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Multiscaling pulse representation of temporal rainfall

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[1] We develop a pulse-based representation of temporal rainfall with multifractal properties in the small-scale limit. The representation combines a traditional model for the exterior process at the synoptic scale with a novel hierarchical pulse model for the event interiors. For validation we apply the model to a temporal rainfall record from Florence, Italy. Although the model has only six parameters (four for the exterior process and two for the event interiors), it accurately reproduces a wide range of empirical statistics, including the distribution of dry and wet periods, the distribution of rainfall intensity up to extreme fractiles, the spectral density, the moment scaling function K(q), and the distribution of the partition coefficients for rainfall disaggregation. The model also reproduces observed deviations of physical rainfall from perfect scaling/multiscaling behavior. *INDEX TERMS:* 1854 Hydrology: Precipitation (3354); 1869 Hydrology: Stochastic processes; 1833 Hydrology: Hydroclimatology; *KEYWORDS:* rainfall, scaling, multifractal

1. Introduction

[2] Like other atmospheric phenomena (turbulence, clouds, etc.), rainfall has long been recognized to have a basic pulse structure. This structure has been exploited by so-called LeCam models [*LeCam*, 1961], which represent rainfall through the superposition of pulses with clustering in space and time. For an early review of these models, see *Waymire and Gupta* [1981a, 1981b, 1981c]. More recent developments include those of *Smith and Karr* [1985], *Sivapalan and Wood* [1987], *Rodriguez-Iturbe et al.* [1987], and *Veneziano and Villani* [1996]. LeCam models have a nested structure, which reflects the physical organization of rainfall into rain bands, mesoscale precipitation areas, convective cells, etc., but they are not scale invariant.

[3] In recent years, a new class of rainfall models has been developed, based on an observed scale invariance property called multifractality. This property is largely inherited from the hierarchical structure of atmospheric turbulence. Multifractality implies that the rainfall process looks statistically the same at small and large scales, except for simple transformations. The literature on multifractal scaling of rainfall has grown rapidly in the past decade. Important contributions include those by *Schertzer and Lovejoy* [1987], *Gupta and Waymire* [1990, 1993], *Tessier et al.* [1993], *Ladoy et al.* [1993], *Over and Gupta* [1994, 1996], *Lovejoy and Schertzer* [1995], *Svensson et al.* [1996], *Perica and Foufoula-Georgiou* [1996], *Marsan et al.* [1996], *Menabde et al.* [1997], *Olsson* [1995, 1998], *Harris et al.* [1998], *Schmitt et al.* [1998], *Venugopal et al.* [1999], *Deidda et al.* [1999], *Menabde and Sivapalan* [2000], and *Deidda* [2000].

[4] With the exception of the models of *Deidda et al.* [1999] and *Deidda* [2000], the proposed multifractal representations of rainfall use discrete or continuous multiplicative cascades or wavelet decompositions and are not pulse-based. Wavelet models [e.g., *Perica and Foufoula-Georgiou*, 1996] are not necessarily positive and therefore are not appealing for rainfall. The models of *Deidda et al.* [1999] and *Deidda* [2000], which use nonnegative pulses, are themselves not very satisfactory due to the geometrical arrangement of the pulses, which is similar to the regular pattern of cells in discrete cascades. The boundaries between pulses are clearly visible in realizations from these processes. Moreover, the regular arrangement of the pulses makes the processes nonstationary.

[5] One problem with scaling approaches to temporal rainfall is that multifractality applies over a limited range of scales, roughly between one hour and a few days. This is for example evident from the results of spectral analysis [*Fraedrich and Larnder*, 1993; *Olsson*, 1995; *Olsson et al.*, 1993]. If interest is over a wider range of scales, then modifications to standard multifractal models must be made [*Menabde et al.*, 1997, *Schmitt et al.*, 1998]. One aspect of lack of exact scaling is that the structure of the rainfall support (the wet/dry process) is nonfractal and inconsistent with so-called beta models. This inconsistency has been documented by *Schmitt et al.* [1998] through a detailed analysis of the rainfall record from Uccle, Belgium.

[6] In summary, no existing representation of rainfall (a) satisfactorily combines the pulse structure and the scaling properties that are separately emphasized by the LeCam and multifractal models, or (b) explains the observed deviations of physical rainfall from perfect scaling. The model proposed here attempts to fill these gaps. We distinguish between the arrival, duration and average intensity of synoptic storms and the variation of rainfall intensity during each synoptic event. Following a standard notation in rainfall modeling, we refer

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to the former as the "exterior process" and to the latter as the "interior process." We use the classical representation of the exterior process as an alternating renewal process with independent mean rainfall intensities for different rainstorms, whereas for the interior process we use a pulse representation of a new type with random number, location and intensity of the pulses and multifractal properties at small scales. We call this an Iterated Random Pulse (IRP) process, due to the hierarchical clustering of the pulse locations; see *Veneziano et al.* [2002] and section 2.2. Pulses at different scales have amplitudes with cascade-like dependence. The IRP process produces a random pattern of wet and dry periods inside the synoptic events. This small-scale lacunarity is nonfractal and, as we shall see, is of key importance for a realistic representation of rainfall.

[7] A different class of multifractal pulse processes, based on the idea of exponentiated fractal sums of pulses (EFSP), has been introduced by *Veneziano et al.* [1995] and *Saucier* [1996]. Both IRP and EFSP processes are stationary and "gridless" in that the pulses are not constrained to discrete locations. However, there are important differences between the two schemes. One is that IRP processes have clustered pulses with a hierarchical organization (like in the LeCam models), whereas the pulses of EFSP have Poisson locations. If rainfall is viewed as a discrete sequence of storm events, then IRP processes provide a more natural representation of the storm interiors than EFSP processes do. A second difference is that, in the subclass of IRP models used here, the rainfall process is made exclusively of pulses at the highest resolution, whereas in the EFSP model pulses of all sizes coexist. A third important difference is that, at locations with no pulse coverage, IRP processes are zero, whereas EFSP processes have nonzero (unit) value. The latter property of EFSP models is not appealing for rainfall.

[8] The proposed IRP-based model has only six parameters, four of which describe the exterior process and two control the lacunarity and small-scale multifractality of the interior process. In spite of its parsimonious parameterization, the model reproduces a wide range of observed rainfall statistics including the spectrum, the duration of wet and dry periods, the near-fractal characteristics of the rain support, the distribution of rainfall intensities for different aggregation periods up to extreme fractiles, the moment-scaling function K(q), and the distribution of the partition coefficients. The partition coefficients are factors used in the disaggregation of rainfall time series [*Harris et al.*, 1998; *Olsson*, 1998]. In addition, the model reproduces several observed deviations of physical rainfall and its support from exact fractal/multifractal behavior.

[9] In section 2 we review the construction and properties of IRP processes and describe our rainfall model. In section 3 we fit the model to a high-resolution rainfall time series from Florence, Italy, and compare statistics of the real time series that are not used for estimation with those of simulations from the fitted model. We also discuss data requirements and alternative parameter estimation strategies. In the concluding section we mention possible model refinements and extensions.

2. Exterior and Interior Rainfall Models

[10] As it is typical in event-based rainfall representations, we distinguish between an exterior model and an interior model [*Eagleson*, 1972]. The exterior model is a coarse representation of rainfall, which characterizes the arrival, duration and average intensity of rainfall events at the synoptic scale. The interior model describes the detailed fluctuations of rainfall intensity at subsynoptic scales. Next we describe these two components of our model.

