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06 May 2024

ON THE OPTIMIZATION OF CONSERVATION LAW MODELS AT A JUNCTION WITH INFLOW AND FLOW DISTRIBUTION CONTROLS[*]

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Abstract. The paper proposes a general framework to analyze control problems for conservation law models on a network. Namely, we consider a general class of junction distribution controls and inflow controls and we establish the compactness in $\mathbf{L}^1$ of a class of flux-traces of solutions. We then derive the existence of solutions for two optimization problems: (I) the maximization of an integral functional depending on the flux-traces of solutions evaluated at points of the incoming and outgoing edges; (II) the minimization of the total variation of the optimal solutions of problem (I). Finally we provide an equivalent variational formulation of the min-max problem (II) and we discuss some numerical simulations for a junction with two incoming and two outgoing edges.

1. Introduction. Fluid-dynamic models of traffic flow on networks, based on conservation laws, have been intensively investigated in the last twenty years. For a general introduction we refer to [15, 39, 41, 54]. We recall that the idea of modeling unidirectional car traffic on a single road in terms of the scalar conservation law

$$(1.1) \qquad\qquad \partial_t u + \partial_x f(u) = 0$$

was first proposed in the seminal papers by Lighthill, Whitham, and Richards (LWR model) [66, 73]. Here, the unknown $u = u(t,x)$ denotes the traffic density taking values in a compact interval Ω, and the flux has the form $f(u) = u \, v(u)$, where $v(u)$ is the average velocity of cars which is assumed to depend on the density alone.

Setting of the problem. In order to determine the evolution of vehicular traffic in an entire network of roads modeled by a directed graph, one has to further assign a set of suitable boundary conditions at road intersections. Due to finite propagation speed, it will be sufficient to analyze the local solution in a neighborhood of each intersection to capture the global behavior on the whole network. To fix the ideas, let us consider a graph composed by a single vertex (or node), with m incoming edges

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I_i, $i \in \mathcal{I} = \{1, \ldots, m\}$, and n outgoing ones I_j, $j \in \mathcal{O} = \{m+1, \ldots, m+n\}$. We may model each incoming edge with the half-line $(-\infty, 0)$ and every outgoing one with the half-line $(0, +\infty)$. In this way, the junction is always sitting at $x = 0$. Denoting with u_ℓ a traffic density in I_ℓ ($\ell \in \mathcal{I} \cup \mathcal{O}$), the conservation law (1.1) on every incoming and outgoing edge must be supplemented with some initial condition $\bar{u}_\ell$, which thus yields the Cauchy problems

$$(1.2) \qquad \begin{cases} \partial_t u_i + \partial_x f(u_i) = 0, & x < 0, \, t > 0, \\ u_i(0, x) = \overline{u}_i(x), & x < 0, \end{cases}$$

for every $i \in \mathcal{I}$, and

$$(1.3) \qquad \begin{cases} \partial_t u_j + \partial_x f(u_j) = 0, & x > 0, \, t > 0, \\ u_j(0, x) = \overline{u}_j(x), & x > 0, \end{cases}$$

for every $j \in \mathcal{O}$, where we may assume that the initial conditions $\overline{u}_\ell : I_\ell \to \Omega$, $\ell \in \mathcal{I} \cup \mathcal{O}$, are given $\mathbf{L}^1$-functions with bounded variation.

The equations in (1.2)–(1.3) are usually coupled through transition conditions prescribed at the boundary $x = 0$ (also called nodal conditions, coupling conditions, or junction conditions). Typically, one introduces such conditions to guarantee that (1.2)–(1.3) admits a unique solution so as to show that the Cauchy problem at a junction is well-posed. Thus, in particular, every $(m+n)$-tuple of initial data $\overline{u}_\ell, \ell \in \mathcal{I} \cup \mathcal{O}$, determines a unique $(m+n)$-tuple of incoming and outgoing fluxes $g_\ell \doteq f(u_\ell(\cdot, 0)), \ell \in \mathcal{I} \cup \mathcal{O}$, of the corresponding solution to (1.2)–(1.3). In this paper, instead, we consider a minimal set of natural coupling conditions and treat as control laws both the flow distribution parameters related to such conditions and the incoming fluxes at the junction (the *inflows*) that are compatible with them. Our intention here is to propose a general setup to analyze optimal control problems on networks with cost functionals depending on flow distribution controls and inflow junction controls.

Review of the literature. Optimization and control issues for network models based on conservation laws have raised an increasing interest in the last decade, motivated by a wide range of applications in various research fields. In fact, beside vehicular traffic (see [12, 17, 28, 37, 46, 47, 56, 57, 72] and references therein), such kinds of problems have been addressed in air traffic [13, 68], supply chains [35, 36, 49, 59], irrigation channels [48], gas pipelines [29], telecommunication and data [23], biomedical [15, 26], blood circulation [20], socioeconomical, and other areas. In these contexts, a crucial role is played by the design, analysis, and numerical implementation of controls acting at the nodes of the network. The investigation of optimal control properties of time varying parameters corresponding to junction distribution coefficients have been considered, for example, in [21, 22, 23, 42, 46, 50, 51, 69] while inflow controls have been analyzed in [28, 36, 45, 65].

To introduce our control framework let's focus again on the LWR model. The transition conditions at a junction, in a realistic car traffic model, are determined by (i) *drivers' turning preferences* and (ii) *relative priority assigned to different incoming roads*.

As described in [25, 41, 53], the nodal condition (i) can be expressed requiring that the flux traces of the solutions to (1.2)–(1.3) at $x = 0$ satisfy some, possibly time varying, distribution rules. Namely, consider an $n \times m$ Markov matrix $A(t) =$

$(a_{ji}(t))_{j,i}$ with
(1.4)

$$0 \le a_{ji}(t) \le 1 \quad \forall\, j \in \mathcal{O}, i \in \mathcal{I}, t > 0, \qquad \sum_{j=m+1}^{m+n} a_{ji}(t) = 1 \qquad \forall\, i \in \mathcal{I}, t > 0,$$

and impose the condition

$$(1.5) \qquad f(u_j(t,0^+)) = \sum_{i=1}^{m} a_{ji}(t) f(u_i(t,0^-)) \qquad \text{for a.e. } t > 0 \qquad \forall\, j \in \mathcal{O},$$

where $u_\ell(t,0^\pm) \doteq \lim_{x \to 0^\pm} u_\ell(t,x)$. Throughout the following we shall simply adopt the notation $u_\ell(t,0)$ for the one-sided limit of $u_\ell(t,\cdot)$ at the boundary $x = 0$. Here $a_{ji}(t)$ represents the fraction of drivers arriving from the ith incoming road that wish to turn on the jth outgoing road at time t. Notice that, because of (1.4), the condition (1.5) in particular implies a Kirchhoff-type formula

$$\sum_{j=m+1}^{m+n} f(u_j(t,0)) = \sum_{i=1}^{m} f(u_i(t,0)) \qquad \text{for a.e. } t > 0,$$

which expresses the conservation of the total flux of cars through the node. Instead, the nodal condition (ii) is expressed in [25, 41] assigning right of way parameters $\eta_i(t) \in [0,1]$, $i \in \mathcal{I}$, with $\sum_i \eta_i(t) = 1$. For example, in an intersection regulated by a traffic signal, the coefficient η_i can be interpreted as the fraction of time in which it is given a green light to cars arriving from the ith road.

However, the nodal conditions (i)–(ii) above described in general are satisfied by infinitely many solutions to (1.2)–(1.3). Various approaches have been proposed in the literature to overcome this ill-posedness of the Cauchy problem. A widely used method is based on the introduction of a junction Riemann solver, i.e., a rule to construct a unique self-similar solution of the Cauchy problem (1.2)–(1.3) when the initial data are constant on each incoming and outgoing edge. Once a junction Riemann solver is given, the solution of the Cauchy problem at a junction for general initial data is then constructed by a standard wavefront tracking technique [25, 41, 43, 53]. A different model, which is not based on the construction of a junction Riemann solver, was proposed in [16]. Here, cars seeking to enter a congested road wait in a buffer of limited capacity and the distribution coefficients $a_{ji}(t)$ in (1.5) are regarded as the boundary values at $x = 0$ of passive scalars transported by a semilinear equation coupled with the LWR equation (1.1). In this model the unique solution of a general Cauchy problem (1.2)–(1.3) is obtained employing an extension of the Lax–Oleĭnik formula to the initial boundary value problem and determining the length of queues inside a buffer as the fixed point of a contractive transformation. In fact, it is shown in [18] that a specific Riemann solver at the junction determines the limiting solution obtained in the presence of a buffer when the buffer's capacity approaches zero. Instead, a junction Riemann solver for a source destination model that incorporates a dynamic description of the car path choices was previously introduced in [40].

We point out that alternative approaches to establish a well-posedness theory of traffic flow on networks, based on the analysis of Hamilton–Jacobi equations at a junction, have been developed in recent years, using either PDE methods (see the papers [55, 67] and references therein) or optimal control interpretation (see [1] and references therein). In [7, 24] an analysis of the well-posedness of (1.2)–(1.3) based on a vanishing viscosity approximation and adoption of a similar framework of the

theory of germs introduced in [10] for conservation laws with discontinuous flux is pursued. Additional models can be found in [39]. We refer also to [34, 61] for traffic engineering references.

The Riemann solver approach. We remark that all the models without buffer proposed in the literature provide a procedure to associate with any given $(m+n)$-tuple of initial conditions $\overline{u}_\ell$, $\ell \in \mathcal{I} \cup \mathcal{O}$, a unique $(m+n)$-tuple of boundary values $\widetilde{u}_\ell$, $\ell \in \mathcal{I} \cup \mathcal{O}$, so that the resulting solutions $u_\ell(t,x)$, $(t,x) \in (0,+\infty) \times I_\ell$ of the Cauchy–Dirichlet problems with initial data $\overline{u}_\ell$ and boundary data $\widetilde{u}_\ell$ satisfy the nodal constraint (1.5). In this paper we adopt a different perspective following a control theoretic approach. Namely, we propose a general framework to treat optimal control problems at a junction, where one regards as junction controls the distribution coefficients $a_{ji}(t)$ and the incoming fluxes $f(u_i(t,0))$ compatible with the transition condition (1.5).

To illustrate our control strategy let's recall the standard construction performed for a Riemann problem at a junction with a constant matrix distribution $A = (a_{ji})_{j,i}$ satisfying (1.4). Namely, consider the Cauchy problem

$$(1.6) \qquad \begin{cases} \partial_t u_\ell + \partial_x f(u_\ell) = 0, & x \in I_\ell,\ t > 0, \\ u_\ell(0,x) = \overline{u}_\ell, & x \in I_\ell, \end{cases} \qquad \ell = 1,\ldots,m+n,$$

where $\overline{u}_1,\ldots,\overline{u}_{m+n} \in \Omega$ are constants. In connection with every $\overline{u}_i$, $i \in \mathcal{I}$, we define the set $\Omega_{\overline{u}_i}^{\leq 0}$ consisting of all states $\widetilde{u}_i \in \Omega$ for which the entropy admissible solution of the classical Riemann problem on the whole real line,

$$(1.7) \qquad \begin{cases} \partial_t u_i + \partial_x f(u_i) = 0, & x \in \mathbb{R},\ t > 0, \\ u_i(0,x) = \begin{cases} \overline{u}_i & \text{if} \quad x < 0, \\ \widetilde{u}_i & \text{if} \quad x > 0, \end{cases} \end{cases}$$

contains only waves with nonpositive characteristic speeds. Similarly, for every $\overline{u}_j$, $j \in \mathcal{O}$, we let $\Omega_{\overline{u}_j}^{\geq 0}$ denote the set of all states $\widetilde{u}_j \in \Omega$ for which the entropy admissible solution of the classical Riemann problem

$$\begin{cases} \partial_t u_j + \partial_x f(u_j) = 0, & x \in \mathbb{R},\ t > 0, \\ u_j(0,x) = \begin{cases} \widetilde{u}_j & \text{if} \quad x < 0, \\ \overline{u}_j & \text{if} \quad x > 0, \end{cases} \end{cases}$$

contains only waves with nonnegative characteristic speeds. Therefore, given $\overline{u} = (\overline{u}_1,\ldots,\overline{u}_{m+n}) \in \Omega^{m+n}$, consider the closed, convex, not empty set

$$(1.8) \quad \Gamma_{\overline{u}} \doteq \left\{ (\gamma_1,\ldots,\gamma_m) \in \prod_{i=1}^m f(\Omega_{\overline{u}_i}^{\leq 0}) \ :\ \left(A \cdot (\gamma_1,\ldots,\gamma_m)^T \right)^T \in \prod_{j=m+1}^{m+n} f(\Omega_{\overline{u}_j}^{\geq 0}) \right\},$$

where γ^T denotes the (column) transpose vector of γ. Clearly $(0,\ldots,0) \in \Gamma_{\overline{u}}$ and so $\Gamma_{\overline{u}}$ is not empty. Moreover it is closed since it is the preimage of a closed set through a continuous function, and the convexity follows from the linearity of the function associated with A. The set $\Gamma_{\overline{u}}$ describes the flux-traces at $x = 0$ on the incoming roads I_i, $i = 1,\ldots,m$ of every self-similar entropy admissible solution to the nodal Riemann problems (1.6) that satisfies the conditions (1.5) for a constant A.

In general, the set $\Gamma_{\overline{u}}$ in (1.8) contains more than one point so, to achieve uniqueness, an optimization criterion is usually imposed, for example, requiring the maximization of the total flux through the junction [25, 41, 43, 53]. An alternative approach to identify a unique Riemann solver at the junction is proposed in [7] in the same spirit of the theory of germs for conservation laws with discontinuous flux.

Once a procedure is provided to associate with any $\overline{u} \in \Omega^{m+n}$ a unique element $\gamma_{\overline{u}} \in \Gamma_{\overline{u}}$ that identifies the boundary flux-traces of the solution of (1.6) selected by the chosen admissibility criterion, one can require that the solution $u(t,x)$ of a general Cauchy problem at the junction (1.2)–(1.3) satisfies the nodal condition

$$\big(f(u_1(t,0)), \ldots, f(u_m(t,0))\big) = \gamma_{u(t,0)} \in \Gamma_{u(t,0)} \qquad \forall\, t > 0,$$

where

$$u(t,0) = (u_1(t,0), \ldots, u_m(t,0), u_{m+1}(t,0), \ldots, u_{m+n}(t,0))$$

denotes the $(m+n)$-tuple of traces at $x = 0$ of $u(t, \cdot)$.

