))$, $2\,\mathrm{div}\,\mathbf{v}=-|D\zeta|^2$ and

$$c_\nu\frac{h^3\,E}{24}\int_\Omega |D^2\zeta|^2+\frac12 h f_h\int_\Omega|D\zeta|^2\le 0\,; \tag{2.27}$$

we emphasize that $\zeta_n=0$ at $\partial\Omega$ entails $⨍_\Omega D\zeta_n=0$, and therefore $|\int_\Omega g_h\zeta_n|\le C\,\|g_h\|_{L^2}\|D^2\zeta_n\|_{L^2}$ for a suitable constant $C=C(\Omega)$, and hence (2.27) can be achieved even without assuming (2.12).

Therefore by taking into account that $\int_\Omega D\zeta=0$ (due to $\zeta\in H^1_0$), Poincaré inequality (2.24) and assumption (2.26) altogether entail

$$c_\nu\frac{h^3\,E}{24K(\Omega)}\int_\Omega|D\zeta|^2+\frac12 hf_h\int_\Omega|D\zeta|^2\le c_\nu\frac{h^3\,E}{24}\int_\Omega|D^2\zeta|^2+\frac12 hf_h\int_\Omega|D\zeta|^2\le 0\,, \tag{2.28}$$

so $D\zeta=0$ and, by $\mathbb{D}(\mathbf{v},\zeta)=\mathbb{O}$, $\mathbb{E}(\mathbf{v})=\mathbb{O}$, that is a contradiction since $\|\mathbb{E}(\mathbf{v}_n)\|_{L^2}=1$ and $\mathbb{E}(\mathbf{v}_n)\to\mathbb{E}(\mathbf{v})$ in $L^2(\Omega,\mathrm{Sym}_{2,2}(\mathbb{R}))$. The claim follows by repeating the last part of the Theorem 2.1 proof: here transverse load balancing (2.12) is not needed, due to boundary condition $\mathcal{A}^1$. ∎

Remark 2.4. By inspection of the proof of Theorem 2.3 we deduce also existence theorems for a plate clamped on a possibly proper subset Γ of the boundary. Precisely, assuming Ω bounded, connected, Lipschitz, (2.9), and (2.25) with $f_h > -(h^2c_\nu E)/(12\,\widetilde K(\Omega,\Gamma))$, where $\widetilde K(\Omega,\Gamma)$ is the best constant s.t. $\int_\Omega|\mathbf{v}|^2\,dx\le K(\Omega,\Gamma))$ $\{\int_\Omega|D\mathbf{v}|^2\,dx+\int_\Gamma|\mathbf{v}|^2\,d\mathcal{H}^1\}$, then $\mathcal{F}_h$ achieves a minimum over $H^1(\Omega,\mathbb{R}^2)\times\mathcal{A}^0(\Gamma)$.

Similar claims in $H^1(\Omega,\mathbb{R}^2)\times\mathcal{A}^1(\Gamma)$ (for plates supported on Γ) fail, even by adding assumption $\int_\Omega x_1g_hd\mathbf{x}=\int_\Omega x_2g_hd\mathbf{x}=0$. Indeed, if $\Omega=(0,1)^2$, $\Gamma=\{0\}\times[0,1]$, $g_h\equiv0$, $\mathbf{f}_h=-\lambda^2h^2\mathbf{n}$, then $\inf\mathcal{F}_h=-\infty$, as shown by $\mathbf{u}_m=-(1/6)(x_1+m)^3\,\mathbf{e}_1$, $w_m=((x_1+m)^2-m^2)/2$, $m\in\mathbb{N}$.

Concerning the existence of minimizers for $\mathcal{F}_h$ in $H^1(\Omega,\mathbb{R}^2)\times\mathcal{A}^i$ for $i=0,1$, when $\Gamma=\partial\Omega$, that is, for clamped and simply supported plates, respectively, at the whole boundary, in the presence of boundary forces which fulfill neither condition (2.13) nor conditions (2.25)–(2.26) we need to state first the following lemma (see also [23, Proposition 9]), which clarifies the link between $\ker\mathbb{D}$ and the solutions of the Monge–Ampère equation in Ω.

LEMMA 2.5. *Let* $\Omega\subset\mathbb{R}^2$ *be an open set and assume that* $\mathbf{u}\in H^1(\Omega,\mathbb{R}^2),\varphi\in H^2(\Omega)$ *satisfy*

$$2\mathbb{E}(\mathbf{u})+D\varphi\otimes D\varphi=0\ \text{ in }\Omega\,.$$

Then $\det D^2\varphi\equiv0$ *in* Ω, *where* $\det D^2\varphi$ *is the pointwise hessian of* φ.

Proof. Since $\mathbb{E}(\mathbf{u})$ satisfies the compatibility equation

$$\mathbb{E}_{11,22}+\mathbb{E}_{22,11}=2\mathbb{E}_{12,12}$$

in the sense of $\mathcal{D}'(\Omega)$, we get

$$\int_\Omega \psi_{,2}(\mathbb{E}_{11,2}-\mathbb{E}_{12,1})+\psi_{,1}(\mathbb{E}_{22,1}-\mathbb{E}_{12,2})\,d\mathbf{x}=0 \qquad \forall\psi\in C_0^\infty(\Omega)\,.$$

Therefore since $D\varphi\otimes D\varphi=-2\mathbb{E}(\mathbf{u})$ we get

$$\begin{aligned}\mathbb{E}_{22,1}&=-\,\varphi_{,2}\,\varphi_{,12}\,,\\ \mathbb{E}_{12,2}&=-\frac{1}{2}\,\varphi_{,2}\,\varphi_{,12}\,-\,\frac{1}{2}\,\varphi_{,1}\,\varphi_{,22}\,,\\ \mathbb{E}_{11,2}&=-\,\varphi_{,1}\,\varphi_{,12}\,,\\ \mathbb{E}_{12,1}&=-\frac{1}{2}\,\varphi_{,2}\,\varphi_{,11}\,-\,\frac{1}{2}\,\varphi_{,1}\,\varphi_{,12}\,.\end{aligned}$$

Summarizing

$$\frac{1}{2}\int_\Omega \psi_{,2}(\varphi_{,11}\varphi_{,2}-\varphi_{,1}\varphi_{,21})+\psi_{,1}(\varphi_{,1}\varphi_{,22}-\varphi_{,2}\varphi_{,21})\,d\mathbf{x}=0 \qquad \forall\psi\in C_0^\infty(\Omega)\,,$$

that is $\operatorname{Det}D^2\varphi=0$, where $\operatorname{Det}D^2\varphi$ is the distributional hessian of φ.

Since $\varphi\in H^2(\Omega)$ we have $\det D^2\varphi=\operatorname{Det}D^2\varphi=0$ in Ω. ∎

In what follows we state and prove an existence theorem for simply supported plates whose proof relies on a result by Pakzad [41] for the degenerate Monge–Ampère equations (see also [25]), which is recalled in the subsequent proposition.

Proposition 2.6 (see [41, Proposition 1.1]). *Assume $\Omega\subset\mathbf{R}^2$ is a bounded open convex set, $\mathbf{h}\in H^1(\Omega,\mathbb{R}^2)$ is a map with symmetric gradient, and the determinant of $D\mathbf{h}$ is vanishing a.e.*

Then for every point $\mathbf{x}\in\Omega$ there exists either a neighborhood U of $\mathbf{x}$ or a segment with endpoints on $\partial\Omega$ and passing through $\mathbf{x}$ where $\mathbf{h}$ is constant.

The above result entails the next crucial consequence.

Lemma 2.7. *Let $\Omega\subset\mathbf{R}^2$ be a bounded open convex set and $\zeta\in H^2(\Omega)\cap H_0^1(\Omega)$ be such that $\det D^2\zeta\equiv 0$ in Ω. Then $\zeta\equiv 0$ in Ω.*

Proof. We prove first that $D\zeta$ is continuous in Ω. Indeed, set $\mathbf{h}_\varepsilon:=D\zeta+\varepsilon(-x_2,x_1)$; then $\mathbf{h}_\varepsilon\in H^1(\Omega,\mathbf{R}^2)$ and $\det D\mathbf{h}_\varepsilon=\varepsilon^2>0$. Hence the continuity of $\mathbf{h}_\varepsilon$ in the whole Ω follows by a result of [47] (see also [34, Theorem 4.4]).

Continuity of $D\zeta$ follows by uniform convergence of $\mathbf{h}_\varepsilon$ to $D\zeta$.

For any pair of points $\mathbf{x}_1,\ \mathbf{x}_2\in\partial\Omega$ and $t\in(0,1)$ we set $\mathbf{x}_t:=t\mathbf{x}_1+(1-t)\mathbf{x}_2$ and we define

$$\begin{aligned}&T:=\{\mathbf{x}\in\Omega:\exists U \text{ open}:\ \mathbf{x}\in U,\ D\zeta \text{ constant in } U\}\\ &S:=\{\mathbf{x}\in\Omega:\exists\mathbf{x}_1,\ \mathbf{x}_2\in\partial\Omega,\ \bar t\in(0,1)\text{ s.t. }\mathbf{x}=\mathbf{x}_{\bar t},\ \ D\zeta(\mathbf{x}_t)\text{ is constant if } t\in(0,1)\}\end{aligned}$$

By Proposition 2.6 we get $\Omega=T\cup S$, $\zeta\equiv 0$ on $S\cup\partial\Omega$ and if either T or S is empty, the thesis follows easily.

Otherwise, $D\zeta$ is continuous and locally constant in the open set T, and hence $D\zeta$ is constant on each connected component C of T. Since $\partial C\subset S\cup\partial\Omega$ we get $\zeta\equiv 0$ on C and the thesis follows. ∎

Theorem 2.8 (simply supported plate). *If $\Omega\subset\mathbb{R}^2$ is a bounded, convex open set and $\mathbf{f}_h$ is an equilibrated in-plane load distribution, say,*

$$\int_{\partial\Omega}\mathbf{f}_h\cdot\mathbf{z}\;d\mathcal{H}^1=0 \qquad \forall\mathbf{z}\in\mathcal{R}\,, \tag{2.29}$$

then, for every fixed $h > 0$, the **FvK** *functional $\mathcal{F}_h$ in* (2.11) *achieves a minimum over $H^1(\Omega, \mathbb{R}^2) \times H^2(\Omega) \cap H^1_0(\Omega)$.*

Proof. Here $\Gamma \equiv \partial\Omega$ so, referring to (2.8), we look for minimizers of $\mathcal{F}_h$ over $H^1(\Omega, \mathbb{R}^2) \times \mathcal{A}^1 = H^1(\Omega, \mathbb{R}^2) \times H^2(\Omega) \cap H^1_0(\Omega)$. The proof will be achieved by showing the existence of a minimizing sequence equibounded in $H^1(\Omega, \mathbb{R}^2) \times H^2(\Omega)$, since $\mathcal{F}_h$ is sequentially l.s.c. with respect to the weak convergence in this space. Due to $\inf_{H^1 \times \mathcal{A}^1} \mathcal{F}_h \le \mathcal{F}_h(\mathbf{0}, 0) \le 0$, if $\mathcal{F}_h(\mathbf{u}_n, w_n) \to \inf_{H^1 \times \mathcal{A}^1} \mathcal{F}_h$ we may suppose $\mathcal{F}_h(\mathbf{u}_n, w_n) \le 1$. So by taking into account (2.29) and (2.7) we get via Korn and Poincaré inequality

(2.30)
$$c_\nu \frac{h^3\,E}{24} \int_\Omega |D^2 w_n|^2 + c_\nu \frac{h\,E}{2} \int_\Omega |\mathbb{D}(\mathbf{u}_n, w_n)|^2$$
$$\le h \int_\Omega \mathbf{f}_h \cdot \mathbf{u}_n + h \int_\Omega g_h w_n + 1$$
$$= h \int_\Omega \mathbf{f}_h \cdot \big(\mathbf{u}_n - \mathcal{R}(\mathbf{u}_n)\big) + h \int_\Omega g_h w_n + 1 \le \|\mathbb{E}(\mathbf{u}_n)\|_{L^2} \|\mathbf{f}_h\|_{L^2} + h\,\|g_h\|_{L^2} \|D w_n\|_{L^2} + 1.$$

Set $\lambda_n := \|\mathbb{E}(\mathbf{u}_n)\|_{L^2}$, assume by contradiction $\lambda_n \to +\infty$, and set $\mathbf{v}_n := \lambda_n^{-1} \mathbf{u}_n$ $\zeta_n := \lambda_n^{-1/2} w_n$. By substituting in (2.30) and dividing by λ_n, via Poincaré inequality in $H^2 \cap H^1_0$, we get

$$\begin{aligned} & c_\nu \frac{h^3\,E}{24} \int_\Omega |D^2 \zeta_n|^2 + \lambda_n c_\nu \frac{h\,E}{2} \int_\Omega |\mathbb{D}(\mathbf{v}_n, \zeta_n)|^2 \\ & \le \|\mathbf{f}_h\|_{L^2} + \lambda_n^{-1/2}\, h \|g_h\|_{L^2} \|D\zeta_n\|_{L^2} + \lambda_n^{-1} \\ & \le C + \lambda_n^{-1/2}\, h \int_\Omega |D\zeta_n|^2 \le C + \lambda_n^{-1/2}\, h \int_\Omega |D^2 \zeta_n|^2, \end{aligned} \tag{2.31}$$

thus obtaining as in the proof of Theorem 2.1

$$c_\nu \frac{h^3\,E}{24} \int_\Omega |D^2 \zeta_n|^2 + \lambda_n c_\nu \frac{h\,E}{2} \int_\Omega |\mathbb{D}(\mathbf{v}_n, \zeta_n)|^2 \le C' \tag{2.32}$$

for a suitable $C' > 0$. Since $\|\mathbb{E}(\mathbf{v}_n)\|_{L^2} = 1$, $D\zeta_n$ are then equibounded in $H^1(\Omega, \mathbb{R}^2)$ so, up to subsequences, $\zeta_n \to \zeta$ weakly in $H^2(\Omega)$, $D\zeta_n \to D\zeta$ strongly in $L^4(\Omega, \mathbb{R}^2)$, $\mathbf{v}_n \to \mathbf{v}$ weakly in $H^1(\Omega, \mathbb{R}^2)$, and $\mathbb{D}(\mathbf{v}_n, \zeta_n) \to 0$ strongly in $L^2(\Omega)$. Hence

$$2\mathbb{E}(\mathbf{v}_n) + D\zeta_n \otimes D\zeta_n \to 2\mathbb{E}(\mathbf{v}) + D\zeta \otimes D\zeta = \mathbb{O} \qquad \text{strongly in } L^2(\Omega, \mathrm{Sym}_{2,2}(\mathbb{R})) \tag{2.33}$$

and $\mathbb{E}(\mathbf{v}_n) \to \mathbb{E}(\mathbf{v})$ strongly in $L^2(\Omega, \mathrm{Sym}_{2,2}(\mathbb{R}))$. Then by Lemma 2.5 we have $\det D^2\zeta = 0$ and by taking into account that Ω is convex and $\zeta = 0$ on the whole $\partial\Omega$, by the uniqueness property of Lemma 2.7 we get $\zeta \equiv 0$ in Ω. This implies $\mathbb{E}(\mathbf{v}) = -\frac{1}{2} D\zeta \otimes D\zeta = \mathbb{O}$, which is a contradiction since $\|\mathbb{E}(\mathbf{v}_n)\|_{L^2} = 1$. Hence $\lambda_n \le C$ for suitable $C > 0$, so $\mathbf{u}_n - \mathcal{R}(\mathbf{u}_n)$ are equibounded in $H^1(\Omega, \mathbb{R}^2)$ and equiboundedness of w_n in $H^2(\Omega)$ follows from (2.32). Existence of minimizers is obtained via direct method. ∎

Remark 2.9. Concerning Theorem 2.8, at first we stated and proved an existence theorem for simply supported strictly convex plates, relying on a result by Rauch and Taylor (see [46, Theorem 5.1]) about the Dirichlet problem for the Monge–Ampère equation; subsequently an anonymous referee drew our attention to the result by

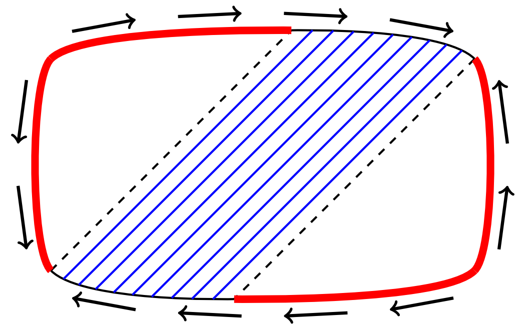

FIG. 1. *Counterexample* 2.10 *:* Γ *represented by the thicker part of boundary.*

Pakzad [41], which allows us to soften the strict convexity assumption into the plain convexity assumption of present Theorem 2.8. Here we take the opportunity to thank the referee for highlighting such issue.

