mathematics

Article

Well-Posedness Results for the Continuum Spectrum Pulse Equation

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Received: 25 September 2019; Accepted: 21 October 2019; Published: 23 October 2019

Abstract: The continuum spectrum pulse equation is a third order nonlocal nonlinear evolutive equation related to the dynamics of the electrical field of linearly polarized continuum spectrum pulses in optical waveguides. In this paper, the well-posedness of the classical solutions to the Cauchy problem associated with this equation is proven.

Keywords: existence; uniqueness; stability; continuum spectrum pulse equation; Cauchy problem

MSC: 35G25; 35K55

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1. Introduction

In this paper, we investigate the well-posedness of the classical solution of the following Cauchy problem:

$$\begin{cases} \partial_t u + 3gu^2\partial_x u - a\partial_x^3 u + q\partial_x(uv) = bP, & t>0,\ x\in\mathbb{R},\\ \partial_x P = u, & t>0,\ x\in\mathbb{R},\\ \alpha\partial_x^2 v + \beta\partial_x v + \gamma v = \kappa u^2, & t>0,\ x\in\mathbb{R},\\ P(t,-\infty)=0, & t>0,\\ u(0,x)=u_0(x), & x\in\mathbb{R}, \end{cases} \tag{1}$$

where $g,\ a,\ q,\ b,\ \alpha,\ \beta,\ \gamma,\ \kappa \in \mathbb{R}$.

On the initial datum, we assume that

$$u_0 \in H^2(\mathbb{R}) \cap L^1(\mathbb{R}), \quad \int_{\mathbb{R}} u_0(x)dx = 0. \tag{2}$$

Following [1–6], on the function

$$P_0(x) = \int_{-\infty}^{x} u_0(y)dy, \tag{3}$$

we assume that

$$\begin{aligned} &\int_{\mathbb{R}} P_0(x)dx = \int_{\mathbb{R}}\left(\int_{-\infty}^{x} u_0(y)dy\right) = 0,\\ &\|P_0\|^2_{L^2(\mathbb{R})} = \int_{\mathbb{R}}\left(\int_{-\infty}^{x} u_0(y)dy\right)^2 dx < \infty. \end{aligned} \tag{4}$$

In addition, we assume that

$$\frac{q\beta}{\kappa} \geq 0, \quad g \neq 0, \quad a \neq 0, \quad \alpha \neq 0. \tag{5}$$

Observe that, since α cannot vanish, we can factorize it and deal with only three constants.

In the physical literature (1) is termed the continuum spectrum pulse equation (see [7–14]). It is used to describe the dynamics of the electrical field u of linearly polarized continuum spectrum pulses in optical waveguides, including fused-silica telecommunication-type or photonic-crystal fibers, as well as hollow capillaries filled with transparent gases or liquids.

The constants a, b, g, q, α, κ, β, γ, in (1), take into account the frequency dispersion of the effective linear refractive index and the nonlinear polarization response, the excitation efficiency of the vibrations, the frequency and the decay time (see [7,8,14]).

Moreover, (1) generalizes the following system:

$$\begin{cases} \partial_t u + q\partial_x(uv) = bP, & t > 0,\ x \in \mathbb{R}, \\ \partial_x P = u, & t > 0,\ x \in \mathbb{R}, \\ \alpha\partial_x^2 v + \beta\partial_x v + \gamma v = \kappa u^2, & t > 0,\ x \in \mathbb{R}, \\ P(t, -\infty) = 0, & t > 0, \\ u(0, x) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{6}$$

whose the well-posedness is studied in [15]. From a mathematical point of view, the presence of the term

$$3gu^2\partial_x u - a\partial_x^3 u$$

in the first equation of (1) makes the analysis of such system more subtle than the one of (6).