Our approach and main result. In the present paper instead, aiming to perform a control theoretic analysis, we consider as *admissible solutions of the nodal Cauchy problem* (1.2)–(1.3) every function $u \doteq (u_{\mathcal{I}}, u_{\mathcal{O}})$ with $u_{\mathcal{I}} : [0, +\infty) \times [0, +\infty) \to \Omega^m$, $u_{\mathcal{O}} : [0, +\infty) \times (-\infty, 0] \to \Omega^n$, that for some $k_\ell : [0, +\infty) \to \Omega$, $\ell \in \mathcal{I} \cup \mathcal{O}$, provides an entropy admissible solution of the mixed initial boundary value problems

$$(1.9) \qquad \begin{cases} \partial_t u_i + \partial_x f(u_i) = 0, & x < 0,\ t > 0, \\ u_i(0,x) = \overline{u}_i(x), & x < 0, \qquad\quad i \in \mathcal{I}, \\ u_i(t,0) = k_i(t), & t > 0, \end{cases}$$

$$(1.10) \qquad \begin{cases} \partial_t u_j + \partial_x f(u_j) = 0, & x > 0,\ t > 0, \\ u_j(0,x) = \overline{u}_j(x), & x > 0, \qquad\quad j \in \mathcal{O}, \\ u_j(t,0) = k_j(t), & t > 0, \end{cases}$$

and satisfies the linear constraint (1.5) (see Definition 2.2). Observe that, in general, the boundary data k_ℓ, $\ell \in \mathcal{I} \cup \mathcal{O}$, are not pointwise attained by the entropy admissible solutions of the Cauchy–Dirichlet problems (1.9), (1.10) because of the presence of boundary layers. In fact, for conservation laws the boundary conditions are enforced in the weak sense of Bardos, Le Roux, and Nédélec [11]. Moreover, by uniqueness, the solutions to (1.9), (1.10) can be equally defined as the entropy admissible solutions of the corresponding mixed problems with assigned initial datum $\overline{u}_\ell$ and boundary flux $f(u_\ell(t,0))$ (cf. [62, proof of Theorem 2.2]). Therefore, we may uniquely identify every admissible solutions of the nodal Cauchy problem (1.2)–(1.3) assigning the matrix valued map $A(\cdot)$ satisfying (1.4) and the m-tuple of boundary incoming flux-traces $\gamma = (f(u_1(\cdot,0)), \ldots, f(u_m(\cdot,0)))$ which, in turn, by (1.5) determines also the n-tuple of boundary outgoing flux-traces $(f(u_{m+1}(\cdot,0)), \ldots, f(u_{m+n}(\cdot,0)))$. We shall denote by $u(t,x; A, \gamma)$ such a solution, where γ denotes an m-tuple of *admissible boundary incoming flux-traces* (see Definition 3.1). Then, for a fixed initial datum $\overline{u} \in \Pi_{\ell=1}^{m+n} \mathbf{BV}(I_\ell; \Omega)$, and for a fixed time $T > 0$, we will address the following optimization problems.

(I) Given a continuous map $\mathcal{J} : \mathbb{R}^m \to \mathbb{R}$, fix $M > 0$ and consider

$$(1.11) \qquad \sup_{A,\,\gamma} \int_0^T \mathcal{J}\big(f(u_1(t,0;\,A,\gamma)), \ldots, f(u_m(t,0;\,A,\gamma))\big) dt,$$

where the supremum is taken over all pairs of matrix valued maps $A(\cdot) = (a_{ji}(\cdot))_{j,i} \in \mathbf{BV}([0,T]; \mathbb{R}^{m \times n})$ satisfying (1.4) and admissible boundary flux-traces $\gamma = (\gamma_1, \ldots, \gamma_m) \in \mathbf{BV}([0,T]; (f(\Omega))^m)$, with total variation bounded by

$$\mathrm{TV}\{a_{ji}\} \leq M \quad \forall j \in \mathcal{O}, i \in \mathcal{I}, \qquad \mathrm{TV}\{\gamma_i\} \leq M \quad \forall i \in \mathcal{I}.$$

(II) The pairs of junction controls $(\widehat{A}, \widehat{\gamma})$ that optimize (1.11) are in general not unique, as shown in section 3. Let $\mathcal{U}_{max}^M$ denote the set of such optimal pairs. In order to restrict the set of optimal solutions, we then consider the minimization problem

$$\inf_{(\widehat{A}(\cdot), \widehat{\gamma}) \in \mathcal{U}_{max}^M} \sum_{i=1}^{m} \mathrm{TV}\left\{ f(u_i(\cdot, 0; \widehat{A}, \widehat{\gamma})) \right\}.$$

The goal of the above problems is to maximize suitable functionals depending on the through flux at the junction in a fixed time interval (for example, the time-average sum or product of the inflows within a day), keeping as small as possible the oscillation of the incoming fluxes. The control parameters determine the percentage of drivers who take the different roads emerging from the junction (*traffic distribution controls*) and regulate the rate at which the vehicles pass through the junction (*inflow controls*). The former can be practically implemented with the use of route information panels giving recommendations to the drivers to take one of the outgoing roads, while the latter can be enforced by ramp meter signals, traffic light timing, yielding and stop signs, integrated vehicular and roadside sensors so as to modulate the amount of incoming fluxes entering the junction. These types of control approaches are of critical importance in traffic management since they allow one to improve the performance of traffic system, alleviate congestion, reduce pollution and accidents without requiring expensive road constructions to increase road capacity.

Notice that an admissible solution of the junction Cauchy problem (1.2)–(1.3) defined as above may exhibit entropy admissible shocks originated at positive times from a point of the line $x = 0$ and stationary nonclassical shocks (see [63]) located at $x = 0$. Moreover, an admissible solution of a junction Riemann problem may well be not self-similar, in the sense that it is not constant along a line exiting from the origin. We may observe that such features of the admissible solutions considered in the present paper occur in various models of conservation laws arising in different contexts: like in the case of discontinuous flux [2, 6], of local [9, 27] or nonlocal [8] constraints, of the LWR model (or also higher order models) coupled with an ODE modeling a buffer dynamics [38].

Our main contributions can be summarized as follows. We shall first establish a compactness result for some classes of flux-traces of solutions, which in turn yields the existence of solutions to the optimization problems (I)–(II) in their general setting.

Next, we provide an equivalent variational formulation of the optimization problem (II) which is useful also for numerical investigations of the optimal solutions. Finally, we analyze the optimization problem (1.11) in the case where the supremum is taken over all pairs of matrix valued maps $A(\cdot)$ and admissible boundary flux-traces with arbitrarily large total variation.

We stress that, in our intention, the approach pursued in this paper is a first step towards building a general strategy to address control problems on networks, which can be applied in various contexts not limited to vehicular traffic models.

Organization of the paper. Section 2 is devoted to preliminary results on boundary value problems for conservation laws and on the Cauchy problem for a junction. In section 3 we derive the compactness properties for flux-traces of solutions and analyze the optimization problems (I)–(II). Finally, in section 4, we discuss some numerical experiments for a junction with two incoming and two outgoing arcs.

2. Preliminaries.

2.1. The Dirichlet and the junction Cauchy problem for conservation laws. Consider a directed graph composed by a single vertex, located at $x = 0$, with m incoming arcs $I_i = (-\infty, 0)$, $i \in \mathcal{I} = \{1, \dots, m\}$, and n outgoing ones $I_j = (0, +\infty)$, $j \in \mathcal{O} = \{m+1, \dots, m+n\}$. On each arc, the evolution of the unknown density $u_\ell(t, x)$ is governed by the scalar conservation law

$$(2.1) \qquad \partial_t u_\ell + \partial_x f(u_\ell) = 0, \qquad t \geq 0, \ x \in I_\ell.$$

The flux function $f : \Omega \to \mathbb{R}$ is defined on a compact interval $\Omega = [0, u^{max}]$ and satisfies the following assumptions:

(A)   1. $f \in \mathbf{C}^1(\Omega; \mathbb{R})$ and is strictly concave.

         2. $f(0) = f(u^{max}) = 0$, $\ f'(u^{max}) < 0 < f'(0)$.

Here $u^{max} > 0$ denotes the maximal possible density inside a road. We denote by $\theta \in \Omega$ the point of global maximum for f, i.e.,

$$\theta \doteq \arg\max_{u \in \Omega} f(u),$$

and, for every $u \in \Omega \setminus \{\theta\}$, we denote by $\pi(u)$ the unique point in Ω such that

$$f(u) = f(\pi(u)) \qquad \text{and} \qquad \pi(u) \neq u,$$

while we set $\pi(\theta) \doteq \theta$. Given an $(m+n)$-tuple of initial data $\overline{u} \in \Pi_{\ell=1}^{m+n} \mathbf{L}^\infty(I_\ell; \Omega)$ and boundary data $k \in \mathbf{L}^\infty\big([0, +\infty); \Omega^{m+n}\big)$, consider, for every $i = 1, \dots, m$, $j = m+1, \dots, m+n$, the mixed initial boundary value problems

$$(2.2) \qquad \begin{cases} \partial_t u_i + \partial_x f(u_i) = 0, & x < 0, \ t > 0, \\ u_i(0, x) = \overline{u}_i(x), & x < 0, \\ u_i(t, 0) = k_i(t), & t > 0, \end{cases}$$

$$(2.3) \qquad \begin{cases} \partial_t u_j + \partial_x f(u_j) = 0, & x > 0, \ t > 0, \\ u_j(0, x) = \overline{u}_j(x), & x > 0. \\ u_j(t, 0) = k_j(t), & t > 0. \end{cases}$$

Remark 2.1. For simplicity we decide to use the same flux function on each edge of the junction, but it is not a restrictive assumption. For real applications, it is natural to consider different flux functions on each road of the network. The extension of all the results of this paper to such a case is straightforward, provided that each flux function satisfies the assumption (A) with possibly different u^{max}.

As observed in the introduction, the Dirichlet conditions in (2.2), (2.3) are, in general, not literally satisfied but must be interpreted in the relaxed sense of Bardos, Le Roux, and Nédélec [11]. In particular, since we assume the strict concavity of the

flux function f, we may equivalently express the boundary condition in (2.2) requiring as in [62] that, for a.e. $t > 0$, there holds

(2.4)
$$\text{either} \qquad u_i(t,0) = \max\{k_i(t), \theta\}$$
$$\text{or} \qquad f'(u_i(t,0)) \geq 0 \quad \text{and} \quad f(u_i(t,0)) \leq f\big(\max\{k_i(t), \theta\}\big),$$

while the boundary condition in (2.3) is equivalent to requiring that, for a.e. $t > 0$, there holds:

(2.5)
$$\text{either} \qquad u_j(t,0) = \min\{k_j(t), \theta\}$$
$$\text{or} \qquad f'(u_j(t,0)) \leq 0 \quad \text{and} \quad f(u_j(t,0)) \leq f\big(\min\{k_i(t), \theta\}\big).$$

Moreover, one can provide an equivalent formulation of (2.4)–(2.5) in terms of boundary flux conditions. Namely, the boundary condition for the problem (2.2) on the negative half-line can be expressed as

(2.6)
$$\text{either} \qquad f'(u_i(t,0)) < 0 \quad \text{and} \quad f'(k_i(t)) < 0, \quad f(u_i(t,0)) = f(k_i(t)),$$
$$\text{or} \qquad f'(u_i(t,0)) \geq 0 \quad \text{and} \quad f'(k_i(t)) > 0,$$
$$\text{or} \qquad f'(u_i(t,0)) \geq 0 \quad \text{and} \quad f'(k_i(t)) \leq 0, \quad f(u_i(t,0)) \leq f(k_i(t)),$$

while the boundary condition for the problem (2.3) on the positive half-line can be expressed as

(2.7)
$$\text{either} \qquad f'(u_j(t,0)) > 0 \quad \text{and} \quad f'(k_j(t)) > 0, \quad f(u_j(t,0)) = f(k_j(t)),$$
$$\text{or} \qquad f'(u_j(t,0)) \leq 0 \quad \text{and} \quad f'(k_j(t)) < 0,$$
$$\text{or} \qquad f'(u_j(t,0)) \leq 0 \quad \text{and} \quad f'(k_j(t)) \geq 0, \quad f(u_j(t,0)) \leq f(k_j(t)).$$

As stated in the introduction, $u_\ell(t,0)$ denotes the one-sided limit at $x = 0$ of the solution $u_\ell(t, \cdot)$ to (2.2), (2.3). We recall that functions of one variable with locally bounded variation ($\mathbf{BV}_{\mathrm{loc}}$) admit left and right limits at every point. Moreover, since an element of $\mathbf{BV}_{\mathrm{loc}}(I_\ell)$ is an equivalence class of locally integrable functions, we will always assume that a function $u \in \mathbf{BV}_{\mathrm{loc}}(I_\ell)$ is left continuous if $\ell \in \mathcal{I}$, right continuous if $\ell \in \mathcal{O}$, possibly modifying its values at its countably many discontinuity points (see [14, Chapter 2, Lemma 2.1]). Here and throughout the paper, for a function $\psi \in \mathbf{BV}([\tau', \tau''])$ we shall always define $\mathrm{TV}\{\psi\}$ as the *essential variation* of ψ on the open interval (τ', τ'') which coincides with the *pointwise variation* on (τ', τ'') of the right continuous or left continuous representative of ψ (see [3, section 3.2]). This implies that, if ψ is a right continuous (or left continuous) function, there holds

(2.8)
$$\mathrm{TV}\{\psi\} = pV(\psi) \doteq \sup_{\tau' < t_0 < t_1 < \cdots < t_N < \tau''} \left\{ \sum_{\ell=1}^{N} |\psi(t_\ell) - \psi(t_{\ell-1})| \right\},$$

where $pV(\psi)$ denotes the pointwise variation of ψ. For a function $\psi \in \mathbf{BV}([0, T])$, and a subinterval $(\tau', \tau'') \subset [0, T]$, we shall also denote as $\mathrm{TV}_{(\tau', \tau'')}\{\psi\} \doteq \mathrm{TV}\big\{\psi\big|_{(\tau', \tau'')}\big\}$ the pointwise variation of the restriction on (τ', τ'') of the right continuous (or left continuous) representative of ψ.

Remark 2.2. Consider a function $\psi \in \mathbf{BV}([0, T])$, let ψ_1 be a left continuous (or right continuous) representative of ψ, and consider any other element ψ_2 of the same

equivalent class in $\mathbf{L^1}\left((0,T)\right)$ of ψ, i.e., such that $\psi_1(t) = \psi_2(t)$ for a.e. $t \in [0,T]$. Then, one has

$$pV(\psi_1) \leq pV(\psi_2).$$

In fact, the pointwise variation defined in (2.8) clearly depends on the choice of the representative of ψ and the infimum is achieved by the left continuous or right continuous representatives of ψ (cf. [3, section 3.2]).

We shall then adopt the following definition of an entropy admissible solution of the mixed initial boundary value problem for a conservation law (2.1) with concave flux (see [11, 30, 33, 58, 60, 71, 74]).

DEFINITION 2.1. *Given* $\overline{u}_\ell \in \mathbf{L}^\infty(I_\ell; \Omega)$ *and* $k_\ell \in \mathbf{L}^\infty\left((0,+\infty); \Omega\right)$, $\ell \in \mathcal{I} \cup \mathcal{O}$, *we say that a function* $u_\ell \in \mathbf{C}\left([0,+\infty); \mathbf{L}^1_{\mathrm{loc}}(I_\ell; \Omega)\right)$ *is an entropy admissible weak solution of* (2.2) *(resp., of* (2.3)*) if*
   (i) *for every* $t > 0$, $u_\ell(t) \in \mathbf{BV}_{\mathrm{loc}}(I_\ell; \Omega)$ *and admits the one-sided limit at* $x = 0$;
   (ii) u_ℓ *is a weak entropy solution of* (2.1) *with initial data* $\overline{u}_\ell$, *i.e., for all* $\varphi \in \mathcal{C}^1_c\left(\mathbb{R} \times I_\ell; \mathbb{R}\right)$ *there holds*

$$(2.9) \qquad \int_0^{+\infty} \int_{I_\ell} \left[u_\ell \, \partial_t \varphi + f(u_\ell) \, \partial_x \varphi\right](t,x) \; dxdt + \int_{I_\ell} \overline{u}_\ell(x) \varphi(0,x) dx \; = \; 0\,,$$

   and the Lax entropy condition [60] *is satisfied,*

$$u_\ell(t, x^-) \leq u_\ell(t, x^+), \qquad t > 0, \; x \in I_\ell;$$

   (iii) *for a.e.* $t > 0$ *the boundary condition at* $x = 0$ *is verified in the sense of* (2.6) *(resp., of* (2.7)*).*

Remark 2.3. Notice that, because of the assumption (A) on the flux function, an entropy admissible weak solution u_ℓ of (2.1) satisfies the one-sided Oleĭnik inequality on the decay of negative waves [70] which, together with the well-known $\mathbf{L}^\infty$ bounds on u_ℓ, yields uniform BV-bounds on $u_\ell(t)$ at any fixed time $t > 0$. Moreover, for every $t > 0$ and $\ell \in \mathcal{I}$, $u_\ell(t)$ admits the one-sided limit $u_\ell(t,0) \doteq \lim_{x \to 0^-} u_\ell(t,x)$. Indeed, if $\lim_{x \to 0^-} u_\ell(t,x)$ does not exist, then one can find two sequences x_n and y_n of negative numbers converging to 0 such that

$$v_1 = \lim_{n \to +\infty} u_\ell\left(t, x_n\right), \qquad v_2 = \lim_{n \to +\infty} u_\ell\left(t, y_n\right), \qquad v_1 < v_2.$$

Since f is concave, then $f'(v_1) > f'(v_2)$ and so, for n sufficiently large, we would deduce that $f'\left(u_\ell(t, x_n)\right) > f'\left(u_\ell(t, y_n)\right)$. By tracing the backward generalized characteristics [32, 33] starting at (t, x_n), (t, y_n), we would then conclude that such characteristics intersect in I_ℓ at some positive time, which is not possible. A similar argument holds for the existence of $\lim_{x \to 0^+} u_\ell(t,x)$, $t > 0$, $\ell \in \mathcal{O}$. Therefore, condition (i) of Definition 2.1 is indeed a consequence of condition (ii). We have included it in the definition for the sake of clarity.