2.1. Exterior Model

[11] We use an exterior model of conventional renewal type, consisting of an alternating sequence of dry and wet periods with independent durations. We take the distribution of the wet periods to be exponential and that of the dry periods to be Weibull. The average rainfall intensities in different wet periods are independent and identically distributed variables, with exponential distribution. These distributions have been previously used by *Eagleson* [1972] and *Molini et al.* [2000], among others. We shall further justify the choice of the Weibull distribution in section 3, using the Florence rainfall record. This very simple alternating renewal process with independent mean event intensities is incapable of representing seasonal effects and other long-term dependencies, but as we shall show in section 3, it is sufficient if the objective is to reproduce rainfall extremes and many properties at subsynoptic scales. Also *Schmitt et al.* [1998] used an alternating renewal process for the dry and wet periods to model temporal rainfall at Uccle, Belgium. However, these authors used the empirical distributions of the dry and wet durations and therefore assumed that the renewal model applies at all scales, including in the interior of the synoptic events. In our case the exterior model is parametric and the lacunarity of rainfall within the precipitation events is modeled separately, as explained below. Our exterior model is characterized by four parameters: the mean duration of the wet periods $m_{\tau_{wet}}$, the mean value $m_{\tau_{dry}}$ and exponent k of the Weibull distribution of the dry periods, and the mean value m_I of the average rainfall intensity during the synoptic events.

2.2. Interior Model

[12] The most innovative aspect of our work is the way we represent rainfall during the rainy periods (interior model). For this purpose we use an iterated random pulse (IRP) process [*Veneziano et al.*, 2002], which generalizes the representation of *Deidda et al.* [1999]. The main feature of the IRP model is that rainfall is represented as the superposition of pulses with a hierarchically nested structure of temporal occurrences and a cascade-like dependence of the intensities. More specifically, the IRP representation of each synoptic event results from the following recursive procedure.

1. At level 0, rainfall intensity during the synoptic event is given by the associated pulse in the exterior process. We denote this pulse by $h(t - t_0)$, where t_0 is a location parameter.

2. At level 1, the level-0 pulse is replaced by a random number N of offspring pulses, contracted by a factor $r > 1$ relative to the parent pulse, with random temporal offsets relative to t_0 and randomly scaled intensities. This replacement has the form

$$h(t - t_0) \Rightarrow \sum_{i=1}^{N} \eta_i \, h(r(t - t_i)) \tag{1}$$

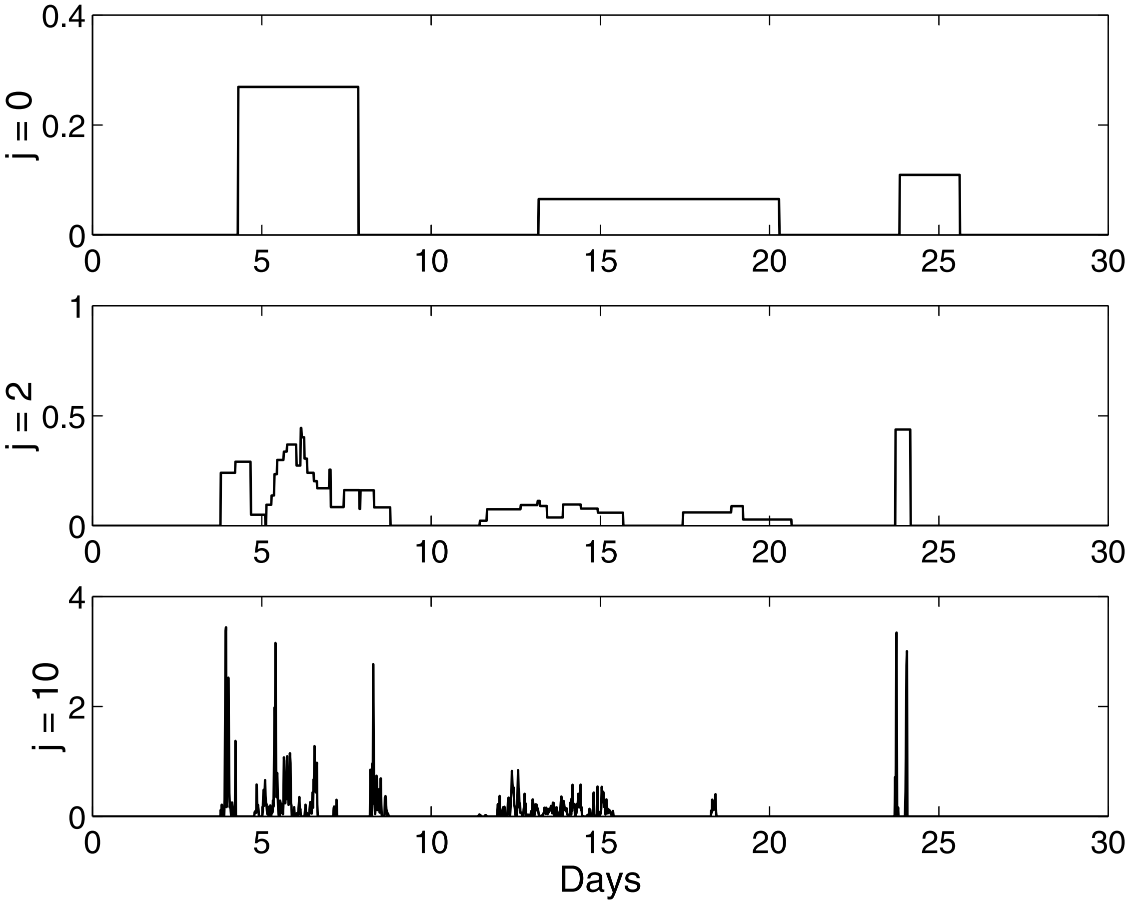

Figure 1. Illustration of the IRP model at different levels of resolution j.

where N has Poisson distribution with mean value r (the same as the contraction factor), the locations $t_1, \ldots, t_N$ are independent and identically distributed random variables with a probability density function $f_t(t) = h(t-t_0)/\int h(\tau)d\tau$ obtained by scaling the parent pulse at level 0, and the intensity factors η_i are independent copies of a nonnegative random variable η with mean value 1. Due to the form of the density function f_t and the fact that $E[N] = r$ and $E[\eta] = 1$, the replacement in equation (1) preserves in approximation the mean temporal evolution of rainfall intensity inside the synoptic event. Equation (1) is applied independently to each pulse at level 0.

3. At subsequent levels $j = 2, 3, \ldots$, each pulse at level $j-1$ is replaced by a Poisson cluster of pulses at level j, using a similar procedure. This means that equation (1) is used with t_0 and $h(t-t_0)$ replaced by the location and intensity of a (j-1)-level pulse.

[13] This process of replacing each pulse with clusters of smaller pulses continues, ideally to infinity. At any finite level j, the rainfall intensity inside a synoptic event is the sum of a random number N_j of level-j pulses. The mean value of N_j is $E[N_j] = r^j$, the duration of the level-j pulses equals the duration of the parent pulse at level 0 divided by r^j, and the amplitudes of the level-j pulses are obtained as the product of the amplitude of the synoptic pulse times j independent copies of the random variable η. It follows from this construction that the locations and amplitudes of the pulses are mutually dependent variables. Figure 1 shows an example realization of the IRP process for a 1 month period that includes three synoptic events. The plot at the top is the exterior process (in this case a series of rectangular pulses that represent the average shape and intensity of the synoptic events). The other two plots show the rainfall intensity generated by the interior IRP process at resolution levels j = 2 and j = 10. As the resolution level increases, the support of rainfall becomes more lacunar and the average intensity during the rainy intervals increases, in such a way that the mean rainfall volume for each synoptic event remains constant (notice that the vertical scale in Figure 1 varies with j).