Existence of minimizers may fail when $\Gamma \not\equiv \partial\Omega$: this happens even if the in-plane load $\mathbf{f}_h$ is equilibrated and the plate is strictly convex, as shown by the next counterexample.

COUNTEREXAMPLE 2.10 (buckling under in-plane shear). *Referring to Figure* 1, *set* $\gamma > 0$, $\varepsilon > 0$, $h^2 < \gamma/(6EC_\nu)$,

$$\Omega_\varepsilon = \left\{(x_1,x_2) : |x_1| < 2 + \varepsilon\left(1 - x_2^2\right),\ |x_2| < 1 + \varepsilon\left(4 - x_1^2\right)\right\},$$

$$\Gamma_\varepsilon = \partial\Omega_\varepsilon \cap \{(x_1,x_2) : |x_1 - x_2| \geq 1\}, \tag{2.34}$$

$$\mathbf{f}_h := \gamma\boldsymbol{\tau}\left(\mathbf{1}_{\Sigma^{2,\pm}} - \mathbf{1}_{\Sigma^{1,\pm}}\right), \tag{2.35}$$

where $\boldsymbol{\tau}$ *denotes the counterclockwise oriented unit vector tangent to* $\partial\Omega_\varepsilon = \Sigma_\varepsilon^{1,\pm} \cup \Sigma_\varepsilon^{2,\pm}$ *and*

$$\Sigma_\varepsilon^{1,\pm} = \left\{(x_1,x_2) : |x_1| \leq 2,\ x_2 = \pm\left(1 + \varepsilon\left(4 - x_1^2\right)\right)\right\},$$
$$\Sigma_\varepsilon^{2,\pm} = \left\{(x_1,x_2) : |x_2| \leq 1,\ x_1 = \pm\left(2 + \varepsilon\left(1 - x_2^2\right)\right)\right\}.$$

We claim that there exists $\widetilde{\varepsilon}$ *such that* $\inf \mathcal{F}_h = -\infty$ *over* $H^1(\Omega_{\widetilde{\varepsilon}}, \mathbb{R}^2) \times \mathcal{A}^1$ *under the assumptions listed above, notwithstanding the strict convexity of* $\Omega_{\widetilde{\varepsilon}}$ *and the fact that condition* (2.29) *holds true.*

Indeed, let $\psi \in C^{1,1}(\mathbb{R})$ *be an even function, with* $\operatorname{spt}\psi \subset [-1,1]$, $\psi' = -1$ *in* $[1/4, 3/4]$ *and* $|\psi''| \leq 4$ *in* $\mathbb{R}$. *We set* $\varphi(x_1,x_2) = \psi(x_1 - x_2)$ *and define* $w_n := \sqrt{n}\varphi$ *and* $\mathbf{u}_n := n\mathbf{u}$, *where*

$$u_2(x_1,x_2) = -u_1(x_1,x_2) = \frac{1}{2}\int_{-1}^{x_1 - x_2} |\psi'(\tau)|^2\, d\tau\ .$$

By setting $\Omega_0 := (-2,2) \times (-1,1) \subset \Omega_\varepsilon$, *there is* $C > 0$ *such that for every* $0 < \varepsilon \leq 1$

$$\left| \int_{\partial\Omega_\varepsilon} \mathbf{f}_h \cdot \mathbf{u}\; d\mathcal{H}^1 - \int_{\partial\Omega_0} \mathbf{f}_h \cdot \mathbf{u}\; d\mathcal{H}^1 \right| \leq C\,\varepsilon\ ,$$

and hence by (2.35) *there exists* $\widetilde{\varepsilon} \in (0,1)$ *such that*

$$\int_{\partial\Omega_{\widetilde{\varepsilon}}} \mathbf{f}_h \cdot \mathbf{u}\; d\mathcal{H}^1 \geq \int_{\partial\Omega_0} \mathbf{f}_h \cdot \mathbf{u}\; d\mathcal{H}^1 - \frac{\gamma}{2} = \gamma \int_{\Omega_0} 2\,\mathbb{E}_{12}(\mathbf{u})\; d\mathbf{x} - \frac{\gamma}{2}\ . \tag{2.36}$$

So

$$
\begin{aligned}
u_{1,1}(x_1,x_2) &= -\frac{1}{2}|\psi'(x_1-x_2)|^2 = -\frac{1}{2}\varphi_{,1}^2,\\
u_{2,2}(x_1,x_2) &= -\frac{1}{2}|\psi'(x_1-x_2)|^2 = -\frac{1}{2}\varphi_{,2}^2,\\
\frac{u_{1,2}+u_{2,1}}{2} &= \frac{1}{2}\left[\frac{1}{2}|\psi'(x_1-x_2)|^2 + \frac{1}{2}|\psi'(x_1-x_2)|^2\right]\\
&= \frac{1}{2}|\psi'(x_1-x_2)|^2 = -\frac{1}{2}\varphi_{,1}\varphi_{,2},
\end{aligned}
$$

that is, $\mathbb{E}(\mathbf{u}_n) = -\frac{1}{2}Dw_n \otimes Dw_n$ *and moreover, by* (2.7), (2.36), *and* $\varphi_{,2} = -\varphi_{,1}$ *we deduce*

$$
\begin{aligned}
&\mathcal{F}_h(\mathbf{u}_n,w_n)\\
&\le C_\nu\,\frac{h^3\,nE}{24}\int_{\Omega_0}|D^2\varphi|^2\,dx + C_\nu\,\frac{h^3\,nE}{24}\int_{\Omega_{\widetilde{\varepsilon}}\setminus\Omega_0}|D^2\varphi|^2 d\mathbf{x} + hn\gamma\int_{\Omega_0}\varphi_{,1}\varphi_{,2}d\mathbf{x} + hn\frac{\gamma}{2}\\
&\le C_\nu\,\frac{8h^3\,nE}{3}\Big(\big|\{(x_1,x_2)\in\Omega_0:\ 4|x_1-x_2|\le 1\ \ or\ \ 3\le 4|x_1-x_2|\le 4\}\big| + |\Omega_{\widetilde{\varepsilon}}\setminus\Omega_0|\Big)\\
&\quad - hn\gamma|\{(x_1,x_2)\in\Omega_0 : 1\le 4|x_1-x_2|\le 3\}| + hn\frac{\gamma}{2}\\
&\le 3C_\nu Eh^3 n - hn\frac{\gamma}{2} \to -\infty
\end{aligned}
\tag{2.37}
$$

as $n\to+\infty$ *whenever* $6EC_\nu\,h^2<\gamma$, *thus proving the claim.* ∎

Clearly Theorems 2.1, 2.3, 2.8 hold for the clamped plate too: minimization in $H^1(\Omega,\mathbb{R}^2)\times\mathcal{A}^0$. Even better, in the case of clamped plate we can drop both convexity assumption on Ω and equilibrated out-of-plane load (2.12) as is shown by the next result.

THEOREM 2.11 (clamped plate). *If* Ω *is a bounded connected Lipschitz open set and* (2.29) *holds, then for every fixed* $h>0$ *the functional* $\mathcal{F}_h$ *in* (2.11) *achieves its minimum over* $H^1(\Omega,\mathbb{R}^2)\times H_0^2(\Omega)$.

Proof. Again we need only to exhibit an equibounded minimizing sequence. Indeed, as in the proof of Theorem 2.8 if $\mathcal{F}_h(\mathbf{u}_n,w_n)\to\inf_{H^1\times H_0^2}\mathcal{F}_h$, we may suppose $\mathcal{F}_h(\mathbf{u}_n,w_n)\le 1$. Then, since $\Gamma=\partial\Omega$ entails $H_0^2(\Omega)=\mathcal{A}^0\subset\mathcal{A}^1$, by setting $\lambda_n := \|\mathbb{E}(\mathbf{u}_n)\|_{L^2}$, $\mathbf{v}_n := \lambda_n^{-1}\mathbf{u}_n$, $\zeta_n := \lambda_n^{-1/2}w_n$ and assuming $\lambda_n\to+\infty$, arguing as in the previous proofs we achieve the estimates (2.30), (2.31), (2.32). Then the sequence $D\zeta_n$ is equibounded in $H^1(\Omega,\mathbb{R}^2)$ so, up to subsequences, $\zeta_n\to\zeta$ weakly in $H^2(\Omega)$, $D\zeta_n\to D\zeta$ in $L^4(\Omega,\mathbb{R}^2)$, $\mathbf{v}_n\to\mathbf{v}$ weakly in $H^1(\Omega,\mathbb{R}^2)$, and $\mathbb{D}(\mathbf{v}_n,\zeta_n)\to\mathbb{O}$ in $L^2(\Omega,\mathrm{Sym}_{2,2}(\mathbb{R}))$. Hence

$$
2\mathbb{E}(\mathbf{v}_n) + D\zeta_n\otimes D\zeta_n \to 2\mathbb{E}(\mathbf{v}) + D\zeta\otimes D\zeta = \mathbb{O} \qquad \text{strongly in } L^2(\Omega,\mathrm{Sym}_{2,2}(\mathbb{R})),
\tag{2.38}
$$

$\mathbb{E}(\mathbf{v}_n)\to\mathbb{E}(\mathbf{v})$ strongly in $L^2(\Omega,\mathrm{Sym}_{2,2}(\mathbb{R}))$, and by Lemma 2.5 we have $\det D^2\zeta = 0$ in the whole Ω. Since $\zeta\equiv\frac{\partial\zeta}{\partial\mathbf{n}}\equiv 0$ on $\partial\Omega$, there exists a disk $\widetilde{\Omega}$ (bounded and convex!) such that $\Omega\subset\widetilde{\Omega}$ and the trivial extension $\widetilde{\zeta}$ of ζ in $\widetilde{\Omega}$ belongs to $H_0^2(\widetilde{\Omega})$. Therefore $\det D^2\widetilde{\zeta}=0$ on $\widetilde{\Omega}$ and still by [46, Theorem 5.1] we get $\widetilde{\zeta}\equiv 0$ in $\widetilde{\Omega}$, and hence $\zeta\equiv 0$ in Ω. Then by (2.38) $\mathbb{E}(\mathbf{v})=\mathbb{O}$, a contradiction since $\|\mathbb{E}(\mathbf{v}_n)\|_{L^2}=1$. ∎

3. Critical points nearby a flat configuration. When existence of global minimizers fails because the energy is unbounded from below, it is natural to investigate the structure of local minimizers or, more in general of critical points. Since the nonlinearity in the FvK functional relies in the interaction between membrane and bending contributions, we will focus in this section on the asymptotic analysis of critical points in the neighborhood of a flat configuration, i.e., we will study the behavior for small out-of-plane displacements. Throughout this section we assume that $h>0$ is fixed and

$$g_h \equiv 0, \tag{3.1}$$

that is, we restrict our analysis to the case of in-plane load acting on a plate of prescribed thickness. Assume $\mathbf{f}_h \in L^2(\partial\Omega, \mathbb{R}^2)$ and (2.29) holds true. For every $(\mathbf{u}, w) \in H^1(\Omega, \mathbb{R}^2) \times H^2(\Omega)$, referring to (2.1)–(2.11), we enclose boundary conditions in the functional, by setting

$$\mathcal{F}_h^i(\mathbf{u}, w) = \begin{cases} \mathcal{F}_h(\mathbf{u}, w) & \text{if } \mathbf{u} \in H^1(\Omega, \mathbb{R}^2),\ w \in \mathcal{A}^i\,, \\ +\infty & \text{otherwise}\ , \end{cases} \tag{3.2}$$

$$\mathcal{F}_{h,\varepsilon}^i(\mathbf{u}, w) = \mathcal{F}_h^i(\mathbf{u}, \varepsilon w) \qquad \forall\, \varepsilon > 0\,. \tag{3.3}$$

By noticing that $\mathcal{F}_{h,0} := \mathcal{F}_{h,0}^i$ actually is independent of i, we also set

$$\mathcal{E}_{h,\varepsilon}^i(\mathbf{u}, w) = \varepsilon^{-2}\left(\mathcal{F}_{h,\varepsilon}^i(\mathbf{u}, w) - \min_{H^1(\Omega,\mathbb{R}^2)} \mathcal{F}_{h,0}\right), \tag{3.4}$$

$$\mathcal{E}_h^i(\mathbf{u}, w) = \begin{cases} F_h^b(w) + \dfrac{h}{2}\displaystyle\int_\Omega J'(\mathbb{E}(\mathbf{u})) : Dw \otimes Dw\, d\mathbf{x} & \text{if } (\mathbf{u}, w) \in \{\operatorname{argmin} \mathcal{F}_{h,0}\} \times \mathcal{A}^i \\ +\infty & \text{else in } \ H^1(\Omega, \mathbb{R}^2) \times H^2(\Omega)\,, \end{cases} \tag{3.5}$$

where

$$J'(\mathbb{A}) = \frac{E}{1+\nu}\mathbb{A} + \frac{E\nu}{1-\nu^2}(\operatorname{Tr}\mathbb{A})\mathbb{I} \tag{3.6}$$

denotes the derivative of J.

