

## Research Article

# Nonstationary First Threshold Crossing Reliability for Linear System Excited by Modulated Gaussian Process

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Received 21 March 2018; Accepted 8 July 2018; Published 5 September 2018

Academic Editor: Salvatore Caddemi

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A widely used approach for the first crossing reliability evaluation of structures subject to nonstationary Gaussian random input is represented by the direct extension to the nonstationary case of the solution based on the qualified envelope, originally proposed for stationary cases. The most convenient way to approach this evaluation relies on working in the time domain, where a common assumption used is to adopt the modulation of stationary envelope process instead of the envelope of modulated stationary one, by utilizing the so-called “preenvelope” process. The described assumption is demonstrated in this work, also showing that such assumption can induce some errors in the envelope mean crossing rate.

## 1. Introduction

One of the most important indexes for engineers and designers is the accurate quantification of structural safety. Its correct evaluation is connected with loads’ nature and failure type. Moreover, a wide class of engineering problems deals with reliability evaluation of structures subject to loads characterized by an intrinsic probabilistic nature and whose correct description could be obtained by means of random process. Besides, there are situations where suitable modelling of these loads requires taking their nonstationary characteristics into account [1–4]. Examples of such conditions include ground acceleration during an earthquake and structural vibration induced by gust loading.

In these cases, the probabilistic characterization of structural loads turns the random dynamic analysis to be the most useful method in achieving an accurate and qualified structural reliability evaluation [5–10]. On the other hand, using this method considerably improves the complexity of analysis and the computational costs for reliability estimation. For instance, exact solutions exist only in few cases. These limitations have substantially reduced the applications

of the stochastic methods from design engineering in real structural safety problems. However, many different approximations have been proposed to overcome these practical restrictions.

In particular for Gaussian inputs and linear systems, the first- and second-order moments completely define the statistics of the response. This problem does not encounter serious difficulties in both spectral and time domains relative to a stationary case and with regard to first crossing failure. However, the application of Rice’s reliability formulation presents some problems related to the required knowledge of hazard function, defined as the probability density function (PDF) of collapse at a given instant, under the restriction that crisis has not happened yet. Due to complexity in its evaluations, a simplified approach is based on the independent crossing hypothesis, using then an unconditioned PDF as hazard function. This hypothesis, which assumes threshold crossing distributed as a Poisson process, can be acceptable only for wideband process and high barrier values, where barrier crossing can be really assumed to be independent. Besides these restrictions, Poisson approach is too poor and conservative for a correct reliability evaluation [11, 12].

A more recent approach for the first passage problem has been developed in [13] for linear and nonlinear systems driven by Poissonian and normal white noise input. In addition, in [14], a numerical path integral solution approach is developed for determining the response and first passage probability density functions (PDFs) of nonlinear oscillators subject to evolutionary broad-band stochastic excitations. A strategy for estimating first excursion probabilities for linear dynamical systems involving uncertain structural parameters subject to Gaussian excitation has been developed recently in [15].

Among different methods, an approach, widely used nowadays, is that proposed by Vanmarcke [16] and is based on the envelope process statistics, which have lower dependencies on threshold crossing events. This approach has been described for probabilistic assessments of structural failure using the barrier crossing rates obtained by using the envelope probabilistic information of a number of parameters derived from the PDF generally known as spectral moments. Their first interpretation has described geometrical moments of the one-sided PSD with respect to the central origin [17]. This frequency definition has been extended to time domain first by Di Paola [18] who has found suitable relationships between the correlation function, its Hilbert transform, and the spectral moments. This approach, also known as “nongeometrical formulation” [19], is able to extend the stationary approach to nonstationary cases, so eliminating the significant difficulties derived by using a geometrical interpretation [20]. It is based on the use of the so-called “preenvelope” process defined as the complex process whose real part is the given process and the imaginary part is the Hilbert transform of the real part,  $A = \sqrt{X^2 + \widehat{X}^2}$  [21]. It is worth mentioning that different definitions have been advanced for envelope processes such as a process joining the peaks of the response [22]. Using a “nongeometrical” approach by covariances of the preenvelope process, defined in time domain, it is possible to obtain the same quantities needed for reliability evaluation derived by a “geometrical” one [19, 23]. The two approaches coincide in a stationary case; however, this result is lost in a nonstationary condition. The reason can be explained as follows: for physical and mathematical questions (different from zero before beginning of input), the exact nonstationary preenvelope process is replaced by a simplified one, the modulation of stationary preenvelope process, which has a more realistic characterization. In addition, this simplification, commonly used by different authors for reliability in nonstationary conditions [24–26], induces some mistake in the evaluation of the joint probability density function (JPDF) of the preenvelope and its first time derivative process. In the present work, the agreement between the approximate analytical formulation and the numerical simulation is shown for different transient conditions and structural damping. Moreover, this study reveals that this simplified approach can be used to transpose important characteristic of covariances matrices needed for reliability evaluation. By utilizing this result and by adopting a differential approach, it is demonstrated that useful simplifications

can be applied to the computation of the mean crossing rate. Finally, a benchmark between simulation and different analytical formulae for mean crossing rate is developed. Also, a simple modified formula is finally proposed to increase the agreement with numerical results.

## 2. Structural Reliability Evaluation by Classical Poisson and Vanmarcke Approaches

Mechanical safety or reliability  $R(t)$  at a fixed time  $t$  throughout the life of a structure subject to static or dynamic loads can be defined as the collapse survival probability, where the collapse is a partial or a total static damage in a fixed time interval  $[t_0, t]$ . Time  $t_0$  is typically the initial observation time assumed as the end of the manufacturing process. In this situation, the collapse probability at time  $t_0$  is usually assumed equal to 0. It is clear that the definition of structural collapse (or in general, failure) plays an important role in reliability assessment, considering the numerous possible meanings (not only of mechanical kind) of this condition. Two of the most common mechanical undertakings are related to fatigue phenomenon and first threshold crossing failure.

In this section, a simple one-degree-of-freedom, viscous-elastic system subject to random dynamic actions is analysed, for which failure is related to displacement barrier crossing. Precisely, the system crisis happens when structural displacement  $X(t)$  exceeds an admissible value  $\beta$  for the first time. In many structural situations, typically if the response is symmetric, the most used definition of reliability  $R(t)$  is given by the probability that the system will not have a failure associated with a bilateral threshold crossing of level  $\beta$  by process displacement  $|X(t)|$  (double barrier problem) in the time interval  $[t_0, t]$ . Its formal definition can be written as

$$R(t) = P[|X(t)| < \beta; t \in [t_0, t]]. \quad (1)$$

An alternative method is using its complementary collapse probability (failure)  $P_f(t) = 1 - R(t)$ , which can be defined as the probability that the double barrier  $\beta$  will be exceeded by the process  $|X(t)|$  during the observation interval [11]. Reliability evaluation requires the knowledge of the hazard function  $h(t)$ , also named as “decay rate function”, which is related to  $R(t)$  according to the following equation:

$$R(t + dt) = [1 - h(t) dt]R(t). \quad (2)$$

The hazard function  $h(t)$  is defined as the quantity whose product with the infinitesimal time interval  $dt$  supplies the structural collapse probability for threshold crossing during  $[t, t + dt]$ , under the condition that at time  $t$ , the response process has not yet shown any other threshold crossing, which is formally represented as

$$h(t)dt = P[C | S], \quad (3)$$

where  $C$  is the event “threshold crossing” in  $[t, t + dt]$  and  $S$  is the absence of barrier excursions before time  $t$ .

Using (2), it is possible to write

$$\begin{aligned}
R'(t) &= \lim_{dt \rightarrow 0} \frac{R(t+dt) - R(t)}{dt} \\
&= \lim_{dt \rightarrow 0} \frac{([1 - h(t)dt] - 1)R(t)}{dt} = -h(t)R(t).
\end{aligned} \tag{4}$$

The formal solution of (4) is  $\int (dR/R) = -\int h(t)dt$ , which gives at last the well-known integral reliability formulation:

$$R(t) = R_0 \exp\left\{-\int_{t_0}^t h(t)dt\right\}. \tag{5}$$

The exact hazard function formulation is still an open question due to the fact that it has been solved only in few cases. In the original Rice's formulation [27], the hazard function has the following form:

$$h(t) = \int_{-\infty}^{+\infty} |\dot{x}| p_{X\dot{X}}(\beta, \dot{x}; t | S) d\dot{x}, \tag{6}$$

where  $p_{X\dot{X}}(\beta, \dot{x}; t | S)$  is the conditioned JPDF of  $X$  and its first time derivative  $\dot{X}(t)$  processes. The difficulties related to its evaluation have induced the development of approximate solutions. One of the most common approximations uses the unconditioned JPDF instead of the conditioned one. Using this approximation, the hazard function  $h(t)$  is replaced by  $v(t)$ , known as unconditioned threshold crossing rate. In addition, the relative reliability is evaluated in the so-called Poisson hypothesis, because the independent threshold crossings assumption means, according to a Poisson process, that they are rare and without memory events. This approximation has been demonstrated to be suitable for wideband processes and high barrier levels, when the threshold crossing is effectively a rare event, so that the distribution of crossings has an asymptotic Poisson distribution in this situation. On the contrary, this approach could be particularly poor for narrowband processes and for relatively low barrier levels, since it is too conservative. In fact, the barrier crossings in these situations tend to happen in clumps, completely neglecting the independence hypothesis.

A frequently used solution to overcome the limitation discussed above is obtained by using the approach of Vanmarcke [16, 17], which is based on a two-state Markov crossing assumption. This formulation uses the statistical information of "envelope process  $A(t)$ ," which is introduced to consider the effects of the clumps, and thus reducing the errors that originate from the independent crossing assumption. The envelope process has the general physical meaning of a nonnegative random process that joins the peaks of  $|X(t)|$ . Using the Vanmarcke formulation, a two-state Markov process is introduced to describe the two possible conditions (over and below) of the envelope process  $A(t)$  with respect to the assigned barrier  $\beta$  and to define the so-called E-crossing event (i.e., an up-crossing of the envelope). The author has developed an analytical formulation for hazard function by using the Poisson assumption for the arrivals of E-crossing and by taking into account only the cases where a D-crossing (up-crossing of  $|X(t)|$ ) follows it (qualified envelope process). The resulting formulation has

a stronger agreement with numerical results as compared to the original simpler one that is based on the Poisson assumption directly applied to the system response process [11, 13].

For a stationary process, the formulation that furnishes the hazard function for a bilateral threshold level  $\beta$  is

$$v_{VM}(\beta) = v_X(\beta) \frac{1 - \exp\{(v_A^+(\beta)/v_X(\beta))\}}{1 - (v_X(\beta)/v_X(0))}, \tag{7}$$

where  $v_X(\beta) = 2v_X^+(\beta)$  is the mean crossing rate for a single barrier evaluated with the Poisson hypothesis:

$$v_X(\beta) = \int_{-\infty}^{+\infty} |\dot{x}| p_{X\dot{X}}(\beta, \dot{x}) d\dot{x}. \tag{8}$$

And the single barrier mean crossing rate of the envelope process  $v_A^+(\beta)$  is defined using the same approach as

$$v_A^+(\beta) = \int_0^{+\infty} \dot{a} p_{A\dot{A}}(\beta, \dot{a}) d\dot{a}. \tag{10}$$

Its evaluation requires the JPDF  $p_{A\dot{A}}(a, \dot{a})$  of  $A(t)$  and  $\dot{A}(t)$ , which means the stochastic characterization of these two processes. Using this solution for a nonstationary process, the hazard function for threshold level  $\beta$  is [11]

$$v_{vm}(\beta, t) = 2v_X^+(\beta, t) \frac{1 - \exp\{(v_A^+(\beta, t)/2v_X^+(\beta, t))\}}{1 - (v_X^+(\beta, t)/v_X^+(0, t))}, \tag{11}$$

where  $v_X^+(\beta, t)$  is the mean crossing rate for a single barrier  $\beta$  evaluated under the Poisson hypothesis.