Observe that, taking $b = \alpha = \beta = 0$, (1) becomes the modified Korteweg-de Vries equation (see [16–20])

$$\partial_t u + \left(g + \frac{q\kappa}{\gamma}\right)\partial_x u^3 - a\partial_x^3 u = 0. \tag{7}$$

In [8,9,21–24], it is proven that (7) is a non-slowly-varying envelope approximation model that describes the physics of few-cycle-pulse optical solitons. In [6,18], the Cauchy problem for (7) is studied, while, in [16,19], the convergence of the solution of (7) to the unique entropy solution of the following scalar conservation law

$$\partial_t u + \left(g + \frac{q\kappa}{\gamma}\right)\partial_x u^3 = 0 \tag{8}$$

is proven.

On the other hand, taking $\alpha = \beta = 0$ in (1), we have the following equations

$$\begin{cases} \partial_t u + \left(g + \dfrac{q\kappa}{\gamma}\right)\partial_x u^3 = bP, & t > 0,\ x \in \mathbb{R}, \\ \partial_x P = u, & t > 0,\ x \in \mathbb{R}, \end{cases} \tag{9}$$

that were deduced by Kozlov and Sazonov [12] for the description of the nonlinear propagation of optical pulses of a few oscillations duration in dielectric media and by Schäfer and Wayne [25] for the description of the propagation of ultra-short light pulses in silica optical fibers. Moreover, (9) is a non-slowly-varying envelope approximation model that describes the physics of few-cycle-pulse optical solitons (see [22–24,26–28]), a particular Rabelo equation which describes pseudospherical surfaces (see [29–32]), and a model for the descriptions of the short pulse propagation in nonlinear metamaterials characterized by a weak Kerr-type nonlinearity in their dielectric response (see [33]).

Finally, (9) was deduced in [34] in the context of plasma physic and that similar equations describe the dynamics of radiating gases [35,36], in [37–40] in the context of ultrafast pulse propagation in a mode-locked laser cavity in the few-femtosecond pulse regime and in [41] in the context of Maxwell equations.

The Cauchy problem for (9) was studied in [42–44] in the context of energy spaces, [4,5,45,46] in the context of entropy solutions. The homogeneous initial boundary value problem was studied in [47–50]. Nonlocal formulations of (9) were analyzed in [15,51] and the convergence of a finite difference scheme proved in [52].

Observe that, taking $\alpha = \beta = 0$, (1) reads

$$\begin{cases} \partial_t u + \left(g + \frac{q\kappa}{\gamma}\right)\partial_x u^3 - a\partial_x^3 u = bP, & t > 0,\ x \in \mathbb{R}, \\ \partial_x P = u, & t > 0,\ x \in \mathbb{R}. \end{cases} \tag{10}$$

It was derived by Costanzino, Manukian and Jones [53] in the context of the nonlinear Maxwell equations with high-frequency dispersion. Kozlov and Sazonov [12] show that (10) is an more general equation than (9) to describe the nonlinear propagation of optical pulses of a few oscillations duration in dielectric media.

Mathematical properties of (10) are studied in many different contexts, including the local and global well-posedness in energy spaces [43,53] and stability of solitary waves [53,54], while, in [6], the well-posedness of the classical solutions is proven.

Observe that letting $a \to 0$ in (10), we obtain (9). Hence, following [19,55,56], in [5,57], the convergence of the solution of (10) to the unique entropy solution of (9).

The main result of this paper is the following theorem.

Theorem 1. *Assume (2), (3), (4) and (5). Fix $T > 0$, there exists an unique solution (u, v, P) of (1) such that*

$$\begin{aligned} &u \in L^\infty(0,T;H^2(\mathbb{R})),\\ &v \in L^\infty(0,T;H^4(\mathbb{R})),\\ &P \in L^\infty(0,T;H^3(\mathbb{R})). \end{aligned} \tag{11}$$