Functionals $\mathcal{E}_{h,\varepsilon}^i$ and $\mathcal{F}_{h,\varepsilon}^i$ are linked via the following result

PROPOSITION 3.1. $\mathcal{E}_h^i = \Gamma \lim_{\varepsilon\to 0_+} \mathcal{E}_{h,\varepsilon}^i$. *Precisely, the following relations hold true:*

(i) *for every* $(\mathbf{u}_\varepsilon, w_\varepsilon) \rightharpoonup (\mathbf{u}, w)$ *in* $w - H^1 \times H^2$ *we have*

$$\liminf_{\varepsilon\to 0} \mathcal{E}_{h,\varepsilon}^i(\mathbf{u}_\varepsilon, w_\varepsilon) \geq \mathcal{E}_h^i(\mathbf{u}, w); \tag{3.7}$$

(ii) *for every* $(\mathbf{u}, w) \in H^1 \times H^2$ *there exists* $(\widetilde{\mathbf{u}}_\varepsilon, \widetilde{w}_\varepsilon) \rightharpoonup (\mathbf{u}, w)$ *in* $w - H^1 \times H^2$ *such that*

$$\lim_{\varepsilon\to 0} \mathcal{E}_{h,\varepsilon}^i(\widetilde{\mathbf{u}}_\varepsilon, \widetilde{w}_\varepsilon) = \mathcal{E}_h^i(\mathbf{u}, w). \tag{3.8}$$

Proof. Let $(\mathbf{u}_\varepsilon, w_\varepsilon) \rightharpoonup (\mathbf{u}, w)$ in $w - H^1 \times H^2$: by convexity of J we have

$$\begin{aligned} \mathcal{F}^i_{h,\varepsilon}(\mathbf{u}_\varepsilon, w_\varepsilon) &\geq \varepsilon^2 F^b_h(w) + h\int_\Omega J(\mathbb{E}(\mathbf{u}_\varepsilon))\,dx \\ &\quad + \frac{h\varepsilon^2}{2}\int_\Omega J'(\mathbb{E}(\mathbf{u}_\varepsilon)) : Dw_\varepsilon \otimes Dw_\varepsilon\,dx - h\int_{\partial\Omega} \mathbf{f}_h\cdot\mathbf{u}_\varepsilon\,d\mathcal{H}^1 \\ &\geq \varepsilon^2 F^b_h(w) + \frac{h\varepsilon^2}{2}\int_\Omega J'(\mathbb{E}(\mathbf{u}_\varepsilon)) : Dw_\varepsilon \otimes Dw_\varepsilon\,dx + \min \mathcal{F}_{h,0} \end{aligned} \tag{3.9}$$

and by taking into account that $Dw_\varepsilon \otimes Dw_\varepsilon \to Dw \otimes Dw$ strongly in $L^2(\Omega, \mathrm{Sym}_{2,2}(\mathbb{R}))$ and $J'(\mathbb{E}(\mathbf{u}_\varepsilon)) \rightharpoonup J'(\mathbb{E}(\mathbf{u}))$ weakly in $L^2(\Omega, \mathrm{Sym}_{2,2}(\mathbb{R}))$, we get

$$\liminf_{\varepsilon\to 0} \mathcal{E}^i_{h,\varepsilon}(\mathbf{u}_\varepsilon, w_\varepsilon) \geq \mathcal{E}^i_h(\mathbf{u}, w)$$

and (i) is proven. The proof of (ii) is achieved by taking $(\widetilde{\mathbf{u}}_\varepsilon, \widetilde{w}_\varepsilon) \equiv (\mathbf{u}, w)$. ◻

We recall that if $\mathcal{I} : X \to \mathbb{R}$ is any C^1 functional defined on a Banach space X, then $\overline{x} \in X$ is a critical point for $\mathcal{I}$ if $\mathcal{I}'(\overline{x}) = 0$, where $\mathcal{I}' : X \to X^*$ denotes the Gateaux differential of $\mathcal{I}$.

Due to formula (3.10) below, $\mathcal{F}^i_{h,\varepsilon}$ is a C^1 functional in the Hilbert space $H^1(\Omega, \mathbb{R}^2) \times \mathcal{A}^i$: precisely, for every $(\mathbf{u}, w) \in H^1(\Omega, \mathbb{R}^2) \times \mathcal{A}^i$ the Gateaux differential of $\mathcal{F}^i_{h,\varepsilon}$ at $(\mathbf{u}, w)$ is given by

$$(\mathcal{F}^i_{h,\varepsilon})'(\mathbf{u}, w)[(\mathbf{z}, \omega)] = \Big(\tau_1(\mathbf{u}, w)[\mathbf{z}]\,,\, \tau_2(\mathbf{u}, w)[\omega]\Big) \qquad \forall \mathbf{z} \in H^1(\Omega, \mathbb{R}^2)\,, \forall \omega \in \mathcal{A}^i\,,$$

where

$$\begin{aligned} \tau_1(\mathbf{u}, w)[\mathbf{z}] &:= h\int_\Omega J'\left(\mathbb{E}(\mathbf{u}) + \frac{\varepsilon^2}{2} Dw \otimes Dw\right) : \mathbb{E}(\mathbf{z}) - h\int_{\partial\Omega} \mathbf{f}_h \cdot \mathbf{z}\,, \\ \tau_2(\mathbf{u}, w)[\omega] &:= \varepsilon^2\,\frac{h^3}{12}\int_\Omega J'(D^2 w) : D^2\omega + \varepsilon^2\,h\int_\Omega J'\left(\mathbb{E}(\mathbf{u}) + \frac{\varepsilon^2}{2} Dw \otimes Dw\right) : Dw \odot D\omega\,. \end{aligned} \tag{3.10}$$

$(\tau_1(\mathbf{u}, w)[\mathbf{z}], \tau_2(\mathbf{u}, w)[\omega])$ is replaced by the shorter notation $(\,\tau_1[\mathbf{z}]\,,\, \tau_2[\omega]\,)$, whenever the dependance on fixed choice for $(\mathbf{u}, w)$ is understood. Actually (3.10) provides the explicit information that $(\mathcal{F}^i_{h,\varepsilon})'(\mathbf{u}, w)$ depends continuously on $(\mathbf{u}, w)$.

Hence the Föppl–von Kármán plate equations in weak form together with boundary conditions can be written as follows:

$$\begin{cases} \mathbf{u},\, w \in H^1(\Omega, \mathbb{R}^2) \times \mathcal{A}^i\,, & \\ \tau_1(\mathbf{u}, w)[\mathbf{z}] = 0 & \forall \mathbf{z} \in H^1(\Omega, \mathbb{R}^2)\,, \\ \tau_2(\mathbf{u}, w)[\omega] = 0 & \forall \omega \in \mathcal{A}^i\,. \end{cases} \tag{3.11}$$

Clearly $(\mathcal{E}^i_{h,\varepsilon})'(\mathbf{u}, w) = \varepsilon^{-2}(\mathcal{F}^i_{h,\varepsilon})'(\mathbf{u}, w)$, hence $\mathcal{F}^i_{h,\varepsilon}$ and $\mathcal{E}^i_{h,\varepsilon}$ have the same critical points. Moreover if $\mathbf{u}_* \in \operatorname{argmin} \mathcal{F}_{h,0}$, then $\tau_2(\mathbf{u}_*, 0) \equiv 0$ and $(\mathbf{u}_*, 0)$ is a critical point for $\mathcal{F}^i_{h,\varepsilon}$.

The next definition tunes the standard notion of the *Palais–Smale sequence* to the present context.

DEFINITION 3.2. *Let X be a Banach space and $\mathcal{I}_\varepsilon : X \to \mathbb{R}$ be a sequence of C^1 functionals. A sequence $\{x_\varepsilon\} \subset X$ is a* ***uniform Palais–Smale sequence*** *if there exists $C > 0$ such that $\mathcal{I}_\varepsilon(x_\varepsilon) \leq C$ and $\|\, \mathcal{I}'_\varepsilon(x_\varepsilon)\,\|_{X^*} \to 0$, as $\varepsilon \to 0_+$.*

Notice that the above definition reduces to the usual notion of Palais–Smale sequences when $\mathcal{I}_\varepsilon \equiv \mathcal{I}$ for every $\varepsilon > 0$. Letting $\mathbf{u}_* \in \operatorname{argmin} \mathcal{F}_{h,0}$, we denote by $\mathcal{K}_h^i(\mathbf{u}_*)$ the set of critical points in $\mathcal{A}^i$ of $\mathcal{E}_h^i(\mathbf{u}_*, \cdot)$, that is,

$$\mathcal{K}_h^i(\mathbf{u}_*) = \{w \in \mathcal{A}^i : \tau_2(\mathbf{u}_*, w)[\omega] = 0 \ \forall\, \omega \in \mathcal{A}^i\} \ . \tag{3.12}$$

The next result shows that any critical point of $\mathcal{E}_h(\mathbf{u}_*, \cdot)$ in $\mathcal{A}^i$ can be approximated by a uniform Palais–Smale sequence of $\mathcal{E}_{h,\varepsilon}^i$ whose energy converges to the energy of the critical point itself.

THEOREM 3.3. *Let* $\mathbf{u}_* \in \operatorname{argmin} \mathcal{F}_{h,0}$, $w \in \mathcal{K}_h^i(\mathbf{u}_*)$ *and* $\mathbf{z}_w \in \operatorname{argmin} \mathcal{Q}_w(\mathbf{z})$, *where*

$$\mathcal{Q}_w(\mathbf{z}) := \int_\Omega J\left(\mathbb{E}(\mathbf{z}) + \frac{1}{2} Dw \otimes Dw\right) d\mathbf{x}\,. \tag{3.13}$$

Then $\{(\mathbf{u}_* + \varepsilon^2 \mathbf{z}_w, w)\}_{\varepsilon > 0}$ *is a uniform Palais–Smale sequence for* $\mathcal{E}_{h,\varepsilon}^i$ *and*

$$\lim_{\varepsilon \to 0_+} \mathcal{E}_{h,\varepsilon}^i\left(\mathbf{u}_* + \varepsilon^2 \mathbf{z}_w, w\right) = \mathcal{E}_h^i(\mathbf{u}_*, w) \ .$$

Proof. We have to prove the following conditions:

(a) $\mathcal{E}_{h,\varepsilon}^i(\mathbf{u}_* + \varepsilon^2 \mathbf{z}_w, w) \ \leq \ C \ < +\infty \quad \forall \varepsilon \in (0, 1]$.

(b) $(\mathcal{E}_{h,\varepsilon}^i)'(\mathbf{u}_* + \varepsilon^2 \mathbf{z}_w, w) \to 0$ strongly in $(H^1(\Omega, \mathbb{R}^2) \times \mathcal{A}^i)^*$.

(c) $\lim_{\varepsilon \to 0_+} \mathcal{E}_{h,\varepsilon}^i(\mathbf{u}_* + \varepsilon^2 \mathbf{z}_w, w) = \mathcal{E}_h^i(\mathbf{u}_*, w)$.

We first prove (c), which implies (a) too. Indeed

$$\begin{aligned}
&\mathcal{E}_{h,\varepsilon}^i(\mathbf{u}_* + \varepsilon^2 \mathbf{z}_w, w)\\
&= \varepsilon^{-2}\left[\mathcal{F}_h^1(\mathbf{u}_* + \varepsilon^2 \mathbf{z}_w, \varepsilon w) - \mathcal{F}_{0,h}(\mathbf{u}_*)\right]\\
&= \varepsilon^{-2}\left[\frac{h^3}{12}\int_\Omega J(\varepsilon D^2 w)\, dx + h \int_\Omega J\left(\mathbb{E}(\mathbf{u}_*) + \varepsilon^2 \mathbb{E}(\mathbf{z}_w) + \frac{\varepsilon^2}{2} Dw \otimes Dw\right) d\mathbf{x}\right]\\
&\quad - \varepsilon^{-2}\left[h \int_\Omega J(\mathbb{E}(\mathbf{u}_*)) + \varepsilon^2 h \int_{\partial\Omega} \mathbf{f}_h \cdot \mathbf{z}_w\right]\\
&= \varepsilon^{-2}\left[\frac{h^3}{12}\varepsilon^2 \int_\Omega J(D^2 w)\, d\mathbf{x} + h \int_\Omega J(\mathbb{E}(\mathbf{u}_*) + \varepsilon^4 h \int_\Omega J\left(\mathbb{E}(\mathbf{z}_w) + \frac{1}{2} Dw \otimes Dw\right) d\mathbf{x}\right]\\
&\quad + \varepsilon^{-2}\left[\varepsilon^2 h \int_\Omega J'(\mathbb{E}(\mathbf{u}_*)) : \left(\mathbb{E}(\mathbf{z}_w) + \frac{1}{2} Dw \otimes Dw\right) d\mathbf{x} - h \int_\Omega J(\mathbb{E}(\mathbf{u}_*)) - \varepsilon^2 \int_{\partial\Omega} \mathbf{f}_h \cdot \mathbf{z}_w\right]\\
&= \frac{h^3}{12}\int_\Omega J(D^2 w)\, d\mathbf{x} + \varepsilon^2 h \int_\Omega J\left(\mathbb{E}(\mathbf{z}_w) + \frac{1}{2} Dw \otimes Dw\right) d\mathbf{x}\\
&\quad + h \int_\Omega J'(\mathbb{E}(\mathbf{u}_*)) : \left(\mathbb{E}(\mathbf{z}_w) + \frac{1}{2} Dw \otimes Dw\right) d\mathbf{x} - h \int_{\partial\Omega} \mathbf{f}_h \cdot \mathbf{z}_w\\
&= \frac{h^3}{12}\int_\Omega J(D^2 w)\, d\mathbf{x} + \varepsilon^2 h \int_\Omega J\left(\mathbb{E}(\mathbf{z}_w) + \frac{1}{2} Dw \otimes Dw\right) d\mathbf{x}\\
&\quad + \frac{h}{2}\int_\Omega J'(\mathbb{E}(\mathbf{u}_*)) : Dw \otimes Dw\, d\mathbf{x}
\end{aligned}$$

since, due to minimality of $\mathbf{u}_*$,

$$\int_\Omega J'(\mathbb{E}(\mathbf{u}_*)) : \mathbb{E}(\mathbf{z}_w)\, d\mathbf{x} - \int_{\partial\Omega} \mathbf{f}_h \cdot \mathbf{z}_w = 0.$$

Hence $\lim_{\varepsilon \to 0} \mathcal{E}_{h,\varepsilon}^i(\mathbf{u}_* + \varepsilon^2 \mathbf{z}_w, w) = \mathcal{E}_h^i(\mathbf{u}_*, w)$ as claimed.

Eventually we prove (b). By recalling (3.4) and (3.10), we get for every $\mathbf{z} \in H^1(\Omega, \mathbb{R}^2)$ and $\omega \in \mathcal{A}^i$

$$(\mathcal{E}^i_{h,\varepsilon})' \left(\mathbf{u}_* + \varepsilon^2 \mathbf{z}_w, w\right) [(\mathbf{z}, \omega)] = \varepsilon^{-2} \, \left(\tau_1 \left(\, \mathbf{u}_* + \varepsilon^2 \mathbf{z}_w, w \,\right) [\mathbf{z}] \, , \, \tau_2 \left(\, \mathbf{u}_* + \varepsilon^2 \mathbf{z}_w, w \,\right) [\omega] \,\right) \; .$$

Since $\mathbf{z}_w \in \operatorname{argmin} \mathcal{Q}(\mathbf{z})$, $\mathbf{u}_* \in \operatorname{argmin} \mathcal{F}_{h,0}$, and $w \in \mathcal{K}^i_w(\mathbf{u}_*)$ we get

$$\tau_1(\mathbf{u}_*, 0)[\mathbf{z}] = 0 \;\; \forall \mathbf{z} \in H^1(\Omega, \mathbb{R}^2) \, , \qquad \tau_2(\mathbf{u}_*, w)[\omega] = 0 \;\; \forall \omega \in \mathcal{A}^i \, ,$$

$$\varepsilon^{-2} \, \tau_2(\mathbf{u}_* + \varepsilon^2 \mathbf{z}_w, w)[\omega] = \varepsilon^2 \int_\Omega J' \left(\mathbb{E}(\mathbf{z}_w) + \frac{1}{2} Dw \otimes Dw \right) : Dw \otimes D\omega \; .$$

The above relationships imply

$$\sup_{\|(\mathbf{z},\omega)\| \le 1} \left| \, (\mathcal{E}^i_{h,\varepsilon})'(\mathbf{u}_* + \varepsilon^2 \mathbf{z}_w, w)[(\mathbf{z}, \omega)] \, \right| \to 0 \quad \text{as} \quad \varepsilon \to 0,$$

where $\|\,(\mathbf{z}, \omega)\,\| = \|\,\mathbf{z}\,\|_{H^1} + \|\,\omega\,\|_{H^2}$, thus proving (b) . ☐

Remark 3.4. Despite the results of this section, the problem of the existence of nonminimizing critical points remains open in the general case, to the best of our knowledge. Nevertheless as far as uniform Palais–Smale sequences are surrogates of critical points for (1.11) in a ε-neighborhood of the flat configuration at the scale ε^2, Theorem 3.3 allows us to recover them starting from critical points of the limit functional $\mathcal{E}^i_h$ (see Examples 3.8 and 3.9).