### 3. Envelope Process Definition

In a stationary condition, the envelope process can be defined as the modulus of a complex process whose real part is the original one and the imaginary part has to be opportunely defined in order to satisfy the following equation:

$$A_s(t) = |U_s(t)| = \sqrt{X_s^2(t) + iY_s^2(t)}. \tag{12}$$

The choice of  $Y_s(t)$  has to guarantee the physical meaning of the envelope process, which implies a smoothly curve able to connect response peaks. Different approaches have been employed to choose a suitable formulation of  $Y_s(t)$  [22]. Usually, the preferred ones are those based on the use of the response process first time derivative  $Y_s(t) = (\dot{X}_s(t)/\omega_c)$  [21], or on its Hilbert transform (Hilbert transform of function  $f(t)$  is  $H[f(t)] = \hat{f}(t) = (1/\pi) \int_{-\infty}^{+\infty} (f(\tau)/(t-\tau))d\tau$ )  $Y_s(t) = \hat{X}_s(t)$  [21]. The latter is the approach adopted in the current work.

White noise or filtered Gaussian uniform modulated processes is a widely used model for representing non-stationary loads [28]. As shown in (13), it is obtained as the product of a stationary Gaussian white noise process  $S(t)$  and a deterministic time modulation amplitude controlling function  $\varphi(t)$ , which must satisfy these conditions:  $\varphi(t) \geq 0$  for  $t > 0$ ;  $\varphi(t) = 0$  for  $t < 0$ ; and  $\max\{\varphi(t)\} = 1$ :

$$N(t) = \varphi(t)S(t). \tag{13}$$

The stationary process is usually represented by a zero-mean white noise process  $w(t)$  ( $R_{WW}(\tau) = 2\pi S_0 \delta(\tau)$ );

the covariance of the corresponding modulated process  $F(t) = \varphi(t)w(t)$  ( $\mu_N(t) = \varphi(t)\mu_S(t)$ ,  $R_{NN}(t_1, t_2) = \varphi(t_1)\varphi(t_2)R_{SS}(t_1, t_2)$ ,  $\sigma_N^2(t) = \varphi^2(t)\sigma_S^2(t)$ ) is given by

$$R_{FF}(t_1, t_2) = 2\pi S_0 \varphi(t_1)\varphi(t_2)\delta(t_2 - t_1), \quad (14)$$

where  $\delta(t)$  is the Dirac delta function.

A multiplicative factor equal to  $1/\sqrt{2}$  has to be taken into account due to the consideration that, for a generic analytical stochastic process  $v(t) = (1/\sqrt{2})[u(t) + i\hat{u}(t)]$  (where  $u(t)$  is real), the covariance is given by

$$R_v(\tau) = \left[ R_u(\tau) + i\hat{R}_u(\tau) \right]. \quad (15)$$

In this case,  $U_s(t)$  becomes a complex process whose imaginary part is the Hilbert transform of the real one; that is, it can be expressed as the system response to the Hilbert transform of a real input modelled as a white noise process  $w_s(t)$ :

$$U_s(t) = \frac{1}{\sqrt{2}} \left( X_s(t) + i\hat{X}_s(t) \right), \quad (16)$$

where  $X_s(t) = \int_0^t h(t-\tau)w_s(\tau)d\tau$  and  $\hat{X}_s(t) = \int_0^t h(t-\tau)w_s(\tau)d\tau$ ;  $h(t)$  is the impulse response function, which for a SDOF system can be written as  $h(t) = (1/\bar{\omega})e^{-\xi\omega t} \sin(\bar{\omega}t)$ ,

where  $\bar{\omega} = \omega\sqrt{1-\xi^2}$ .

In a more compact form,  $U_s(t)$  can be defined as the analytical process response of a linear system to the analytical input process:

$$F_s(t) = \frac{1}{\sqrt{2}} \left( w_s(t) + i\hat{w}_s(t) \right). \quad (17)$$

The covariance matrix of the stationary vector process  $U_s(t)$  can be expressed as  $\mathbf{R}_{U_s U_s}(\tau) = \mathbf{R}_{X_s X_s}(\tau) + i\mathbf{R}_{X_s \hat{X}_s}(\tau)$ , due to the following properties, generally true only in the stationary case [28]:

$$\mathbf{R}_{X_s X_s}(\tau) = \mathbf{R}_{\hat{X}_s \hat{X}_s}(\tau), \quad (18a)$$

$$\mathbf{R}_{X_s \hat{X}_s}(\tau) = -\mathbf{R}_{\hat{X}_s X_s}(\tau) = \hat{\mathbf{R}}_{X_s X_s}(\tau). \quad (18b)$$

In a nonstationary condition, the structural response envelope can be defined in a way similar to the stationary case, by introducing a nonstationary complex process  $U(t)$ . However, a different approach is necessary to define the complex part, due to physical and analytical reasons.

Using a nonstationary modulated process input  $F(t) = \varphi(t)w(t)$ , the direct Hilbert transforms applied to the linear system response

$$\hat{X}(t) = \int_0^t h(t-\tau)H[\varphi(\tau)w(\tau)]d\tau, \quad (19)$$

is different from zero in the time interval  $[-\infty, 0]$ , and the corresponding envelope too, conflicting with the physical meaning of an event that actually begins at time  $t_0 = 0$ . For this reason, the modulated Hilbert transform input process  $\hat{F}(t) = \varphi(t)\hat{w}(t)$ , first proposed by Di Paola and Muscolino

[18, 29, 30], is used to define the so-called analytical preenvelope input process:

$$F(t) = \frac{1}{\sqrt{2}} \varphi(t) (w(t) + i\hat{w}(t)). \quad (20)$$

which corresponds to the analytical preenvelope response:

$$U(t) = \frac{1}{\sqrt{2}} (X(t) + i\hat{X}(t)), \quad (21)$$

where the real and imaginary parts are

$$X(t) = \int_0^t h(t-\tau)\varphi(\tau)w(\tau)d\tau, \quad (22a)$$

$$\hat{X}(t) = \int_0^t h(t-\tau)\varphi(\tau)\hat{w}(\tau)d\tau. \quad (22b)$$

In this way, it can be written in [18]:

$$\mathbf{R}_{UU}(t_1, t_2) = \mathbf{R}_{XX}(t_1, t_2) + i\mathbf{R}_{X\hat{X}}(t_1, t_2). \quad (23)$$

By extending the following properties to a nonstationary case:

$$\mathbf{R}_{XX}(t_1, t_2) = \mathbf{R}_{\hat{X}\hat{X}}(t_1, t_2), \quad (24a)$$

$$\mathbf{R}_{X\hat{X}}(t_1, t_2) = -\mathbf{R}_{\hat{X}X}(t_1, t_2). \quad (24b)$$

which are in general not true using the direct Hilbert transformation when applied to the modulated input process.

#### 4. Standard Space State Covariance Response

As a main direct consequence of the above assumptions, working with the two-state space vectors is expressed in [6]:

$$Z(t) = \begin{pmatrix} X(t) \\ \dot{X}(t) \end{pmatrix}, \quad (25a)$$

$$\tilde{Z}(t) = \begin{pmatrix} \hat{X}(t) \\ \tilde{\dot{X}}(t) \end{pmatrix}. \quad (25b)$$

The correlation matrix is given by

$$\langle \bar{Z}(t)\bar{Z}(t) \rangle = \mathbf{R}_{ZZ}(t) = \begin{pmatrix} \sigma_X^2(t) & \gamma_{X\dot{X}}(t) \\ \gamma_{\dot{X}X}(t) & \sigma_{\dot{X}}^2(t) \end{pmatrix}. \quad (26)$$

and is symmetric as it is well known, being  $\langle X(t)\dot{X}(t) \rangle = \gamma_{\dot{X}X}(t) = \gamma_{X\dot{X}}(t)$ .

Similarly the matrix

$$\langle \bar{Z}(t)\hat{\bar{Z}}(t) \rangle = \mathbf{R}_{Z\hat{Z}}(t) = \begin{pmatrix} 0 & \gamma_{X\hat{X}}(t) \\ \gamma_{\hat{X}X}(t) & 0 \end{pmatrix}. \quad (27)$$

is asymmetric, being  $\langle \dot{X}(t)\hat{X}(t) \rangle = \gamma_{\hat{X}X}(t) = -\gamma_{X\hat{X}}(t)$ .

Then, it is possible to write the differential equations in the space state as

$$\dot{\bar{Z}}(t) = \mathbf{A}\bar{Z}(t) + \bar{B}(t), \quad (28a)$$

$$\dot{\hat{\bar{Z}}}(t) = \mathbf{A}\hat{\bar{Z}}(t) + \hat{\bar{B}}(t), \quad (28b)$$

where the state matrix  $\mathbf{A}$  in the case of a linear SDF system is

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -\omega_0^2 & -2\xi_0\omega_0 \end{pmatrix}. \quad (29)$$

And the input vectors  $\bar{\mathbf{B}}(t)$  and  $\overline{\bar{\mathbf{B}}}(t)$  are, respectively

$$\bar{\mathbf{B}}(t) = \varphi(t) \begin{pmatrix} 0 \\ w(t) \end{pmatrix}, \quad (30a)$$

$$\overline{\bar{\mathbf{B}}}(t) = \varphi(t) \begin{pmatrix} 0 \\ \widehat{w}(t) \end{pmatrix}. \quad (30b)$$

As shown earlier in (10), the nonstationary envelope process mean rate threshold crossing  $v_A(\beta, t)$  needs the JPDF of  $A(t)$  and  $\dot{A}(t)$ , which are related to both processes  $U(t)$  and  $\dot{U}(t)$ , being  $\dot{A}(t) = |\dot{U}(t)| = (1/\sqrt{2})(\dot{X}(t) + i\dot{\hat{X}}(t))$ . It is then possible to introduce the analytical envelope state vector process as

$$\bar{\Psi}(t) = \begin{pmatrix} U(t) \\ \dot{U}(t) \end{pmatrix} = \begin{pmatrix} X(t) \\ \dot{X}(t) \end{pmatrix} + i \begin{pmatrix} \hat{X}(t) \\ \dot{\hat{X}}(t) \end{pmatrix}. \quad (31)$$

The covariance matrix is given by

$$\begin{aligned} \mathbf{R}_{\bar{\Psi}\bar{\Psi}}(t) &= \langle \bar{\Psi}(t)\bar{\Psi}(t)^T \rangle = \left\langle \begin{pmatrix} \bar{Z}(t) \\ \overline{\bar{Z}}(t) \end{pmatrix} \begin{pmatrix} \bar{Z}(t)^T & \overline{\bar{Z}}(t)^T \end{pmatrix} \right\rangle \\ &= \begin{pmatrix} \langle \bar{Z}\bar{Z}^T \rangle & \langle \bar{Z}\overline{\bar{Z}}^T \rangle \\ \langle \overline{\bar{Z}}\bar{Z}^T \rangle & \langle \overline{\bar{Z}}\overline{\bar{Z}}^T \rangle \end{pmatrix}. \end{aligned} \quad (32)$$

It can be rewritten in a more compact form by decomposing the  $4 \times 4$  matrix into four  $2 \times 2$  submatrices [6]:

$$\mathbf{R}_{\bar{\Psi}\bar{\Psi}}(t) = \begin{pmatrix} \mathbf{R}_{ZZ}(t) & \mathbf{R}_{Z\widehat{Z}}(t) \\ \mathbf{R}_{\widehat{Z}Z}(t) & \mathbf{R}_{\widehat{Z}\widehat{Z}}(t) \end{pmatrix}. \quad (33)$$

Using the mean value Hilbert properties previously shown, it is possible to demonstrate that the 4 submatrices are two by two equal ( $\mathbf{R}_{ZZ}(t) = \mathbf{R}_{\widehat{Z}\widehat{Z}}(t)$ ) and opposite ( $\mathbf{R}_{Z\widehat{Z}}(t) = -\mathbf{R}_{\widehat{Z}Z}(t)$ ).

Therefore,  $\mathbf{R}_{\bar{\Psi}\bar{\Psi}}(t)$  can be defined just knowing the two  $2 \times 2$  submatrices  $\mathbf{R}_{ZZ}(t)$  and  $\mathbf{R}_{\widehat{Z}\widehat{Z}}(t)$ .