Remark 2.4. For every fixed point $x \in I_\ell$, $\ell \in \mathcal{I} \cup \mathcal{O}$, the flux map $t \mapsto f\left(u_\ell(t,x)\right)$ of an entropy admissible weak solution u_ℓ of (2.1) is nonincreasing in the presence of shock discontinuities. Hence, since one may derive Oleĭnik-type inequalities on the positive variation of $f\left(u_\ell(\cdot, x)\right)$ as for the negative variation of $u_\ell(t, \cdot)$, it follows that $f\left(u_\ell(\cdot, x)\right) \in \mathbf{BV}_{\mathrm{loc}}\left((0, +\infty); \Omega\right)$.

As observed in [62], it is not restrictive to assume that the boundary data have characteristics always entering the domain, i.e., that

$$(2.10) \qquad \begin{aligned} k_i(t) &\geq \theta \qquad \text{if} \quad i \in \mathcal{I}, \\ k_j(t) &\leq \theta \qquad \text{if} \quad j \in \mathcal{O}. \end{aligned}$$

In fact, the entropy admissible weak solutions u_ℓ of (2.2), (2.3) with boundary data k_ℓ, $\ell \in \mathcal{I} \cup \mathcal{O}$, can be as well obtained replacing k_ℓ with the *normalized boundary data*

$$\widetilde{k}_\ell(t) \doteq \begin{cases} \max \big\{ u_\ell(t,0), \, \pi(u_\ell(t,0)) \big\} & \text{if} \quad \ell \in \mathcal{I}, \\ \min \big\{ u_\ell(t,0), \, \pi(u_\ell(t,0)) \big\} & \text{if} \quad \ell \in \mathcal{O}, \end{cases}$$

which satisfy (2.10) and

$$f\big(\widetilde{k}_\ell(t)\big) = f\big(u_\ell(t,0)\big) \qquad \forall \, t > 0 \, .$$

We next recall a general property of weak solutions of (2.1) that will be useful later.

PROPOSITION 2.1. *Given $\overline{u}_\ell \in \mathbf{BV}(I_\ell; \Omega)$ and $k_\ell \in \mathbf{L}^\infty\big((0, +\infty); \Omega\big)$, $\ell \in \mathcal{I} \cup \mathcal{O}$, let u_ℓ be an entropy admissible weak solution of (2.2) if $\ell \in \mathcal{I}$ or (2.3) if $\ell \in \mathcal{O}$. Then, for every $0 \leq t_1 < t_2$, $x_0 \in I_\ell$, one has*

$$(2.11) \quad \int_{t_1}^{t_2} f\left(u_\ell(t,0)\right) dt - \int_{t_1}^{t_2} f\left(u_\ell(t, x_0^\pm)\right) dt = \int_{x_0}^{0} u_\ell(t_1, x) dx - \int_{x_0}^{0} u_\ell(t_2, x) dx.$$

Proof. For simplicity we consider only the case $\ell \in \mathcal{I}$, the other case being completely similar; thus $x_0 < 0$. Notice first that

$$(2.12) \qquad \int_{t_1}^{t_2} f(u_\ell(t, x_0^+)) dt = \int_{t_1}^{t_2} f(u_\ell(t, x_0^-)) dt.$$

Indeed, by the Rankine–Hugoniot condition, the function $f(u_\ell(t,x))$ does not admit discontinuities of zero slope. Moreover if t_0 is a point of continuity for $t \to f(u_\ell(t, x_0))$, then also x_0 must be a point of continuity for $x \to f(u_\ell(t_0, x))$. Therefore

$$\big\{ t > 0 \, | f(u_\ell(t, x_0^-)) \neq f(u_\ell(t, x_0^+)) \big\} \subset \big\{ t > 0 \, | f(u_\ell(t^+, x_0)) \neq f(u_\ell(t^-, x_0)) \big\} \, .$$

The latter set has Lebesgue measure zero, since $f(u_\ell(\cdot, x_0))$ is a function of BV (see Remark 2.4). Thus also the first set has Lebesgue measure zero, which implies (2.12).

Fix now the domain $D \doteq (t_1, t_2) \times (x_0, 0)$ and a sequence $\varphi_\nu \in C_c^1 (\mathbb{R} \times I_\ell; \mathbb{R})$, $\nu \in \mathbb{N}$, such that $\mathrm{Supp}\,(\varphi_\nu) \subseteq D$ for every $\nu \in \mathbb{N}$ and φ_ν converges to the characteristic function of D as $\nu \to +\infty$. The divergence theorem to the BV vector field $(u_\ell \varphi_\nu, f(u_\ell)\varphi_\nu)$ on D in combination with the integral equality (2.9) implies that

$$\begin{aligned} 0 &= \int \int_D \mathrm{div}\, (u_\ell(t,x)\varphi_\nu(t,x), f(u_\ell(t,x))\varphi_\nu(t,x)) \, dx \, dt \\ &= \int \int_D \mathrm{div}\, (u_\ell(t,x), f(u_\ell(t,x))) \, \varphi_\nu(t,x) dx \, dt \end{aligned}$$

for every $\nu \in \mathbb{N}$. Passing to the limit as $\nu \to +\infty$, we deduce that

$$\int \int_D \mathrm{div}\, (u_\ell(t,x), f(u_\ell(t,x))) \, dx \, dt = 0$$

which implies (2.11). $\qquad \square$

Since we consider finite horizon optimization problems (see section 3), we now introduce the concept of a solution to the Cauchy problem for a node of a network on $[0, T]$.

DEFINITION 2.2. *Let $T > 0$ and $A(\cdot) = (a_{ji}(\cdot))_{j,i} \in \mathbf{L}^\infty((0, T); \mathbb{R}^{m \times n})$ be a matrix valued map satisfying* (1.4). *Given $\overline{u} \in \Pi_{\ell=1}^{m+n} \mathbf{L}^\infty(I_\ell; \Omega)$, we say that an entropy admissible weak solution to the nodal Cauchy problem* (1.2)–(1.3) *on $[0, T]$, is a function $u \in \mathbf{C}([0, T]; \Pi_{\ell=1}^{m+n} \mathbf{L}^1_{\mathrm{loc}}(I_\ell; \Omega))$ that, for some $k \in \mathbf{L}^\infty((0, T); \Omega^{m+n})$, enjoys the following properties:*

1. *for every $i = 1, \ldots, m$, the ith component u_i of u is an entropy admissible weak solution of* (2.2) *on $[0, T] \times I_i$, in the sense of Definition 2.1;*
2. *for every $j = m + 1, \ldots, m + n$, the jth component u_j of u is an entropy admissible weak solution of* (2.3) *on $[0, T] \times I_j$, in the sense of Definition 2.1;*
3. *for a.e. $t \in (0, T]$ condition* (1.5) *is satisfied.*

3. Optimization problems. In this section we first introduce the general notation of admissible controls and corresponding solutions. Then we establish a compactness property for a class of flux-traces of solutions. Next, we analyze the optimization problems (I)–(II) described in the introduction. We shall first consider the maximization of a functional, defined in (3.28), related to the integral of the flux at the junction. Then, among all solutions which maximize this functional, we choose the ones whose flux at the junction enjoy minimal total variation (see the minimization problem (3.32)). Finally, we provide also an equivalent variational formulation of this min-max problem in terms of the functional (3.50), which will be useful in the numerical analysis of the optimal solutions.

3.1. General setting. Fix $T > 0$. Given $\overline{u} \in \Pi_{\ell=1}^{m+n} \mathbf{L}^\infty(I_\ell; \Omega)$, for every $i \in \mathcal{I}$ and $j \in \mathcal{O}$, recalling Definition 2.1, consider the sets

(3.1)
$$\mathcal{F}_i \doteq \mathcal{F}_i(\overline{u}_i) \doteq \left\{ f(u_i(\cdot, 0)) \middle| \begin{array}{c} u_i \text{ is a weak entropy admiss. sol. of (2.2) on} \\ [0, T] \times I_i \text{ with boundary data } k_i \in \mathbf{L}^\infty((0, T); \Omega) \end{array} \right\}$$
$$\mathcal{F}_j \doteq \mathcal{F}_j(\overline{u}_j) \doteq \left\{ f(u_j(\cdot, 0)) \middle| \begin{array}{c} u_j \text{ is a weak entropy admiss. sol. of (2.3) on} \\ [0, T] \times I_j \text{ with boundary data } k_j \in \mathbf{L}^\infty((0, T); \Omega) \end{array} \right\},$$

which consist of the boundary flux-traces of all possible entropy admissible weak solutions to (2.2) and to (2.3) with initial data $\overline{u}_i$, $\overline{u}_j$, respectively. We recall that the flux-traces are defined as the one-sided limit $f(u_i(t, 0)) \doteq \lim_{x \to 0^-} f(u_i(t, x))$ for the incoming arcs I_i, $i \in \mathcal{I}$, and as the one-sided limit $f(u_j(t, 0)) \doteq \lim_{x \to 0^+} f(u_j(t, x))$ for the outgoing arcs I_j, $j \in \mathcal{O}$. Then, define the set of admissible matrix valued maps fulfilling (1.4) for a.e. t,

(3.2) $\quad \mathcal{A} \doteq \left\{ A = (a_{ji}(\cdot))_{j,i} \in \mathbf{BV}([0, T]; \mathbb{R}^{m \times n}) \middle| \text{cond. (1.4) holds for a.e. } t \in (0, T] \right\},$

(3.3) $\quad \mathcal{A}^M \doteq \left\{ A = (a_{ji}(\cdot))_{j,i} \in \mathcal{A} \mid \mathrm{TV}\{a_{ji}\} \le M \quad \forall i, j \right\}, \qquad M > 0.$

Admissible controls. As observed in the introduction, we may uniquely identify all admissible solutions of the nodal Cauchy problem (1.2)–(1.3) by assigning the matrix valued map $A \in \mathcal{A}$ and the m-tuple of boundary incoming flux-traces $\gamma = (f(u_1(\cdot, 0)), \ldots, f(u_m(\cdot, 0)))$ which, in turn, by conditions (1.5), determines also the n-tuple of boundary outgoing flux-traces $(f(u_{m+1}(\cdot, 0)), \ldots, f(u_{m+n}(\cdot, 0)))$. We then regard (γ, A) as a pair of *junction controls* and we shall adopt the following definition.

DEFINITION 3.1. *Given $\overline{u} \in \Pi_{\ell=1}^{m+n} \mathbf{L}^\infty(I_\ell; \Omega)$ and $A = (a_{ji}(\cdot))_{j,i} \in \mathcal{A}$, we say that*

$$\gamma = (\gamma_1, \ldots, \gamma_m) \in \mathbf{L}^\infty\big((0, T); [f(\Omega)]^m\big)$$

is an m-tuple of A-admissible boundary inflow controls if there exists a boundary datum $k \in \mathbf{L}^\infty((0, T); \Omega^{m+n})$ such that the entropy admissible weak solutions u_i, $i \in \mathcal{I}$, and u_j, $j \in \mathcal{J}$, of (2.2) and of (2.3), respectively, satisfy
   1. $f(u_\ell(\cdot, 0)) \in \mathcal{F}_\ell$ for every $\ell \in \mathcal{I} \cup \mathcal{O}$;
   2. $\gamma_i(t) = f(u_i(t, 0))$ for a.e. $t \in [0, T]$ and all $i \in \mathcal{I}$;
   3. $\sum_{i=1}^m a_{ji}(t) \gamma_i(t) = f(u_j(t, 0))$ for a.e. $t \in [0, T]$ and all $j \in \mathcal{O}$.

We denote by $u(\cdot, \cdot; A, \gamma)$ the entropy admissible weak solution of (1.2)–(1.3) determined by γ and A. The components $u_\ell(\cdot, \cdot; A, \gamma)$, $\ell \in \mathcal{I} \cup \mathcal{O}$, are an entropy admissible weak solution of (2.2), (2.3), with normalized boundary data $k \in \mathbf{L}^\infty\big((0, T); \Omega^{m+n}\big)$ defined by

$$k_\ell(t) \doteq \begin{cases} f_+^{-1}\big(\gamma_\ell(t)\big) & if \quad \ell \in \mathcal{I}, \\ f_-^{-1}\big(\gamma_\ell(t)\big) & if \quad \ell \in \mathcal{O}, \end{cases}$$

where we denote as f_-, f_+ the restrictions of f to the intervals $[0, \theta]$ and $[\theta, u^{max}]$, respectively.

We then define the sets of admissible controls as

(3.4)

$$\mathcal{U} \doteq \mathcal{U}(\overline{u}) \doteq \left\{ (A, \gamma) \in \mathcal{A} \times \mathbf{L}^1((0, T); \mathbb{R}^m) \;\middle|\; \begin{array}{c} \gamma \text{ is an } m\text{-tuple of } A\text{-admissible} \\ \text{boundary inflow controls} \end{array} \right\},$$

$$\mathcal{U}^M \doteq \mathcal{U}^M(\overline{u}) \doteq \left\{ ((a_{ji})_{j,i}, \gamma) \in \mathcal{U}(\overline{u}) \;\middle|\; \begin{array}{ll} \mathrm{TV}\{\gamma_i\} \leq M & \forall\, i \in \mathcal{I} \\ \mathrm{TV}\{a_{ji}\} \leq M & \forall\, i \in \mathcal{I}, \forall\, j \in \mathcal{O} \end{array} \right\}$$

for all $M \geq 0$.

Remark 3.1. One can easily verify that the sets of admissible controls defined by (3.4) are nonempty. Indeed, given $\overline{u} \in \Pi_{\ell=1}^{m+n} \mathbf{L}^\infty(I_\ell; \Omega)$, consider the $(m + n)$-tuple of boundary data $k_\ell \in \mathbf{L}^\infty([0, T]; \Omega)$, $\ell = 1, \ldots, m + n$, defined by

$$k_i(t) = u^{max} \qquad \forall\, t \in [0, T] \qquad \forall\, i \in \mathcal{I},$$

$$k_j(t) = 0 \qquad \forall\, t \in [0, T] \qquad \forall\, j \in \mathcal{O},$$

and let u_i, $i \in \mathcal{I}$, u_j, $j \in \mathcal{O}$, denote the corresponding entropy admissible weak solution of (2.2) and (2.3), respectively. Observe that, because of the assumption (A) on the flux function, the boundary conditions (2.6), (2.7) imply

$$u_\ell(t, 0) \in \{0, u^{max}\} \qquad \forall\, t \in [0, T] \qquad \forall\, \ell \in \mathcal{I} \cup \mathcal{O}.$$

Therefore, we have

$$\gamma_i(t) \doteq f(u_i(t, 0)) = 0 \qquad \forall\, t \in [0, T] \qquad \forall\, i \in \mathcal{I}$$

$$f(u_j(t, 0)) = 0 \qquad \forall\, t \in [0, T] \qquad \forall\, j \in \mathcal{O},$$

which shows that the m-tuple $\gamma \doteq (\gamma_1, \ldots, \gamma_m)$ satisfies condition (3) of Definition 3.1, for every matrix $A \doteq (a_{ji}(\cdot))_{j,i} \in \mathcal{A}$, or $A \doteq (a_{ji}(\cdot))_{j,i} \in \mathcal{A}^M$, and that $\mathrm{TV}\{\gamma_i\} \leq M$ for any $i \in \mathcal{I}$. This proves that if $A \in \mathcal{A}$ one has $(A, \gamma) \in \mathcal{U}(\overline{u})$, while if $A \in \mathcal{A}^M$ one has $(A, \gamma) \in \mathcal{U}^M(\overline{u})$ S for every $M > 0$.