In addition we emphasize that that existence of nonminimizing critical points cannot be deduced here by applying the asymptotic mini-max of [27, Theorem 4.4], since its compactness condition can be violated, as we show in Counterexample 3.7 below.

Remark 3.5. Let $\mathbf{u}_* \in \operatorname{argmin} \mathcal{F}_{h,0}$, $w \in \mathcal{K}^i_h(\mathbf{u}_*)$ and then

$$0 = \mathcal{E}^i_h(\mathbf{u}_*, w)'[(\mathbf{0}, w)] = \frac{h^3}{12} \int_\Omega J'(D^2 w) \cdot D^2 w \, d\mathbf{x} + h \int_\Omega J' \left(\mathbb{E}(\mathbf{u}_*)\right) : Dw \otimes Dw, \tag{3.14}$$

that is, $\mathcal{E}^i_h(\mathbf{u}_*, w) = 0$ and $\mathcal{E}^i_{h,\varepsilon}(\mathbf{u}_* + \varepsilon^2 \mathbf{z}_w, w) = \varepsilon^2 h \min \mathcal{Q}_w$.

Remark 3.6. In Theorem 3.3 we have shown that every critical point for $\mathcal{E}^i_h$ of the kind $(\mathbf{u}_*, w)$ with $\mathbf{u}_* \in \operatorname{argmin} \mathcal{F}_{h,0}$ and $w \in \mathcal{K}^i_h(\mathbf{u}_*)$ can be approximated (in the strong convergence of $H^1(\Omega, \mathbb{R}^2) \times H^2(\Omega)$) by uniform Palais–Smale sequences of $\mathcal{E}^i_{h,\varepsilon}$. Actually the displacement pair sequence can be chosen explicitly of the kind $(\mathbf{u}_* + \varepsilon^2 \mathbf{z}_w, w)$, say, with fixed out-of-plane component and in-plane displacement approximated by an infinitesimal correction tuned by the out-of-plane component. Nevertheless we cannot expect that every uniform Palais–Smale sequence of $\mathcal{E}^i_{h,\varepsilon}$ is equibounded in $H^1(\Omega, \mathbb{R}^2) \times \mathcal{A}^i$, as we are going to show in the next counterexample.

Counterexample 3.7 (a uniform Palais–Smale sequence lacking compactness). *Referring to Figure* 2 *if* $\Omega = (0, a) \times (0, 1)$, $\Gamma \equiv \partial\Omega$, *and* $\mathbf{f}_h = \gamma \, \mathbf{e}_2(\mathbf{1}_{(0,a)\times\{0\}}) - \mathbf{1}_{(0,a)\times\{1\}})$, *where* γ *is a suitable constant to be chosen later, then the unboundedness may develop.*

So by Theorem 2.11 *(clamped plate), for all* $h > 0$, *for all* $\varepsilon > 0$ *there exists* $(\mathbf{u}_\varepsilon, w_\varepsilon) \in \operatorname{argmin} \mathcal{E}^0_{h,\varepsilon}$. *Hence* $(\mathbf{u}_\varepsilon, w_\varepsilon)$ *is a uniform Palais–Smale sequence for* $\mathcal{E}^0_{h,\varepsilon}$, *and moreover we show below that such a sequence must lack weak compactness in* $H^1(\Omega, \mathbb{R}^2) \times H^2(\Omega)$ *for big* γ. *Indeed, if compactness were true, we would obtain (up to subsequences) that* $(\mathbf{u}_\varepsilon, w_\varepsilon) \rightharpoonup (\mathbf{u}, w) \in \operatorname{argmin} \mathcal{E}^0_h$, *due to Proposition* 3.1. *Eventually we show that* $\inf \mathcal{E}^0_h = -\infty$, *thus obtaining a contradiction.*

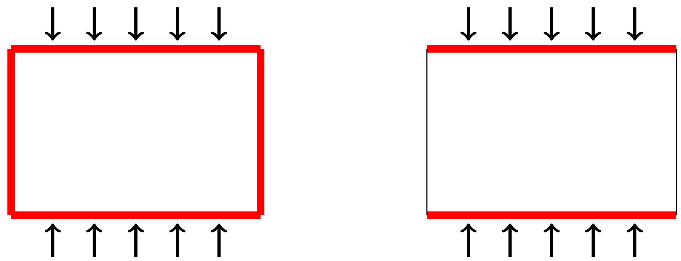

FIG. 2. *Plates of Counterexample* 3.7 *(left) and Example* 3.8 *(right): Γ is represented by the thicker part of the boundary.*

Actually, due to Euler equations

$$\int_\Omega J'\big(\mathbb{E}(\mathbf{u})\big) : \mathbb{E}(\mathbf{v}) = \int_{\partial\Omega} \mathbf{f}_h \cdot \mathbf{v} = -\,\gamma \int_\Omega v_{2,2} \qquad \forall \mathbf{v} \in H^1(\Omega, \mathbb{R}^2)\,, \tag{3.15}$$

so, for every $\mathbf{u} \in \operatorname{argmin} \mathcal{F}_{h,0}$, $J'(\mathbb{E}(\mathbf{u})) = -\gamma \mathbf{e}_2 \otimes \mathbf{e}_2$, $\mathbf{u} = 2\frac{\gamma\nu}{E}\frac{1+\nu}{1+3\nu}(x_1\mathbf{e}_1 + x_2\mathbf{e}_2) + r$, $r \in \mathcal{R}$, *and by* (2.2)

$$\mathcal{E}_h^0(\mathbf{u}, w) = \begin{cases} \dfrac{h^3}{12}\displaystyle\int_\Omega J(D^2 w) - \dfrac{h\gamma}{2}\displaystyle\int_\Omega |w_{,2}|^2 d\mathbf{x} & \textit{if}\, \mathbf{u} \in \operatorname{argmin} \mathcal{F}_{h,0}(\cdot, 0), w \in \mathcal{A}^0\,, \\ +\infty & \textit{otherwise in} \;\; H^1(\Omega) \times H^2(\Omega)\,. \end{cases} \tag{3.16}$$

Hence, if $\mathbf{u} \in \operatorname{argmin} \mathcal{F}_{h,0}$, $w \in \mathcal{A}^0$, *we get*

$$\mathcal{E}_h^0(\mathbf{u}, w) \le \frac{C_\nu\, E\, h^3}{24} \int_\Omega |D^2 w|^2 d\mathbf{x} - \frac{h}{2}\gamma \int_\Omega |w_{,2}|^2\, d\mathbf{x}.$$

Set $w(x_1, x_2) = \alpha(x_1)\beta(x_2)$ *with* $\alpha \in H_0^2(0, a)$ *and* $\beta \in H_0^2(0, 1)$. *Then* $w \in H_0^2(\Omega)$ *and*

$$\mathcal{E}_h^2(\mathbf{u}, w) \le (A_0 C_0 + A_1 C_1 + A_2)\, C_\nu\, \frac{Eh^3}{24} \int_0^1 |\beta''|^2\, dx_2 - \frac{A_2 h\gamma}{2} \int_0^1 |\beta'|^2 dx_2, \tag{3.17}$$

where

$$A_0 = \int_0^1 |\alpha''|^2 dx_1, \quad A^1 = 2\int_0^1 |\alpha'|^2 dx_1, \quad A_2 = \int_0^1 \alpha^2 dx_1,$$

and C_0, C_1 *are the best constants such that*

$$\int_0^1 \beta^2 dx_2 \le C_0 \int_0^1 |\beta''|^2 dx_2, \quad \int_0^1 |\beta'|^2 dx_2 \le C_1 \int_0^1 |\beta''|^2 dx_2 \quad \forall \beta \in H_0^2(0,1).$$

If $\xi \in H_0^2(0,1)$ *is the eigenfunction fulfilling the equality* $\int_0^1 |\xi'|^2 dx_2 = C_1 \int_0^1 |\xi''|^2 dx_2$ *and*

$$\gamma > \frac{1}{6}\,(A_0 C_0 + A_1 C_1 + A_2)\; C_\nu\, E\, h^2/(A_2 C_1)\,,$$

setting $\beta_n := n\xi \in H_0^2(0,1)$ *and* $w = \alpha\beta_n$, *the right-hand side of* (3.17) *goes to* $-\infty$ *as* $n \to \infty$.

In the previous counterexample we have shown that some uniform Palais–Smale sequence may be not converging to any critical point, while in the next examples we show how Theorem 2.3 can be used to detect buckled configurations of the plate (associated to critical points for FvK) by means of uniform Palais–Smale sequences for the approximating functionals.

Example 3.8 (buckling of a rectangular plate under compressive load). Referring to Figure 2, set $\Omega = (0,a)\times(0,1)$, $\mathbf{f}_h = \gamma\,\mathbf{e}_2(\mathbf{1}_{(0,a)\times\{0\}}) - \mathbf{1}_{(0,a)\times\{1\}})$, and $\Gamma = \Sigma_+\cup\Sigma_-$ with $\Sigma_+ = [0,1]\times\{1\}$, $\Sigma_- = [0,1]\times\{0\}$.

Now $\Gamma \neq \partial\Omega$: by arguing as the in previous counterexample we find noncompact uniform Palais–Smale sequences together with energy of admissible configurations unbounded from below.

In the present case we push forward the analysis: as before we find that if $\mathbf{u} \in \operatorname{argmin} \mathcal{F}_{h,0}$ and $w \in \mathcal{A}^i$, i=0,1,2, then $J'(\mathbb{E}(\mathbf{u})) = -\gamma\mathbf{e}_2\otimes\mathbf{e}_2$, so that

$$\mathcal{E}_h^i(\mathbf{u},w) = \frac{h^3}{12}\int_\Omega J(D^2w)d\mathbf{x} - \frac{h\gamma}{2}\int_\Omega |w_{,2}|^2 d\mathbf{x} \qquad \text{if } \mathbf{u}\in \operatorname{argmin}\mathcal{F}_{h,0}^i(\cdot,0),\ w\in\mathcal{A}^i\,.$$

We look for critical points in the form $w = w(x_2)$ under the following conditions:

$$\begin{aligned} &w(0) = w(1) = w'(0) = w'(1) = 0 && \text{if } i = 0\,;\\ &w(0) = w(1) = 0 && \text{if } i = 1\,;\\ &w(0)'' = w''(1) = w'''(0) = w'''(1) = 0 && \text{if } i = 2\,. \end{aligned}$$

Since $J(\mathbf{e}_2\otimes\mathbf{e}_2) = \frac{E}{2(1-\nu^2)}$, we have

$$\mathcal{E}_h^i(\mathbf{u},w) = \frac{Eh^3}{24(1-\nu^2)}\int_0^1 |w''(x_2)|^2\,dx_2 - \frac{h\gamma\,a}{2}\int_0^1 |w'(x_2)|^2\,dx_2,$$

whose nontrivial critical points can be easily computed, via the ODE

$$w'''' + \frac{12\gamma a(1-\nu^2)}{Eh^2}w'' = 0.$$

Theorem 3.3 allows us to recover Palais–Smale sequences for $\mathcal{E}_{h,\varepsilon}^i\,,\ i = 0,1,2.$

In the clamped case ($i = 0$) the nontrivial buckled solutions occur for discrete choices of h (e.g., see Figure 3):

$$\begin{aligned} h_n &= \frac{1}{2\,n\,\pi}\sqrt{\frac{12\,\gamma\,a\,(1-\nu^2)}{E}}\,,\\ w_n(x_2) &= 1 + \sin\left(\sqrt{\frac{12\,\gamma\,a\,(1-\nu^2)}{E}}\,\frac{1}{h}\,(x_2 - \pi/2)\right), \qquad n\in\mathbb{N}; \end{aligned}$$

else, for any other choice of h, $w \equiv 0$.

The associated Palais–Smale sequence is

$$(\,2\frac{\gamma\nu}{E}\frac{1+\nu}{1+3\nu}(x_1\mathbf{e}_1 + x_2\mathbf{e}_2) + \varepsilon^2\mathbf{z}_{w_n}(x_1,x_2)\,,\, w_n(x_2))\,,$$

where $\mathbf{z}_{w_n}(x_1,x_2) = (\,0\,,\,1/2\int_0^{x_2}|w_n'(t)|^2dt\,)$ and w_n is given above.

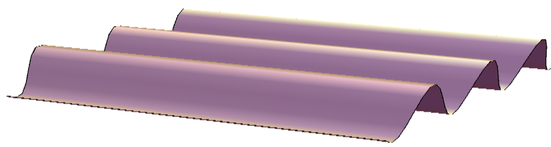

FIG. 3. *One solution* (w_3) *of ODE with* $i=0$ *in Example* 3.8.

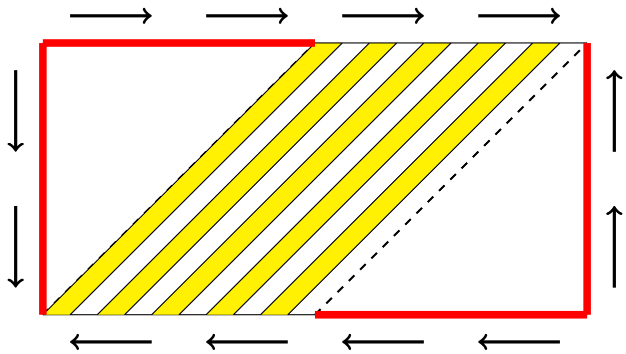

FIG. 4. *Example* 3.9 *:* Γ *is represented by the thicker part of the boundary.*

Example 3.9 (buckling of a rectangular plate under shear load). Referring to Figure 4, set $\Omega=(-2,2)\times(-1,1)$, $i=0$, and $\Gamma=\Sigma^{1,\pm}\cup\Sigma^{2,\pm}$, where

$$\Sigma^{1,+}=[-2,0]\times\{1\},\Sigma^{1,-}=[0,2]\times\{-1\},\Sigma^{2,+}=\{2\}\times[-1,1],\Sigma^{2,-}=\{-2\}\times[-1,1].$$

Assume $\mathbf{f}_h=\gamma\boldsymbol{\tau}(\mathbf{1}_{S^{2,\pm}}-\mathbf{1}_{S^{1,\pm}})$, where $S^{2,\pm}=\Sigma^{2,\pm}$, $S^{1,\pm}=[-2,2]\times\{\pm1\}$, $\gamma>0$, $\boldsymbol{\tau}$ is the counterclockwise oriented tangent unit vector to $\partial\Omega=S^{1,\pm}\cup S^{2,\pm}$. See Figure 4.