The first one can be obtained using the well-known Lyapunov differential matrix covariance equation:

$$\dot{\mathbf{R}}_{ZZ} = \mathbf{A}\mathbf{R}_{ZZ} + \mathbf{R}_{ZZ}\mathbf{A}^T + \langle \mathbf{B}\mathbf{Z}^T \rangle + \langle \mathbf{B}\mathbf{Z}^T \rangle^T, \quad (34)$$

where the last two matrices, in the case of modulated white noise input, can be expressed as

$$\langle \mathbf{B}\mathbf{Z}^T \rangle + \langle \mathbf{B}\mathbf{Z}^T \rangle^T = \mathbf{P}(t) = 2\pi S_0 \varphi^2(t) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \quad (35)$$

Due to the symmetry of  $\mathbf{R}_{ZZ}$ , it can be replaced by a vectorial form by the three first-order differential equations

$$\dot{\lambda}_1 = 2\lambda_2, \quad (36a)$$

$$\dot{\lambda}_2 = -\omega_1^2 \lambda_1 - 2\xi\omega \lambda_2 + \lambda_3, \quad (36b)$$

$$\dot{\lambda}_3 = -2\omega \lambda_2 - 4\xi\omega \lambda_3 + 2\pi S_0 \varphi^2(t), \quad (36c)$$

where  $\lambda_1 = \sigma_X^2(t)$ ,  $\lambda_2 = \gamma_{X\dot{X}}(t)$ , and  $\lambda_3 = \sigma_{\dot{X}}^2(t)$ .

## 5. Hilbert Transform Space State Covariance Response

By applying a similar approach, it is possible to obtain the matrix  $\mathbf{R}_{\widehat{Z}\widehat{Z}}(t) = \langle \widehat{Z}\widehat{Z}^T \rangle$ .

By making its first-order temporal derivation, it can be written as

$$\frac{d}{dt} \{ \mathbf{R}_{\widehat{Z}\widehat{Z}}(t) \} = \dot{\mathbf{R}}_{\widehat{Z}\widehat{Z}}(t) = \langle \dot{Y}(t)\widehat{Y}(t)^T + Y(t)\widehat{Y}(t)^T \rangle. \quad (37)$$

Moreover, by adopting the two-state vector differential equation [6], the left quantities can be expressed as

$$\dot{\widehat{Z}}\widehat{Z}^T = \mathbf{A}\widehat{Z}\widehat{Z}^T + \mathbf{B}\widehat{Z}^T, \quad (38a)$$

$$\widehat{Z}\widehat{Z}^T = \widehat{Z}\widehat{Z}^T \mathbf{A}^T + \mathbf{Z}\widehat{\mathbf{B}}^T. \quad (38b)$$

So finally obtaining the following 1st order differential equation matrix:

$$\dot{\mathbf{R}}_{\widehat{Z}\widehat{Z}}(t) = \mathbf{A}\mathbf{R}_{\widehat{Z}\widehat{Z}}(t) + \mathbf{R}_{\widehat{Z}\widehat{Z}}(t)\mathbf{A}^T + \langle \mathbf{B}\widehat{\mathbf{Y}}^T \rangle + \langle \mathbf{Y}\widehat{\mathbf{B}}^T \rangle. \quad (39)$$

Since the two last matrices are asymmetrical, it results as

$$\mathbf{B}\widehat{\mathbf{Y}}^T + \mathbf{Y}\widehat{\mathbf{B}}^T = \tilde{\mathbf{P}}(t) = \begin{pmatrix} 0 & b(t) \\ -b(t) & 0 \end{pmatrix}, \quad (40)$$

where  $b(t)$  is the convolution integral:

$$b(t) = 2S_0 \varphi(t) \int_0^t \frac{h(u)}{u} \varphi(t-u) du. \quad (41)$$

Since  $\mathbf{R}_{\widehat{Z}\widehat{Z}}(t)$  is asymmetrical, the following single first-order differential equation is obtained:

$$\dot{\gamma}_{X\widehat{X}}(t) = -2\xi\omega \gamma_{X\widehat{X}}(t) + b(t). \quad (42)$$

whose solution depends on the instantaneous value of  $b(t)$ . Its evaluation can be quite expensive from a computational point of view, due to the involvement of a convolution integral; in fact, it has to be reevaluated at each instant  $t$  of the considered time interval. This problem can be solved if the modulation function  $\varphi(t)$  is a separable one. This means that  $\phi(t-u)$  can be expressed as a summation of finite functions just depending on  $t$  or  $u$ :

$$\varphi(t-u) = \sum_{k=1}^m B_k(t) C_k(u). \quad (43)$$

In this case,  $b(t)$  assumes the following form:

$$b(t) = 2S_0\varphi(t) \sum_{r=1}^m \mathbf{N}_r(t) = 2S_0\varphi(t) \sum_{r=1}^m B_r(t)I_r(t), \quad (44)$$

where  $\mathbf{N}_r(t) = B_r(t)I_r(t)$  while  $I_r(t) = \int_0^t (h(\rho)/\rho)C_r(\tau)d\rho^4$  is a standard integral.

Equation (42) can be so rewritten as:

$$\dot{\gamma}_{\widehat{X}\widehat{X}}(t) = -2\xi\omega\gamma_{\widehat{X}\widehat{X}}(t) + 2S_0\varphi(t) \sum_{r=1}^m B_r(t)I_r(t) \quad (45)$$

The covariance matrix  $\mathbf{R}_{\overline{\mathbf{V}}}(t)$  is then completely evaluated, by taking into account only the four evolutionary parameters  $\lambda_1 = \sigma_X^2(t)$ ,  $\lambda_2 = \gamma_{X\widehat{X}}(t)$ ,  $\lambda_3 = \sigma_{\widehat{X}}^2(t)$ , and  $\lambda_4 = \gamma_{\widehat{X}\widehat{X}}(t)$ , and by using the following four first-order differential equations:

$$\dot{\lambda}_1 = 2\lambda_2, \quad (46a)$$

$$\dot{\lambda}_2 = -\omega_1^2\lambda_1 - 2\xi\omega\lambda_2 + \lambda_3, \quad (46b)$$

$$\dot{\lambda}_3 = -2\omega\lambda_2 - 4\xi\omega\lambda_3 + 2\pi S_0\varphi^2(t), \quad (46c)$$

$$\dot{\lambda}_4 = -2\xi\omega\lambda_4 + 2S_0\varphi(t) \sum_{r=1}^p A_r(t)I_r(t), \quad (46d)$$

where only the first three are coupled.

## 6. Envelope JPDF Evaluation

The covariance matrix  $\mathbf{R}_{\overline{\mathbf{U}}}(t)$  is equal to the matrix  $\mathbf{R}_{\overline{\mathbf{U}}}(t)$ , corresponding to the vector  $\overline{\mathbf{U}}(t) = (X(t), \dot{X}(t), \widehat{X}(t), \dot{\widehat{X}}(t))^T$ ,

$$\mathbf{R}_{\overline{\mathbf{U}}} = \langle \overline{\mathbf{U}}\overline{\mathbf{U}}^T \rangle = \begin{pmatrix} \sigma_X^2(t) & \gamma_{X\widehat{X}}(t) & 0 & \gamma_{X\dot{\widehat{X}}}(t) \\ \gamma_{X\widehat{X}}(t) & \sigma_{\widehat{X}}^2(t) & -\gamma_{\widehat{X}\dot{\widehat{X}}}(t) & 0 \\ 0 & -\gamma_{\widehat{X}\dot{\widehat{X}}}(t) & \sigma_X^2(t) & \gamma_{X\dot{\widehat{X}}}(t) \\ \gamma_{X\dot{\widehat{X}}}(t) & 0 & \gamma_{X\dot{\widehat{X}}}(t) & \sigma_{\dot{\widehat{X}}}^2(t) \end{pmatrix}. \quad (47)$$

whose JPDF is:

$$p_{\overline{\mathbf{U}}}(x, \dot{x}, \widehat{x}, \dot{\widehat{x}}; t) = \frac{1}{4\pi^2 |\mathbf{R}_{\overline{\mathbf{U}}}(t)|^{1/2}} \cdot \exp\left\{-\frac{1}{2} \overline{\mathbf{U}}^T(t) \mathbf{R}_{\overline{\mathbf{U}}}^{-1}(t) \overline{\mathbf{U}}(t)\right\}. \quad (48)$$

As demonstrated before, a central point of Vanmarcke formulation for hazard function evaluation is the knowledge of the preenvelope process mean threshold crossing rate  $v_A^+(\beta, t)$ . Its determination requires the knowledge of the JPDF  $p_{a\dot{a}}(a, \dot{a}; t)$ , as follows:

$$v_A^+(\beta, t) = \int_0^{+\infty} |\dot{a}| p_{A\dot{A}}(\beta, \dot{a}; t) d\dot{a}. \quad (49)$$

By applying the definition of the preenvelope process through the original process  $X(t)$  and its nonstationary Hilbert transform  $\widehat{X}(t)$ , it is useful to introduce the 4th order

evolutionary zero mean random vector  $\overline{\mathbf{U}}(t)$ , whose joint PDF can be written as

$$p_{\overline{\mathbf{U}}}(x, \dot{x}, \widehat{x}, \dot{\widehat{x}}) = \frac{1}{4\pi^2 \sqrt{|\mathbf{R}_{\overline{\mathbf{U}}}|}} \exp\left(-\frac{1}{2} \overline{\mathbf{U}}^T \mathbf{R}_{\overline{\mathbf{U}}}^{-1} \overline{\mathbf{U}}\right). \quad (50)$$

By introducing the following vector:

$$\overline{\mathbf{V}}(t) = (A(t) \quad \dot{A}(t) \quad \theta(t) \quad \dot{\theta}(t))^T, \quad (51)$$

where  $\theta = \tan^{-1}(\widehat{X}(t)/X(t))$ , it is possible to obtain the JPDF  $p_{\overline{\mathbf{V}}}(\overline{\mathbf{v}}, t)$  [23] by using the well-known rule  $p_{\overline{\mathbf{V}}}(a, \dot{a}, \theta, \dot{\theta}; t) = |J| p_{\overline{\mathbf{U}}}(x, \dot{x}, \widehat{X}, \dot{\widehat{x}}; t)$ ,  $|J| = |\partial \overline{\mathbf{U}} / \partial \overline{\mathbf{V}}| = a^2$  being the norm of the transformation. The joint density of  $\overline{\mathbf{V}}(t)$  is thus given by

$$p_{\overline{\mathbf{V}}}(a, \dot{a}, \theta, \dot{\theta}; t) = \frac{(a\sigma_x\sigma_{\dot{x}})^2}{4\pi^2 \Delta_A^2} \exp\left\{-\frac{(a\sigma_x\sigma_{\dot{x}})^2}{2\Delta_A^2} \cdot \left(a^2\sigma_{\dot{x}}^2 + a^2\dot{\theta}^2\sigma_x^2 + \dot{a}^2\sigma_x^2 - 2a\dot{a}\gamma_{x\dot{x}} - 2\dot{a}^2\dot{\theta}\gamma_{x\dot{x}}\right)\right\}, \quad (52)$$

where  $\Delta_A(t) = \sqrt{1 - \rho_{X\widehat{X}}^2(t) - \rho_{\widehat{X}\dot{\widehat{X}}}^2(t)}$ . The phase  $\theta$  does not appear in (52), and then having a range over  $[0, 2\pi]$ , it is independent from the other three vector components, characterized by a uniform marginal distribution over the same interval. By applying the saturation with regard to both  $\theta$  and  $\dot{\theta}$ , the required second-order JPDF can be written as

$$p_{A\dot{A}}(a, \dot{a}; t) = \int_0^{2\pi} \int_{-\infty}^{+\infty} p_{\overline{\mathbf{V}}}(a, \dot{a}, \theta, \dot{\theta}; t) d\theta d\dot{\theta} = aB_a(t) \exp\left\{-\frac{1}{2} \left[a^2 C_a(t) - 2a\dot{a} D_a(t) + \dot{a}^2 E_a(t)\right]\right\}, \quad (53)$$

where  $B_a$ ,  $C_a$ ,  $D_a$ , and  $E_a$  are reported in Appendix A as obtained in [25]. It is hence straightforward to obtain the marginal distribution of  $a$  and  $\dot{a}$  making saturation with respect to the other variable at each time, as expressed by

$$p_A(a; t) = \frac{a}{\sigma_x^2(t)} \exp\left\{-\frac{a^2}{2\sigma_x^2(t)}\right\}, \quad (54a)$$

$$p_{\dot{A}}(\dot{a}; t) = \frac{\dot{a}}{\sqrt{2\pi}(q(t) + \rho_{x\dot{x}}^2(t))\sigma_{\dot{x}}(t)} \exp\left\{-\frac{\dot{a}}{2q(t)\sigma_{\dot{x}}^2}\right\}. \quad (54b)$$