Admissible flux traces. In connection with the above sets of admissible controls we introduce now the classes of flux-traces of solutions that we will analyze in the optimization problems considered in this paper. Namely, we first define the set of all junction flux-traces in the incoming arcs for the entropy weak solutions of (1.2)–(1.3) associated with the above classes of admissible controls:

$$(3.5) \quad \mathcal{D} \doteq \mathcal{D}(\overline{u}) \doteq \left\{ (g_1, \ldots, g_m) \in \mathbf{L}^1((0,T); \mathbb{R}^m) \;\middle|\; \begin{array}{l} \exists\, (A, \gamma) \in \mathcal{U}(\overline{u}) \text{ s.t.} \\ g_i = f(u_i(\cdot, 0;\, A, \gamma)) \;\forall\, i \in \mathcal{I} \end{array} \right\},$$

$$(3.6) \quad \mathcal{D}^M \doteq \mathcal{D}^M(\overline{u}) \doteq \left\{ (g_1, \ldots, g_m) \in \mathbf{L}^1((0,T); \mathbb{R}^m) \;\middle|\; \begin{array}{l} \exists\, (A, \gamma) \in \mathcal{U}^M(\overline{u}) \text{ s.t.} \\ g_i = f(u_i(\cdot, 0;\, A, \gamma)) \;\forall\, i \in \mathcal{I} \end{array} \right\}.$$

Next, in view of considering more general optimization problems (see Remark 3.4), for every given $x_j \in I_j$, $j \in \mathcal{O}$, we consider the sets
$$(3.7)$$
$$\mathcal{F}_{j, x_j} \doteq \mathcal{F}_{j, x_j}(\overline{u}_j) \doteq \left\{ f(u_j(\cdot, x_j)) \;\middle|\; \begin{array}{l} u_j \text{ is a weak entropy admiss. sol. of } (2.3) \\ \text{on } [0,T] \times I_j \ \text{ for some } \ k_j \in \mathbf{L}^\infty((0,T); \Omega) \end{array} \right\}.$$

Then, given every $x^{\mathcal{O}} \doteq (x_{m+1}, \ldots, x_{m+n}) \in [0, +\infty)^n$, we define the sets of flux-traces of solutions evaluated at the points x_j of the outgoing arcs:

$$(3.8)$$
$$\mathcal{G}_{x^{\mathcal{O}}} \doteq \mathcal{G}_{x^{\mathcal{O}}}(\overline{u}) \doteq \left\{ (g_{m+1}, \ldots, g_{m+n}) \in \prod_{j=m+1}^{m+n} \mathcal{F}_{j, x_j} \;\middle|\; \begin{array}{l} \exists\, (A, \gamma) \in \mathcal{U}(\overline{u}) \quad \text{s.t.} \\ g_j = f(u_j(\cdot, x_j;\, A, \gamma)) \;\forall j \in \mathcal{O} \end{array} \right\},$$

$$(3.9)$$
$$\mathcal{G}_{x^{\mathcal{O}}}^M \doteq \mathcal{G}_{x^{\mathcal{O}}}^M(\overline{u}) \doteq \left\{ (g_{m+1}, \ldots, g_{m+n}) \in \prod_{j=m+1}^{m+n} \mathcal{F}_{j, x_j} \;\middle|\; \begin{array}{l} \exists\, (A, \gamma) \in \mathcal{U}^M(\overline{u}) \quad \text{s.t.} \\ g_j = f(u_j(\cdot, x_j;\, A, \gamma)) \;\forall j \in \mathcal{O} \end{array} \right\}.$$

3.2. Compactness of admissible flux traces. We provide here the compactness of the sets $\mathcal{D}^M$, $\mathcal{G}_{x^{\mathcal{O}}}^M$ with respect to the $\mathbf{L}^1$ topology, which is the standard setting for one dimensional conservation laws. With the same techniques, one can achieve compactness in $\mathbf{L}^p$ with $p > 1$, if required by particular models. This property then yields the existence of optimal solutions for cost functionals depending on the flux-traces of solutions to the nodal Cauchy problem (1.2)–(1.3) evaluated at the intersection $x = 0$ or at points x_j of the outgoing arcs I_j, $j \in \mathcal{O}$.

THEOREM 3.1. *Fix $\overline{u} \in \Pi_{\ell=1}^{m+n} \mathbf{L}^\infty(I_\ell; \Omega)$. Then, for every $M > 0$ the sets $\mathcal{D}^M$, $\mathcal{G}_{x^{\mathcal{O}}}^M$ in (3.6), (3.9) are compact subsets of $\mathbf{L}^1((0,T); \mathbb{R}^m)$, $\mathbf{L}^1((0,T); \mathbb{R}^n)$, respectively.*

Proof. We shall first show the compactness of $\mathcal{D}^M$ and then derive as a consequence the compactness of $\mathcal{G}_{x^{\mathcal{O}}}^M$.

1. Observe that the set $\mathcal{A}^M$ in (3.3) is compact in the $\mathbf{L}^1$ topology by Helly's theorem. On the other hand, setting

$$(3.10) \qquad \mathcal{F}_\ell^M \doteq \left\{ f(u_\ell(\cdot, 0)) \in \mathcal{F}_\ell \mid \mathrm{TV}\left\{ f(u_\ell(\cdot, 0)) \right\} \leq M \right\}, \qquad \ell \in \mathcal{I} \cup \mathcal{O},$$

where $\mathcal{F}_\ell$ are the sets defined in (3.1), by Definition 3.1 and definitions (3.1), (3.4), (3.6), we have

$$\mathcal{D}^M = \left\{ (g_1, \ldots, g_m) \in \prod_{i=1}^m \mathcal{F}_i^M \;\middle|\; \exists\, (a_{ji})_{ji} \in \mathcal{A}^M \;\; \text{s.t.} \;\; \sum_{i=1}^m a_{ji}\, g_i \in \mathcal{F}_j \;\; \forall\, j \in \mathcal{O} \right\}.$$

Since $\mathcal{A}^M$ is compact, in order to establish the compactness of $\mathcal{D}^M$ it will be sufficient to show that the sets $\mathcal{F}_i^M$, $i \in \mathcal{I}$, are compact and that the sets $\mathcal{F}_j$, $j \in \mathcal{O}$, are closed with respect to the $\mathbf{L}^1$ topology. We shall provide only the proof of the compactness of the set $\mathcal{F}_i^M$, $i \in \mathcal{I}$, in (3.10), the proof of the closureness of the sets $\mathcal{F}_j$, $j \in \mathcal{O}$, being entirely similar.

2. Fix $i \in \mathcal{I}$ and let $(g^\nu)_\nu$ be a sequence in $\mathcal{F}_i^M$. Then, for every ν one has

$$(3.11) \qquad g^\nu(t) = f\big(u_i^\nu(t,0)\big) \qquad a.e. \ t \in [0,T]$$

for some weak entropy admissible solution u_i^ν to (2.2) with boundary data $k_i^\nu \in \mathbf{L}^\infty\big((0,T);\Omega\big)$. Moreover, there holds

$$\mathrm{TV}\left\{g^\nu\right\} \le M \qquad \forall\, \nu\,.$$

Since all g^ν take values in the bounded set $f(\Omega)$, applying Helly's compactness theorem we deduce that there exists a function $\psi \in \mathbf{BV}([0,T]; f(\Omega))$ with

$$(3.12) \qquad \mathrm{TV}\{\psi\} \le M,$$

so that (up to a subsequence) there holds

$$(3.13) \qquad g^\nu \quad \to \quad \psi \qquad \text{in} \quad \mathbf{L}^1((0,T);\mathbb{R})\,.$$

On the other hand, consider the map $\widetilde{k}_i \in \mathbf{L}^\infty\big([0,T];\Omega\big)$ defined by

$$(3.14) \qquad \widetilde{k}_i(t) = f_+^{-1}(\psi(t)) \qquad \forall\, t \in [0,T]\,.$$

Then, letting $u_i \in \mathbf{C}\big([0,T]; \mathbf{L}^1((-\infty,0);\Omega)\big)$ be the entropy admissible weak solution of (2.2) with boundary data $\widetilde{k}_i$, by definition (3.1) we have $f(u_i(\cdot,0)) \in \mathcal{F}_i$. Therefore, in order to complete the proof of the compactness of $\mathcal{F}_i^M$ it will be sufficient to show that (up to a subsequence)

$$(3.15) \qquad f(u_i^\nu(\cdot,0)) \quad \to \quad f(u_i(\cdot,0)) \qquad \text{in} \quad \mathbf{L}^1((0,T);\mathbb{R}).$$

In fact, (3.11), (3.13), (3.15) together imply

$$(3.16) \qquad f(u_i(t,0)) = \psi(t) \qquad \text{for a.e.} \ \ t \in [0,T]\,.$$

Hence, recalling Remark 2.2, we deduce from (3.12), (3.16), that the essential variation of ψ satisfies the bound

$$\mathrm{TV}\left\{f\big(u_i(\cdot,0)\big)\right\} = \mathrm{TV}\{\psi\} \le M\,,$$

which implies $f(u_i(\cdot,0)) \in \mathcal{F}_i^M$ by the definition of (3.10).

3. Towards a proof of (3.15) observe first that, as recalled in subsection 2.1, we may identify u_i^ν as the entropy admissible solution to (2.2) with normalized boundary data

$$\widetilde{k}_i^\nu(t) = \max\{u_i^\nu(t,0), \pi(u_i^\nu(t,0))\} \qquad \forall\, t \in [0,T] \qquad \forall\, \nu\,.$$

Then, by (3.11), one has

$$(3.17) \qquad f\big(\widetilde{k}_i^\nu(t)\big) = g^\nu(t) \qquad \forall\, t \in [0,T] \qquad \forall\, \nu\,.$$

Hence, (3.13), (3.14), (3.17) together imply

$$(3.18) \qquad f\big(\widetilde{k}_i^\nu\big) \quad \to \quad f\big(\widetilde{k}_i\big) \qquad \text{in} \quad \mathbf{L}^1((0,T);\mathbb{R}).$$

Therefore, by virtue of the $\mathbf{L}^1$ Lipschitz continuous dependence of the solution to (2.2) on the flux of the normalized boundary data (cf. [5, Theorem 4]), and because of (3.18), one has

$$(3.19) \qquad u_i^\nu(t,\cdot) \quad \to \quad u_i(t,\cdot) \qquad \text{in} \quad \mathbf{L}^1_{loc}(I_i;\Omega) \qquad \forall\, t \in [0,T].$$

On the other hand, recalling that $\Omega = [0,u^{max}]$, notice that, for any solution to (2.2), the backward generalized characteristics (see [32]) issuing from points (t,x), $x < f'(u^{max}) \cdot T$, $t \in [0,T]$, reach the x-axis without crossing the boundary line $x = 0$. Therefore, all solutions u_i^ν are uniquely determined by the the initial data $\overline{u}$ on the region $[0,T] \times (-\infty, f'(u^{max}) \cdot T)$. Hence, one has

$$(3.20) \qquad u_i^\nu(t,x) = u_i(t,x) \qquad \forall\, t \in [0,T],\ x < f'(u^{max}) \cdot T \qquad \forall\, \nu.$$

Hence, invoking Proposition 2.1 for every u_i^ν and, thanks to (3.19), (3.20), we find that for all $x_0 < f'(u^{max}) \cdot T$ and $t_1, t_2 \in (0,T)$, there holds
$$(3.21)$$
$$
\begin{aligned}
\lim_\nu \int_{t_1}^{t_2} f(u_i^\nu(t,0))dt &= \lim_\nu \left[\int_{t_1}^{t_2} f(u_i^\nu(t,x_0))dt + \int_{x_0}^0 u_i^\nu(t_1,x)dx - \int_{x_0}^0 u_i^\nu(t_2,x)dx \right] \\
&= \int_{t_1}^{t_2} f(u_i(t,x_0))dt + \int_{x_0}^0 u_i(t_1,x)dx - \int_{x_0}^0 u_i(t_2,x)dx \\
&= \int_{t_1}^{t_2} f(u_i(t,0))dt.
\end{aligned}
$$

By the arbitrariness of $t_1, t_2 \in (0,T)$, and since all $f(u_i^\nu(\cdot,0))$ take values in the bounded set $f(\Omega)$, we recover from (3.21) the convergence (3.15) completing the proof of the compactness of $\mathcal{D}^M$.

    4. With the same analysis of the previous points we deduce the compactness of the set $\mathcal{G}_{x^\mathcal{O}}^M$ when $x^\mathcal{O} = 0_n \doteq (0,\dots,0) \in \mathbb{R}^n$. Next, consider $x^\mathcal{O} = (x_{m+1},\dots,x_{m+n}) \neq 0_n$ and observe that by definitions (3.7), (3.8), (3.10), we have
$$(3.22)$$
$$
\mathcal{G}_{x^\mathcal{O}}^M = \left\{ (g_{m+1},\dots,g_{m+n}) \in \mathbf{L}^1((0,T);\mathbb{R}^n) \left| \begin{array}{l} \forall\, j \in \mathcal{O}\ \exists\, u_j \text{ weak entr. admiss.} \\ \text{soln. of (2.3) on } [0,T]\times[0,+\infty) \\ \text{for some } k_j \in \mathbf{L}^\infty\big((0,T);\Omega\big) \\ \text{s.t. } g_j = f(u_j(\cdot,x_j)) \text{ and} \\ \big(f(u_{m+1}(\cdot,0)),\dots,f(u_{m+n}(\cdot,0))\big) \in \mathcal{G}_{0_n}^M \end{array} \right. \right\}.
$$

Then, letting $(g^\nu)_\nu$ be a sequence in $\mathcal{G}_{x^\mathcal{O}}^M$, there will be a sequence $(u^\nu)_\nu$ of n-tuples of entropy admissible weak solutions of (2.3) with boundary data $k^\nu \in \mathbf{L}^\infty\big((0,T);\Omega^n\big)$, so that, for every ν, one has

$$(3.23) \quad g_j^\nu = f(u_j^\nu(\cdot,x_j)) \quad \forall\, j \in \mathcal{O}, \qquad \big(f(u_{m+1}^\nu(\cdot,0)),\dots,f(u_{m+n}^\nu(\cdot,0))\big) \in \mathcal{G}_{0_n}^M.$$