Since $\mathbf{u}\in\operatorname{argmin}\mathcal{F}_{h,0}$, by exploiting Euler–Lagrange equations as before, we obtain $J'(\mathbb{E}(\mathbf{u}))=\gamma(\mathbf{e}_1\otimes\mathbf{e}_2+\mathbf{e}_2\otimes\mathbf{e}_1)$ and by (2.2)

$$\mathcal{E}_h^0(\mathbf{u},w)=\frac{h^3}{12}\int_\Omega J(D^2w)dx+h\gamma\int_\Omega w_{,1}\,w_{,2}\,d\mathbf{x}.$$

We look for critical points in the form

$$w=\begin{cases}\psi(x_1-x_2) & \text{if } \ (x_1,x_2)\in\Omega,\ \ |x_1-x_2|\le1,\\ 0 & \text{else in } \Omega,\end{cases}\tag{3.18}$$

and satisfying $\psi(\pm1)=\psi'(\pm1)=0$.

By $J(\mathbf{e}_1\otimes\mathbf{e}_1+\mathbf{e}_2\otimes\mathbf{e}_2-\mathbf{e}_1\otimes\mathbf{e}_2-\mathbf{e}_2\otimes\mathbf{e}_1)=\frac{2E}{1-\nu^2}$ we obtain

$$\mathcal{E}_h^0(\mathbf{u},w)=\frac{h^3E}{3(1-\nu^2)}\int_{-1}^{1}|\psi''(t)|^2\,dt-2h\gamma\int_{-1}^{1}|\psi'(t)|^2\,dt,$$

whose nontrivial critical points can be easily computed, via the ODE

$$\psi''''+\frac{6\gamma(1-\nu^2)}{Eh^2}\,\psi''=0\,,\qquad \psi(\pm1)=\psi'(\pm1)=0\,.$$

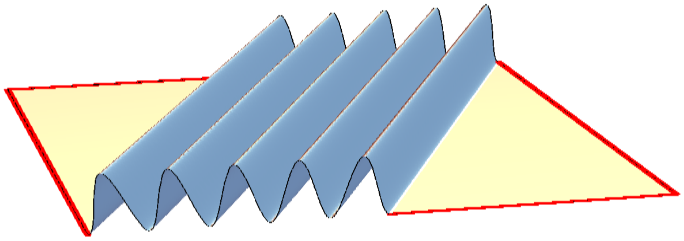

FIG. 5. *Example* 3.9 $w_5(x_1,x_2) = \psi_5(x_1 - x_2)$.

Therefore even now the nontrivial buckled solutions occur for (different) discrete choices of h (e.g., see Figure 5):

$$w = w_n(x_1,x_2) = \psi_n(x_1 - x_2) := 1 + \sin\left(\sqrt{\frac{12\,\gamma\,a\,(1-\nu^2)}{E}}\,\frac{1}{h_n}\,(x_1 - x_2 + 1/2)\right)$$

$$\text{if} \qquad h_n = \frac{1}{n\,\pi}\sqrt{\frac{12\,\gamma\,a\,(1-\nu^2)}{E}} \qquad \text{with} \quad n \in \mathbb{N};$$

else, we have the flat solution $w \equiv 0$ for any other choice of h .

The associated Palais–Smale sequence is $(\,\mathbf{u}(x_1,x_2) + \varepsilon^2 \mathbf{z}_{w_n}(x_1,x_2)\,,\, w_n(x_1,x_2)\,)$, where

$$\mathbf{u}(x_1,x_2) = \gamma\frac{1+\nu}{E}(x_2,x_1)\,,$$
$$\mathbf{z}_{w_n}(x_1,x_2) = \left(-(1/2)\int_{-1}^{x_1-x_2} |w_n'(t)|^2\,dt\,,\ (1/2)\int_{-1}^{x_1-x_2} |w_n'(t)|^2\,dt\right)\,.$$

Remark 3.10. In Examples 3.8, 3.9, when nontrivial solutions exist the period of the oscillations has order h. By scaling loads, that is, by taking $\mathbf{f}_h = h^\alpha \mathbf{f}$, we get $J'(\mathbb{E}(\mathbf{u})) = -h^\alpha\gamma(\mathbf{e}_2 \otimes \mathbf{e}_2)$ and $J'(\mathbb{E}(\mathbf{u})) = h^\alpha\gamma(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1)$, respectively, while related limit functionals become, respectively,

$$\mathcal{E}_h^i(\mathbf{u},w) = \frac{Eh^3}{24(1-\nu^2)}\int_0^1 |w''(x_2)|^2\,dx_2 - \frac{h^{\alpha+1}\gamma\,a}{2}\int_0^1 |w'(x_2)|\,dx_2\,, \qquad i = 0,1,2,$$
$$\mathcal{E}_h^0(\mathbf{u},w) = \frac{h^3 E}{3(1-\nu^2)}\int_{-1}^1 |w''(t)|^2\,dt - 2h^{\alpha+1}\gamma\int_{-1}^1 |w'(t)|^2\,dt,$$

whose nontrivial critical points obviously exhibit oscillation period of order $h^{1-\alpha/2}$.

Computations in Remark 3.10 prove useful in the next section when studying asymptotics of the problem as the thickness tends to 0_+ .

4. Scaling Föppl–von Kármán energy. We recall that the thickness of the plate is $s := hs_0$, where h is an adimensional parameter, s_0 is a physical dimension, and we have chosen measure units such that $s_0 = 1$.

Here we focus on the asymptotic analysis of the mechanical problems for FvK plate as $h \to 0_+$. To highlight properties of the limit solution we examine the behavior of suitably scaled energy: all along this section we assume that there is no transverse load, say, $g_h \equiv 0$, while we refer to a parameter α characterizing different asymptotic regimes of in-plane load $\mathbf{f}_h$, say,

$$\mathbf{f}_h = h^\alpha \mathbf{f}, \quad \text{where} \quad \alpha \geq 0 \quad \text{and} \quad \mathbf{f} \in L^2(\partial\Omega, \mathbb{R}^2)\,. \tag{4.1}$$

The next result and subsequent counterexample show how parameter α may influence the asymptotic behavior of functionals $\mathcal{F}_h$ when $h \to 0_+$; precisely there is a threshold concerning α: if $\alpha > 2$, then there is a scaling of displacements weakly convergent in $H^1(\Omega, \mathbb{R}^2) \times H^2(\Omega)$ such that related energies (after suitable scaling) are convergent too (see Theorem 4.1 and formula (4.2) therein); if $\alpha \in [0, 2)$, then the rescaled energies may be unbounded from below as $h \to 0_+$ for all cases: free, simply supported, and clamped plate (see Counterexample 4.4 and Remark 4.5).

THEOREM 4.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded connected Lipschitz open set, $\alpha \geq 2$, and $i = 0, 1, 2$.*

If $i = 0$ (clamped plate), assume (2.29) *and $\Gamma = \partial\Omega$ (as in Theorem* 2.11*) .*

If $i = 1$ (simply supported plate), assume (2.29)*, Ω convex, $\Gamma = \partial\Omega$ (as in Theorem* 2.8*).*

If $i = 2$ (free plate), assume (2.12) *and* (2.13) *(as in Theorem* 2.1*).*

Set

$$\mathcal{F}^{i,\alpha}(\mathbf{v}, \zeta) = \begin{cases} \mathcal{F}_1^i(\mathbf{v}, \zeta) & \text{if } \alpha = 2, \\ \\ \mathcal{F}_1^i(\mathbf{v}, \zeta) + \chi_{\{D^2\zeta\equiv 0\}}(\zeta) & \text{if } \alpha > 2 \,, \end{cases} \tag{4.2}$$

where $\chi_{\{D^2\zeta\equiv 0\}}(\zeta) = 0$ if $D^2\zeta \equiv 0$, $= +\infty$ else.

Fix $i \in \{0, 1, 2\}$ and a sequence $(\mathbf{u}_h, w_h)$ in $\operatorname{argmin} \mathcal{F}_h^i$*.*

Then there exists $(\mathbf{v}, \zeta) \in \operatorname{argmin} \mathcal{F}^{i,\alpha}$ such that, up to subsequences,

$$(h^{-\alpha}\mathbf{u}_h, h^{-\alpha/2}w_h) \to (\mathbf{v}, \zeta) \qquad \text{weakly in } H^1(\Omega, \mathbb{R}^2) \times H^2(\Omega)\,, \ \text{as } h \to 0_+ \,. \tag{4.3}$$

Moreover

$$h^{-2\alpha-1}\mathcal{F}_h^i(\mathbf{u}_h, w_h) \to \mathcal{F}^{i,\alpha}(\mathbf{v}, \zeta)\,, \ \text{as } h \to 0_+ \,. \tag{4.4}$$

Proof. The case $\alpha = 2$ is trivial since $(\mathbf{u}_h, w_h) \in \operatorname{argmin} \mathcal{F}_h^i$ if and only if $(h^{-2}\mathbf{u}_h, h^{-1}w_h) \in \operatorname{argmin} \mathcal{F}_1^i$ for every h.

If $\alpha > 2$, $i = 0, 1$ and $(\mathbf{u}_h, w_h) \in \operatorname{argmin} \mathcal{F}_h^i$, set $\mathbf{v}_h := h^{-\alpha}\mathbf{u}_h,\ \zeta_h := h^{-\alpha/2}w_h,\ \lambda_h = \|\mathbb{E}(\mathbf{v}_h\|_{L^2}$ and assume by contradiction $\lambda_h \to +\infty$. Then by taking into account the minimality of $(\mathbf{u}_h, w_h)$, (2.7), (2.29) and setting $\varphi_h = \lambda_h^{-1/2}\zeta_h,\ \mathbf{z}_h = \lambda_h^{-1}\mathbf{v}_h$ we get

$$c_\nu \frac{h^{2-\alpha}\,E}{24} \int_\Omega |D^2\varphi_h|^2 + \lambda_h\, c_\nu\, \frac{E}{2} \int_\Omega |\mathbb{D}(\mathbf{z}_h, \varphi_h)|^2 \leq \int_{\partial\Omega} \mathbf{f}\cdot\mathbf{z}_h \leq C. \tag{4.5}$$

Hence $|D^2\varphi_h| \to 0$ in $L^2(\Omega, \mathrm{Sym}_{2,2}(\mathbb{R}))$ and by taking into account that $\varphi_h = 0$ on $\partial\Omega$ we get $\varphi_h \to 0$ in $H^2(\Omega)$; therefore $\mathbb{E}(\mathbf{z}_h) \to \mathbb{O}$ in $L^2(\Omega, \mathbb{R}^2)$, a contradiction since $\|\mathbb{E}(\mathbf{z}_h)\|_{L^2} = 1$. Then λ_h is bounded from above and by taking into account the minimality of $(\mathbf{u}_h, w_h)$, (2.7), (2.29) we get

$$c_\nu \frac{h^{2-\alpha}\,E}{24} \int_\Omega |D^2\zeta_h|^2 + c_\nu\, \frac{E}{2} \int_\Omega |\mathbb{D}(\mathbf{v}_h, \zeta_h)|^2 \leq \int_{\partial\Omega} \mathbf{f}\cdot\mathbf{v}_h \leq \|\mathbf{f}\|\lambda_h \leq C, \tag{4.6}$$

which entails $D^2\zeta_h \to 0$ in $L^2(\Omega)$ and equiboundedness of $D\zeta_h$ in $L^4(\Omega, \mathbb{R}^2)$.

When $i = 2$ we take again $\lambda_h = \|\mathbb{E}(\mathbf{v}_h)\|_{L^2}$ and assume by contradiction $\lambda_h \to +\infty$. Then estimate (4.5) continues to hold and as before $|D^2\varphi_h| \to 0$ in $L^2(\Omega)$, which entails $\varphi_h - \fint_\Omega \varphi_h \to 0$ in $L^2(\Omega)$, $D\varphi_h \to \mathbf{c}$ in L^4, and $2\mathbb{E}(\mathbf{z}_h) \to -\mathbf{c}\otimes\mathbf{c}$ strongly in $L^2(\Omega, \mathrm{Sym}_{2,2}(\mathbb{R}))$ for a suitable $\mathbf{c} \in \mathbb{R}^2$. Therefore (2.13), (4.5) yield

$$0 \leq \lim_{h\to 0_+} \int_{\partial\Omega} \mathbf{f}\cdot\mathbf{z}_h = \lim_{h\to 0_+} f \int_{\partial\Omega} \mathbf{n}\cdot\mathbf{z}_h = \lim_{h\to 0_+} f \int_\Omega \operatorname{div}\mathbf{z}_h = -\frac{f}{2}|\Omega||\mathbf{c}|^2, \tag{4.7}$$

that is, $\mathbf{c} = \mathbf{0}$ so $\mathbb{E}(\mathbf{z}_h) \to \mathbb{O}$ in $L^2(\Omega, \mathrm{Sym}_{2,2}(\mathbb{R}))$ as in the previous cases, again a contradiction. Thus equiboundedness holds in this case too. Since, for $0 < h \leq 1$, the w.l.s.c. functionals $\mathcal{F}^{i,\alpha}$ fulfill $\mathcal{F}^{i,\alpha} \leq h^{-2\alpha-1}\mathcal{F}^i_h$, the proof can be completed by a standard argument in Γ convergence. ∎

Remark 4.2. It is worth noticing that, if $\alpha = 2$, then the limit energy is still the FvK energy of a plate of thickness s_0; indeed h^{-5} is exactly the scaling factor of the hierarchy in [23] for the derivation of FvK plate model.

Remark 4.3. We emphasize that, if $D^2 w \equiv \mathbb{O}$, then

$$\mathcal{F}_1(\mathbf{v}, w) = \mathcal{F}_1(\mathbf{v}, 0) \qquad \text{if } i = 0, 1\,, \tag{4.8}$$

$$\mathcal{F}_1(\mathbf{v}, w) = \mathcal{F}_1(\mathbf{v}, \boldsymbol{\xi}\cdot\mathbf{x}) = \int_\Omega J\left(\mathbb{E}(\mathbf{v}) + \frac{1}{2}\boldsymbol{\xi}\otimes\boldsymbol{\xi}\right) - \int_{\partial\Omega} \mathbf{f}_h\cdot\mathbf{v} \quad \text{for} \quad w = \boldsymbol{\xi}\cdot\mathbf{x} \ \text{ if } i = 2. \tag{4.9}$$

Theorem 4.1 is optimal in the sense that if $\alpha < 2$, we cannot expect neither that $h^{-2\alpha-1}\min_{\mathcal{A}^i}\mathcal{F}_h$ are bounded from below nor that minimizers are equibounded in $H^1(\Omega, \mathbb{R}^2) \times W^{1,4}(\Omega)$ when we let $h \to 0_+$. This phenomenon may take place even if Ω is a rectangle as shown by the next counterexample, where we consider a plate with the same geometry and load of Counterexample 3.7 (see Figure 2, at left); nevertheless here we push further the analysis of this case.