By using the preenvelope density function  $p_{A\dot{A}}(a, \dot{a}; t)$ , it is possible to establish the analytical form of the mean threshold crossing rate by solving the following integral:

$$v_A^+(\beta, t) = \frac{\eta(t)}{\sqrt{2}\Delta_A(t)} \frac{\sigma_{\dot{x}}(t)}{\sigma_X(t)} \exp\left\{-\frac{1}{2}\eta^2(t)\right\} \chi[d_A(t)], \quad (55)$$

where the adimensional threshold  $\eta(t)$  is defined as the evolutionary function  $\eta(t) = (\beta/(\sigma_X(t)))$ . The function  $\chi(y)$  is given by  $\chi(y) = \exp(-(y^2/2)) + \sqrt{2\pi}y\Phi(y)$ , where  $\Phi(y)$  is the normal cumulative function  $\Phi(y) = (1/\sqrt{2\pi}) \int_{-\infty}^y \exp(-u^2/2) du$ , and  $d_A(t) = (1/\sqrt{2})\eta(t) ((\rho_{\widehat{X}\dot{\widehat{X}}}(t)/\Delta_A(t)))$ .

By extending the original approach proposed by Vanmarcke [16, 17] and by using the nonstationary mean threshold crossing rate for the Gaussian process  $X(t)$  in the Poisson hypothesis, it results:

$$v_X^+(\beta, t) = \frac{1}{\sqrt{\pi}} \Delta_X(t) \frac{\sigma_{\dot{X}}(t)}{\sigma_X(t)} \exp\left\{-\frac{1}{2}\eta^2(t)\right\} \chi[d_X(t)], \quad (56)$$

where  $\Delta_X(t) = \sqrt{1 - \rho_{X\dot{X}}^2(t)}$  and  $d_X(t) = (1/\sqrt{2})\eta(t)$  ( $\rho_{X\dot{X}}(t)/\Delta_X(t)$ ). With reference to the formulation from Michelov et al. [23]

$$v_{\text{VLM}}^+(\xi, t) = \frac{1}{\pi} \sqrt{1 - \rho_{X\dot{X}}^2} \frac{\sigma_{\dot{X}}(t)}{\sigma_X(t)} \cdot \frac{1 - \exp((v_A^+(\xi, t)/(2v_X^+(\xi, t))) \exp[(\eta^2(t)/2)] - 1)}{\exp[(\eta^2(t)/2)] - 1}. \quad (57)$$

It can be rewritten as:

$$v_{\text{VLM}}^+(\xi, t) = \frac{1}{\pi} \Delta_X(t) \frac{\sigma_{\dot{X}}(t)}{\sigma_X(t)} \chi[d_X(t)] \cdot \frac{1 - \exp(-\sqrt{\pi/2} (\Delta_A(t)/\Delta_X(t)) (\chi[d_A(t)]/\chi[d_X(t)]))}{\exp[(\eta^2(t)/2)] - \chi[d_X(t)]}, \quad (58)$$

which is the same relation proposed in [23] except for  $\chi[d_X(t)]$  whose complete expression is

$$\chi(\vartheta, t) = e^{-\vartheta^2(t)} + \sqrt{\pi}\vartheta(t)\Phi\left(\frac{\vartheta(t)}{\sqrt{2}}\right), \quad (59)$$

where

$$\vartheta(t) = \eta(t) \frac{\rho_{X\dot{X}}(t)}{\sqrt{1 - \rho_{X\dot{X}}^2(t)}}, \quad (60)$$

Expression in (60) is function of time through both the dimensionless threshold value  $\eta(t)$  and the ratio ( $\rho_{X\dot{X}}(t)/\sqrt{1 - \rho_{X\dot{X}}^2(t)}$ ). Moreover, since  $\vartheta(t)$  can assume just positive values (in stationary case, it is exactly the dimensionless threshold  $\eta$ ), its values are close to the unit, that is,  $\vartheta(t)$  is smaller than 1, growing up for greater values (Figure 1). In particular, in practical cases, it is quite close to one, as shown in the numerical applications.

This formulation can be improved by introducing the parameter  $k$ , originally proposed by Vanmarcke [16, 17] for the stationary case, to enhance the agreement between the numerical simulation and the analytical formulation. This parameter is an empirical exponent of the “equivalent bandwidth factor”  $q_{X_s}$  defined for the stationary case. Its value has been evaluated to be equal to 1.2 and has been subsequently used in different studies.

In the nonstationary case, the parameter  $k$  has to be taken into account as the exponent of

$$q(t) = (\Delta_A(t)/\Delta_X(t)) = \sqrt{1 - \frac{\gamma_{X\dot{X}}^2(t)}{1 - \gamma_{X\dot{X}}^2(t)}}. \quad (61)$$

which coincides with the stationary value when  $\rho_{X\dot{X}}^2(t) = 0$ .

The mean threshold crossing rate  $v_{\text{VLM}}^+(\xi, t)$  can be finally rewritten as

$$v_{\text{VLM}}^+(\xi, t) = \frac{1}{\pi} \Delta_X(t) \frac{\sigma_{\dot{X}}(t)}{\sigma_X(t)} \chi[d_X(t)] \cdot \frac{1 - \exp(-\sqrt{\pi/2} q^k(t) (\chi[d_A(t)]/\chi[d_X(t)]))}{\exp[(\eta^2(t)/2)] - e^{-\vartheta^2(t)} + \sqrt{\pi}\vartheta(t)\Phi((\vartheta(t)/\sqrt{2}))}. \quad (62)$$

## 7. Numerical Results

Different modulation functions have been proposed in the scientific literature to model the behavior of nonstationary loads in a suitable way [5]. These functions typically depend on two or three control parameters. Aiming at achieving an accurate reliability evaluation, the present work uses a single parameter parabolic modulation function, whose equation is

$$\varphi_p(t) = -\left(\frac{t}{t_m}\right)^2 + 2\left(\frac{t}{t_m}\right). \quad (63)$$

for a time interval equal to  $[0, 2t_m]$ . The control of the amplitude modulation variation velocity is done by setting the time  $t_m$ , which is the time at which the modulation reaches its maximum value, to be equal to one, as shown in Figure 2.

Regarding the specific modulation function in (61), the nonstationary structural responses for an SDOF system have been attained by integrating the stationary white noise simulations, which are generated and then modulated.

Structural characteristics differ for damping ratio ( $\xi_0 = 0.01 - 0.05 - 0.10$ ), while the modulation control parameter is expressed as the dimensionless ratio between the time of maximum modulation and the natural structural system period  $T_0$ :

$$\tau_m = \frac{t_m}{T_0}. \quad (64)$$

with  $\tau = t/T_0$ . The values of  $\tau_m$  used in this study are 2.5, 5, 10, and 15.

In addition to evaluating the structural response in the “classical” space state  $\bar{Z}(t)$  (25) by the integration of the generated  $\varphi(t)w(t)$ , the determination of this response is carried out in the “exact” and “approximate” space state  $\bar{Z}(t)$  and  $\bar{\bar{Z}}(t)$  (26), by integrating  $H[\varphi(t)w(t)]$  and  $\varphi(t)H[w(t)]$ , respectively. The Hilbert transforms of the generated stationary white noise and the modulated one are accomplished by MATLAB standard algorithms. Figures 3 and 4 show displacement results and their envelopes evaluated by “exact” and “approximate” ways in the cases of damping ratio equal to 0.01 and 0.10. It is quite evident that there is an acceptable agreement between the two different envelopes, except for a few instants at the beginning of the excitation. This is an effect exactly of the difference of the proposed methodology that has a zero value at the starting time. This consideration, which has originally been suggested by Borino et al. [31], is not exactly correct when referring to the first time derivative of the envelope, as

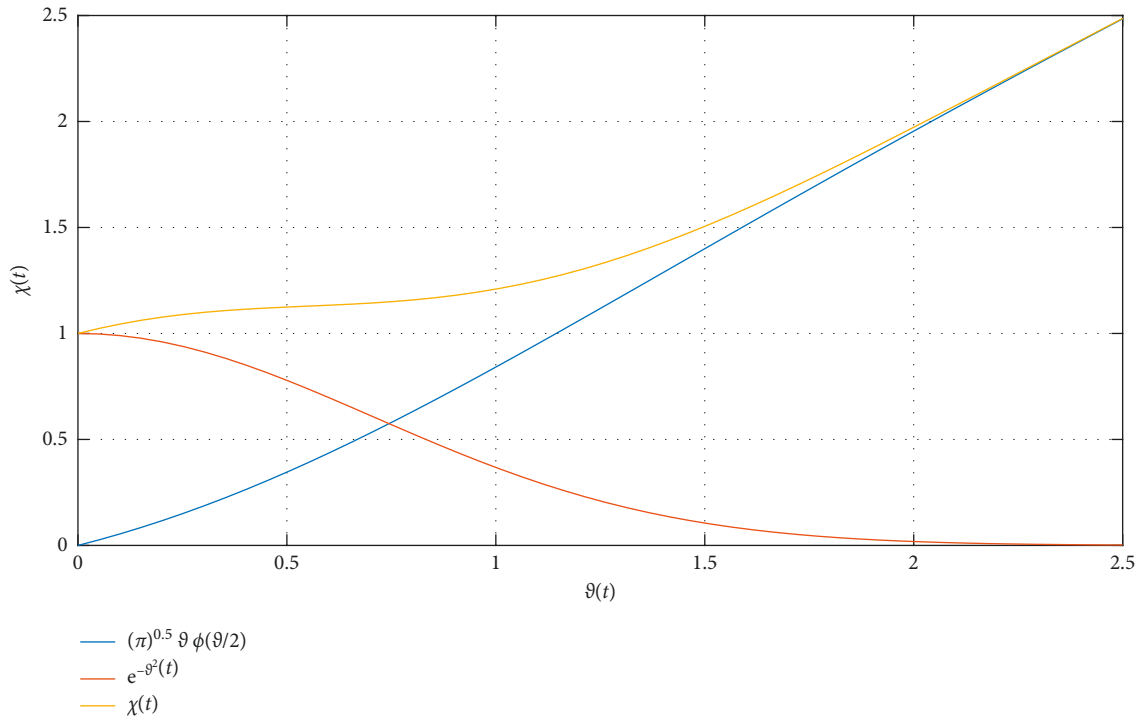


FIGURE 1: Factor  $\chi$ .