By the compactness of $\mathcal{G}_{0_n}^M$ and relying on the analysis performed at previous points, it follows that there exists an n-tuple $(u_{m+1},\dots,u_{m+n})$ of entropy admissible weak solutions of (2.3) with boundary data $k \in \mathbf{L}^\infty\big([0,T];\Omega^n\big)$, such that

$$(3.24) \qquad \big(f(u_{m+1}(\cdot,0)),\dots,f(u_{m+n}(\cdot,0))\big) \in \mathcal{G}_{0_n}^M$$

and a subsequence $\left(\left(f(u^{\nu'}_{m+1}(\cdot,0)),\dots,f(u^{\nu'}_{m+n}(\cdot,0))\right)\right)_{\nu'}$ so that there holds

$$f(u^{\nu'}_j(\cdot,0)) \quad \to \quad f(u_j(\cdot,0)) \qquad \text{in} \quad \mathbf{L}^1((0,T);\mathbb{R}),$$

$$u^{\nu'}_j(t,\cdot) \quad \to \quad u_j(t,\cdot) \qquad \text{in} \quad \mathbf{L}^1_{loc}(I_j;\Omega) \qquad \forall\ t \in [0,T]$$

$$\forall\ j \in \mathcal{O}\,.$$

Hence, invoking Proposition 2.1 for every $u^{\nu'}_j$, we find

(3.25)

$$\begin{aligned}
\lim_{\nu'} \int_{t_1}^{t_2} f(u^{\nu'}_j(t,x_j))dt &= \lim_{\nu'} \left[\int_{t_1}^{t_2} f(u^{\nu'}_j(t,0))dt + \int_0^{x_j} u^{\nu'}_j(t_1,x)dx - \int_0^{x_j} u^{\nu'}_j(t_2,x)dx \right] \\
&= \int_{t_1}^{t_2} f(u_j(t,0))dt + \int_0^{x_j} u_j(t_1,x)dx - \int_0^{x_j} u_j(t_2,x)dx \\
&= \int_{t_1}^{t_2} f(u_j(t,x_j))dt.
\end{aligned}$$

By the arbitrariness of $t_1, t_2 \in (0,T)$, and since all $f(u^{\nu'}_i(\cdot,x_j))$ take values in the bounded set $f(\Omega)$, we derive from (3.23), (3.25) the convergence

$$g^{\nu'}_j \quad \to \quad f(u_j(\cdot,x_j)) \qquad \text{in} \quad \mathbf{L}^1((0,T);\mathbb{R}) \qquad \forall\ j \in \mathcal{O}.$$

On the other hand, because of (3.22), (3.24), we have

$$\left(f(u_{m+1}(\cdot,x_j)),\dots,f(u_{m+n}(\cdot,x_)))\right) \in \mathcal{G}^M_{x_\mathcal{O}},$$

completing the proof of the compactness of $\mathcal{G}^M_{x_\mathcal{O}}$. $\qquad\qquad\square$

Remark 3.2. By the proof of Theorem 3.1 and relying on Helly's compactness theorem it follows that, if $\{\left(A^\nu, f(u^\nu_1(\cdot,0)),\dots f(u^\nu_m(\cdot,0))\right)\}_\nu$ is a sequence in $\mathcal{A}^M \times \mathcal{D}^M$, then letting $\psi \in \mathbf{BV}([0,T]; f(\Omega))$ be a map such that

$$f(u^\nu_i(\cdot,0)) \quad \to \quad \psi \qquad \text{in} \quad \mathbf{L}^1((0,T);\mathbb{R})\,,$$

and denoting by u_i, $i \in \mathcal{I}$, the entropy admissible weak solutions of (2.2) with boundary data $\widetilde{k}_i$ defined according with (3.14), there exist $A \in \mathcal{A}^M$ so that (up to a subsequence) one has

$$A^\nu(t) \quad \to \quad A(t) \qquad \text{for a.e.}\ \ t \in [0,T]$$

and

$$f(u^\nu_i(t,0)) \quad \to \quad f(u_i(t,0)) \qquad \text{for a.e.}\ \ t \in [0,T]\,.$$

Remark 3.3. We underline that, in order to achieve the compactness of a set of flux-traces, one can alternatively consider a class of admissible controls defined as a set of uniformly $\mathbf{BV}$ bounded boundary data. Unfortunately this choice makes the analysis more involved, due to the lack of convergence of the trace of solutions. In fact, if we consider a sequence $(u^\nu)_\nu$ of solutions to (2.2) that converge in $\mathbf{L}^1_{loc}$ to some solution u, then the boundary traces of u^ν do not converge in general to the boundary trace of the limiting solution u.

As an example, given the flux function $f(u) = u(1-u)$, $u \in [0,1]$, consider the solutions u^ν_i to (2.2) with initial data

(3.26)
$$\overline{u}_i(x) = \begin{cases} \frac{1}{8} & \text{if}\quad x < -1\,, \\ \frac{1}{4} & \text{if}\quad -1 < x < 0\,, \end{cases}$$
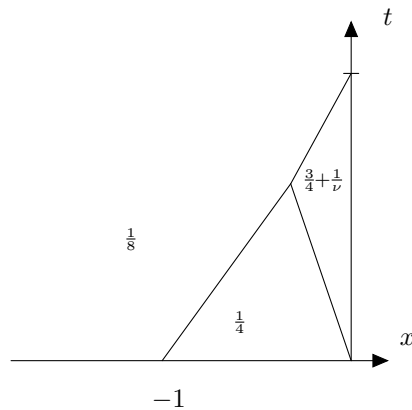
FIG. 1. *The representation of the function u_i^ν in* (3.27) *of Remark* 3.3.

and boundary data

$$k_i^\nu(t) = \frac{3}{4} + \frac{1}{\nu} \qquad \forall\, t \quad \forall\, \nu > 8\,.$$

By a direct computation (see Figure 1), one can verify that
(3.27)

$$u_i^\nu(t,x) = \begin{cases} \frac{1}{8} & \text{if} \begin{cases} x < -1 + \frac{5t}{8}, & t \le \frac{8\nu}{5\nu+8} \\[4pt] \qquad\qquad \text{or} \\[4pt] x < -\frac{8}{5\nu+8} + \frac{(\nu-8)\big((5\nu+8)t-8\nu\big)}{8\nu(5\nu+8)}, & \frac{8\nu}{5\nu+8} \le t \le \frac{8\nu^2}{(5\nu+8)(\nu-8)} \\[4pt] \qquad\qquad \text{or} \\[4pt] x < 0, & t \ge \frac{8\nu^2}{(5\nu+8)(\nu-8)}, \end{cases} \\[30pt] \frac{1}{4} & \text{if } -1 + \frac{5t}{8} < x < -\frac{t}{\nu}, \qquad t < \frac{8\nu}{5\nu+8}, \\[16pt] \frac{3}{4} + \frac{1}{\nu} & \text{if} \begin{cases} -\frac{t}{\nu} < x < 0, & t < \frac{8\nu}{5\nu+8} \\[4pt] \qquad\qquad \text{or} \\[4pt] \frac{(\nu-8)\big((5\nu+8)t-8\nu\big)}{8\nu(5\nu+8)} - \frac{8}{5\nu+8} < x < 0, & \frac{8\nu}{5\nu+8} < t < \frac{8\nu^2}{(5\nu+8)(\nu-8)}\,. \end{cases} \end{cases}$$

The sequence $(u_i^\nu)_\nu$ converges in $\mathbf{L}_{loc}^1\big((0,+\infty) \times (-\infty,0)\big)$ to the solution u_i of (2.2) with initial data (3.26) and boundary data

$$k_i(t) = \frac{3}{4} \qquad \forall\, t\,,$$

since $(k_i^\nu)_\nu$ converges to k_i in $\mathbf{L}_{loc}^1(0,\infty)$. Notice that

$$u_i(t,x) = \begin{cases} \frac{1}{8} & \text{if} \begin{cases} x < -1 + \frac{5t}{8}, & t \le \frac{8}{5}, \\[4pt] \qquad\quad \text{or} \\[4pt] x < 0, & t \ge \frac{8}{5}, \end{cases} \\[18pt] \frac{1}{4} & \text{if} \quad -1 + \frac{5t}{8} < x < 0, \qquad t < \frac{8}{5}, \end{cases}$$

and hence its boundary trace is

$$u_i(t,0) = \begin{cases} \frac{1}{4} & \text{if} \quad t < \frac{8}{5}, \\ \frac{1}{8} & \text{if} \quad t \geq \frac{8}{5}. \end{cases}$$

On the other hand the boundary traces of u_i^ν are

$$u_i^\nu(t,0) = \begin{cases} \frac{1}{4} & \text{if} \quad t = 0, \\ \frac{3}{4} + \frac{1}{\nu} & \text{if} \quad 0 < t < \frac{8\nu^2}{(5\nu+8)(\nu-8)}, \\ \frac{1}{8} & \text{if} \quad t \geq \frac{8\nu^2}{(5\nu+8)(\nu-8)}, \end{cases}$$

which pointwise converge to the function

$$\psi(t) = \begin{cases} \frac{1}{4} & \text{if} \quad t = 0, \\ \frac{3}{4} & \text{if} \quad 0 < t < \frac{8}{5}, \\ \frac{1}{8} & \text{if} \quad t \geq \frac{8}{5}. \end{cases}$$

Instead, the sequence of flux-traces $(f(u_i^\nu(\cdot,0)))_\nu$ clearly converges to $f(u_i(\cdot,0))$.

3.3. Maximization of flux depending functionals. Let $\mathcal{J} : \mathbb{R}^m \to \mathbb{R}$ be a continuous map. Given $\overline{u} \in \Pi_{\ell=1}^{m+n} \mathbf{L}^\infty(I_\ell; \Omega)$, and $T, M > 0$, we consider the optimization problem

$$(3.28) \qquad \sup_{(A,\gamma) \in \mathcal{U}^M(\overline{u})} \int_0^T \mathcal{J}\big(f^{\mathcal{I}}(u(t,0;\, A,\gamma))\big) dt,$$

where $\mathcal{U}^M(\overline{u})$ is the set of admissible controls defined in (3.4) and we set

$$(3.29) \qquad f^{\mathcal{I}}(u(t,0;\, A,\gamma)) \doteq \big(f(u_1(t,0;\, A,\gamma)),\ldots,f(u_m(t,0;\, A,\gamma))\big)$$

with the notations of Definition 3.1. Typical examples of the map $\mathcal{J}$ are

$$\mathcal{J}_1(\gamma_1,\ldots,\gamma_m) = \sum_{i=1}^m \gamma_i = \gamma_1 + \cdots + \gamma_m, \qquad \mathcal{J}_2(\gamma_1,\ldots,\gamma_m) = \prod_{i=1}^m \gamma_i = \gamma_1 \cdot \cdots \cdot \gamma_m,$$

which are commonly considered in the optimization rules introduced for the definition of various Riemann solvers (see [19, 41]). Thanks to Theorem 3.1 we immediately deduce that the optimization problem (3.28) admits a solution.

THEOREM 3.2. *Given $\overline{u} \in \Pi_{\ell=1}^{m+n} \mathbf{L}^\infty(I_\ell; \Omega)$ and $T > 0$, for every fixed $M > 0$ there exists $(\widehat{A}, \widehat{\gamma}) \in \mathcal{U}^M(\overline{u})$ such that*

$$(3.30) \qquad \int_0^T \mathcal{J}\big(f^{\mathcal{I}}(u(t,0;\, \widehat{A}, \widehat{\gamma}))\big)\, dt = \sup_{(A,\gamma) \in \mathcal{U}^M(\overline{u})} \int_0^T \mathcal{J}\big(f^{\mathcal{I}}(u(t,0;\, A,\gamma))\big)\, dt.$$

Proof. Since the set $\mathcal{D}^M$ is compact in $\mathbf{L}^1$ by Theorem 3.1, the conclusion follows noting that the map $g \mapsto \int_0^T \mathcal{J}(g(t))dt$ from $(\mathbf{L}^1((0,T);\mathbb{R}))^m$ to $\mathbb{R}$ is continuous. $\square$

The optimal solutions provided by Theorem 3.2 determine the solutions of the nodal Cauchy problem (1.2)–(1.3) whose boundary incoming flux-traces solve the optimization problem (3.28). Namely, an immediate consequence of Theorem 3.2 is the following corollary.

COROLLARY 3.1. *Given* $\overline{u} \in \Pi_{\ell=1}^{m+n} \mathbf{L}^\infty(I_\ell; \Omega)$, *and* $T, M > 0$, *let* $((\widehat{a}_{ji})_{ji}, \widehat{\gamma}) \in \mathcal{U}^M(\overline{u})$ *be an optimal pair of controls for the maximization problem* (3.30). *Then,* $\widehat{u} \doteq u(\cdot, \cdot; (\widehat{a}_{ji})_{ji}, \widehat{\gamma})$ *is an entropy admissible weak solution of the nodal Cauchy problem* (1.2)–(1.3) *on* $[0, T]$ *that satisfies*

$$f\big(\widehat{u}_i(t, 0)\big) = \widehat{\gamma}_i(t) \qquad a.e. \ t \in [0, T] \quad \forall \, i \in \mathcal{I},$$

$$f(\widehat{u}_j(t, 0)) = \sum_{i=1}^{m} \widehat{a}_{ji} \, f(\widehat{u}_i(t, 0)) \qquad a.e. \ t \in [0, T] \quad \forall \, j \in \mathcal{O}.$$

Remark 3.4. Relying on the compactness of the set $\mathcal{G}_{x^\mathcal{O}}^M$ in (3.9) established in Theorem 3.1, we may derive the existence of optimal solutions for more general cost functionals than the one considered in (3.28) related both to junction fluxes of the incoming edges and to the fluxes of solutions at fixed points of the outgoing edges. Namely, given a continuous map $\mathcal{J} : \mathbb{R}^{m+n} \to \mathbb{R}$, initial data $\overline{u} \in \Pi_{\ell=1}^{m+n} \mathbf{L}^\infty(I_\ell; \Omega)$, and $T, M > 0$, $x^\mathcal{O} = (x_{m+1}, \ldots, x_{m+n}) \in [0, +\infty)^n$, setting

$$f^\mathcal{O}(u(t, x^\mathcal{O}; A, \gamma)) \doteq \big(f(u_{m+1}(t, x_{m+1}; A, \gamma)), \ldots, f(u_{m+n}(t, x_{m+n}; A, \gamma))\big),$$

consider the optimization problem

$$(3.31) \qquad \sup_{(A, \gamma) \in \mathcal{U}^M(\overline{u})} \int_0^T \mathcal{J}\Big(f^\mathcal{I}(u(t, 0; A, \gamma)), f^\mathcal{O}(u(t, x^\mathcal{O}; A, \gamma))\Big) dt,$$

where $f^\mathcal{I}$ is defined as in (3.29). With the same arguments as in Theorem 3.2 we deduce that (3.31) admits an optimal solution and the supremum is achieved as a maximum.

3.4. Minimization of the total variation of optimal solutions. It is quite easy to realize that the solution of the optimization problem (3.28) is in general not unique. This is illustrated by an example discussed in [4] where it is considered the special case of a junction with one incoming and one outgoing arc and two solutions of (3.28) are provided for the functional $\mathcal{J}(\gamma) = \gamma$.

On the other hand, in many applications (e.g., see [28] for vehicular traffic modeling) it would be desirable to select those optimal solutions that keep as small as possible the total variation of the incoming fluxes, i.e., that minimize $\sum_{i=1}^{m} \mathrm{TV}\{f(u_i(\cdot, 0))\}$. Therefore, for every $M > 0$, we define the set of optimal pairs

$$\mathcal{U}_{max}^M = \Big\{(\widehat{A}, \widehat{\gamma}) \in \mathcal{U}^M : \ (3.30) \ \text{holds}\Big\},$$

which is nonempty because of Theorem 3.2, and we consider the optimization problem

$$(3.32) \qquad \inf_{(\widehat{A}, \widehat{\gamma}) \in \mathcal{U}_{max}^M} \sum_{i=1}^{m} \mathrm{TV}\Big\{f\big(u_i(\cdot, 0; \widehat{A}, \widehat{\gamma})\big)\Big\}.$$

In the same spirit as (3.32), we also address an optimization problem that includes, in the same cost, the integral functional in (3.28) and the total variation of the flux

in (3.32). Namely, for every fixed $\delta > 0$, consider the maximization problem
(3.33)
$$\sup_{(A,\,\gamma)\in\mathcal{U}}\left(\int_0^T \mathcal{J}\big(f^{\mathcal{I}}(u(t,0;\,A,\gamma))\big)dt - \delta\sum_{i=1}^m \mathrm{TV}\left\{f\big(u_i(\cdot,0;\,A,\gamma)\big)\right\} - \delta\,\mathrm{TV}\left\{A\right\}\right),$$

where $\mathrm{TV}\left\{A\right\} = \mathrm{TV}\left\{(a_{ji})_{j,i}\right\} \doteq \sum_{j,i}\mathrm{TV}\{a_{ji}\}$. Here, the admissible controls are
BV functions with arbitrarily large total variation.