COUNTEREXAMPLE 4.4. *Let* $a > EC_\nu$, $\alpha \in [0, 2)$, $\mathbf{f}_h = h^\alpha \mathbf{f}$ *with*

$$\Omega = (0, a)\times(0, 1)\,, \quad \Gamma = \partial\Omega\,, \quad g_h \equiv 0\,, \quad \mathbf{f} = \left(\mathbf{1}_{\{y=0\}} - \mathbf{1}_{\{y=1\}}\right)\mathbf{e}_2\,. \tag{4.10}$$

Then for any sequence $(\mathbf{u}_h, w_h) \in \arg\min \mathcal{F}^0_h$ *(such sequences do exist due to Theorem* 2.11*), the scaled sequence* $(h^{-\alpha}\mathbf{u}_h, h^{-\alpha/2}w_h)$ *is not equibounded in* $H^1(\Omega, \mathbb{R}^2) \times W^{1,4}(\Omega)$. *Moreover,* $\inf h^{-2\alpha-1}\mathcal{F}^0_h \to -\infty$ *as* $h \to 0_+$.

Indeed we can set $\mathbf{v}_h := h^{-\alpha}\mathbf{u}_h$, $\zeta_h := h^{-\alpha/2}w_h$, *and*

$$\mathcal{W}_h(\mathbf{v}_h, \zeta_h) := h^{-1-2\alpha}\mathcal{F}_h(\mathbf{u}_h, w_h) = \frac{h^{2-\alpha}}{12}\int_\Omega J(D^2\zeta_h) + \int_\Omega J(\mathbb{D}(\mathbf{v}_h, \zeta_h)) - \int_{\partial\Omega}\mathbf{f}\cdot\mathbf{v}_h\,, \tag{4.11}$$

$$\mathcal{I}^+(\mathbf{v}, \zeta) := \inf\left\{\limsup_{h\to 0_+} \mathcal{W}_h(\mathbf{v}_h, \zeta_h) \ :\ \mathbf{v}_h \overset{w-H^1}{\rightharpoonup} \mathbf{v}\,,\ \zeta_h \overset{w-W^{1,4}}{\rightharpoonup} \zeta\right\}\,, \tag{4.12}$$

$$\mathcal{I}^-(\mathbf{v}, \zeta) := \inf\left\{\liminf_{h\to 0_+} \mathcal{W}_h(\mathbf{v}_h, \zeta_h) \ :\ \mathbf{v}_h \overset{w-H^1}{\rightharpoonup} \mathbf{v}\,,\ \zeta_h \overset{w-W^{1,4}}{\rightharpoonup} \zeta\right\}, \tag{4.13}$$

$$\mathcal{J}(\mathbb{B}, \boldsymbol{\eta}) = \frac{E}{8(1+\nu)}\left|\mathbb{B} + \mathbb{B}^T + \boldsymbol{\eta}\otimes\boldsymbol{\eta}\right|^2 + \frac{E\nu}{8(1-\nu^2)}\left|\mathrm{Tr}\left(\mathbb{B} + \mathbb{B}^T + \boldsymbol{\eta}\otimes\boldsymbol{\eta}\right)\right|^2\,. \tag{4.14}$$

Then by arguing as in [14, Lemma 4.1] *we get*

$$\mathcal{I}^+(\mathbf{v}, \zeta) \leq \Lambda(\mathbf{v}, \zeta) := \int_\Omega \mathcal{J}(\mathbb{D}(\mathbf{v}, \zeta))\,dx - \int_{\partial\Omega}\mathbf{f}\cdot\mathbf{v}\,d\mathcal{H}^1. \tag{4.15}$$

Then by denoting with $Q\mathcal{J}$ *the quasiconvex envelope of* $\mathcal{J}$, *since* $\mathcal{I}^+$ *is sequentially l.s.c. in* $w - H^1 \times w - W^{1,4}$, *we obtain*

$$\mathcal{I}^+(\mathbf{v}, \zeta) \leq \int_\Omega Q\mathcal{J}\left(D\mathbf{v}, D\zeta\right)\,dx - \int_{\partial\Omega}\mathbf{f}\cdot\mathbf{v}\,d\mathbf{x}. \tag{4.16}$$

On the other hand for every $\mathbf{v}_h \rightharpoonup^{w-H^1} \mathbf{v}$, $\zeta_h \rightharpoonup^{w-W^{1,4}} \zeta$ *we get*

$$\liminf_{h\to 0_+} h^{-1-2\alpha}\, \mathcal{F}_h(h^\alpha \mathbf{v}_h, h^{\alpha/2}\zeta_h) \geq \int_\Omega Q\mathcal{J}(D\mathbf{v}, D\zeta) - \int_{\partial\Omega} \mathbf{f}\cdot\mathbf{v},$$

that is,

$$\mathcal{I}^-(\mathbf{v},\zeta) \geq \int_\Omega Q\mathcal{J}(D\mathbf{v}, D\zeta) - \int_{\partial\Omega} \mathbf{f}\cdot\mathbf{v}.$$

By

$$\mathcal{I}(\mathbf{v},\zeta) := \int_\Omega Q\mathcal{J}(D\mathbf{v}, D\zeta) - \int_{\partial\Omega} \mathbf{f}\cdot\mathbf{v} \geq \mathcal{I}^+(\mathbf{v},\zeta) \geq \mathcal{I}^-(\mathbf{u},w) \geq \mathcal{I}(\mathbf{v},\zeta) \tag{4.17}$$

we get

$$\Gamma\lim_{h\to 0_+} \mathcal{W}_h = \mathcal{I}\ . \tag{4.18}$$

Therefore, if $(h^{-\alpha}\mathbf{u}_h^*, h^{-\alpha/2}w_h^*)$ *were equibounded in* $H^1(\Omega,\mathbb{R}^2)\times W^{1,4}(\Omega)$, *then*

$$h^{-1-2\alpha}\mathcal{F}_h(\mathbf{u}_h^*, w_h^*) \to \min \mathcal{I} = \inf \Lambda$$

since Λ *is the relaxed functional of* $\mathcal{I}$, *and we will show that this leads to a contradiction. Indeed, we choose*

$$\zeta_n(x,y) = \frac{1}{\sqrt{n}}\varphi(ny)\,\psi_n(x), \quad \mathbf{v}_n(x,y) = \left(0, \frac{-n}{2}y\right) \tag{4.19}$$

with

$$\varphi:\mathbb{R}\to\mathbb{R}\,,\quad 1\text{-periodic}\,, \varphi(y) = \frac{1}{2}(1-|1-2y|)\ \forall y\in(0,1), \tag{4.20}$$

$$\psi_n(x) = nx\mathbf{1}_{\{[0,1/n]\}} + \mathbf{1}_{\{[1/n,a-1/n]\}} - n(x-a)\mathbf{1}_{\{[a-1/n,a]\}}\,. \tag{4.21}$$

We get

$$\mathbb{E}(\mathbf{v}_n) = \begin{bmatrix} 0 & 0 \\ 0 & -\dfrac{n}{2} \end{bmatrix},$$

$$\mathbb{D}(\mathbf{v}_n,\zeta_n) = \begin{bmatrix} \dfrac{1}{2n}\big(\psi_n'(x)\big)^2\big(\varphi(ny)\big)^2 & \dfrac{1}{2}\psi_n(x)\psi_n'(x)\varphi(ny)\varphi'(ny) \\ \dfrac{1}{2}\psi_n(x)\psi_n'(x)\varphi(ny)\varphi'(ny) & \dfrac{n}{2}\big(\psi_n^2(x)|\varphi'(ny)|^2 - 1\big) \end{bmatrix}$$

and by taking into account (2.6), (2.7) *and that* $2|\varphi|\leq 1$, $|\varphi'|=1$, $|\psi|\leq 1$, $|\psi_n'|\leq n$, $\operatorname{spt}\psi_n' \subset [0,1/n]\cup[a-1/n,a]$, $|\psi_n| = 1$ *on* $[1/n,a-1/n]$, $a > EC_\nu$,

$$\begin{aligned}
&\Lambda(\mathbf{v}_n,\zeta_n) \\
&\quad = \int_0^a\int_0^1 J(\,\mathbb{D}(\mathbf{v}_n,\zeta_n)\,)\; dx\,dy - \int_{\partial\Omega} \mathbf{f}\cdot\mathbf{v}_n\; dx\,dy \\
&\quad \leq \int_0^a\int_0^1 \frac{EC_\nu}{8}\Bigg(n^{-2}|\psi_n'(x)|^4|\varphi(ny)|^4 + 2|\psi_n(x)|^2|\psi_n'(x)|^2|\varphi(ny)|^2|\varphi'(ny)|^2 \\
&\qquad\qquad + \ n^2\left(\psi_n^2(x)|\varphi'(ny)|^2 - 1\right)^2\Bigg) - \frac{na}{2} \\
&\quad \leq \int_0^a\int_0^1 \frac{EC_\nu}{8}\left(n^2\mathbf{1}_{[0,1/n]\cup[a-1/n,a]} + n^2\left(\psi_n^2(x)-1\right)^2\right) - \frac{na}{2} \leq \frac{nEC_\nu}{2} - \frac{na}{2} \to -\infty
\end{aligned}$$

leads to a contradiction.

So $(h^{-\alpha}\mathbf{u}_h^*, h^{-\alpha/2}w_h^*)$ *are not equibounded in* $H^1(\Omega,\mathbb{R}^2)\times W^{1,4}(\Omega)$ *and the first claim follows.*

Eventually we prove the second claim. By (4.15) *there exists* $(\mathbf{v}_{n,h},\zeta_{n,h}) \to (\mathbf{v}_n,\zeta_n)$ *weakly in* $H^1(\Omega,\mathbb{R}^2)\times H^2(\Omega)$ *such that* $\limsup \mathcal{W}_h(\mathbf{v}_{n,h},\zeta_{n,h}) \le \mathcal{I}(\mathbf{v}_n,\zeta_n) \le -Kn$ *for suitable* $K>0$*, and hence by using a diagonal argument we achieve the claim.*

Remark 4.5. If $a > EC_\nu,\ \alpha\in[0,2),\ \mathbf{f}_h = h^\alpha \mathbf{f}$ with

$$\Omega = (0,a)\times(0,1)\,, \qquad g_h \equiv 0,\ \mathbf{f} = \big(\mathbf{1}_{\{y=0\}} - \mathbf{1}_{\{y=1\}}\big)\,\mathbf{e}_2,\quad \Gamma = \partial\Omega\,. \tag{4.22}$$

Then $h^{-1-2\alpha}\inf\mathcal{F}_h^i \to -\infty$ as $h\to 0_+$ *holds true also for* $i=1,2$.

Indeed, though existence of minimizers of $\mathcal{F}_h^i$, $(i=1,2)$ *may fail, nevertheless* $\inf\mathcal{F}_h^i \le \inf\mathcal{F}_h^0$ *for* $i=1,2$*; hence the claim follows by previous counterexample.*

5. Prestressed plates: Oscillating versus flat equilibria. Counterexample 4.4 and Remark 3.4 show that the Föppl–von Kármán functional might not be suitable for studying equilibria of plates when thickness $h\to 0_+$, at least in the presence of in-plane loads scaling as h^α, when $\alpha\in[0,2)$) and h is the scale factor for the plate thickness.

To circumvent this difficulty, as in the case of many practical engineering applications, we assume that our plate-like structure is initially prestressed and undergoes a transverse displacement about the prestressed state.

In this section the minimization with respect to the out-of-plane displacement alone is performed via relaxation techniques (Theorem 5.1, Lemma 5.2, Proposition 5.3, and Remark 5.4): this approach allows us to clarify the structure of the asymptotic sequence of minimizers as h vanishes. By subsequent Examples 5.5–5.11 we show a way to recover the geometry of the asymptotic minimizers by studying the Lamé problem in presence of compressive and tension forces: in particular, whenever there is a region of positive measure where prestress has at least one negative eigenvalue, the asymptotic sequence of minimizers exhibit oscillations with a period which can be easily estimated.

Precisely, in this section we fix $g_h\equiv 0$, $\mathbf{f}\in L^2(\partial\Omega,\mathbb{R}^2)$, $\alpha\in[0,2)$ and we assume that the prestressed state is caused by the (scaled) force field $\mathbf{f}_h = h^\alpha\mathbf{f}$ and is given by every $\mathbf{u}^*\in H^1(\Omega,\mathbb{R}^2),\ \mathbf{u}^* = h^\alpha\mathbf{v}^*$, where $\mathbf{v}^*$ is a minimizer of the functional

$$\mathcal{F}(\mathbf{v}) := \int_\Omega J(\mathbb{E}(\mathbf{v})) - \int_{\partial\Omega}\mathbf{f}\cdot\mathbf{v}\,. \tag{5.1}$$

The transverse displacement w is chosen such that the pair $(\mathbf{u}^*, w)$ minimizes the functional $\mathcal{G}_h$ over $H^1(\Omega,\mathbb{R}^2)\times\mathcal{A}^i$, defined by

$$\mathcal{G}_h(\mathbf{u},w) = \begin{cases} \mathcal{F}_h(\mathbf{u},w) & \text{if } \mathbf{u}=\mathbf{u}^* \text{ and } w\in\mathcal{A}^i,\\ +\infty & \text{else}\,. \end{cases}$$

Moreover we have $\mathcal{G}_h(\mathbf{u},w) = \widetilde{\mathcal{G}}_h(\mathbf{v},\zeta)$ when setting $\mathbf{v} := h^{-\alpha}\mathbf{u},\ \zeta := h^{-\alpha/2}w$, and

$$\widetilde{\mathcal{G}}_h(\mathbf{v},\zeta) = \begin{cases} h^\alpha F_h^b(\zeta) + h^{2\alpha+1}\displaystyle\int_\Omega J(\mathbb{D}(\mathbf{v},\zeta)) - h^{2\alpha+1}\int_{\partial\Omega}\mathbf{f}\cdot\mathbf{v} & \text{if } \mathbf{v}\in\operatorname{argmin}\mathcal{F}, \zeta\in\mathcal{A}^i\,,\\ +\infty & \text{else in } \ H^1(\Omega)\times\mathcal{A}^i\,. \end{cases}$$

We aim to capture the nature of the transverse minimizer through a detailed study of the asymptotic behavior of minimizers of $\widetilde{\mathcal{G}}_h$ as $h\to 0_+$. A first hint in this perspective is the next result.

THEOREM 5.1. *For every* $\mathbf{v} \in \arg\min \mathcal{F}$, *let* $I_{\mathbf{v}}^{**}(\mathbf{x},\cdot)$ *be the convex envelope of* $I_{\mathbf{v}}(\mathbf{x},.)$, *where* $I_{\mathbf{v}}(\mathbf{x},\boldsymbol{\xi}) := J(\mathbb{E}(\mathbf{v})(\mathbf{x}) + \frac{1}{2}\boldsymbol{\xi}\otimes\boldsymbol{\xi})$, *and*

$$\mathcal{G}^{**}(\mathbf{v},\zeta) := \int_\Omega I_{\mathbf{v}}^{**}(\mathbf{x}, D\zeta)\, d\mathbf{x} - \int_{\partial\Omega} \mathbf{f}\cdot\mathbf{v}\, d\mathcal{H}^1 \qquad \forall\, \zeta \in W^{1,4}(\Omega)\,. \tag{5.2}$$

Then, for every $\alpha \in [0,2)$,

$$h^{-2\alpha-1}\min_{\mathcal{A}^i} \widetilde{\mathcal{G}}_h \to \begin{cases} \min\left\{\mathcal{G}^{**}(\mathbf{v},\zeta):\ \zeta\in W^{1,4}(\Omega),\ \zeta = 0 \text{ on } \Gamma\right\} & \text{if } i = 0,1, \\ \min\{\mathcal{G}^{**}(\mathbf{v},\zeta):\ \zeta \in W^{1,4}(\Omega)\} & \text{if } i = 2\,. \end{cases} \tag{5.3}$$

Moreover if $(\mathbf{v},\zeta_h) \in \arg\min_{\mathcal{A}^i}\widetilde{\mathcal{G}}_h$, *then* $\zeta_h \to \zeta$ *weakly in* $W^{1,4}(\Omega)$, *up to subsequences, with* $(\mathbf{v},\zeta) \in \arg\min \mathcal{G}^{**}$.