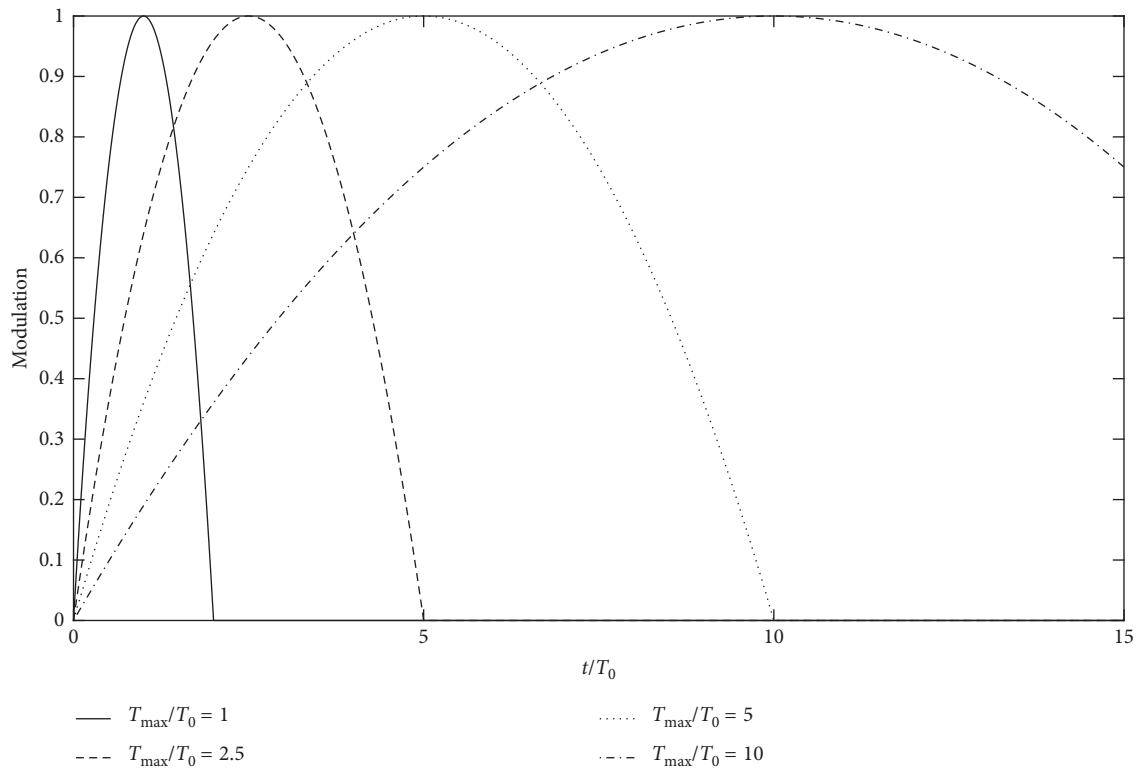


FIGURE 2: Parabolic modulation function for different values of  $t_m$ .

illustrated in Figures 5 and 6. In these two figures, it can be observed that some discrepancies exist between the approximate and exact nonstationary solutions, and they increase as the damping ratio diminishes. In this case, it is evident that at the end of the process, there are discrepancies,

especially for case 2 ( $T_m = 10T_0$ ) and for low damping. In this specific case, there is a separation between the two results at the end when the approximate solution has a strongly oscillatory shape. This result is inferred more clearly in Figures 7–10 (each for different values of damping ratio:



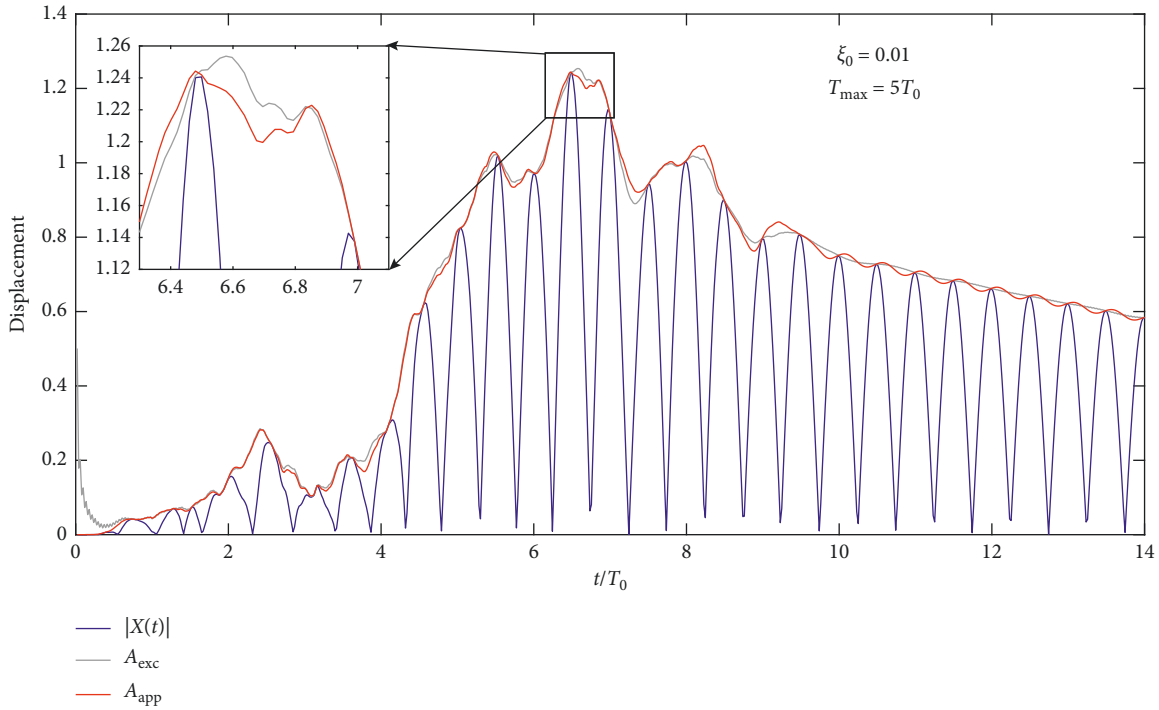


FIGURE 3: Structural displacement  $X(t)$  and its envelope  $A(t)$  evaluated by “exact” and “approximate” ways, for damping ratio equal to 0.01.

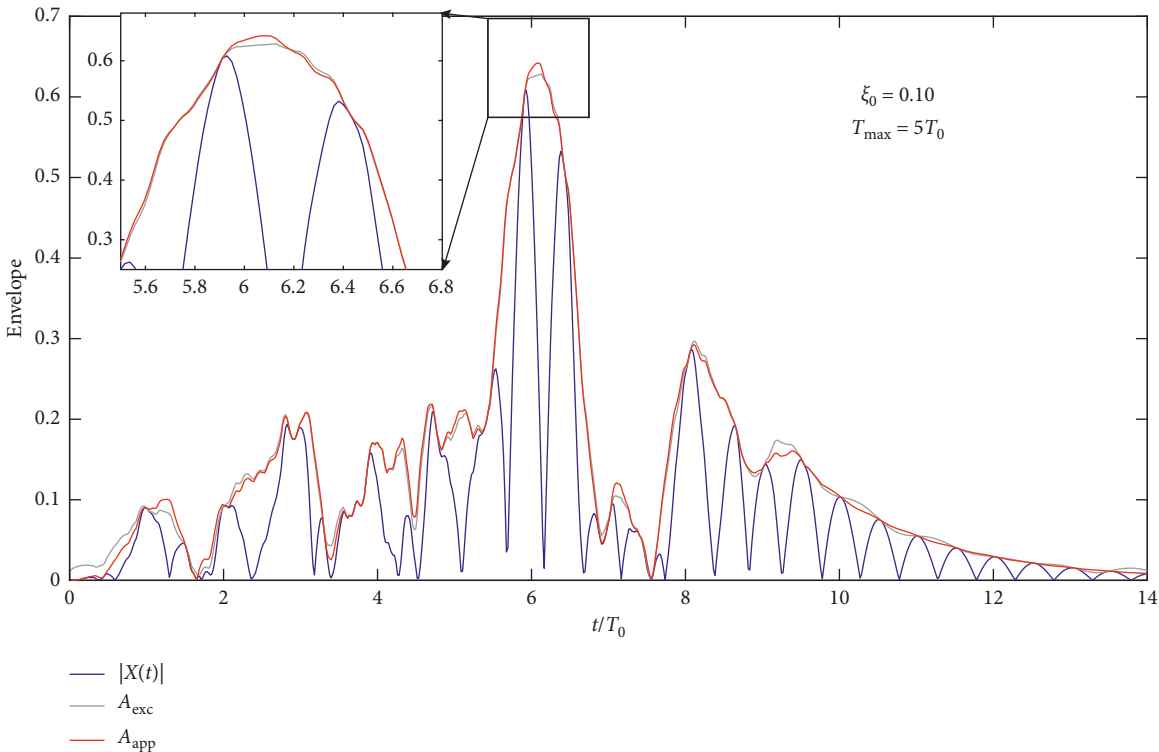


FIGURE 4: Structural displacement  $X(t)$  and its envelope  $A(t)$  evaluated by “exact” and “approximate” ways, for damping ratio equal to 0.1.

0.01, 0.05, and 0.10), where the JPDF of  $A(t)$  and  $\dot{A}(t)$  is shown for different values of  $\tau/\tau_m = 0.5 - 1.0 - 1.5$ .

Furthermore, the good accordance between the approximate and exact numerical evaluations of  $p_A(a; t)$  is obvious in all cases. However, this agreement does not

hold for the PDF of  $\dot{A}(t)$ , where the two cases tend to diverge especially for high values of the envelope velocity. This can be attributed reasonably to numerical reasons. This condition (differences between  $p_{\dot{A}}(\dot{a}; t)$  evaluated by the two different approaches) can lead to a nonperfect

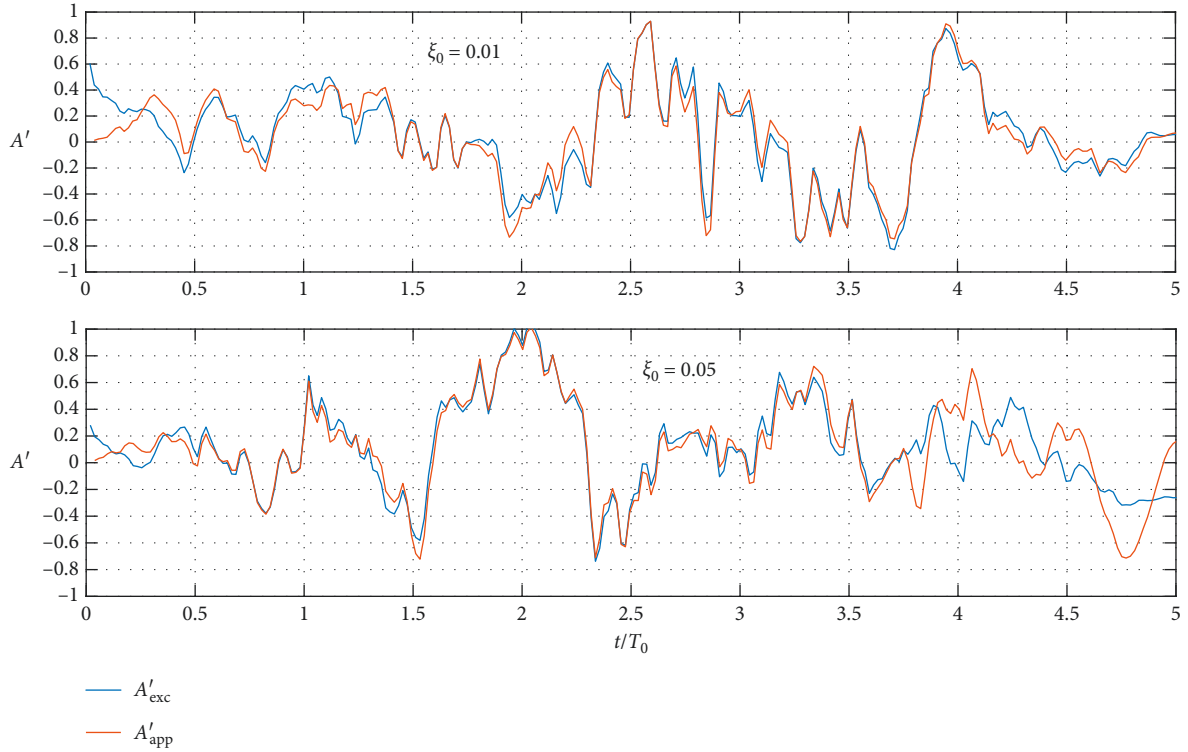


FIGURE 5: Approximate and exact nonstationary solutions for  $T_{\max} = 2.5T_0$ .