In particular, we have that

$$\int_{\mathbb{R}} u(t,x)dx = 0, \quad t \geq 0. \tag{12}$$

Moreover, if (u_1, v_1, P_1) and (u_2, v_2, P_2) are two solutions of (1), we have that

$$\begin{aligned} \|u_1(t,\cdot) - u_2(t,\cdot)\|_{L^2(\mathbb{R})} &\leq e^{C(T)t}\|u_{1,0} - u_{2,0}\|_{L^2(\mathbb{R})},\\ \|v_1(t,\cdot) - v_2(t,\cdot)\|_{H^2(\mathbb{R})} &\leq e^{C(T)t}\|u_{1,0} - u_{2,0}\|_{L^2(\mathbb{R})},\\ \|P_1(t,\cdot) - P_2(t,\cdot)\|_{H^1(\mathbb{R})} &\leq e^{C(T)t}\|P_{1,0} - P_{2,0}\|_{H^1(\mathbb{R})}, \end{aligned} \tag{13}$$

where,

$$P_{1,0}(x) = \int_{-\infty}^x u_{1,0}(y)dy, \quad P_{2,0}(x) = \int_{-\infty}^x u_{2,0}(y)dy, \tag{14}$$

for some suitable $C(T) > 0$, and every $0 \leq t \leq T$.

The proof of Theorem 1 is based on the Aubin–Lions Lemma (see [58–60]).

The paper is organized as follows. In Section 2, we prove several a priori estimates on a vanishing viscosity approximation of (1). Those play a key role in the proof of our main result, that is given in

Section 3. Appendix A is an appendix, where we prove the posedness of the classical solutions of (1), under the assumption

$$u_0 \in L^1(\mathbb{R}) \cap H^3(\mathbb{R}). \tag{15}$$

2. Vanishing Viscosity Approximation

Our existence argument is based on passing to the limit in a vanishing viscosity approximation of (1).

Fix a small number $0 < \varepsilon < 1$ and let $u_\varepsilon = u_\varepsilon(t,x)$ be the unique classical solution of the following mixed problem [19,61,62]:

$$\begin{cases} \partial_t u_\varepsilon + 3gu_\varepsilon^2 \partial_x u_\varepsilon - a\partial_x^3 u_\varepsilon + q\partial_x(v_\varepsilon u_\varepsilon) = bP_\varepsilon - \varepsilon\partial_x^4 u_\varepsilon, & t>0,\ x\in\mathbb{R},\\ \partial_x P_\varepsilon = u_\varepsilon, & t>0,\ x\in\mathbb{R},\\ \alpha\partial_x^2 v_\varepsilon + \beta\partial_x v_\varepsilon + \gamma v_\varepsilon = \kappa u_\varepsilon^2, & t>0,\ x\in\mathbb{R},\\ P_\varepsilon(t,-\infty) = 0, & t>0,\\ u_\varepsilon(0,x) = u_{\varepsilon,0}(x), & x\in\mathbb{R}, \end{cases} \tag{16}$$

where $u_{\varepsilon,0}$ is a C^∞ approximation of u_0 such that

$$\begin{aligned} &\|u_{\varepsilon,0}\|_{H^2(\mathbb{R})} \le \|u_0\|_{H^2(\mathbb{R})}, \quad \int_{\mathbb{R}} u_{\varepsilon,0}dx = 0,\\ &\|P_{\varepsilon,0}\|_{L^2(\mathbb{R})} \le \|P_0\|_{L^2(\mathbb{R})}, \quad \int_{\mathbb{R}} P_{\varepsilon,0}dx = 0. \end{aligned} \tag{17}$$

Let us prove some a priori estimates on u_ε, P_ε and v_ε. We denote with C_0 the constants which depend only on the initial data, and with $C(T)$, the constants which depend also on T.

Lemma 1. *For each $t \ge 0$,*

$$P_\varepsilon(t,\infty) = 0. \tag{18}$$

In particular, we have that

$$\int_{\mathbb{R}} u_\varepsilon(t,x)dx = 0. \tag{19}$$

Proof. We begin by proving (18). Thanks to the regularity of u_ε and the first equation of (16), we have that

$$\lim_{x\to\infty}\left(\partial_t u_\varepsilon + 3gu_\varepsilon^2\partial_x u_\varepsilon - a\partial_x^3 u_\varepsilon + q\partial_x(v_\varepsilon u_\varepsilon) - \varepsilon\partial_x^5 u_\varepsilon\right) = bP_\varepsilon(t,\infty) = 0,$$

which gives (18).