THEOREM 3.3. *Given $\overline{u} \in \Pi_{\ell=1}^{m+n}\,\mathbf{L}^\infty(I_\ell;\Omega)$, and $T > 0$, the following hold:*
(i) *For every fixed $M > 0$ there exists $(A^*,\gamma^*) \in \mathcal{U}_{max}^M(\overline{u})$ such that*
(3.34)
$$\sum_{i=1}^m \mathrm{TV}\left\{f\big(u_i(\cdot,0;\,A^*,\gamma^*)\big)\right\} = \inf_{(\widehat{A},\widehat{\gamma})\in\mathcal{U}_{max}^M(\overline{u})}\sum_{i=1}^m \mathrm{TV}\left\{f\big(u_i(\cdot,0;\,\widehat{A},\widehat{\gamma})\big)\right\}.$$

(ii) *For every fixed $\delta > 0$ there exists $(A^\delta,\gamma^\delta) \in \mathcal{U}(\overline{u})$ such that*
(3.35)
$$\int_0^T \mathcal{J}\big(f^{\mathcal{I}}(u(t,0;\,A^\delta\gamma^\delta))\big)dt - \delta\sum_{i=1}^m \mathrm{TV}\left\{f\big(u_i(\cdot,0;\,A^\delta,\gamma^\delta)\big)\right\} - \delta\,\mathrm{TV}\left\{A^\delta\right\}$$
$$= \sup_{(A,\gamma)\in\mathcal{U}(\overline{u})}\left(\int_0^T \mathcal{J}\big(f^{\mathcal{I}}(u(t,0;\,A,\gamma))\big)dt - \delta\sum_{i=1}^m \mathrm{TV}\left\{f\big(u_i(\cdot,0;\,A,\gamma)\big)\right\} - \delta\,\mathrm{TV}\left\{A\right\}\right).$$

Proof. (i) Consider a minimizing sequence $\big((A^\nu,g^\nu)\big)_\nu$ for (3.32) such that

$$(3.36)\qquad g_i^\nu = f\big(u_i(\cdot,0;\,A^\nu,\gamma^\nu)\big), \qquad (A^\nu,\gamma^\nu) \in \mathcal{U}_{max}^M, \qquad i \in \mathcal{I}, \qquad \forall\,\nu,$$

$$(3.37)\quad \int_0^T \mathcal{J}\big(g_1^\nu(t),\ldots,g_m^\nu(t)\big)\,dt = \max_{(A,\,\gamma)\in\mathcal{U}^M}\int_0^T \mathcal{J}\big(f^{\mathcal{I}}(u(t,0;\,A,\gamma))\big)\,dt, \qquad \forall\,\nu,$$

$$(3.38)\qquad \sum_{i=1}^m \mathrm{TV}\left\{g_i^\nu\right\} \quad\to\quad \inf_{(\widehat{A},\widehat{\gamma})\in\mathcal{U}_{max}^M}\sum_{i=1}^m \mathrm{TV}\left\{f\big(u_i(\cdot,0;\,\widehat{A},\widehat{\gamma})\big)\right\}.$$

Since by (3.6) we have $(A^\nu,g^\nu) \in \mathcal{D}^M$ for all ν, applying Theorem 3.1 and relying on
Remark 3.2, we deduce that there exists a subsequence, again denoted by $\big((A^\nu,g^\nu)\big)_\nu$,
and an element $(A^*,g^*) \in \mathcal{D}^M$ with $g^* = f^{\mathcal{I}}(u(\cdot,0;\,A^*,\gamma^*))$, $(A^*,\gamma^*) \in \mathcal{U}^M$, such
that

$$(3.39)\qquad\qquad\qquad g^\nu \quad\to\quad g^* \qquad \text{in}\quad \mathbf{L}^1.$$

Hence, (3.37) together with (3.39) yields

$$\int_0^T \mathcal{J}\big(g^*(t)\big)\,dt = \max_{(A,\,\gamma)\in\mathcal{U}^M}\int_0^T \mathcal{J}\big(f^{\mathcal{I}}(u(t,0;\,A,\gamma))\big)\,dt,$$

showing that $(A^*,\gamma^*) \in \mathcal{U}_{max}^M$. On the other hand, by the lower semicontinuity
property of the essential variation with respect to the $\mathbf{L}_{\mathrm{loc}}^1$-topology (see [3, sections
3.1–3.2]), and by the property of the liminf operation with respect to the sum, we

derive
(3.40)

$$\sum_{i=1}^{m} \mathrm{TV}\left\{g_i^*\right\} \le \sum_{i=1}^{m} \liminf_{\nu} \mathrm{TV}\left\{g_i^\nu\right\} \le \liminf_{\nu}\left(\sum_{i=1}^{m} \mathrm{TV}\left\{g_i^\nu\right\}\right)$$

$$= \lim_{\nu}\left(\sum_{i=1}^{m} \mathrm{TV}\left\{g_i^\nu\right\}\right) = \inf_{(\widehat{A},\widehat{\gamma})\in\mathcal{U}_{max}^M} \sum_{i=1}^{m} \mathrm{TV}\left\{f\big(u_i(\cdot,0;\,\widehat{A},\widehat{\gamma})\big)\right\}.$$

Since $(A^*,\gamma^*)\in\mathcal{U}_{max}^M$ it follows from (3.40) that (3.34) holds concluding the proof of (i).

(ii) We shall provide a proof of the existence of $(A^\delta,\gamma^\delta)\in\mathcal{U}(\overline{u})$ satisfying (3.35) only in the case where the supremum in (3.33) is strictly positive. In fact, in the case where such a supremum is nonpositive, we may always consider the problem (3.33) with a new integrand function $\mathcal{J}^\sharp = \alpha + \mathcal{J}$, α being a positive constant chosen so that the supremum in (3.35) is strictly positive. Clearly, a maximizer for (3.33) with $\mathcal{J}^\sharp$ in place of $\mathcal{J}$ also provides a maximizer for the original problem (3.33).

Then, assume that the supremum in (3.33) is strictly positive and consider a maximizing sequence $\big((A^\nu,g^\nu)\big)_\nu$ for (3.33) such that, for all ν, there holds

$$(3.41) \qquad g_i^\nu = f\big(u_i(\cdot,0;\ A^\nu,\gamma^\nu)\big) \qquad (A^\nu,\gamma^\nu)\in\mathcal{U}, \qquad i\in\mathcal{I},$$

and satisfying
(3.42)

$$\lim_{\nu}\left(\int_0^T \mathcal{J}\big(g_1^\nu(t),\ldots,g_m^\nu(t)\big)\,dt - \delta\sum_{i=1}^{m}\mathrm{TV}\left\{g_i^\nu\right\} - \delta\,\mathrm{TV}\left\{A^\nu\right\}\right)$$

$$= \sup_{(A,\gamma)\in\mathcal{U}}\left(\int_0^T \mathcal{J}\big(f^{\mathcal{I}}(u(t,0;\ A,\gamma))\big)dt - \delta\sum_{i=1}^{m}\mathrm{TV}\left\{f\big(u_i(\cdot,0;\ A,\gamma)\big)\right\} - \delta\,\mathrm{TV}\left\{A\right\}\right).$$

Since all g^ν take values in the bounded set $[f(\Omega)]^m$ and $\mathcal{J}$ is continuous, and by virtue of the strictly positive assumption on the supremum in (3.33), we may find a constant $C_1 > 0$ such that, for all ν sufficiently large, there hold

$$\int_0^T \mathcal{J}\big(g_1^\nu(t),\ldots,g_m^\nu(t)\big)\,dt \le C_1\,,$$

(3.43)

$$\int_0^T \mathcal{J}\big(g_1^\nu(t),\ldots,g_m^\nu(t)\big)\,dt - \delta\sum_{i=1}^{m}\mathrm{TV}\left\{g_i^\nu\right\} - \delta\,\mathrm{TV}\left\{A^\nu\right\} > 0\,.$$

Thus, (3.43) implies that, for ν sufficiently large, one has

$$(3.44) \qquad \mathrm{TV}\left\{g_i^\nu\right\} \le \frac{C_1}{\delta} \qquad \forall\,i\in\mathcal{I}\,, \qquad\qquad \mathrm{TV}\left\{a_{ji}\right\} \le \frac{C_1}{\delta} \qquad \forall\,j,i\,.$$

Recalling that by Definition 3.1 and because of (3.41) we have

$$g_i^\nu = \gamma_i^\nu \qquad \forall\,i\in\mathcal{I} \qquad \forall\,\nu \ \text{large}\,,$$

we deduce from (3.44) that $(A^\nu,g^\nu)\in\mathcal{D}^{M_0}$ with $M_0 \doteq \frac{C_1}{\delta}$, for all ν sufficiently large. Therefore, applying Theorem 3.1 and relying on Remark 3.2, we deduce that there

exist a subsequence, again denoted by $\big((A^\nu, g^\nu)\big)_\nu$, and an element $(A^\delta, g^\delta) \in \mathcal{D}^{M_0} \subset \mathcal{D}$, with $g^\delta = f^{\mathcal{I}}(u(\cdot, 0;\, A^\delta, \gamma^\delta))$, $(A^\delta, \gamma^\delta) \in \mathcal{U}^{M_0} \subset \mathcal{U}$, such that

$$(3.45) \qquad A^\nu(t) \quad \to \quad A^\delta(t) \qquad \forall\, t \in [0, T]\,,$$

$$(3.46) \qquad g^\nu \quad \to \quad g^\delta \qquad \text{in} \quad \mathbf{L}^1\,.$$

Then, relying on (3.45), (3.46), we deduce as in the proof of point (i) that

$$(3.47) \qquad \int_0^T \mathcal{J}\big(g_1^\delta(t), \ldots, g_m^\delta(t)\big)\, dt = \lim_\nu \int_0^T \mathcal{J}\big(g_1^\nu(t), \ldots, g_m^\nu(t)\big)\, dt\,,$$

$$(3.48) \qquad \sum_{i=1}^m \mathrm{TV}\left\{g_i^\delta\right\} \le \liminf_\nu \sum_{i=1}^m \mathrm{TV}\left\{g_i^\nu\right\}\,,$$

$$(3.49) \qquad \sum_{j,i=1}^m \mathrm{TV}\left\{a_{ji}^\delta\right\} \le \liminf_\nu \sum_{j,i=1}^m \mathrm{TV}\left\{a_{ji}^\nu\right\}\,.$$

Hence, by the property of the liminf and limsup operations with respect to the sum and by virtue of (3.42), (3.47)–(3.49), we find

$$
\sup_{(A,\gamma)\in\mathcal{U}} \left(\int_0^T \mathcal{J}\big(f^{\mathcal{I}}(u(t, 0;\, A, \gamma))\big) dt - \delta \sum_{i=1}^m \mathrm{TV}\left\{f\big(u_i(\cdot, 0;\, A, \gamma)\big)\right\} - \delta\, \mathrm{TV}\left\{A\right\} \right)
$$

$$
\le \lim_\nu \int_0^T \mathcal{J}\big(g_1^\nu(t), \ldots, g_m^\nu(t)\big)\, dt + \limsup_\nu \left(-\delta \sum_{i=1}^m \mathrm{TV}\left\{g_i^\nu\right\} - \delta\, \mathrm{TV}\left\{A^\nu\right\} \right)
$$

$$
\le \lim_\nu \int_0^T \mathcal{J}\big(g_1^\nu(t), \ldots, g_m^\nu(t)\big)\, dt - \delta \sum_{i=1}^m \liminf_\nu \mathrm{TV}\left\{g_i^\nu\right\} - \delta \sum_{j,i=1}^m \liminf_\nu \mathrm{TV}\left\{a_{ji}^\nu\right\}
$$

$$
= \int_0^T \mathcal{J}\big(g_1^\delta(t), \ldots, g_m^\delta(t)\big)\, dt - \delta \sum_{i=1}^m \mathrm{TV}\left\{g_i^\delta\right\} - \delta \sum_{j,i=1}^m \mathrm{TV}\left\{a_{ji}^\delta\right\}\,,
$$

proving that $(A^\delta, \gamma^\delta) \in \mathcal{U}$ satisfies (3.35), completing the proof of the theorem. $\qquad\square$

3.5. Equivalent variational formulations. We present here an equivalent variational formulation of the optimization problem (3.32) which may be useful also for numerical investigations of the optimal solutions as discussed in section 4. Namely, for every fixed $M > 0$, consider the function
(3.50)

$$
I_M(\delta) = \max_{(A,\gamma)\in\mathcal{U}^M(\overline{u})} \left(\int_0^T \mathcal{J}\big(f^{\mathcal{I}}(u(t, 0;\, A, \gamma))\big) dt - \delta \sum_{i=1}^m \mathrm{TV}\left\{f\big(u_i(\cdot, 0;\, A, \gamma)\big)\right\} \right)
$$

which is well-defined for $\delta > 0$ by the same arguments as the proof of Theorem 3.3(ii). We shall analyze its limit when the argument vanishes.

THEOREM 3.4. *Given* $\overline{u} \in \Pi_{\ell=1}^{m+n} \mathbf{L}^\infty(I_\ell; \Omega)$ *and* $T > 0$, *for every fixed* $M > 0$ *one has*

$$(3.51) \qquad \lim_{\delta \to 0} I_M(\delta) = \max_{(A,\gamma)\in\mathcal{U}^M(\overline{u})} \int_0^T \mathcal{J}\big(f^{\mathcal{I}}(u(t, 0;\, A, \gamma))\big)\, dt\,.$$

Moreover, given any sequence $(A^\nu, \gamma^\nu) \in \mathcal{U}^M(\overline{u})$ *of maximizers for* $I_M(\delta^\nu)$, *with* $\delta^\nu \to$
0, *there exist a subsequence* $((A^{\nu'}, \gamma^{\nu'}))_{\nu'}$ *and* $(A^\sharp, \gamma^\sharp) \in \mathcal{U}^M_{max}(\overline{u})$, *satisfying*

$$(3.52) \qquad f^{\mathcal{I}}\big(u(\cdot, 0;\ A^{\nu'}, \gamma^{\nu'})\big) \quad \to \quad f^{\mathcal{I}}\big(u(\cdot, 0;\ A^\sharp, \gamma^\sharp)\big) \qquad in \qquad \mathbf{L}^1$$

and

$$(3.53) \quad \sum_{i=1}^m \mathrm{TV}\left\{f\big(u_i(\cdot, 0;\ A^\sharp, \gamma^\sharp)\big)\right\} = \min_{(\widehat{A}, \widehat{\gamma}) \in \mathcal{U}^M_{max}(\overline{u})} \sum_{i=1}^m \mathrm{TV}\left\{f\big(u_i(\cdot, 0;\ \widehat{A}, \widehat{\gamma})\big)\right\}.$$

Proof. In order to establish the theorem it will be sufficient to show that, given
$\delta^\nu \to 0$ and every sequence $(A^\nu, \gamma^\nu) \in \mathcal{U}^M$ such that
(3.54)

$$I_M(\delta^\nu) = \int_0^T \mathcal{J}\big(f^{\mathcal{I}}(u(t, 0;\ A^\nu, \gamma^\nu))\big) dt - \delta^\nu \sum_{i=1}^m \mathrm{TV}\left\{f\big(u_i(\cdot, 0;\ A^\nu, \gamma^\nu)\big)\right\} \qquad \forall\, \nu,$$

there exist a subsequence, again denoted by $((A^\nu, \gamma^\nu))_\nu$, and $(A^\sharp, \gamma^\sharp) \in \mathcal{U}^M_{max}$, such
that there hold (3.52), (3.53), and

$$(3.55) \qquad \lim_\nu I_M(\delta^\nu) = \max_{(A, \gamma) \in \mathcal{U}^M(\overline{u})} \int_0^T \mathcal{J}\big(f^{\mathcal{I}}(u(t, 0;\ A, \gamma))\big)\, dt.$$