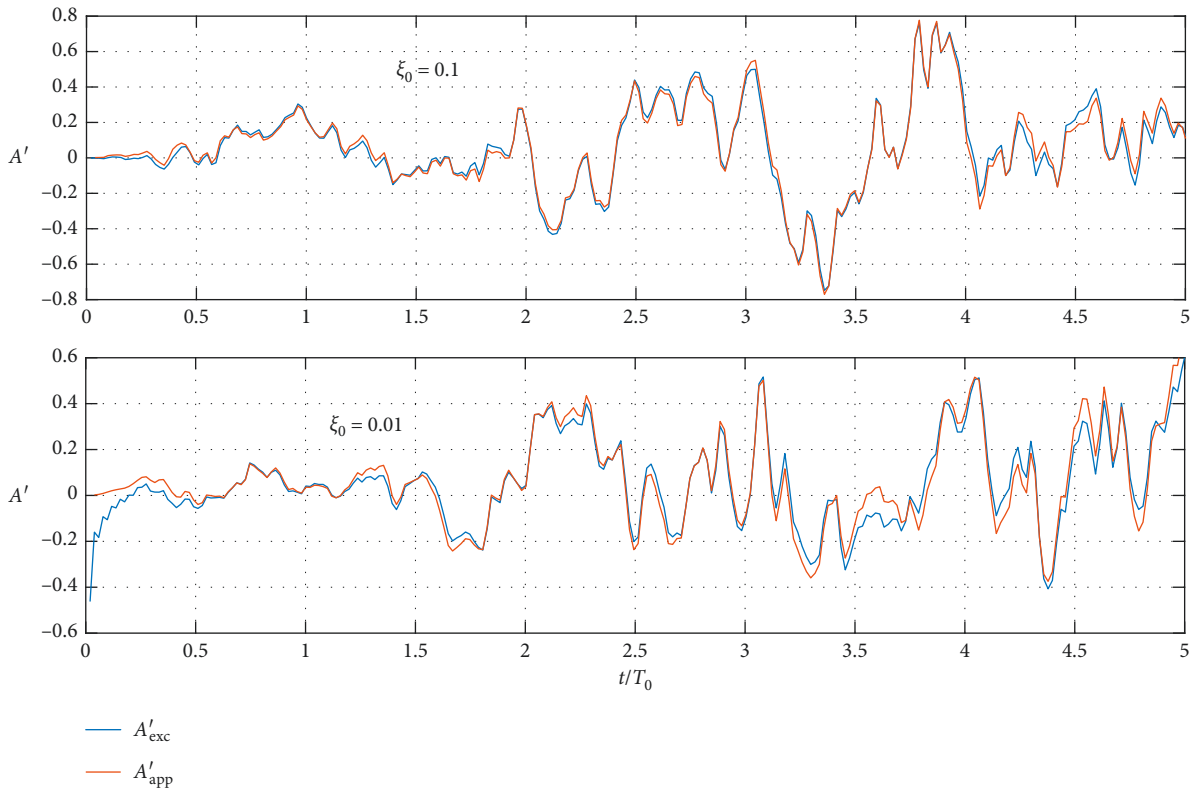


FIGURE 6: Approximate and exact nonstationary solutions for  $T_{\max} = 10T_0$ .

agreement between the approximate and exact approaches for the mean crossing rate of the envelope process, which depends on the JPDF of  $A(t)$  and  $\dot{A}(t)$ .

Additionally, and as stated above, the adoption of the approximate solution in (22b) instead of those in (19) leads to a direct and serious reduction of the analytical and

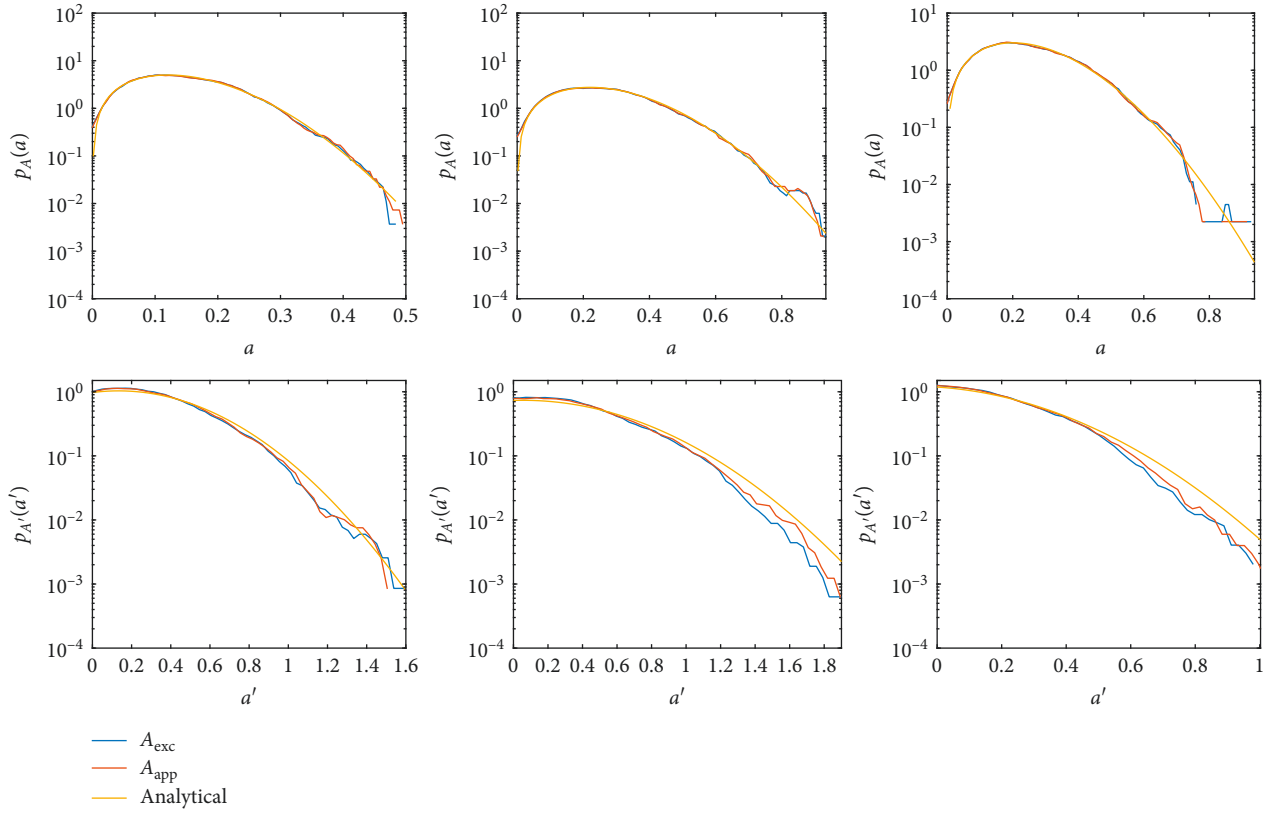


FIGURE 7: JPDF of  $A(t)$  and  $\dot{A}(t)$  for  $\xi=0.1$  and  $T_{max}=2.5T_0$ .

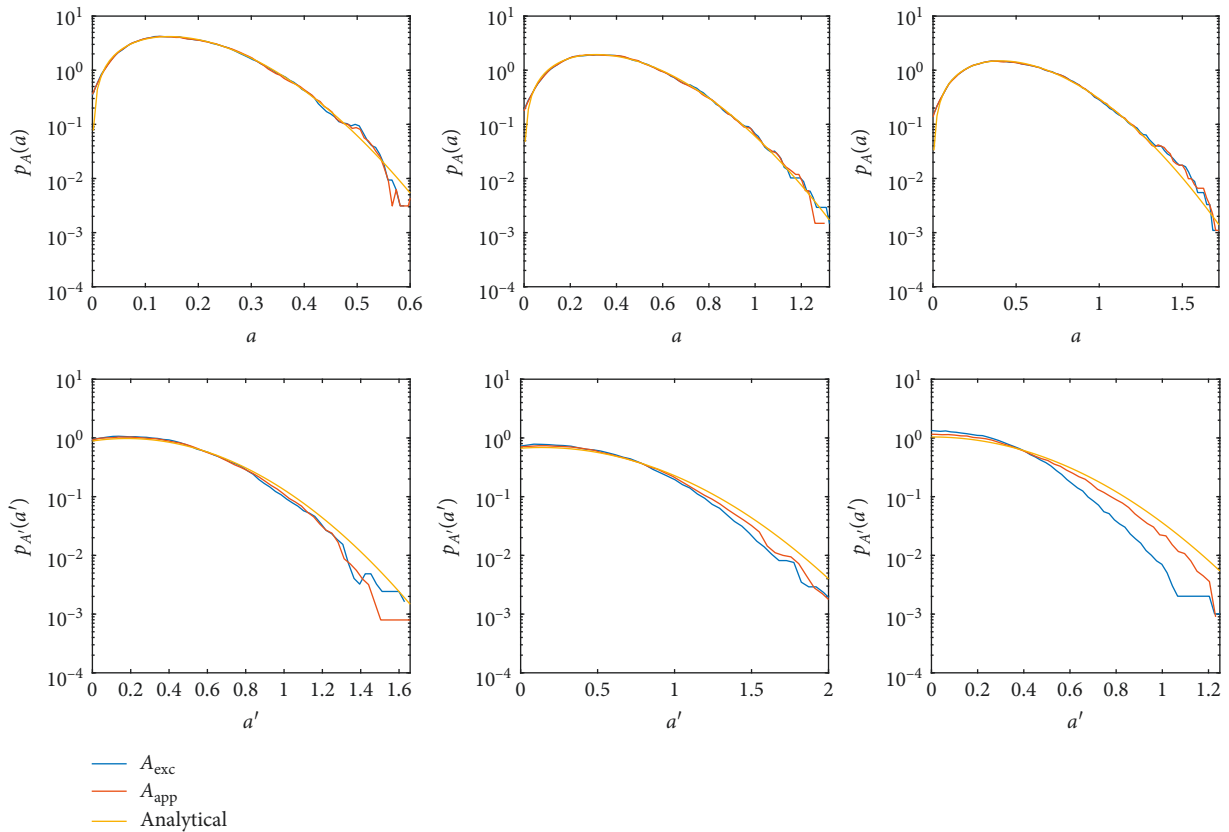


FIGURE 8: JPDF of  $A(t)$  and  $\dot{A}(t)$  for  $\xi=0.01$  and  $T_{max}=2.5T_0$ .

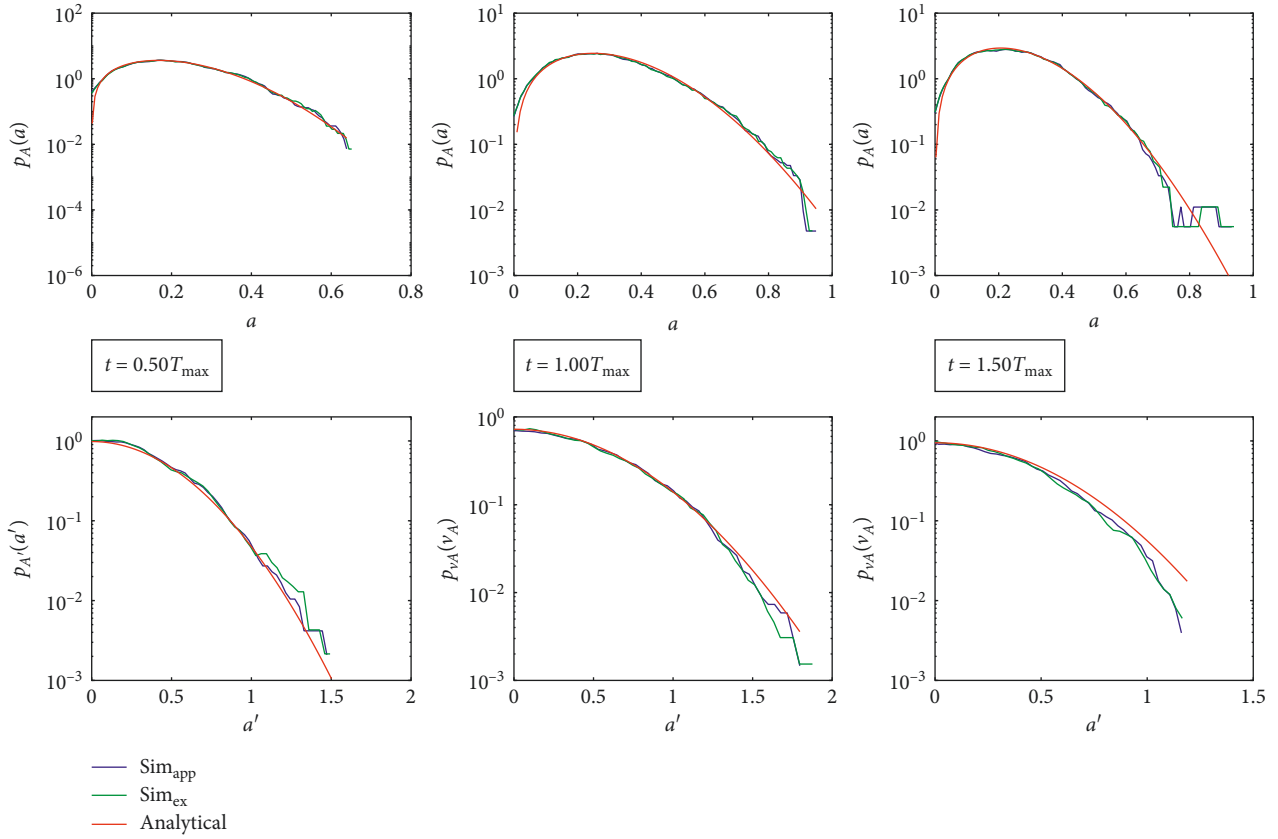


FIGURE 9: JPDF of  $A(t)$  and  $\dot{A}(t)$  for  $\xi=0.1$  and  $T_{\max}=10T_0$ .