Finally, we prove (19). Integrating the second equation of (16) on $(-\infty,x)$, by (16), we have that

$$P_\varepsilon(t,x) = \int_{-\infty}^{x} u_\varepsilon(t,y)dy. \tag{20}$$

Equation (19) follows from (18) and (20). □

Arguing as in ([15], Lemma 2.2), we have the following result.

Lemma 2. *Assume (5). We have that*

$$\int_{\mathbb{R}} u_\varepsilon^2 \partial_x v_\varepsilon dx = \begin{cases} \dfrac{\beta}{\kappa}\|\partial_x v_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}^2, & \text{if } \beta \neq 0,\\ 0, & \text{if } \beta = 0. \end{cases} \tag{21}$$

Lemma 3. *Assume* (5)*. If $\beta \neq 0$, then for each $t \geq 0$, there exists a constant $C_0 > 0$, independent on ε, such that*

$$\begin{aligned} \|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} &+ \frac{q\beta}{\kappa}\int_0^t \|\partial_x v_\varepsilon(s,\cdot)\|^2_{L^2(\mathbb{R})} ds \\ &+ 2\varepsilon\int_0^t \|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} ds \leq C_0. \end{aligned} \tag{22}$$

If $\beta = 0$, then for each $t \geq 0$,

$$\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\varepsilon\int_0^t \|\partial_x u_\varepsilon(s,\cdot)\|^2_{L^2(\mathbb{R})} ds \leq \|u_0\|^2_{L^2(\mathbb{R})}. \tag{23}$$

In particular, we have

$$\begin{aligned} \|\partial_x v_\varepsilon(t,\cdot)\|_{L^\infty(\mathbb{R})}, \|\partial_x v_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})} \leq & C_0, \\ \|v_\varepsilon(t,\cdot)\|_{L^\infty(\mathbb{R})}, \|v_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})} \leq & C_0. \end{aligned} \tag{24}$$

Moreover, fixed $T > 0$, there exists a constant $C(T) > 0$, independent on ε, such that

$$\varepsilon\int_0^t \|\partial_x u_\varepsilon(s,\cdot)\|^2_{L^2(\mathbb{R})} ds \leq C(T). \tag{25}$$

Proof. Multiplying by $2u_\varepsilon$ the first equation of (16), an integration on $\mathbb{R}$ gives

$$\begin{aligned} \frac{d}{dt}\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} =& 2\int_{\mathbb{R}} u_\varepsilon \partial_t u_\varepsilon dx \\ =& -6g\int_{\mathbb{R}} u_\varepsilon^3 \partial_x u_\varepsilon dx - 2q\int_{\mathbb{R}} u_\varepsilon \partial_x(u_\varepsilon v_\varepsilon) dx + 2b\int_{\mathbb{R}} P_\varepsilon u_\varepsilon dx \\ &+ 2a\int_{\mathbb{R}} u_\varepsilon \partial_x^3 u_\varepsilon dx - 2\varepsilon\int_{\mathbb{R}} u_\varepsilon \partial_x^4 u_\varepsilon dx \\ =& -2q\int_{\mathbb{R}} u_\varepsilon \partial_x(u_\varepsilon v_\varepsilon) dx - 2a\int_{\mathbb{R}} \partial_x u_\varepsilon \partial_x^2 u_\varepsilon dx \\ &+ 2b\int_{\mathbb{R}} P_\varepsilon u_\varepsilon dx + 2\varepsilon\int_{\mathbb{R}} \partial_x u_\varepsilon \partial_x^3 u_\varepsilon dx \\ =& -2q\int_{\mathbb{R}} u_\varepsilon \partial_x(u_\varepsilon v_\varepsilon) dx - 2a\int_{\mathbb{R}} \partial_x u_\varepsilon \partial_x^2 u_\varepsilon dx \\ &+ 2b\int_{\mathbb{R}} P_\varepsilon u_\varepsilon dx - 2\varepsilon\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}. \end{aligned}$$