[14] The IRP construction has clear analogies with multiplicative cascade models and with the pulse process of *Deidda et al.* [1999]. In particular, the parameter r corresponds to the multiplicity of the cascade and the random variable η is analogous to the cascade generator. However, there are also important differences. Contrary to previous models, each pulse of the IRP process has a random number of offspring pulses, which are randomly located relative to their parent. This construction produces hierarchical clustering of the pulses inside the synoptic events, manifested in bursts of high precipitation intensity separated by dry intervals; see Figure 1. As previously noted, the lacunarity within the synoptic events increases as the level j increases. One can show that, in the limit as $j \to \infty$, the probability that it rains at a generic point in time vanishes and the expected intensity when it rains diverges, in such a way that the mean rainfall intensity remains constant nonzero [*Veneziano et al.*, 2002]. These features are absent from the

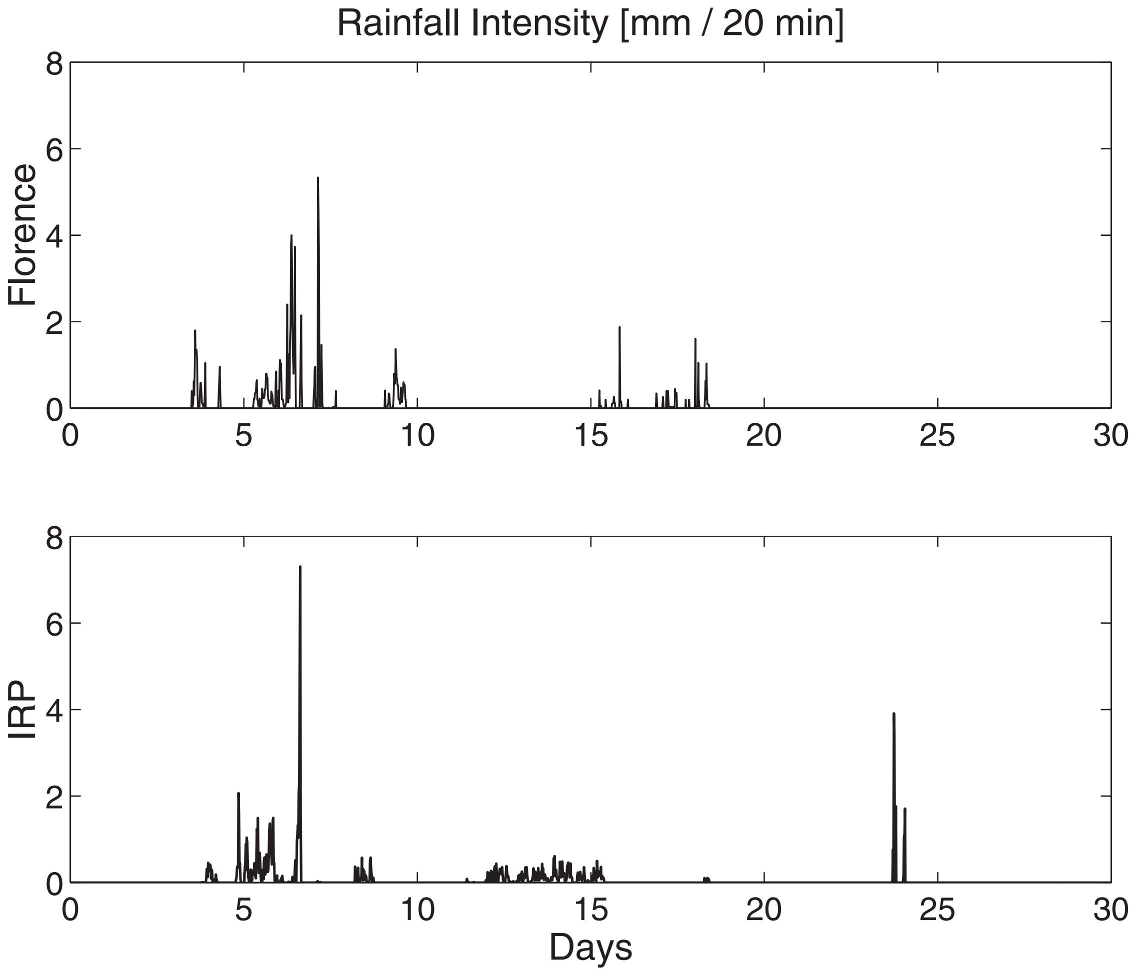

Figure 2. Comparison of 1 month segment of the Florence record with a similar segment from an IRP simulation.

models of *Deidda et al.* [1999] and *Schmitt et al.* [1998]. In section 3, we shall show that the lacunarity of the IRP process matches that of the rainfall record in Florence, Italy. Another result obtained by *Veneziano et al.* [2002] is that, for $j \rightarrow \infty$ and at small scales, IRP processes have the same multifractal properties as discrete cascades with multiplicity r and generator η; in particular, their moment scaling function K(q) is given by $K(q) = \log_r E[\eta^q]$.

[15] In section 3, we shall assume that the distribution of η is lognormal. In this case the interior process has just two parameters, r and the so-called codimension $C_1 = 0.5\log_r(E[\eta^2])$.

3. Parameter Estimation and Model Validation

[16] Next we show how the exterior and interior parameters are estimated through an application to the rainfall record from the Osservatorio Ximeniano in Florence, Italy [*Becchi and Castelli*, 1989]. The record has a temporal resolution of 5 min and covers a period of 24 years from 1962 to 1985. The tipping bucket gage used for measuring precipitation has a mechanical tip event counter that records the time of each 0.2 mm of rainfall depth. The instrument cannot resolve very low rates or determine the exact beginning and ending times of rainfall events. For these reasons and in consultation with the data preparers (F. Castelli, personal communication, 2000), we aggregated the original 5 min data into 20 min intervals. The gage resolution of 0.2 mm makes it impossible to resolve rainfall intensities below 0.6 mm/h (1 bucket tip in 20 min). Lower rainfall values are present in the data due to a manual interpolation operation applied to the tip counts prior to digitization; see *Becchi and Castelli* [1989]. We considered these low rain rates in the wet-dry statistics but ignored them as numerically unreliable when analyzing the marginal distribution of rainfall intensity.

[17] An accurate representation of rainfall should of course consider seasonal variations, but for simplicity we present only results for the entire annual series. We have found that the performance of the model is virtually the same when the record is broken down into seasons and the parameters are estimated separately for each season. Figure 2 shows a 1 month segment from the Florence record and compares it with a similar period from an IRP simulation with parameters estimated as explained below. The two processes are visually similar both in the fluctuations of rainfall intensity and in the pattern of on-off periods. To validate the model, we compare various actual and model-simulation statistics that were not used for parameter estimation.

[18] In discussing the parameter fitting procedure and present model validation results, we divide the character-

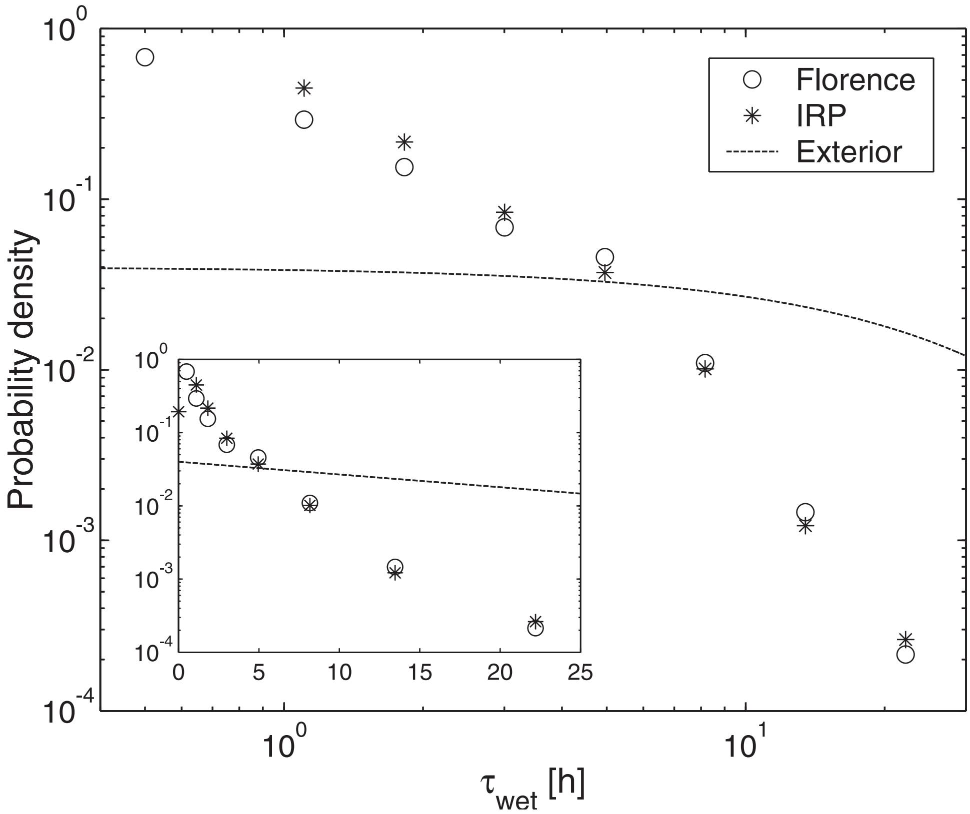

Figure 3. Comparison of the distribution of wet durations for Florence, the exterior model, and the complete IRP process. In the exterior process the wet duration has exponential distribution with mean value $m_{\tau_{wet}} = 25$ hours.