Towards this goal, in connection with the sequence $(A^\nu, \gamma^\nu) \in \mathcal{U}^M$ satisfying (3.54),
set

$$g_i^\nu = f\big(u_i(\cdot, 0;\ A^\nu, \gamma^\nu)\big), \qquad i \in \mathcal{I}, \qquad \forall\, \nu,$$

and consider the sequence of $(A^\nu, g^\nu) \in \mathcal{D}^M$. Then, invoking Theorem 3.1 and Re-
mark 3.2 we deduce that there exists a subsequence, again denoted by $\big((A^\nu, g^\nu)\big)_\nu$,
and an element $(A^\sharp, g^\sharp) \in \mathcal{D}^M$ with $g^\sharp = f^{\mathcal{I}}(u(\cdot, 0;\ A^\sharp, \gamma^\sharp))$, $(A^\sharp, \gamma^\sharp) \in \mathcal{U}^M$, so that
we have

$$(3.56) \qquad\qquad g^\nu \quad \to \quad g^\sharp \qquad in \quad \mathbf{L}^1.$$

Next, relying on Theorem 3.3(i), consider $(A^*, \gamma^*) \in \mathcal{U}^M_{max}$ such that (3.34) holds and
set $g^* = f^{\mathcal{I}}(u(\cdot, 0;\ A^*, \gamma^*))$. Since $(A^*, \gamma^*) \in \mathcal{U}^M$, by definition (3.50) we have

$$(3.57) \quad \int_0^T \mathcal{J}(g^*(t)) dt - \delta_\nu \sum_{i=1}^m \mathrm{TV}\left\{g_i^*\right\} \le \int_0^T \mathcal{J}(g^\nu(t)) dt - \delta_\nu \sum_{i=1}^m \mathrm{TV}\left\{g_i^\nu\right\} \qquad \forall\, \nu.$$

Notice that $\mathrm{TV}\{g_i^\nu\} \le M$ for all $i \in \mathcal{I}$ and for all ν because $(A^\nu, g^\nu) \in \mathcal{D}^M$. Hence,
taking the limit in (3.57) when $\delta^\nu \to 0$, and relying on (3.56), we derive

$$\int_0^T \mathcal{J}(g^*(t)) dt \le \int_0^T \mathcal{J}(g^\sharp(t)) dt$$

showing that also $(A^\sharp, \gamma^\sharp) \in \mathcal{U}^M_{max}$. Thus, one has

$$(3.58) \qquad \int_0^T \mathcal{J}(g^\sharp(t)) dt = \max_{(A, \gamma) \in \mathcal{U}^M(\overline{u})} \int_0^T \mathcal{J}\big(f^{\mathcal{I}}(u(t, 0;\ A, \gamma))\big)\, dt.$$

With the same arguments, relying on (3.56), and taking the limit in (3.54), we deduce

$$(3.59) \qquad \lim_{\nu} I_M(\delta^\nu) = \int_0^T \mathcal{J}(g^\sharp(t))dt.$$

Therefore we recover (3.52), (3.55) from (3.56), (3.58), (3.59). In order to complete the proof of the theorem it remains to establish (3.53). To this end notice that, since $(A^\nu, \gamma^\nu) \in \mathcal{U}^M$ for all ν and $(A^*, \gamma^*) \in \mathcal{U}_{max}^M$, we have

$$\int_0^T \mathcal{J}(g^\nu(t))dt \le \int_0^T \mathcal{J}(g^*(t))dt \qquad \forall\, \nu \,,$$

which in turn, because of (3.57), implies

$$(3.60) \qquad \sum_{i=1}^m \mathrm{TV}\left\{g_i^\nu\right\} \le \sum_{i=1}^m \mathrm{TV}\left\{g_i^*\right\} \qquad \forall\, \nu \,.$$

Thus, by the property of the liminf operation and relying on (3.56) as in the proof of Theorem 3.3, we derive from (3.60) the estimates

$$(3.61) \qquad \begin{aligned} \sum_{i=1}^m \mathrm{TV}\left\{g_i^\sharp\right\} &\le \sum_{i=1}^m \liminf_\nu \mathrm{TV}\left\{g_i^\nu\right\} \\ &\le \liminf_\nu \left(\sum_{i=1}^m \mathrm{TV}\left\{g_i^\nu\right\}\right) \le \sum_{i=1}^m \mathrm{TV}\left\{g_i^*\right\}. \end{aligned}$$

Since (3.34) holds for (A^*, γ^*), we deduce from (3.61) that $(A^\sharp, \gamma^\sharp)$ verifies (3.53), completing the proof of the theorem. $\qquad\square$

We next consider the function
$$(3.62)$$
$$I(\delta) = \max_{(A,\gamma) \in \mathcal{U}(\overline{u})} \left(\int_0^T \mathcal{J}\big(f^{\mathcal{I}}(u(t,0;A,\gamma))\big)dt - \delta \sum_{i=1}^m \mathrm{TV}\{f(u_i(\cdot,0;A,\gamma))\} - \delta\,\mathrm{TV}\{A\} \right),$$

which is well defined for $\delta > 0$ by Theorem 3.3(ii). We shall analyze its relation with the optimization problem

$$(3.63) \qquad \sup_{(A,\gamma)\in\mathcal{U}(\overline{u})} \int_0^T \mathcal{J}\big(f^{\mathcal{I}}(u(t,0;A,\gamma))\big)dt\,.$$

Here, we are maximizing the integral functional of the flux-traces among all admissible controls without imposing a uniform bound on the total variation.

THEOREM 3.5. *Given* $\overline{u} \in \Pi_{\ell=1}^{m+n} \mathbf{L}^\infty(I_\ell; \Omega)$ *and* $T > 0$, *one has*

$$(3.64) \qquad \lim_{\delta\to 0} I(\delta) = \sup_{(A,\gamma)\in\mathcal{U}(\overline{u})} \int_0^T \mathcal{J}\big(f^{\mathcal{I}}(u(t,0;A,\gamma))\big)dt\,.$$

More precisely, one of the following two cases holds:
(i) *there exists* $M_0 > 0$ *such that*

$$\lim_{\delta\to 0} I(\delta) = \max_{(A,\gamma)\in\mathcal{U}^M(\overline{u})} \int_0^T \mathcal{J}\big(f^{\mathcal{I}}(u(t,0;A,\gamma))\big)dt \qquad \forall\, M > M_0\,,$$

and the supremum in (3.63) *is attained as a maximum;*

(ii)

$$\lim_{\delta \to 0} I(\delta) > \max_{(A,\,\gamma)\,\in\,\mathcal{U}^M(\overline{u})} \int_0^T \mathcal{J}\big(f^{\mathcal{I}}(u(t,0;\,A,\gamma))\big)dt \qquad \forall\, M > 0\,,$$

and the optimization problem (3.63) *does not admit a maximum.*

Proof. Recalling the definitions (3.4) and because of Theorem 3.2 it follows that
(3.65)

$$\sup_{(A,\,\gamma)\,\in\,\mathcal{U}(\overline{u})} \int_0^T \mathcal{J}\big(f^{\mathcal{I}}(u(t,0;\,A,\gamma))\big)dt = \sup_{M>0}\ \max_{(A,\,\gamma)\,\in\,\mathcal{U}^M(\overline{u})} \int_0^T \mathcal{J}\big(f^{\mathcal{I}}(u(t,0;\,A,\gamma))\big)dt\,,$$

and that

$$M \quad \mapsto \quad \max_{(A,\,\gamma)\,\in\,\mathcal{U}^M(\overline{u})} \int_0^T \mathcal{J}\big(f^{\mathcal{I}}(u(t,0;\,A,\gamma))\big)dt$$

is a nondecreasing map. Hence, in order to establish the theorem it will be sufficient to prove (3.64). To this end notice first that the map $\delta \to I(\delta)$ is nonincreasing. In fact, let $0 < \delta_1 < \delta_2$ and, by Theorem 3.3(ii), consider a maximizer $(A^{\delta_2}, \gamma^{\delta_2}) \in \mathcal{U}$ for $I(\delta_2)$. Then, by definition (3.62), we find

$$I(\delta_2) = \int_0^T \mathcal{J}\big(f^{\mathcal{I}}(u(t,0;\,A^{\delta_2},\gamma^{\delta_2}))\big)dt - \delta_2 \sum_{i=1}^m \mathrm{TV}\big\{f\big(u_i\big(\cdot,0;\,A^{\delta_2},\gamma^{\delta_2}\big)\big)\big\} - \delta_2\,\mathrm{TV}\big\{A^{\delta_2}\big\}$$

$$\leq \int_0^T \mathcal{J}\big(f^{\mathcal{I}}(u(t,0;\,A^{\delta_2},\gamma^{\delta_2}))\big)dt - \delta_1 \sum_{i=1}^m \mathrm{TV}\big\{f\big(u_i\big(\cdot,0;\,A^{\delta_2},\gamma^{\delta_2}\big)\big)\big\} - \delta_1\,\mathrm{TV}\big\{A^{\delta_2}\big\}$$

$$\leq I(\delta_1)\,.$$

Therefore, the limit in (3.64) exists. Next observe that by definitions (3.50), (3.62) we have

$$I(\delta) \geq I_M(\delta) \qquad \forall\, M,\delta > 0\,.$$

Hence, taking first the limit as $\delta \to 0$ on both sides of the inequality, next considering the supremum over $M > 0$ on the right-hand side, and relying on (3.51), (3.65), we find

(3.66) $$\lim_{\delta \to 0} I(\delta) \geq \sup_{(A,\,\gamma)\,\in\,\mathcal{U}(\overline{u})} \int_0^T \mathcal{J}\big(f^{\mathcal{I}}(u(t,0;\,A,\gamma))\big)dt\,.$$

On the other hand, by definitions (3.50), (3.62) there hold

(3.67) $$I(\delta) \leq \sup_M I_M(\delta)\,,$$

(3.68) $$I_M(\delta) \leq \max_{(A,\,\gamma)\,\in\,\mathcal{U}^M(\overline{u})} \int_0^T \mathcal{J}\big(f^{\mathcal{I}}(u(t,0;\,A,\gamma))\big)\,dt \qquad \forall\, M,\delta > 0\,.$$

Thus, taking first the supremum over $M > 0$ on both sides of (3.68) and relying on (3.65), (3.67), next considering the limit as $\delta \to 0$ on the left-hand side, we find

$$\lim_{\delta \to 0} I(\delta) \leq \sup_{(A,\,\gamma)\,\in\,\mathcal{U}(\overline{u})} \int_0^T \mathcal{J}\big(f^{\mathcal{I}}(u(t,0;\,A,\gamma))\big)dt\,,$$

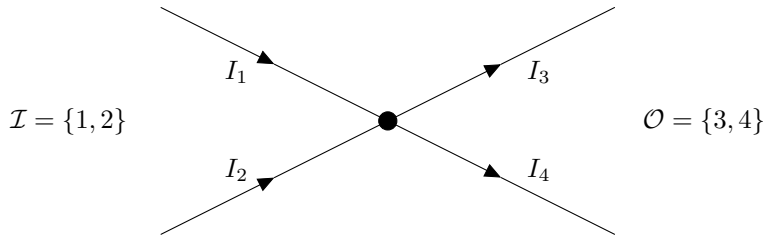which together with (3.66) yields (3.64), completing the proof of the theorem. $\square$

FIG. 2. *The junction considered in section* 4. *The incoming roads are denoted by* I_1 *and* I_2, *while the outgoing ones are* I_3 *and* I_4.

4. Numerical simulations for a node with two incoming and two outgoing arcs. This section is devoted to presenting a few numerical simulations in the case of a node J with two incoming and two outgoing arcs. These numerical results seem to indicate the fact that the maximization problem (3.28) may well have no optimal solution within the ones constructed by the standard junction Riemann solver [19, 25, 41, 43, 53].

In every numerical example, we model the incoming roads I_1 and I_2 through the real interval $(-5,0)$, while the outgoing roads I_3 and I_4 are described by the interval $(0,5)$, so that the node is located at $x = 0$; see Figure 2.

In each arc, the LWR model (1.1) is considered with the flux function given by $f(u) = 4u(1-u)$, so that the set Ω of all the possible densities is the interval $[0,1]$. The solution in each arc is computed by the Godunov method; see, for example, [64, section 12.1], [44, Chapter III], or [52, Chapter 3]. We use a uniform spatial mesh with length $\Delta x = 0.05$ and a nonuniform time mesh with length Δt, calculated in such a way the classic CFL condition is satisfied; see [31]. Note that it is also possible using a uniform time mesh. However, in view of future numerical studies, we chose a nonuniform one for improving convergence. The simulations are done in the time interval $(0,T)$.

Concerning the optimization of (3.28), we assume that the distributional matrix A satisfying (1.4) is a priori fixed. In such a way, in the optimization problem (3.28) we only regard as junction controls the incoming fluxes $g = (g_1, g_2)$. In order to numerically approximate an optimal control, we perform a heuristic recursive procedure based on the following steps:

1. Consider a piecewise constant initial control $g = (g_1, g_2)$ with a fixed number, namely, M, of points of discontinuity.
2. On each time interval where the control is constant, we perform a set of variations. For each variation, we compute the corresponding solution and cost (4.1).
3. Among all the possible variations, we select one that maximizes (4.1). In this way, we obtain a piecewise constant control $\overline{g}$, which acts not worse than g.

According to the variational formulation of the min-max problem given by Theorem 3.4, for finding a solution which maximizes (3.28) and minimizes (3.32), we shall maximize the cost

$$(4.1) \qquad \int_0^T \mathcal{J}(f(u_1(t,0)), f(u_2(t,0)))dt - \delta \sum_{i=1}^2 \mathrm{TV}_{[0,T]} f(u_i(\cdot,0))$$

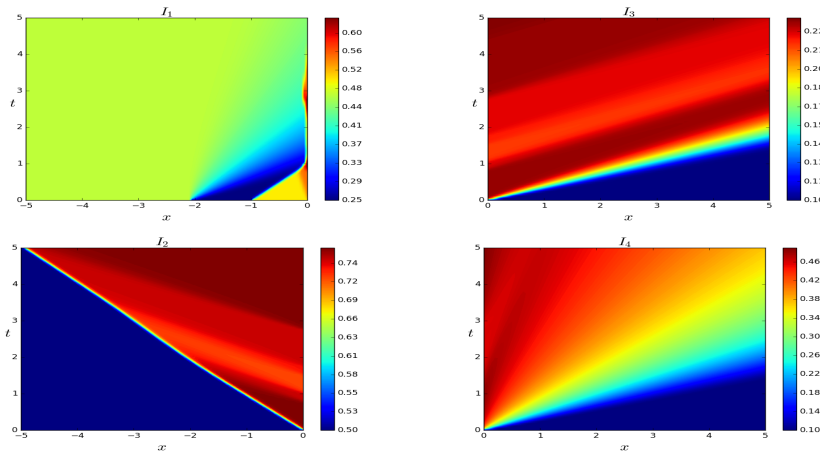with $\delta > 0$ sufficiently small or $\delta = 0$.