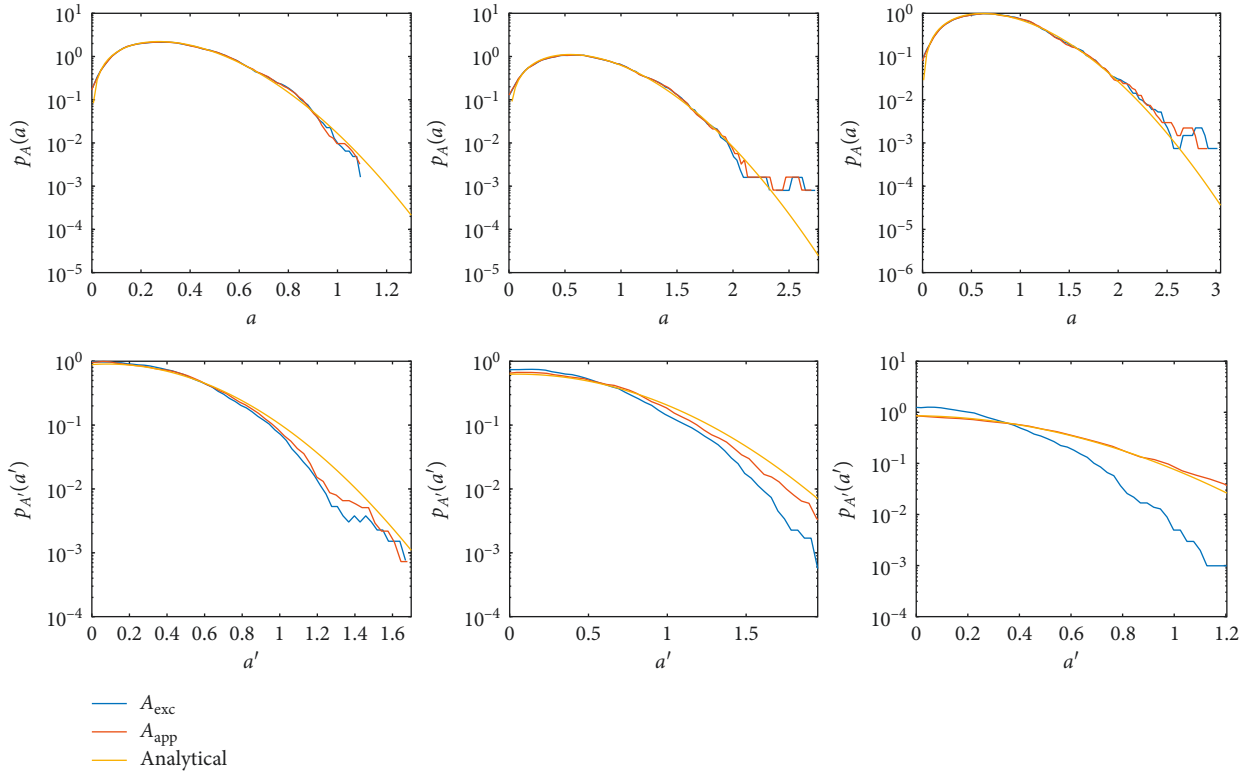


FIGURE 10: JPDF of  $A(t)$  and  $\dot{A}(t)$  for  $\xi=0.01$  and  $T_{\max}=10T_0$ .

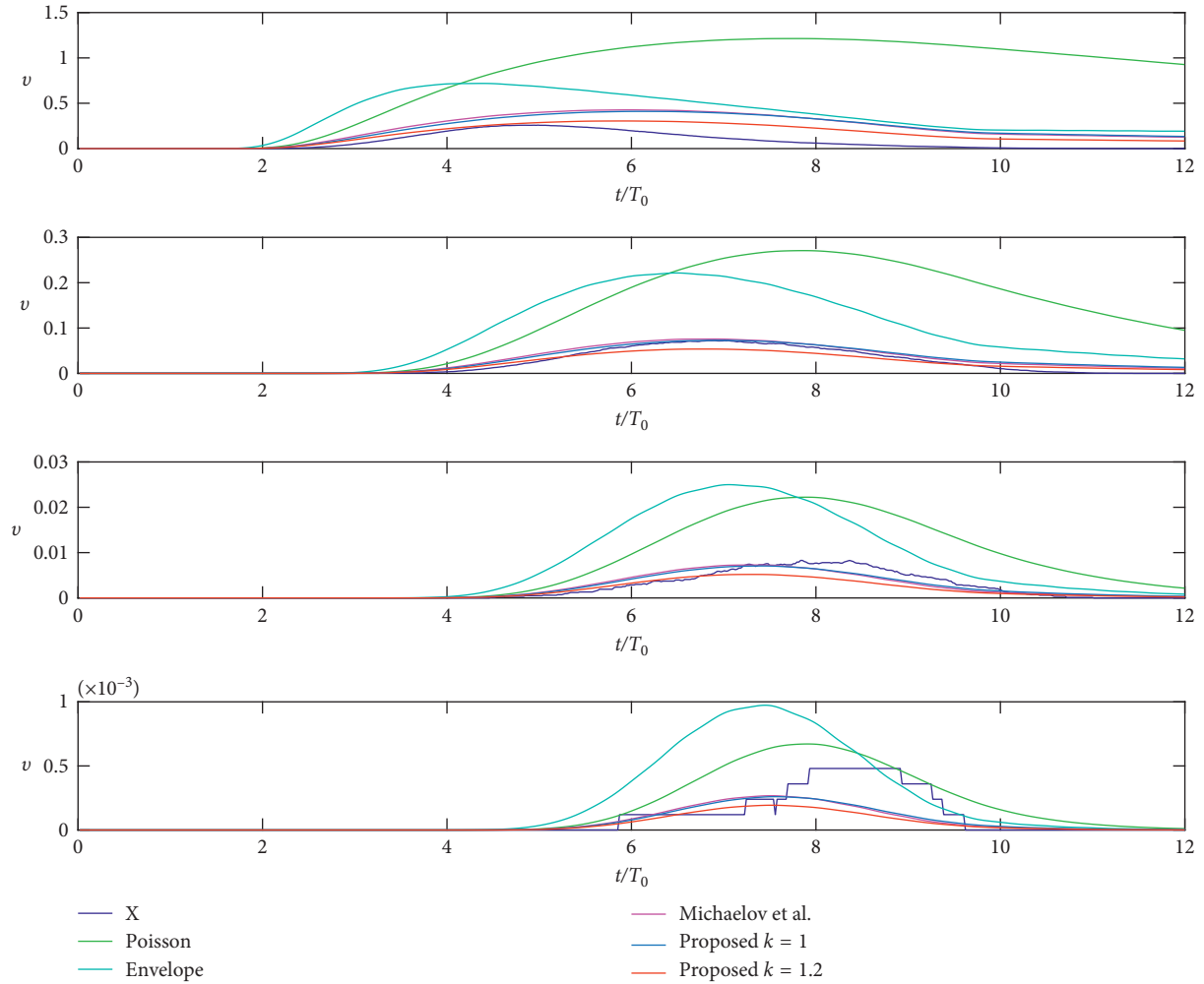


FIGURE 11: Mean threshold crossing rate for  $\xi = 0.01$  and  $T_{\max} = 5T_0$ .

numerical efforts in covariance “preenvelope” evaluations and thus in reliability.

For a better evaluation of differences for threshold crossing problem deriving by the various approaches, in Figures 11 and 12, the numerical results for conditioned crossing rates are shown for different values of the barrier. The barriers are in a range between 1 and 4 in Figure 11 (short phenomena) and 1 and 3 in Figure 12 (long phenomena). In both cases, there is a reasonable agreement with the proposed method, which is anyway quite close to the Michelov et al. [19] formulation, if an exponent  $k$  equal to one is adopted. Using a value equal to 1.2 gives some results just a little bit smaller, but the difference is usually negligible. It should also be noticed that the Michelov and the proposed formulations are in a good agreement with numerical results in the first part of the phenomenon, during the growing phase, but generally underestimate the real threshold crossing rate in the decay part.

Finally, in order to evaluate the empirical correction exponent  $k$  in (62), numerical and analytical results are compared for different values of  $k$  in Figure 13 (it is commonly assumed as 1.2 in stationary case, as originally proposed by Vanmarcke).

## 8. Conclusions

This study dealt with the first crossing reliability evaluation of structures subject to nonstationary Gaussian random input. The solution based on the qualified envelope, originally derived for stationary cases, has been extended to the nonstationary case. At this aim, the modulation of stationary envelope has been adopted by utilizing the so-called “preenvelope” process. A modified reliability formulation has also been proposed, based on the extension of the formulation of the empirical bandwidth factor exponent developed under the stationary hypothesis.