istics of temporal rainfall into two groups and separately deal with each group in sections 3.1 and 3.2. The first group of statistics refers to the alternation of wet and dry periods (distribution of dry and wet durations, box-counting properties of the rainfall support, on/off component of the partition coefficients). The second group includes statistics that reflect also the rainfall intensity during the wet periods (spectral density of the process, marginal distribution of the average intensity in intervals of given duration, IDF curves, moment-scaling function K(q) of multifractal analysis, and intensity component of the partition coefficients). For parameter estimation, we select a set of statistics with the criteria that each statistic be robust and sensitive to one parameter. In this way we effectively decouple the estimation of different parameters.

3.1. On/off Process of Temporal Rainfall

[19] The model parameters that affect the on/off properties of rainfall are the mean durations of the wet and dry periods, $m_{\tau_{wet}}$ and $m_{\tau_{dry}}$, and the exponent k of the Weibull distribution for the dry periods. A fourth parameter is the multiplicity r of the interior IRP process.

[20] The empirical distributions of the duration of rainy periods for the Florence record and a synthetic 10 year time series generated from the IRP process are compared in Figure 3. In order to aid the visual assessment of whether the distributions are exponential or hyperbolic, the probability densities are shown in both log-log and semi-log scale (the latter in the inset). The model parameter to which the distribution is most sensitive is the mean duration of the synoptic events, $m_{\tau_{wet}}$, which in Figure 3 has been set to 25 hours. The effect of $m_{\tau_{wet}}$ is however indirect, as the duration of the wet periods is strongly influenced by the lacunarity of the IRP process at subsynoptic scales. For comparison, Figure 3 shows as a dashed line the probability density of the wet duration for the exterior process (the exponential distribution with a mean value of 25 hours). It is clear that the small-scale lacunarity of the IRP process is essential for the correct reproduction of the distribution of wet durations. The estimate $m_{\tau_{wet}} = 25$ hours has been obtained by repeating the analysis of Figure 3 for different $m_{\tau_{wet}}$ and then selecting the value that from visual inspection best reproduces the empirical distribution for Florence.

[21] A similar comparison for the dry periods is made in Figure 4. In this case the model results depend on the distribution of the dry periods between synoptic events, which is assumed to be Weibull, with mean value $m_{\tau_{dry}}$ and shape parameter k. In principle, one should find the values of $m_{\tau_{wet}}$ and k through repeated simulation and comparison with the Florence statistics, as we have done in Figure 3 for $m_{\tau_{wet}}$. However, a simpler approach is made possible here by the fact that, in the upper tail and except for a multiplicative factor, the distribution of the dry periods in the exterior process is nearly the same as that in the complete model. Therefore, $m_{\tau_{dry}}$ and k can be obtained through direct fit to the upper tail of the dry distribution for Florence (the Weibull distribution itself was selected based on the upper tail of the distribution for Florence shown in Figure 4). The model simulations in Figure 4 are for $m_{\tau_{dry}} = 80$ hours and k = 0.62. These are maximum likelihood estimates using the

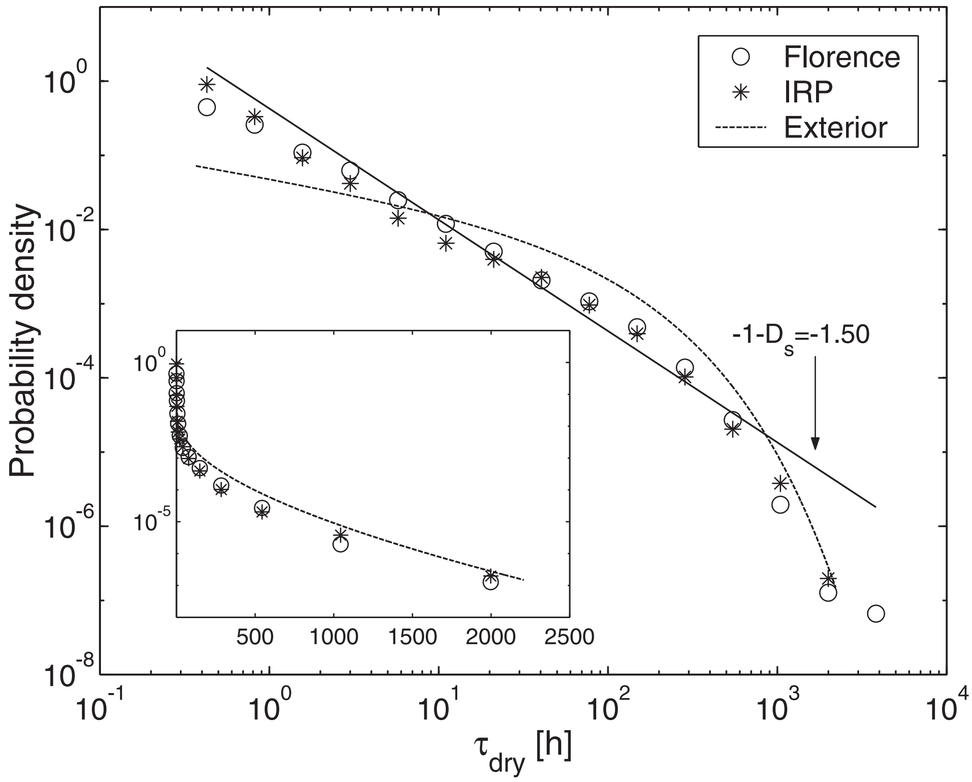

Figure 4. Comparison of the distribution of dry durations for Florence, the exterior model, and the complete IRP process. In the exterior process the dry duration has Weibull distribution with mean value $m_{\tau_{dry}}$ = 80 hours and exponent k = 0.62. The straight line has slope 1 + D_s, where D_s is the fractal dimension estimated from Figure 5.

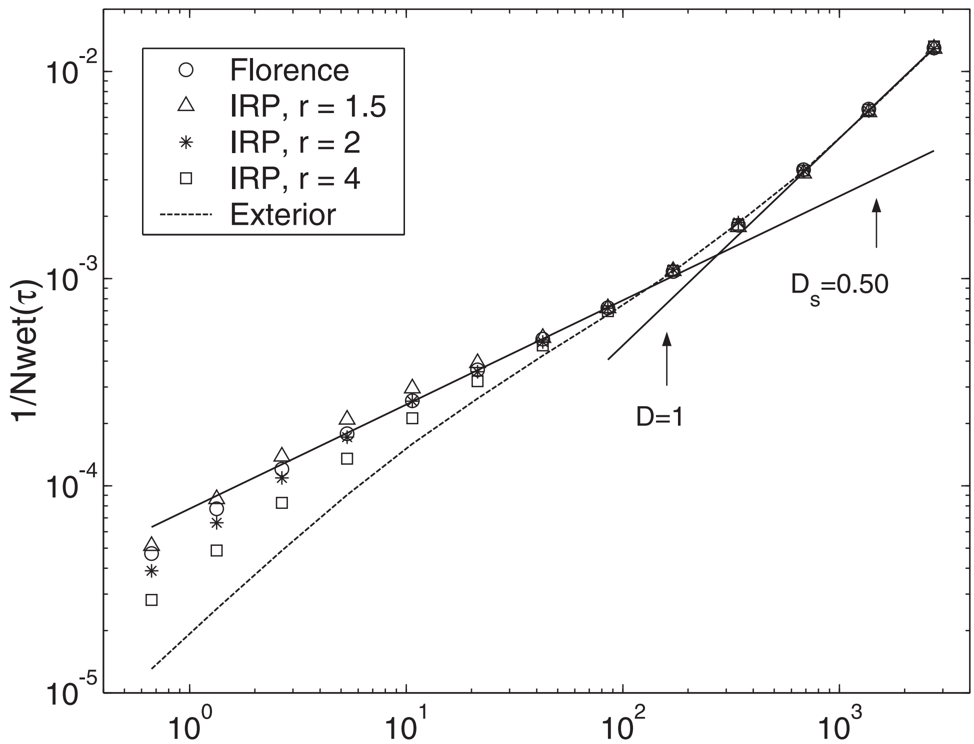

Figure 5. Box-counting analysis of the wet support for Florence, the exterior process, and the complete IRP process. For the complete process, different values of the multiplicity parameter r are used.