FIG. 3. *The numerical optimal solution obtained in subsection* 4.1. *In the first line, the solutions in I_1 and I_3. In the second line, the solutions in I_2 and I_4. (Color online)*

4.1. Rarefaction and shock approaching the node (case 1). Here we consider the case a rarefaction and a shock in an incoming arc interact with the node. The initial data are given by

$$\overline{u}_1(x) = \begin{cases} 0.47 & \text{if } x < -2.1, \\ 0.25 & \text{if } -2.1 < x < -1, \\ 0.5 & \text{if } x > -1, \end{cases} \quad \overline{u}_2(x) = 0.5, \quad \overline{u}_3(x) = 0.1, \quad \overline{u}_4(x) = 0.1,$$

the distribution matrix A is given by

$$A = \begin{pmatrix} 0.5 & 0.3 \\ 0.5 & 0.7 \end{pmatrix},$$

and $\mathcal{J}(f_1, f_2) = f_1 + f_2$. Moreover, we consider the following choice of parameters: $T = 5$, $M = 20$, and $\delta = 0.2$. In Figure 3 the numerical optimal solution is represented. Note that the solution is qualitatively different from that obtained with the classical Riemann solver at the node; see Figure 4. Indeed at about $t \sim 1$ a shock wave with negative speed in the arc I_1 is generated. This wave has almost zero speed, at time $t \sim 2.5$ enters a little more into the domain, and comes back to the boundary at time $t \sim 4$. The numeric cost (4.1) for the optimal solution is approximately 8.415, since

$$\int_0^T \mathcal{J}(f(u_1(t,0)), f(u_2(t,0)))dt \sim 8.498, \quad \begin{array}{l} \text{TV}_{[0,T]} f(u_1(\cdot,0)) \sim 0.250, \\ \text{TV}_{[0,T]} f(u_2(\cdot,0)) \sim 0.163, \end{array}$$

while the cost for the solution obtained with the corresponding Riemann solvers (see Figure 4), is approximately 8.389. In Figures 5 and 6, respectively, the optimal fluxes and the fluxes obtained with the Riemann solver are drawn. Note in particular that the optimal fluxes have less total variation than the fluxes obtained with the Riemann solver.

4.2. Rarefaction and shock approaching the node (case 2). As in subsection 4.1, we present the case of a rarefaction and a shock approaching the node from
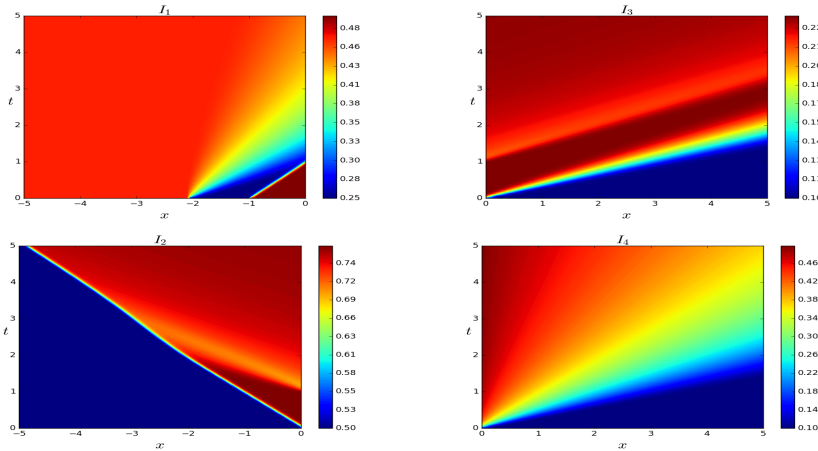
FIG. 4. *The solution, for subsection* 4.1, *obtained with the Riemann solver. In the first line, the solutions in* I_1 *and* I_3. *In the second line, the solutions in* I_2 *and* I_4. *(Color online)*
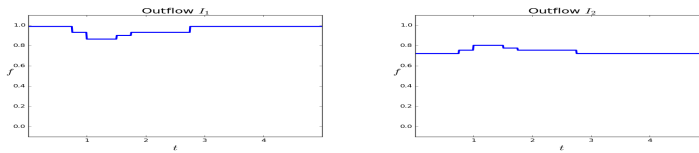

FIG. 5. *The numerical optimal fluxes at the node in the incoming arcs* I_1 *and* I_2, *obtained in subsection* 4.1.
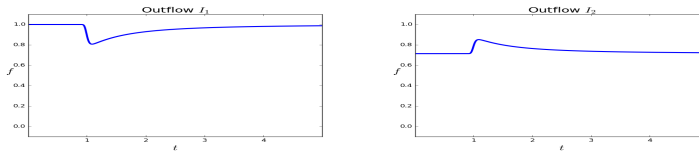

FIG. 6. *The fluxes at the node in the incoming arcs of solution, for subsection* 4.1, *obtained with the Riemann solver.*

the incoming arc I_1. More precisely, we have that the initial data are given by

$$\overline{u}_1(x) = \left\{ \begin{array}{ll} 0.25 & \text{if } x < -2 \text{ or } -1.5 < x < -1, \\ 0 & \text{if } -2 < x < -1.5 \text{ or } -1 < x < -.5, \\ 0.5 & \text{if } x > -.5, \end{array} \right. \qquad \begin{array}{l} \overline{u}_2(x) = 0.5, \\ \overline{u}_3(x) = 0.1, \\ \overline{u}_4(x) = 0.1, \end{array}$$

and the distribution matrix is given by

$$A = \left(\begin{array}{cc} 0.5 & 0.3 \\ 0.5 & 0.7 \end{array} \right).$$

We here maximize the product of the incoming fluxes, i.e., $\mathcal{J}(f_1, f_2) = 50 \cdot f_1 f_2$. Moreover, we consider the following parameters: $T = 1.1$, $M = 20$, and $\delta = 0.4$. In Figure 7 the numerical optimal solution is represented. Note that two shocks are generated at the junction at times $t \sim 0.2$ and $t \sim 0.4$. These shocks interact with shocks and rarefaction waves generated by the initial datum and come back to the
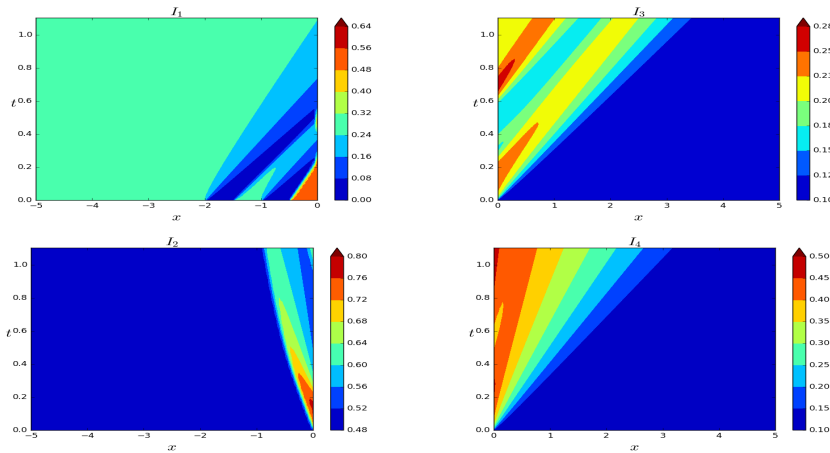
FIG. 7. *The numerical optimal solution obtained in subsection* 4.2. *In the first line, the solutions in I_1 and I_3. In the second line, the solutions in I_2 and I_4. (Color online)*
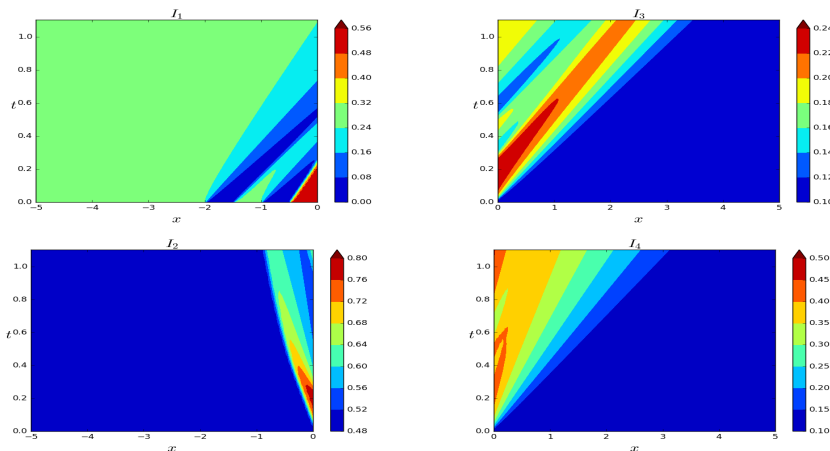

FIG. 8. *The solution, for subsection* 4.2, *obtained with the Riemann solver. In the first line, the solutions in I_1 and I_3. In the second line, the solutions in I_2 and I_4. (Color online)*

junction. The numeric cost (4.1) for this solution is approximately 32.39, since

$$\int_0^T \mathcal{J}(f(u_1(t,0)), f(u_2(t,0)))dt \sim 32.68, \qquad \begin{aligned} \text{TV}_{[0,T]} f(u_1(\cdot,0)) &\sim 0.91, \\ \text{TV}_{[0,T]} f(u_2(\cdot,0)) &\sim 0.53, \end{aligned}$$

while the cost for the solution obtained with the corresponding Riemann solvers (see Figure 8), is approximately 30.73. In Figures 9 and 10, respectively, the optimal fluxes and the fluxes obtained with the Riemann solver are drawn. Again the optimal fluxes have less total variation than the fluxes obtained with the Riemann solver. We observe that, in this case compared with case 1, the optimal cost has a larger difference from the cost associated with the solution constructed with the classical Riemann solver. This is due both to the choice of $\mathcal{J}$ and to the choice of the initial datum, producing a more complex wave pattern.
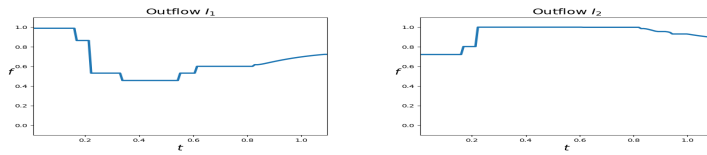
FIG. 9. *The numerical optimal fluxes at the node in the incoming arcs I_1 and I_2, obtained in subsection* 4.2.
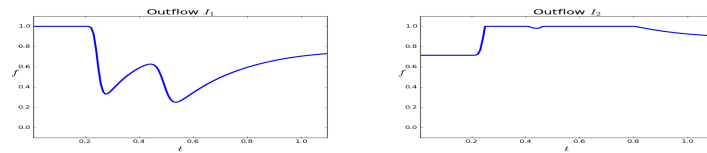

FIG. 10. *The fluxes at the node in the incoming arcs of solution, for subsection* 4.2, *obtained with the Riemann solver.*
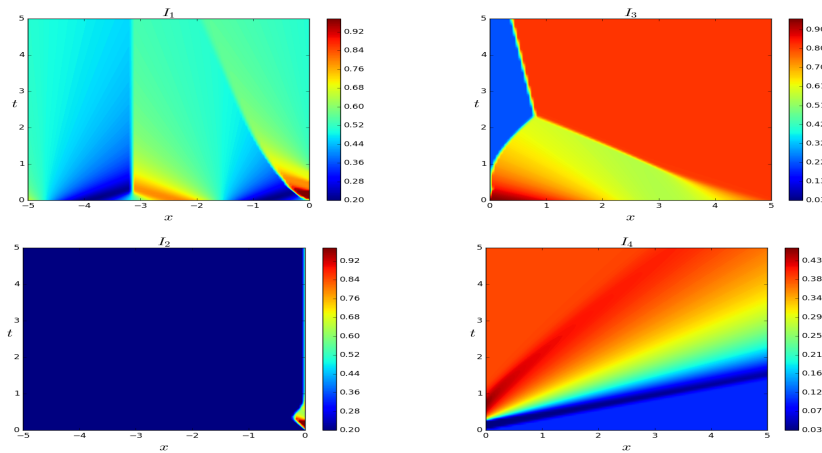

FIG. 11. *The numerical optimal solution obtained in subsection* 4.3. *In the first line, the solutions in I_1 and I_3. In the second line, the solutions in I_2 and I_4. (Color online)*

4.3. Sinusoidal initial conditions. Here we consider the case of a sinusoidal initial condition. Indeed we consider the initial datum

$$\overline{u}_1(x) = 0.5 + 0.3 \cdot \sin(2x), \quad \overline{u}_2(x) = 0.2, \quad \overline{u}_3(x) = 0.75 + 0.2 \cdot \cos(x), \quad \overline{u}_4(x) = 0.1,$$

the distributional matrix

$$A = \left(\begin{array}{cc} 0.5 & 0.3 \\ 0.5 & 0.7 \end{array} \right),$$

and $\mathcal{J}(f_1, f_2) = f_1 f_2$. Moreover, we consider the following choice of parameters: $T = 5$, $M = 20$, and $\delta = 0$, so that the total variation of the fluxes is not taken into account. In Figure 11 the numerical optimal solution is represented. Note that in the arc I_2 a shock with negative speed appears at time $t \sim 0$ and comes back to the node at time $t \sim 1$. Moreover in the arc I_1 the right-hand shock behaves in a different way with respect to the right-hand shock of the solution obtained with the Riemann solver; see Figure 12. The numeric cost (4.1) for the optimal solution is approximately
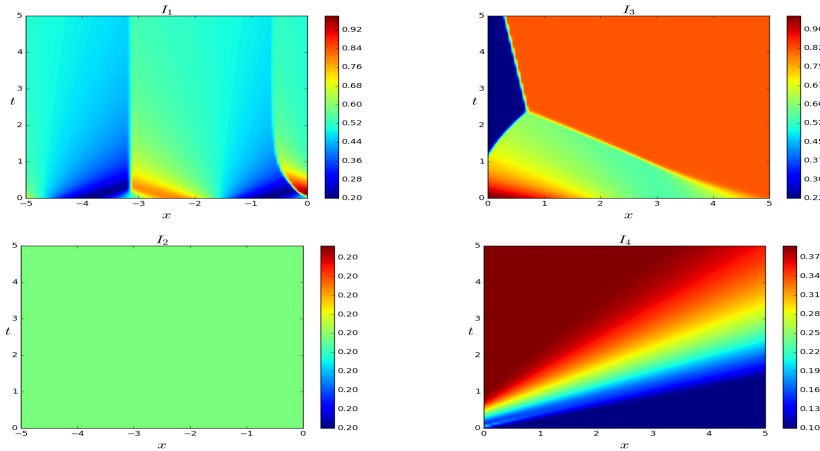
FIG. 12. *The solution, for subsection* 4.3*, obtained with the Riemann solver. (Color online)*
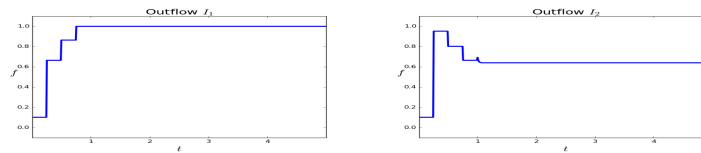

FIG. 13. *The numerical optimal fluxes at the node in the incoming arcs* I_1 *and* I_2*, obtained in subsection* 4.3*.*
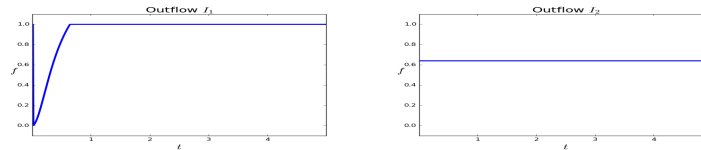

FIG. 14. *The fluxes at the node in the incoming arcs of solution, for subsection* 4.3*, obtained with the Riemann solver.*

3.053 while the cost for the solution obtained with the corresponding Riemann solvers (see Figure 12), is approximately 3.015. In Figures 13 and 14 respectively the optimal fluxes and the fluxes obtained with the Riemann solver are drawn. Here we observe that the total variation of the optimal fluxes is bigger than the total variation of the solution obtained through the Riemann solver. This is not an unexpected feature, since optimal costs are achieved with the price of bigger oscillations.

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