Proof. The claim is a straightforward consequence of techniques developed in [14, Lemma 4.1] and standard relaxation of integral functionals. ∎

In order to characterize equilibrium configurations of $\widetilde{\mathcal{G}}_h$, additional information about minimizers of functional $\mathcal{G}^{**}$ is needed: actually a careful use of Theorem 5.1 allows us to show explicit examples capturing the qualitative behavior of minimizers and their dependence on the thickness h.

To this aim, if $\mathbb{A}\in \mathrm{Sym}_{2,2}(\mathbb{R})$, we denote its ordered eigenvalues by $\lambda_1(\mathbb{A}) \le \lambda_2(\mathbb{A})$ and by $\mathbf{v}_1(\mathbb{A}), \mathbf{v}_2(\mathbb{A})$ their corresponding normalized eigenvectors, which afterward will be denoted shortly with $\lambda_1, \lambda_2, \mathbf{v}_1, \mathbf{v}_2$ whenever there is no risk of confusion.

For every $\nu \neq 1$, $\boldsymbol{\xi} \in \mathbb{R}^2$ and $\mathbb{A}\in\mathrm{Sym}_{2,2}(\mathbb{R})$ we set

$$g_{\mathbb{A}}(\boldsymbol{\xi}) = |\mathbb{A} + \boldsymbol{\xi}\otimes\boldsymbol{\xi}|^2 + \frac{\nu}{(1-\nu)}\left(\mathrm{Tr}\,\mathbb{A} + |\boldsymbol{\xi}|^2\right)^2. \tag{5.4}$$

We notice that there are two possibilities, either tension or compression. Nevertheless compression is given by condition $\nu\lambda_2 + \lambda_1 < 0$ and of course if $\lambda_1 \le \lambda_2 < 0$, then $\nu\lambda_2 + \lambda_1 \le \nu\lambda_2 + \lambda_2 < 0$, due to $\nu > -1$: we make explicit the implications of this algebraic relationship in the next lemma.

LEMMA 5.2. *If* $\nu \in (-1, 1/2)$, *then*

$$\min_{\boldsymbol{\xi}\in\mathbb{R}^2} g_{\mathbb{A}}(\boldsymbol{\xi}) = \begin{cases} g_{\mathbb{A}}(\mathbf{0}) & \text{if } \nu\lambda_2 + \lambda_1 \ge 0\,, \\ (1+\nu)(\lambda_2(\mathbb{A}))^2 & \text{if } \nu\lambda_2+\lambda_1 < 0\,. \end{cases} \tag{5.5}$$

Proof. We write shortly λ_1, λ_2 instead of $\lambda_1(\mathbb{A}(\mathbf{v})), \lambda_2(\mathbb{A}(\mathbf{v}))$. It is worth noticing that the minimum in (5.5) is achieved since $g_{\mathbb{A}} \in C(\mathbb{R}^2)$ and $g_{\mathbb{A}}(\boldsymbol{\xi}) \to +\infty$ as $|\boldsymbol{\xi}| \to +\infty$. Let $\mathbb{M} \in O(2)$ be such that $\mathbb{M}^T\mathbb{A}\mathbb{M} = \mathrm{diag}(\lambda_1,\lambda_2)$. Then it is readily seen that by setting $x := \boldsymbol{\xi}\cdot\mathbf{v}_1$, $y := \boldsymbol{\xi}\cdot\mathbf{v}_2$ we have

$$\widetilde{g}_{\mathbb{A}}(x,y) := g_{\mathbb{A}}(\boldsymbol{\xi}) = (x^2+\lambda_1)^2 + (y^2+\lambda_2)^2 + 2x^2y^2 + \frac{\nu}{1-\nu}\left(\lambda_1+\lambda_2+x^2+y^2\right)^2$$

and an easy computation shows that if $\nu\lambda_2+\lambda_1 \ge 0$, then the minimum is attained at $(x,y) = (0,0)$. Else, if $\nu\lambda_2 + \lambda_1 < 0$, then either $\nu\lambda_1 + \lambda_2 \ge 0$ or $\nu\lambda_2 + \lambda_1 \le \nu\lambda_1 + \lambda_2 < 0$.

In the first case $D\widetilde{g}_{\mathbb{A}}(x,y) = (0,0)$ if and only if $(x,y) \in \{(\pm\sqrt{-\nu\lambda_2-\lambda_1}, 0), (0,0)\}$ and $\widetilde{g}_{\mathbb{A}}(x,y) = (1+\nu)\lambda_2^2$ or $g_{\mathbb{A}}(x,y) = g_{\mathbb{A}}(0,0) > (1+\nu)\lambda_2^2$; in the latter one

$D\widetilde{g}_{\mathbb{A}}(x,y)=(0,0)$ also at $(x_*,\pm y_*)=(0,\pm\sqrt{-\nu\lambda_2-\lambda_1})$ with $\widetilde{g}_{\mathbb{A}}(x_*,\pm y_*)=(1+\nu)\lambda_1^2$. Hence

$$\min_{\boldsymbol{\xi}\in\mathbb{R}^2} g_{\mathbb{A}}(\boldsymbol{\xi})=(1+\nu)\lambda_2^2$$

if $\nu\lambda_2+\lambda_1<0\le\nu\lambda_1+\lambda_2$ and

$$\min_{\boldsymbol{\xi}\in\mathbb{R}^2} g_{\mathbb{A}}(\boldsymbol{\xi})=(1+\nu)\min\{\lambda_2^2,\lambda_1^2\}$$

if $\nu\lambda_2+\lambda_1\le\nu\lambda_1+\lambda_2<0$. In the latter case if $\nu\in(-1,0)$, then $\lambda_1\le\lambda_2\le-\nu\lambda_1$, and hence $\lambda_1\le\lambda_2\le 0$ and $|\lambda_1|\ge|\lambda_2|$. If $\nu\in[0,1/2)$, then $\lambda_1<0$ and either $\lambda_2>|\lambda_1|>0$ or $\lambda_1\le\lambda_2\le 0$. In the first case we get necessarily $\nu>0$ and $|\lambda_1|>\nu^{-1}(1-\nu)\lambda_2>\lambda_2$, a contradiction. Therefore $|\lambda_2|\le|\lambda_1|$ and

$$\min_{\boldsymbol{\xi}\in\mathbb{R}^2} g_{\mathbb{A}}(\boldsymbol{\xi})=(1+\nu)\lambda_2^2$$

whenever $\nu\lambda_2+\lambda_1<0$, thus proving the thesis. ∎

Lemma 5.2 proves quite useful in the perspective of the next proposition and the subsequent examples, since the two alternatives in the right-hand side of (5.5) correspond, respectively, to locally flat or oscillating equilibrium configurations.

PROPOSITION 5.3. *If* $\mathbf{v}_*\in\operatorname{argmin}\mathcal{F}$ *and the ordered eigenvalues* $\lambda_1\le\lambda_2$ *of* $\mathbb{E}(\mathbf{v}_*)$ *fulfill* $\nu\lambda_2+\lambda_1\ge 0$ *in the whole set* Ω, *then*

$$\widetilde{\mathcal{G}}_h(\mathbf{v}_*,\zeta)\ge\widetilde{\mathcal{G}}_h(\mathbf{v}_*,0). \tag{5.6}$$

If in addition $\nu\lambda_2+\lambda_1>0$ *in a set of positive measure, then the inequality in* (5.6) *is strict for every* $\zeta\not\equiv 0$.

Proof. Due to (5.5) in Lemma 5.2, $\nu\lambda_2+\lambda_1\ge 0$ entails $g_{2\mathbb{E}(\mathbf{u}_*)}(\boldsymbol{\xi})\ge g_{2\mathbb{E}(\mathbf{u}_*)}(\mathbf{0})$, and moreover $\nu\lambda_2+\lambda_1>0$ entails $g_{2\mathbb{E}(\mathbf{u}_*)}(\boldsymbol{\xi})>g_{2\mathbb{E}(\mathbf{u}_*)}(\mathbf{0})$. Hence

$$J\big(\mathbb{D}(\mathbf{v}_*,\zeta)\big)=\frac{E}{8(1+\nu)}\,g_{2\mathbb{E}(\mathbf{v}_*)}(Dw)\ge J\big(\mathbb{E}(\mathbf{v}_*)\big)$$

and, for $\zeta\in\mathcal{A}^i$,

$$\begin{aligned}\widetilde{\mathcal{G}}_h(\mathbf{v}_*,\zeta)&=h^\alpha F_h^b(\zeta)+h^{2\alpha+1}\int_\Omega J\big(\mathbb{D}(\mathbf{v}_*,\zeta)\big)-h^{2\alpha+1}\int_{\partial\Omega}\mathbf{f}\cdot\mathbf{v}_*\\&\ge h^\alpha F_h^b(\zeta)+h^{2\alpha+1}\int_\Omega J\big(\mathbb{E}(\mathbf{v}_*)\big)-h^{2\alpha+1}\int_{\partial\Omega}\mathbf{f}\cdot\mathbf{v}_*\\&\ge\widetilde{\mathcal{G}}_h(\mathbf{v}_*,0)\ .\end{aligned}$$

Moreover the first inequality in the last computation is strict whenever $\nu\lambda_2+\lambda_1>0$ in a set of positive measure. ∎

Remark 5.4. Notice that $s_1:=\frac{E}{1-\nu^2}(\nu\lambda_2+\lambda_1)$ is the smallest eigenvalue of the stress tensor $\mathbb{T}(\mathbf{v})=J'\big(\mathbb{E}(\mathbf{v})\big)$, where we write shortly $\lambda_1:=\lambda_1(\mathbb{E}(\mathbf{v}))$, $\lambda_2:=\lambda_2(\mathbb{E}(\mathbf{v})$. (This notation is used in all subsequent examples too.) Therefore Proposition 5.3 shows that, if the eigenvalues of the stress tensor are both strictly positive almost everywhere, then we can expect only one flat minimizer ($\zeta\equiv 0$). On the other hand, the possible occurrence of oscillating configurations requires the presence of a compressive state on a region of positive measure: that is to say, the stress tensor must have at least one negative eigenvalue on a set of positive measure.

We show some examples clarifying how the asymptotic behavior of functionals $\widetilde{\mathcal{G}}_h$ provides useful information about minimizers when Ω is an annular set. (The corresponding minimization is also known as the *Lamé problem* in physics literature [7], [22].)

Set $0 < R_1 < R_2,\ p_1, p_2 \in \mathbb{R},\ \ \Omega := B_{R_2} \setminus B_{R_1}$, and consider uniform in-plane normal traction/compression at each component of the boundary.

$$\mathbf{f} = -p_1 \frac{\mathbf{x}}{R_1} \mathbf{1}_{\{|\mathbf{x}|=R_1\}} + p_2 \frac{\mathbf{x}}{R_2} \mathbf{1}_{\{|\mathbf{x}|=R_2\}}.$$

Therefore $\mathbf{v} \in \arg\min \mathcal{F}_{1,0}$ entails

$$\mathbf{v}(\mathbf{x}) = (a + b|\mathbf{x}|^{-2})\mathbf{x}, \tag{5.7}$$

and exploiting polar coordinates $\mathbf{x} = (r\cos\theta, r\sin\theta)$ we obtain

$$\mathbb{E}(\mathbf{v}) = \begin{bmatrix} a - \dfrac{b}{r^2}\cos 2\theta & -\dfrac{b}{r^2}\sin 2\theta \\ \\ -\dfrac{b}{r^2}\sin 2\theta & a + \dfrac{b}{r^2}\cos 2\theta \end{bmatrix}.$$

By using Neumann boundary condition $J'(\mathbb{E}(\mathbf{v}))\mathbf{n} = \mathbf{f}$ on $\partial\Omega$, we get

$$p_i = E(1+\nu)^{-1}(a(1+\nu)(1-\nu)^{-1} - bR_i^{-2}), \quad i = 1, 2, \tag{5.8}$$

that is,

$$a = \frac{(1-\nu)(p_2R_2^2 - p_1R_1^2)}{E(R_2^2 - R_1^2)}; \quad b = \frac{(1+\nu)(p_2 - p_1)R_1^2R_2^2}{E(R_2^2 - R_1^2)}. \tag{5.9}$$

It is worth noticing that $a - br^{-2},\ a + br^{-2}$ are the eigenvalues of $\mathbb{E}(\mathbf{v})$ and $(\cos\theta, \sin\theta)$, $(-\sin\theta, \cos\theta)$ the corresponding normalized eigenvectors for all $r \in [R_1, R_2]$; order may change according to sign(b).

We examine several different cases which may occur. In the first one we show occurrence of asymptotic radially oscillating equilibria under compressive forces.

Example 5.5 (radially oscillating minimizers). Set $\Gamma = \partial\Omega,\ \nu \in (-1, 1/2),\ i = 0$, and either $p_1 \le p_2 < 0$ or $p_2 \le p_1 < 0$. In the first case we get $b \ge 0$ and in the second one $b \le 0$. However in both cases $\nu\lambda_2 + \lambda_1 < 0$ in the whole annular set.

Set also $\mathbf{v}(\mathbf{x}) = (a + b|\mathbf{x}|^{-2})\mathbf{x} \in \arg\min \mathcal{F}_{0,1}$, so that (5.9) holds true.