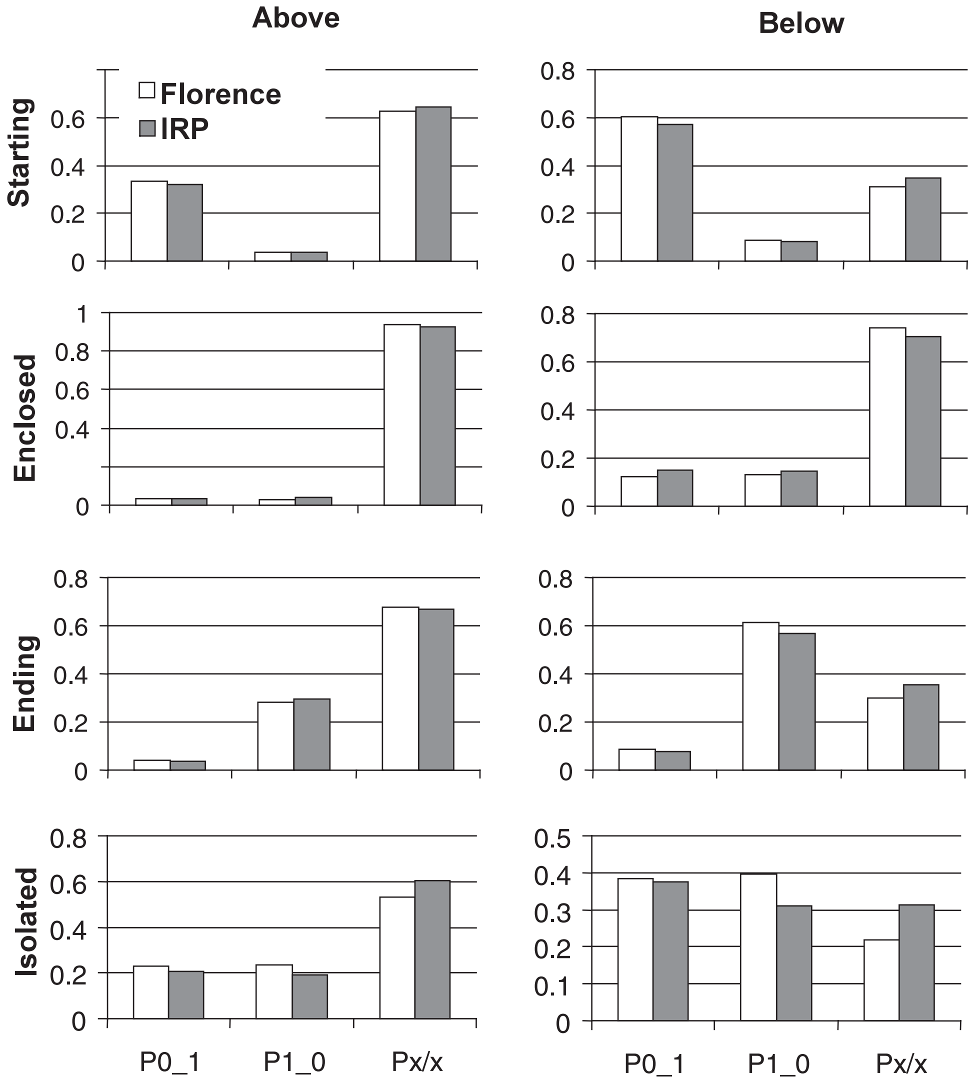

Figure 6. Comparison of the partition coefficients P_{01}, P_{10}, and P_{xx} for Florence and the complete IRP model. All statistics refer to interval durations T between 20 min and 85.33 hours. The interval classes correspond to those of *Olsson* [1998], and "above" and "below" refer to the mean precipitation intensity being above or below the median value, respectively.

dry periods with duration above 30 hours in the Florence record. The fact that the model reproduces very well the whole distribution for Florence validates the present choice of a Weibull distribution type and our parameter estimation procedure. The log likelihood for $\tau > \tau_{\min}$ in terms of the usual (α, k) parameterization is

$$L(\alpha, k; \tau_{\min}) = \sum_{\tau_i > \tau_{\min}} \log\left[\frac{e^{-\alpha\tau_i^k} - e^{-\alpha(\tau_i+\Delta\tau)^k}}{1 - e^{-\alpha\tau_{\min}^k}}\right] \quad (2)$$

where $\Delta\tau$ is the discretization interval. The function in Equation (2) has been maximized with respect to α and k for $\Delta\tau = 1/3$ h $\tau_{\min} = 30$ h and then $m_{\tau_{dry}}$ has been calculated from

$$m_{\tau_{dry}} = \alpha^{\frac{1}{k}}\Gamma\left(1 + \frac{1}{k}\right) \quad (3)$$

[22] Several authors, for example *Schmitt et al.* [1998], have observed that the number of wet (i.e., not completely dry) intervals of duration τ, $N_{wet}(\tau)$, follows in approximation a power law for small τ, $N_{wet}(\tau) \propto \tau^{-D_s}$, indicating that at small scales the support of temporal rainfall is fractal, with a fractal dimension $D_s < 1$. In reality, one observes deviations from perfect fractal behavior in the sense that the rainfall support becomes increasingly compact at scales below a few hours. These characteristics are evident in Figure 5, which compares statistics from Florence and simulations of the exterior and complete model. For the complete model, results are shown for different values of the multiplicity parameter r. Although the macroscopic behavior of $N_{wet}(\tau)$ is already captured by the exterior model, a good fit to the Florence statistics is obtained only after consideration of the breakup of the rain support inside the synoptic events. The closest fit is obtained for r = 2, which is the value we have used in all the IRP simulations. Interestingly, the IRP model reproduces well not only the first-order fractal behavior (with D_s around 0.5, which is close to the value 0.55 reported by *Schmitt et al.* [1998] for Uccle), but also the curvature of the $N_{wet}(\tau)$ plot for Florence at small τ. The fractal dimension D_s is reflected