To evaluate the accuracy of the reliability evaluation achieved by using the previous assumption, the mean crossing rate and the envelope JPDF obtained by numerical simulations for a SDOF system subject to Gaussian white noise have been compared to the ones obtained in analytical way. Different damping ratios and velocities in modulation amplitude variation have been taken into account. Results show that only a partial agreement is obtained, based not directly on the final reliability, but also on a suitable measure of accuracy of hazard function. These analyses are carried

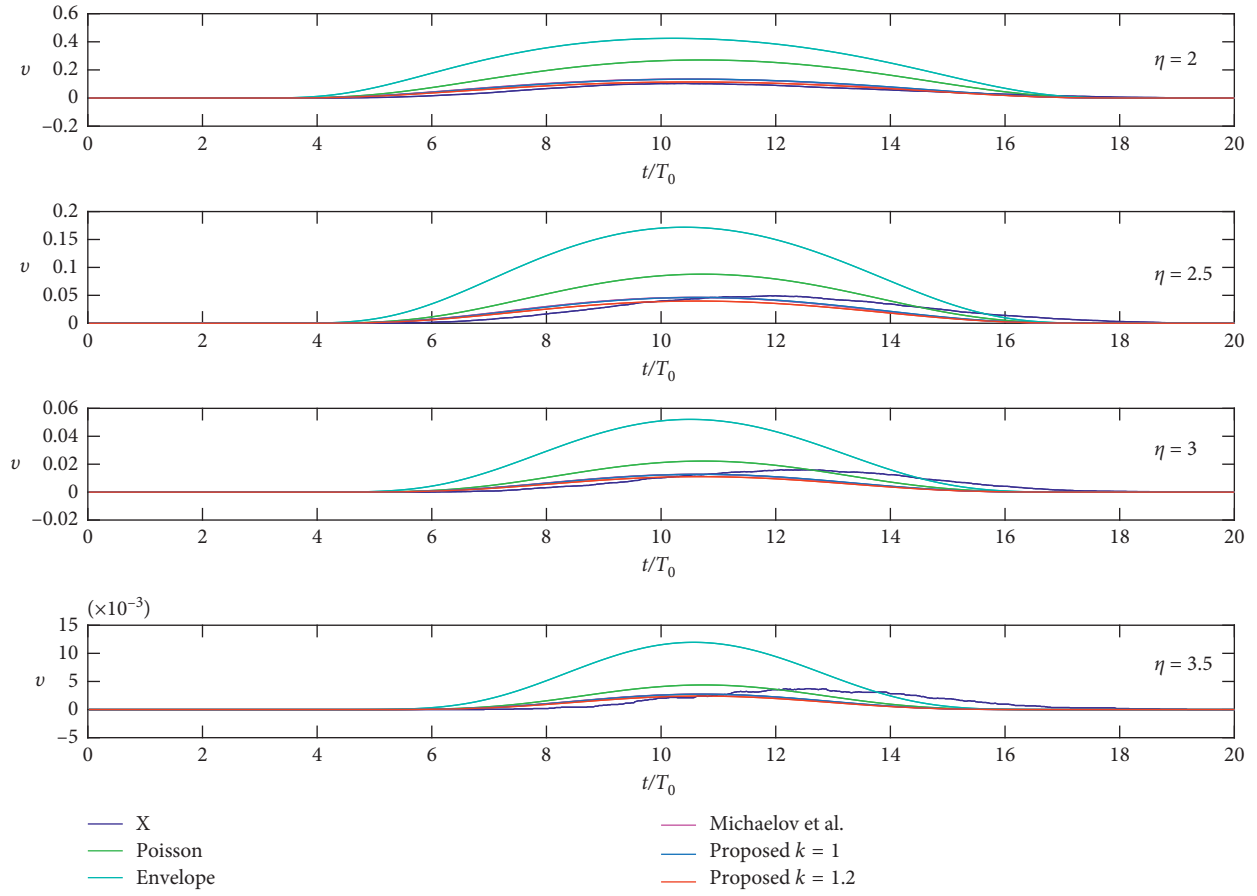
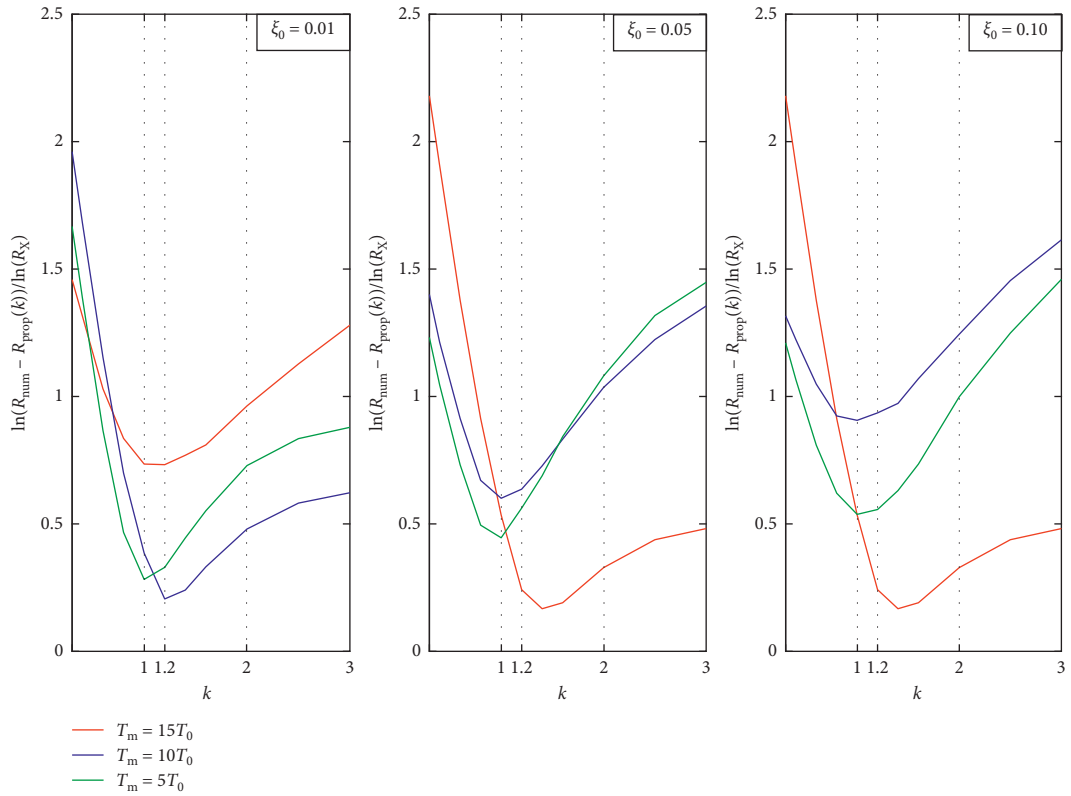
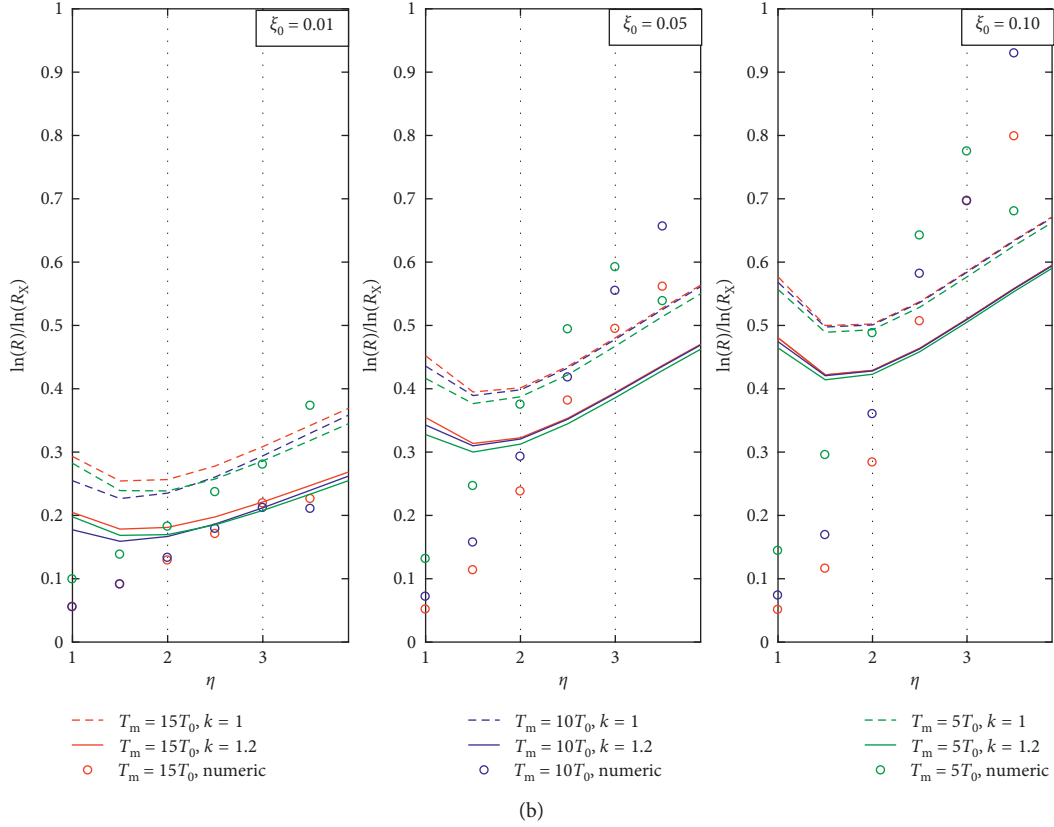


FIGURE 12: Mean threshold crossing rate for  $\xi = 0.1$  and  $T_{\max} = 10T_0$ .



(a)

FIGURE 13: Continued.

FIGURE 13: Numerical and analytical results in function of  $k$ .

out on a wide number of cases taking into account different damping factors and nonstationary levels.

## Appendix

### A

$$\begin{aligned}
 B_a(t) &= \frac{1}{\sqrt{2\pi}\sigma_x^2\sigma_{\dot{x}}\Delta_A}, \\
 C_a(t) &= \left(1 + \frac{\rho_{X\dot{X}}^2(t)}{\Delta_A^2(t)}\right) \frac{1}{\sigma_x^2(t)}, \\
 D_a(t) &= \frac{\rho_{X\dot{X}}(t)}{\sigma_x(t)\sigma_{\dot{x}}(t)\Delta_A^2(t)}, \\
 F_a(t) &= \frac{1}{\sigma_{\dot{x}}^2(t)\Delta_A^2(t)}.
 \end{aligned}
 \tag{A.1}$$

### B. Parabolic Modulation Function

Aiming at performing a parametric analysis on nonstationary input characterization, in the case of load conditions such as

earthquakes [32, 33] a modulation function controlled by a single parameter is used as follows:

$$\varphi(t) = -\left(\frac{t}{t_m}\right)^2 + 2\left(\frac{t}{t_m}\right), \tag{B.1}$$

where  $t_m$  is the maximum intensity time for which  $\varphi(t_m) = 1$ . This function is symmetric, and then its total duration is  $2t_m$ . In this case, it results:

$$\phi(t - \tau) = -\bar{t}^2 + 2\bar{t} - \bar{\tau}^2 - 2\bar{\tau} + 2\bar{t}\bar{\tau}, \tag{B.2}$$

where  $\bar{t} = t/t_m$ ;  $\bar{\tau} = \tau/t_m$ .

Therefore, it is possible to rewrite  $b(t)$  in the following form:

$$\begin{aligned}
 b(t) &= 2S_0 \left( -\left(\frac{t}{t_m}\right)^2 + 2\left(\frac{t}{t_m}\right) \right) \int_0^t \frac{h(\tau)}{\tau} \\
 &\quad \cdot \left[ -\bar{t}^2 + 2\bar{t} - \bar{\tau}^2 - 2\bar{\tau}(1 + \bar{\tau}) \right] d\tau \\
 &= I_1(t) + I_2(t) + I_3(t),
 \end{aligned}
 \tag{B.3}$$

where

$$\begin{aligned}
I_1(t) &= 2S_o \left( -\left(\frac{t}{t_m}\right)^2 + 2\frac{t}{t_m} \right) \int_0^t \frac{e^{-\xi\omega u}}{u} \sin(\bar{\omega}u) du \\
&= \frac{2S_o}{t_m} (-t^2 + 2tt_m) \Psi(t, \bar{\omega}, -\xi\omega), \\
I_2(t) &= \frac{2S_o}{t_m^2} \left( -\left(\frac{t}{t_m}\right)^2 + 2\frac{t}{t_m} \right) \int_0^t \tau e^{-\xi\omega\tau} \sin(\bar{\omega}\tau) d\tau \\
&= \frac{2S_o}{t_m^4} (-t^2 + 2tt_m) \left( 1 - 2\xi^2 + e^{-\xi\omega t} \right. \\
&\quad \cdot \left. \left[ (-\xi\omega t - 2\xi^2 + 1) \cos(\bar{\omega}t) + (\omega t - 2\xi) \sqrt{1 - \xi^2} \sin(\bar{\omega}t) \right] \right), \\
I_3(t) &= \frac{2S_o}{t_m} \left( -\left(\frac{t}{t_m}\right)^2 + 2\frac{t}{t_m} \right) \left( 1 + \frac{t}{t_m} \right) \int_0^t e^{-\xi\omega\tau} \sin(\bar{\omega}\tau) d\tau \\
&= \frac{2S_o}{t_m^4} [(-t^2 + 2tt_m)(t_m - 1)] \\
&\quad \cdot \frac{1}{\omega} \left[ \sqrt{1 - \xi^2} - e^{-\xi\omega t} \left( \xi \sin(\bar{\omega}t) + \sqrt{1 - \xi^2} \cos(\bar{\omega}t) \right) \right].
\end{aligned} \tag{B.4}$$

## Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

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