Choose $\sigma_h \to 0_+,\ \beta_h \to +\infty,\ \psi_h : \mathbb{R} \to \mathbb{R}\ \ (R_2 - R_1)$-periodic such that

$$\psi_h(t) = \max\{0, \min\{t - R_1 - \sigma_h, R_2 - \sigma_h - t\}\} \tag{5.10}$$

and set $\psi_h^* := \psi_h * \rho_h$ being ρ_h mollifiers such that $\operatorname{spt}\rho_h \subset [-\sigma_h, \sigma_h]$. Then by denoting the floor of a real number (maximum integer not exceeding the number) with $\lfloor\cdot\rfloor$, setting $r = |x|$ and

$$\zeta_h(r) = \begin{cases} \lfloor\beta_h\rfloor^{-1}\sqrt{2(1-\nu)br^{-2} - 2a(\nu+1)}\,\psi_h^*(R_1 + (r - R_1)\lfloor\beta_h\rfloor) & \text{if } p_1 \le p_2 < 0, \\ \\ \lfloor\beta_h\rfloor^{-1}\sqrt{2(\nu-1)br^{-2} - 2a(\nu+1)}\,\psi_h^*(R_1 + (r - R_1)\lfloor\beta_h\rfloor) & \text{if } p_2 \le p_1 < 0, \end{cases}$$

$\zeta_h' := \partial\zeta/\partial r,\ \ D\zeta_h = (\zeta_{h,1}, \zeta_{h,2}) = (x_1/r, x_2/r)\,\zeta_h'$ and

$$\mathbb{M}(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \\ \sin\theta & \cos\theta. \end{bmatrix} \quad \mathbb{S}(\theta) = \begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix} = (\zeta_h')^{-2}\, D\zeta_h \otimes D\zeta_h\ .$$

So $\mathbb{M}^T\mathbb{S}\,\mathbb{M} = \mathbf{e}_1 \otimes \mathbf{e}_1$ and there exists $\Omega_h \subset \Omega$ with $|\Omega_h| \sim \sigma_h$ such that $|(\psi_h^*)'| = 1$ on $\Omega \setminus \Omega_h$. Then referring to (5.4) and (5.7), for every $x \in \Omega \setminus \Omega_h$ we have

$$\begin{aligned} &g_{2\mathbb{E}(\mathbf{v})}(D\zeta_h) \\ &\quad = |2\mathbb{E}(\mathbf{v}) + D\zeta_h \otimes D\zeta_h|^2 + \frac{\nu}{1-\nu}\left|2\operatorname{div}\mathbf{v} + |D\zeta_h|^2\right|^2 \\ &\quad = \left|2\mathbb{M}^T E(\mathbf{v})\mathbb{M} + \mathbb{M}^T D\zeta_h \otimes D\zeta_h \mathbb{M}\right|^2 + \frac{\nu}{1-\nu}\left|4a + |D\zeta_h|^2\right|^2 \\ &\quad = \left|2(a - br^{-2})\mathbf{e}_1 \otimes \mathbf{e}_1 + 2(a + br^{-2})\mathbf{e}_2 \otimes \mathbf{e}_2 + |\zeta_h'|^2 \mathbb{M}^T\mathbb{S}\,\mathbb{M}\right|^2 + \frac{\nu}{1-\nu}\left|4a + |\zeta_h'|^2\right|^2 \\ &\quad = (2a - 2br^{-2} + |\zeta_h'|^2)^2 + 4(a + br^{-2})^2 + \frac{\nu}{1-\nu}\left|4a + |\zeta_h'|^2\right|^2 . \end{aligned}$$

If $p_1 \le p_2 < 0$, we have $b \ge 0$, $|\zeta_h'|^2 = 2(1-\nu)br^{-1} - 2a(\nu+1) + O(\lfloor\beta_h\rfloor^{-2})$ on $\Omega \setminus \Omega_h$, and hence

$$\begin{aligned} g_{2\mathbb{E}(\mathbf{v})}(D\zeta_h) &= 4(1+\nu)(a + br^{-2})^2 + O(\lfloor\beta_h\rfloor^{-1}), \\ \int_\Omega I_{\mathbf{v}}(\mathbf{x}, D\zeta_h)\, d\mathbf{x} &= \int_{\Omega\setminus\Omega_h} I_{\mathbf{v}}(\mathbf{x}, D\zeta_h)\, d\mathbf{x} + \int_{\Omega_h} I_{\mathbf{v}}(\mathbf{x}, D\zeta_h)\, d\mathbf{x} \\ &= \frac{E}{2(1-\nu)} \int_{\Omega\setminus\Omega_h} \left\{(a + b|\mathbf{x}|^{-2})^2 + O(\beta_h^{-1})\right\}\, d\mathbf{x} + O(\sigma_h) \\ &\to \frac{E}{2(1-\nu)} \int_\Omega (a + b|\mathbf{x}|^{-2})^2\, dx . \end{aligned}$$

Analogously, if $p_2 \le p_1 < 0$, then $b \le 0$ and $|\zeta_h'|^2 = 2(\nu-1)br^{-2} - 2a(\nu+1) + O(\lfloor\beta_h\rfloor^{-1})$ on $\Omega \setminus \Omega_h$, and hence

$$\begin{aligned} g_{2\mathbb{E}(\mathbf{v})}(D\zeta_h) &= 4(1+\nu)(a - br^{-2})^2 + O(\lfloor\beta_h\rfloor^{-1}), \\ \int_\Omega I_{\mathbf{v}}(\mathbf{x}, D\zeta_h)\, d\mathbf{x} &\to \frac{E}{2(1-\nu)} \int_\Omega (a - b|\mathbf{x}|^{-2})^2\, d\mathbf{x} . \end{aligned}$$

By Lemma 5.2 we know that

$$\min_{\boldsymbol{\xi}\in\mathbb{R}^2} I_{\mathbf{v}}(x, \boldsymbol{\xi}) = \begin{cases} \dfrac{E}{2(1-\nu)}(a + b|\mathbf{x}|^{-2})^2 \text{ if } p_2 \le p_1 < 0, \\[2ex] \dfrac{E}{2(1-\nu)}(a - b|\mathbf{x}|^{-2})^2 \text{ if } p_1 \le p_2 < 0, \end{cases}$$

and therefore in both cases we have proved that

$$\int_\Omega I_{\mathbf{v}}(\mathbf{x}, D\zeta_h)\, dx \to \min\{\mathcal{G}^{**}(\mathbf{v}, \zeta) : \zeta \in W^{1,4}(\Omega),\ \zeta = 0 \text{ in } \partial\Omega\} .$$

Moreover

$$\begin{aligned} h^{-2\alpha-1}\widetilde{\mathcal{G}}(\mathbf{v}, \zeta_h) &= h^{-\alpha-1}F_h^b(\zeta_h) + \int_\Omega I_{\mathbf{v}}(x, D\zeta_h)\, d\mathbf{x} - \int_{\partial\Omega} \mathbf{f}\cdot\mathbf{v}\, d\mathcal{H}^1 , \\ h^{-\alpha-1}F_h^b(\zeta_h) &\sim h^{2-\alpha}\beta_h\sigma_h^{-1} . \end{aligned}$$

Therefore by Theorem 5.1 for every choice of β_h, σ_h satisfying the conditions detailed before, $(\mathbf{v}, \zeta_h)$ can be viewed as an asymptotically minimizing sequence of $\widetilde{\mathcal{G}}_h$

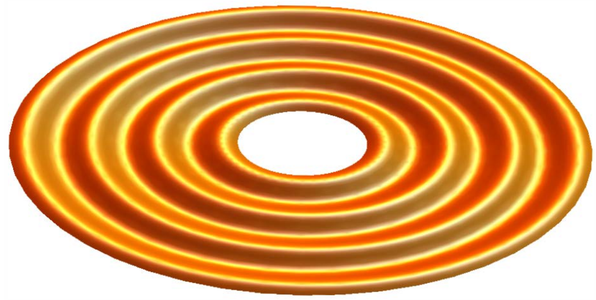

FIG. 6. *Radially oscillating minimizers in Example* 5.5.

whose out-of-plane component exhibits periodic oscillations (period: β_h^{-1}; asymptotic amplitude: $\sqrt{2(1-\nu)br^{-2}-2a(\nu+1)}$ if $p_1 \le p_2 < 0$ and $\sqrt{2(\nu-1)br^{-2}-2a(\nu+1)}$ if $p_2 \le p_1 < 0$) in the radial direction in the whole annular set. The optimal choice of β_h can be determined heuristically as follows: previous estimates show that

$$h^{-2\alpha-1}\widetilde{\mathcal{G}}(\mathbf{v},\zeta_h) - \min\mathcal{G}^{**} = R_h\,,$$

where $R_h \sim h^{2-\alpha}\beta_h\sigma_h^{-1} + \beta_h^{-1} + \sigma_h$. So, approximatively, we have to minimize the last term. A direct calculation shows that the best choice corresponds to ${\beta_h}^{-1} \sim h^{2/3-\alpha/3}$, $\sigma_h \sim h^{5/3(2-\alpha)}$. See Figure 6.

Next example shows that, if prestress has both strictly positive eigenvalues, then the flat configuration is the only admissible asymptotic sequence of minimizers.

Example 5.6 (flat minimizer). Assume $\Gamma = \partial\Omega$, $\nu \in [0,1/2)$, $i=0$ or $i=1$, $p_1 \ge 0$, and (5.9), so that $R_1^2 a \ge (1-2\nu)b$ and by Lemma 5.2 we get

$$\min\{\mathcal{G}^{**}(\mathbf{v},\zeta):\ \zeta \in W^{1,4}(\Omega),\ \zeta=0 \text{ in } \partial\Omega\} = \int_\Omega I_{\mathbf{v}}(\mathbf{x},0)\,d\mathbf{x}.$$

Obviously the minimum is attained at $\zeta \equiv 0$, that is, we have a flat minimizer.

Remark 5.7. Assume $\Gamma = \partial\Omega$, $\nu \in (-1,1/2)$, i=0, $p_1 < 0 \le p_2$, and (5.9).

Hence $a>0$, $b>0$, $\nu\lambda_2 + \lambda_1 = a - br^{-2} + \nu(a+br^{-2}) \ge 0$ in the annular set $A_1 = \{\overline{R} := \sqrt{(1-\nu)(1+\nu)^{-1}ba^{-1}} \le r \le R_2\}$ and <0 in the annular set $A_2 = \{R_1 \le r < \overline{R}\}$. Then by the same computations performed in previous examples we can build minimizers which are flat in A_1 and oscillating in A_2.

The next example shows the occurrence of asymptotic tangentially oscillating equilibria under tension forces.

Example 5.8 (tangentially oscillating minimizers). Let $\Gamma = \partial B_{R_1}$, $\nu \in (-1,1/2)$, $i=1$ and choose $p_1>0$, $p_2>0$ such that $p_2R_2^2 = p_1R_1^2$. If $\mathbf{v} \in \arg\min \mathcal{F}_{1,0}$, we find again $\mathbf{v}(\mathbf{x}) = (a+b|\mathbf{x}|^{-2})\mathbf{x}$, now with

$$a=0, \quad b = -(1+\nu)E^{-1}p_1R_1^2 < 0. \tag{5.11}$$

Hence $\lambda_1 = br^{-2} < 0 < -br^{-2} = \lambda_2$ are the eigenvalues of $\mathbb{E}(\mathbf{v})$ and $\mathbf{v}_1 = (-\sin\theta,\cos\theta)$, $\mathbf{v}_2 = (\cos\theta,\sin\theta)$ the corresponding normalized eigenvectors.

Choose $\sigma_h \to 0_+$, $\beta_h \to +\infty$, $\phi_h : \mathbb{R} \to \mathbb{R}$, 2π-periodic defined by

$$\phi_h(t) = \max\{0, \min\{t-\sigma_h, 2\pi-\sigma_h-t\}\} \tag{5.12}$$

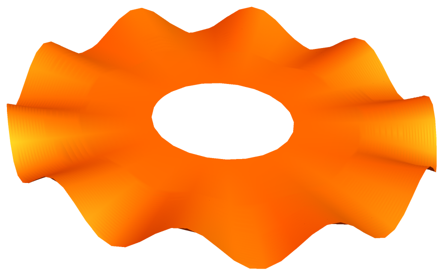

FIG. 7. *Tangentially oscillating minimizers in Example* 5.8.

and set $\phi_h^* := \phi_h * \rho_h$ being ρ_h mollifiers such that $\operatorname{spt}\rho_h \subset [-\sigma_h, \sigma_h]$. Let

$$\zeta_h(r,\theta) = \sqrt{-2b(1-\nu)}\,\lfloor\beta_h\rfloor^{-1}\phi_h^*(\lfloor\beta_h\rfloor\theta)\left(\delta_h^{-1}(r-R_1)\mathbf{1}_{[R_1,R_1+\delta_h]}(r)+\mathbf{1}_{[R_1+\delta_h,R_2]}(r)\right)$$

with $\delta_h \to 0_+,\ \beta_h^{-1}\delta_h^{-1} \to 0$. Then there exists $\Omega_h \subset \Omega$ with $|\Omega_h| \sim \sigma_h$ such that for every $x \in \Omega\setminus\Omega_h$ we have $|(\phi_h^*)'| = 1$ on $\Omega\setminus\Omega_h$. Therefore referring to (5.4) and (5.7) and by setting

$$R_*(\theta) = \begin{bmatrix} -\sin\theta & \cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}$$

we get

$$\begin{aligned}
&\int_{\Omega\setminus\Omega_h}\left(|2E(\mathbf{v}) + D\zeta_h\otimes D\zeta_h|^2 + \frac{\nu}{1-\nu}|D\zeta_h|^4\right)d\mathbf{x}\\
&\quad= \int_{\Omega\setminus\Omega_h}\left(\left|2R_*^T E(\mathbf{v})R_* + R_*^T D\zeta_h\otimes D\zeta_h R_*\right|^2 + \frac{\nu}{1-\nu}|D\zeta_h|^4\right)d\mathbf{x}\\
&\quad= \int_{\Omega\setminus\Omega_h} 4(1+\nu)b^2|\mathbf{x}|^{-4}\,d\mathbf{x} + O(\lfloor\beta_h\rfloor^{-1}\delta_h^{-1}) + O(\sigma_h) + O(\delta_h).
\end{aligned}$$

By using now Lemma 5.2 and by arguing as in Example 5.5 we get

$$\int_\Omega I_{\mathbf{v}}(\mathbf{x}, D\zeta_h)\,dx - \int_{\partial\Omega}\mathbf{f}\cdot\mathbf{v}\,d\mathcal{H}^1 \to \min\{\mathcal{G}^{**}(\mathbf{v},\zeta) : \zeta\in W^{1,4}(\Omega),\ \zeta = 0 \text{ in } \partial\Omega\},$$

$$h^{-2\alpha-1}\widetilde{\mathcal{G}}(\mathbf{v},\zeta_h) \to \min\mathcal{G}^{**} = \frac{Eb^2}{2(1+\nu)}\int_\Omega |\mathbf{x}|^{-4}\,d\mathbf{x} - \int_{\partial\Omega}\mathbf{f}\cdot\mathbf{v}\,d\mathcal{H}^1.$$

Moreover, since $h^{-\alpha-1}F_h^b(\zeta_h) \sim h^{2-\alpha}\beta_h\sigma_h^{-1}$, we get

$$\begin{aligned}
h^{-2\alpha-1}\widetilde{\mathcal{G}}(\mathbf{v},\zeta_h) &= h^{-\alpha-1}F_h^b(\zeta_h) + \int_\Omega I_{\mathbf{v}}(x, D\zeta_h)\,d\mathbf{x} - \int_{\partial\Omega}\mathbf{f}\cdot\mathbf{v}\,d\mathcal{H}^1\\
&= h^{2-\alpha}\beta_h\sigma_h^{-1} + O(\lfloor\beta_h\rfloor^{-1}\delta_h^{-1}) + O(\sigma_h) + O(\delta_h).
\end{aligned}$$

Hence, here the optimal choice is $\beta_h{}^{-1} \sim h^{1-\alpha/2},\ \delta_h \sim \beta_h^{-1/2},\ \sigma_h \sim h^{1-\alpha/2}\beta_h^{1/2}$. See Figure 7.

Remark 5.9. Thanks to Lemma 5.2 and Proposition 5.3, Examples 5.5, 5.6, 5.8 constitute a paradigm for the construction of oscillating versus flat approximated minimizers.

Moreover we sketch another technique to devise new ones, by this procedure: first take a boundary force field, construct the corresponding prestressed state (in two dimensions there are a lot of significant classical examples; see, for instance, those of Examples 3.8, 3.9) and look at the eigenvalues of the strain matrix: it is not difficult to obtain examples according to either $\nu\lambda_2 + \lambda_1 \geq 0$ or $\nu\lambda_2 + \lambda_1 < 0$ in the whole plate.

In the first case through Lemma 5.2 and Proposition 5.3 we argue that there is only a flat minimizer, and in the second one a careful use of Lemma 5.2 on the pattern of Examples 5.5, 5.8 allows an easy construction of oscillating minimizers.

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