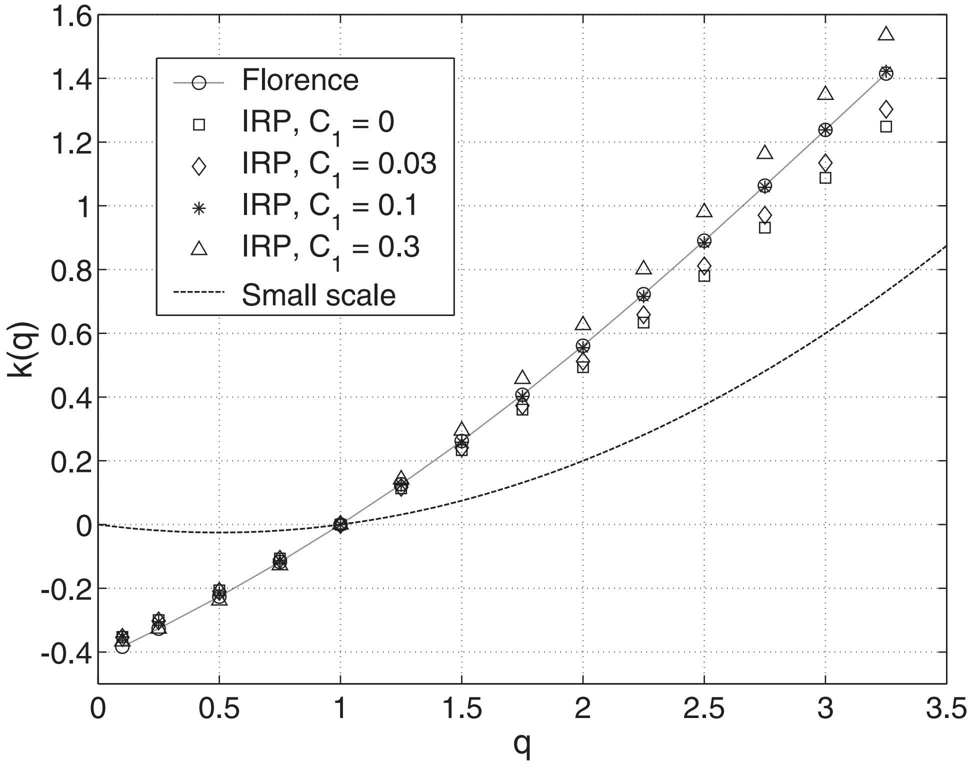

Figure 7. Comparison of the moment scaling functions K(q) for Florence and the complete IRP process for different values of C_1. The dashed line is the theoretical function K(q) of the IRP process at very small scales.

also in the distribution of the dry periods τ_{dry}: as *Schmitt et al.* [1998] have shown, for small durations the probability density of τ_{dry} should follow a power law with exponent $-(1 + D_s)$; see Figure 4.

[23] The last on/off statistic we consider is the distribution of the partition coefficients, which are used to disaggregate rainfall volumes from long to short time intervals [*Olsson*, 1998; *Olsson and Berndtsson*, 1998; *Harris et al.*, 1998]. For this purpose, it is typical to use a binary disaggregation scheme in which the rainfall volume $V > 0$ in a time interval D of duration T is decomposed into a rainfall volume V_1 in the first half interval and a volume $V_2 = V - V_1$ in the second half interval. In order to obtain V_1 and V_2 from V, one must first specify the random on/off pattern of rainfall in the two subintervals. This is done through the probability P_{10} that only the first half interval is rainy (hence $V_1 = V$ and $V_2 = 0$) and the probability P_{01} that only the second half interval is rainy (hence $V_2 = V$ and $V_1 = 0$). The complement $P_{xx} = 1 - P_{10} - P_{01}$ is the probability that both subintervals are rainy. In this case one must further give the distribution of $W_1 = V_1/V$. *Olsson* [1998] has shown that both the probabilities (P_{10}, P_{01}, P_{xx}) and the distribution of W_1 depend strongly on the mean intensity $\bar{I} = V/T$ and on whether D is a starting, ending, enclosed, or isolated interval. This classification is based on the dry/wet nature of the intervals of duration T that immediately precede and follow D.

1. If the preceding interval is dry and the following interval is wet, then D is a starting interval.
2. If the preceding interval is wet and the following interval is dry, then D is an ending interval.
3. If both the preceding and following intervals are wet, then D is an enclosed interval.
4. If both the preceding and following intervals are dry, then D is an isolated interval.

[24] Here we consider how well the IRP model reproduces the probabilities (P_{10}, P_{01}, P_{xx}) for Florence. The distribution of W_1 has to do with rainfall intensity and will be considered later in section 3.2. The probabilities (P_{10}, P_{01}, P_{xx}) vary continuously with the average intensity $\bar{I}$, but here we follow *Olsson* [1998] and evaluate these probabilities only for $\bar{I}$ above or below the median value. Statistics from the Florence record and a 10 year IRP simulation are compared in Figure 6. The histograms refer to aggregations over periods of duration between 20 min and 85.33 hours. The agreement is very good for all the intensity and interval categories.

3.2. Rainfall Intensity

[25] Next we consider statistics related to precipitation intensity. In the model, rainfall intensity depends mainly on the distribution of the intensity of the synoptic events and the parameter C_1 of the interior IRP process. The synoptic events have been assumed to have exponentially distributed amplitudes [*Eagleson*, 1972], with mean value m_I chosen to reproduce the average rainfall intensity over the entire Florence record. Hence m_I has been found from

$$m_I = R\frac{m_{\tau_{wet}} + m_{\tau_{dry}}}{m_{\tau_{wet}}} \qquad (4)$$

where R is the mean annual rainfall intensity over the entire record.

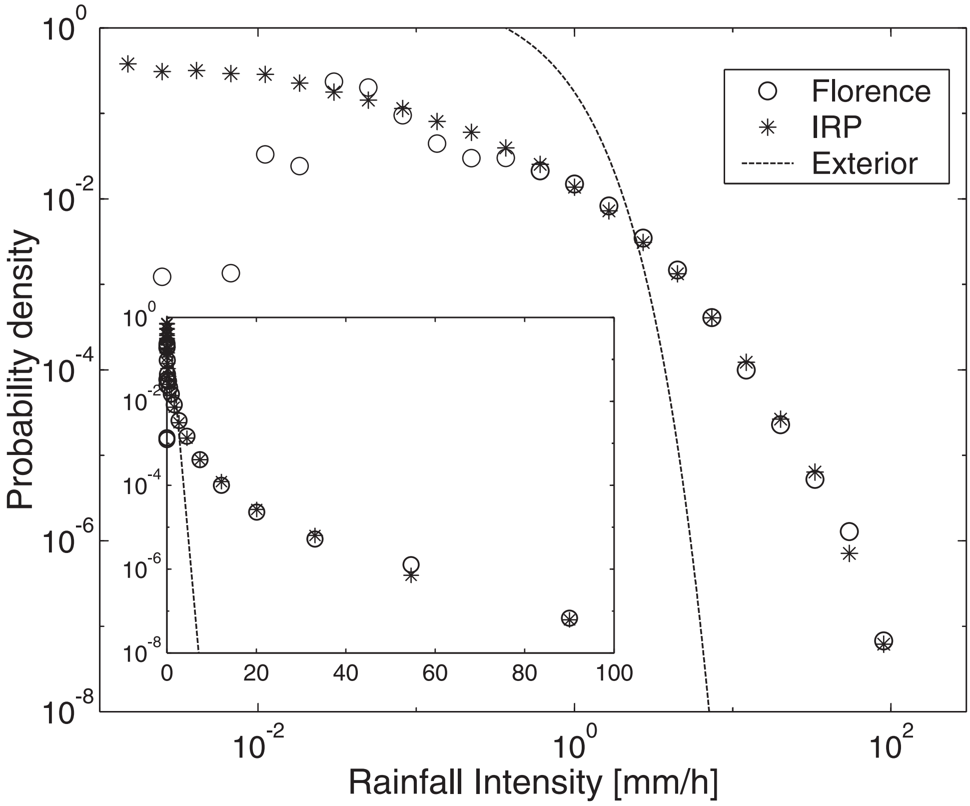

Figure 8. Comparison of the probability density function of the average rainfall intensity in 20-min intervals for Florence, the exterior process, and the interior process. The values for Florence are reliable only for intensities above 0.6 mm/h (thick circles).

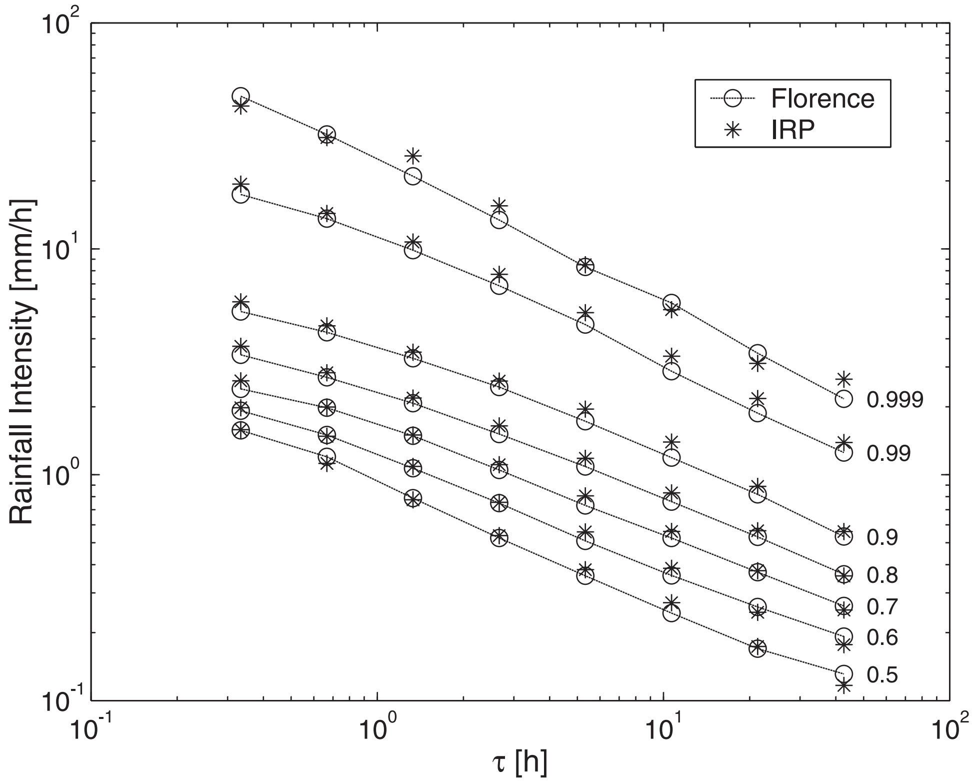

Figure 9. Comparison of fractiles of the average intensity inside periods of different duration τ, conditional on the intensity exceeding 0.6 mm/h. The numbers on the right are the nonexceedance probabilities.

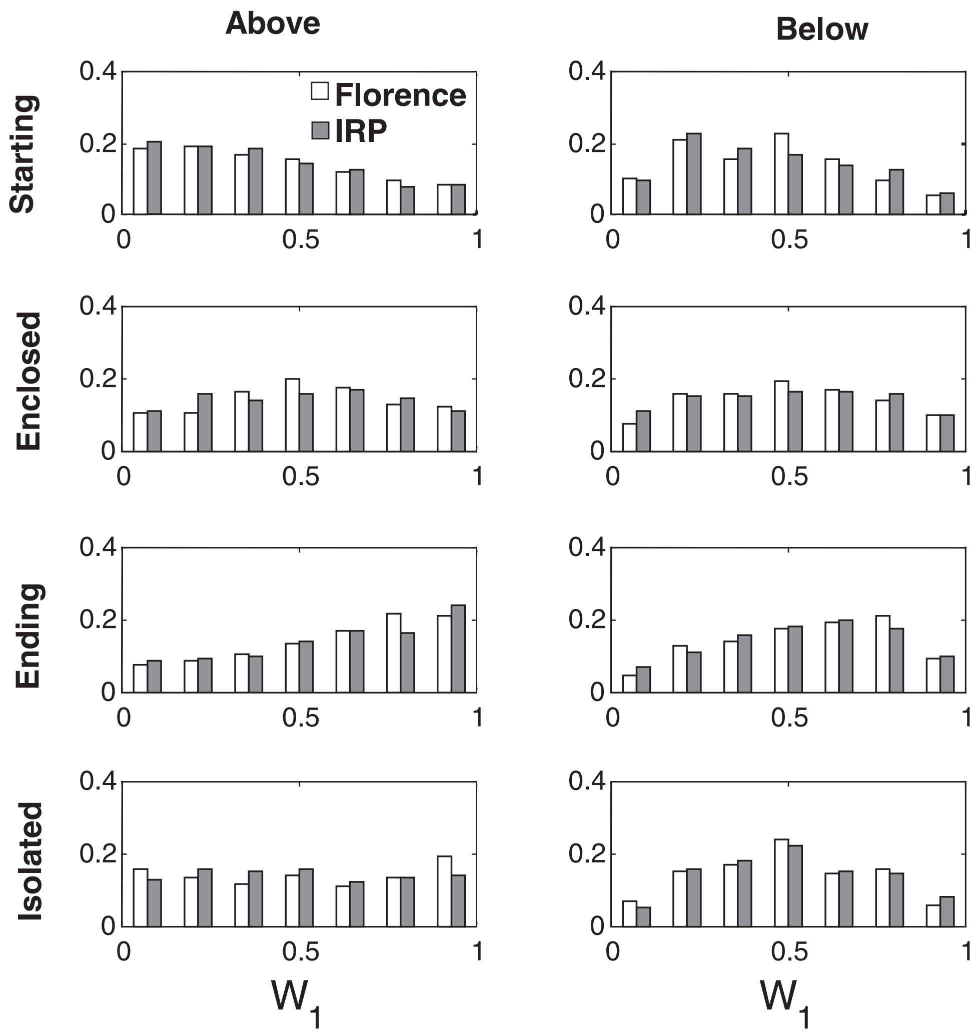

Figure 10. Comparison of the distribution of the partition factor W_1 for Florence and the complete IRP process. The interval and intensity classification is the same as in Figure 6.

[26] The parameter C_1 controls the multifractal properties of rainfall at small scales. Hence we have estimated C_1 from the moment scaling function K(q), which is defined such that

$$E\left[I_\tau^q\right] \sim \tau^{-K(q)} \tag{5}$$

where I_τ is the average intensity in an interval of duration τ. Figure 7 shows the functions K(q) for Florence and for simulations of the IRP process, the latter using different values of C_1. In all cases K(q) was obtained through least squares regression on the empirical moments of I_τ for τ in the range from 20 min to about 85 hours (3.5 days). A close fit is obtained for C_1 = 0.1. This value of C_1 has been used in all the model simulations. An interesting feature of Figure 7 is the dramatic difference between the empirical K(q) function in the above range of scales and the theoretical K(q) function of the IRP process at very small scales. The latter is given by $K(q) = 0.1(q^2 - q)$ and is shown in Figure 7 as a dashed line. The deviation of the symbols from this theoretical function is due to the quasi-fractal characteristics of the rainfall support in the range of scales (20 min to 3.5 days) used to estimate K(q). This is a confirmation of the observation in the Introduction that rainfall has multifractal properties over only a limited range of scales.

[27] To validate the model, Figure 8 compares the marginal distribution of the average intensity I_{20min} for the fitted model and for the Florence record. While the distributions for the exterior process and the complete IRP process are extremely different, the latter is very close to the distribution for Florence for I > 0.6 mm/h (thick circles). The threshold of 0.6 mm/h corresponds to the resolution of the rain gage; hence discrepancies at lower intensity levels (thin circles) are to be expected.

[28] A broader comparison is shown in Figure 9. In this case we have calculated fractiles of the conditional distribution of $[I_\tau | I_\tau > 0.6 \text{ mm/h}]$ for selected values of τ between 20 min and about 2 days. The agreement between the fractiles for Florence and the IRP process is very good up to extreme values. This is a significant result, because the high fractiles are related to the values of the intensity-duration-frequency curves and to rainfall intensities commonly used in engineering practice. Notice that having estimated the parameter C_1 from the empirical moments does not prevent one from using the marginal distribution of

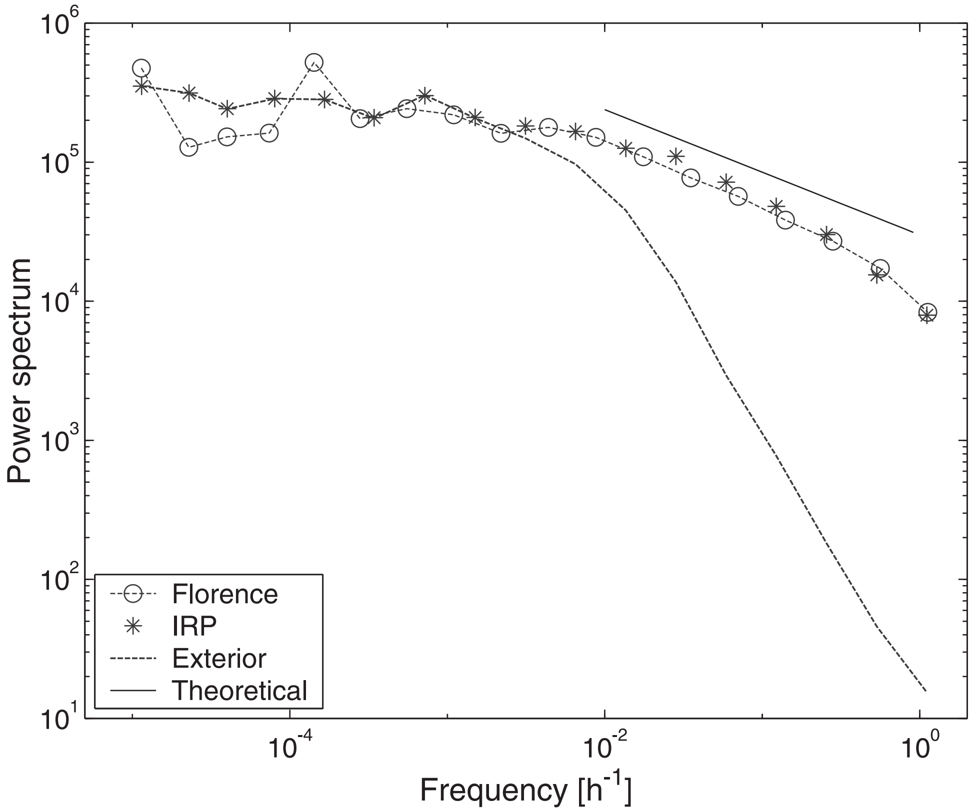

Figure 11. Comparison of the spectral density functions for Florence, the exterior model and the complete IRP process. The straight line segment has slope 1 − K(2) = 0.45, with K(2) from Figure 7 for Florence.

I_τ as a validation statistic, because C_1 controls the scaling of the moments with the aggregation interval τ, not their absolute values.

[29] Next, we complete the analysis of the partition coefficients started in section 3.1 by comparing the empirical distributions of W_1 for Florence and the IRP process. The results are shown in Figure 10, separately for each class of intervals (starting, ending, enclosed, or isolated) and for average rainfall intensities above or below the median value. As one would expect, the distributions for intervals of the starting and ending types are approximately triangular and specularly symmetric, with maximum probability density near 0 and 1, respectively. For the enclosed and isolated intervals the distribution is symmetrical, typically with a mode near 0.5. In all cases the correspondence between the observed and model distributions is very good.

[30] As a final comparison, Figure 11 shows the spectral density functions of the real and simulated series. While at low frequencies the behavior of the spectrum is already captured by the exterior process, a good match of the whole spectrum for Florence is obtained only with the complete IRP model. The slope of the straight line is 1 − K(2) = 0.45. This is the theoretical slope of the spectral density of a multifractal process with the K(q) function for Florence in Figure 7.

3.3. Comments on Parameter Estimation and Data Requirements

[31] Of the six parameters of the model, the mean rainfall intensity during synoptic events, m_I, and the parameters $m_{\tau_{dry}}$ and k of the Weibull distribution of dry duration between such events can be accurately estimated from very few years of rainfall data. These parameters do not necessitate high sampling rates. Since $m_{\tau_{dry}}$ and k are found from the upper tail of the distribution of dry durations, coarse series of daily rainfall are adequate; see *Iacobellis et al.* [2001].

[32] The parameter C_1 has been estimated from the empirical moment scaling function K(q) using average rainfall intensities in time intervals varying from 20 min to a few days. Also this function is well constrained by a few years of rainfall records. However, an attractive alternative is to estimate C_1 with the objective of best reproducing rainfall extremes, as given by intensity-duration-frequency (IDF) curves. Reasons for doing so are the following: (1) As is well known [e.g., *Lovejoy and Schertzer*, 1995], in multifractal processes, there is a direct relation between extremes and the K(q) function. (2) The accurate reproduction of extremes is one of the critical features of a rainfall model. (3) At many locations, extreme statistics in the form of IDF curves are available, whereas reliable continuous records are not. The authors have been studying this alternative estimation strategy; preliminary results have been reported by *Iacobellis et al.* [2001].

[33] The remaining two parameters, the multiplicity r of the interior IRP process and the mean duration of the synoptic events $m_{\tau_{wet}}$, are less critical and in many applications could be set to nominal values (e.g., r = 2 and $m_{\tau_{wet}}$ = 24–48 hours). The parameter r controls the quasi-fractal behavior of the rainfall support at small scales (see Figure 5).

This property is rarely needed in hydrologic applications. The mean duration of the synoptic events is also not very important because the actual duration of rainy intervals is dominated by the wet-dry alternation at smaller temporal scales. The only hydrologic statistics that might be affected by an incorrect setting of $m_{\tau_{wet}}$ are the extreme rainfalls during long time periods. Extremes over shorter durations are dominated by the fluctuations of rainfall intensity inside the synoptic events and are nearly independent of $m_{\tau_{wet}}$.

4. Conclusions

[34] Our purpose has been to develop a model of temporal rainfall that has a pulse structure, multifractal scale invariance at small scales, and deviations from fractal/multifractal behavior similar to those observed in rainfall data. We have achieved this objective by using a conventional exterior process for the arrival, duration and intensity of the synoptic events and a process of the iterated random pulse (IRP) type for the storm interiors. Important features of the IRP process are that the pulse locations have hierarchical clustering in time and the pulse amplitudes are generated by a multiplicative cascade mechanism. These features distinguish the present model from previous multifractal characterizations of rainfall.

[35] The proposed model has four parameters for the exterior rainfall process and two for the storm interiors. The parameters of the exterior process control the marginal distributions of the duration, intensity and separating distance between synoptic events, whereas the parameters of the interior model determine the small-scale lacunarity of rainfall and its small-scale multifractal properties. Through an application to the rainfall record of Florence, Italy, we have shown how the parameters can be estimated so that key statistics are reproduced. We have also validated the model by comparing characteristics of rainfall that were not used for parameter estimation. In spite of its few parameters, the model is able to reproduce a wide range of properties of the on/off process as well as statistics that reflect the intensity fluctuations of precipitation. These include rainfall extremes, renormalization properties, the alternation of wet/dry periods, and the partition coefficients for rainfall downscaling. While at small scales the IRP processes are multifractal, at larger scales they are not. We have found that, in the transient regime from large to small scales, the deviations from perfect scaling correspond closely with those of actual rainfall and its support.

[36] The model is susceptible to refinements and extensions. One aspect that can be easily included and would make the model more realistic is to allow the pulses to have stochastic shape. This is done by treating the functions $h_i(t)$ as random processes. In the work of *Veneziano et al.* [2002] we have included this generalization. Another obvious and important extension is to use the IRP modeling idea for space-time rainfall. In our analysis we have ignored seasonal fluctuations and other long-term dependencies; these features can be included by allowing the model parameters to vary slowly in time (and in space, for nonhomogeneous spatial rainfall fields). In addition, the parameters r and C_1 of the interior process could depend on the intensity of the synoptic event or on other physical characteristics of the rainstorm; such dependencies have been observed in actual rainfall records [*Perica and Foufoula-Georgiou*, 1996; *Over and Gupta*, 1994, 1996; *Olsson and Berndtsson*, 1998]. Finally, one could modify the IRP construction to better reproduce observed rainfall statistics at very small scales, where scaling concepts might no longer apply. For this purpose one may generate the pulse amplitudes in the IRP model using bounded cascades [*Menabde et al.*, 1997, *Menabde and Sivapalan*, 2000], in which the dispersion of the cascade generator is progressively reduced as the resolution level increases.

[37] All these extensions would invariably require more extensive parameterization. What this study has shown is that, in the range of scales from about 20 min to a few days, even a very simple IRP representation of temporal rainfall suffices to reproduce many important rainfall properties.

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