Therefore,

$$\frac{d}{dt}\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\varepsilon\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} = 2b\int_{\mathbb{R}} P_\varepsilon u_\varepsilon dx - 2q\int_{\mathbb{R}} u_\varepsilon \partial_x(u_\varepsilon v_\varepsilon) dx.$$

Arguing as in ([15], Lemma 2.2), we have (22), (23) and (24).
Finally, arguing as in ([6], Lemma 2.3), we have (25). □

Lemma 4. *Assume* (5)*. Fix $T > 0$. There exists a constant $C_0 > 0$, independent on ε, such that*

$$\left\|\partial_x^2 v_\varepsilon\right\|_{L^\infty((0,T)\times\mathbb{R})} \leq C_0\left(1 + \|u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})}\right). \tag{26}$$

Proof. Let $0 \leq t \leq T$. Thanks to the third equation of (16), we have that

$$\alpha\partial_x^2 v_\varepsilon = \kappa u_\varepsilon^2 - \beta\partial_x v_\varepsilon - \gamma v_\varepsilon. \tag{27}$$

Therefore, by (24),

$$\begin{aligned}|\alpha||\partial_x^2 v_\varepsilon| =&|\kappa u_\varepsilon^2 - \beta\partial_x v_\varepsilon - \gamma v_\varepsilon| \leq |\kappa|u_\varepsilon^2 + |\beta||\partial_x v_\varepsilon| + |\gamma||v_\varepsilon|\\ \leq&|\kappa|\,\|u_\varepsilon\|^2_{L^2((0,T)\times\mathbb{R})} + |\beta|\,\|\partial_x v_\varepsilon(t,\cdot)\|_{L^\infty(\mathbb{R})} + |\gamma|\,\|v_\varepsilon(t,\cdot)\|_{L^\infty(\mathbb{R})}\\ \leq&|\kappa|\,\|u_\varepsilon\|^2_{L^2((0,T)\times\mathbb{R})} + C_0 \leq C_0\left(1+\|u_\varepsilon\|^2_{L^2((0,T)\times\mathbb{R})}\right),\end{aligned}$$

which gives (26). □

Arguing as in ([6], Lemma 2.2), we have the following result.

Lemma 5. *For each $t \geq 0$, we have that*

$$\int_0^{-\infty} P_\varepsilon(t,x)dx = A_\varepsilon(t), \tag{28}$$

$$\int_0^{\infty} P_\varepsilon(t,x)dx = A_\varepsilon(t), \tag{29}$$

where

$$A_\varepsilon(t) = -\frac{1}{b}\partial_t P_\varepsilon(t,0) - \frac{g}{b}u_\varepsilon^3(t,0) - \frac{a}{b}\partial_x^2 u_\varepsilon(t,0) - \frac{q}{b}u_\varepsilon(t,0)v_\varepsilon(t,0) + \frac{\varepsilon}{b}\partial_x u_\varepsilon(t,0).$$

In particular, we have

$$\int_{\mathbb{R}} P_\varepsilon(t,x)dx = 0. \tag{30}$$

Lemma 6. *Assume (5). Fix $T > 0$. There exists a constant $C(T) > 0$, independent on ε, such that*

$$\begin{aligned}\|P_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} &+2\varepsilon e^{C_0 t}\int_0^t e^{-C_0 s}\,\|\partial_x u_\varepsilon(s,\cdot)\|^2_{L^2(\mathbb{R})}\,ds\\ &\leq C(T)\left(1+\|u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})}\right).\end{aligned} \tag{31}$$

for every $0 \leq t \leq T$. In particular, we have that

$$\|P_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})} \leq \sqrt[4]{C(T)\left(1+\|u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})}\right)}. \tag{32}$$

Proof. Let $0 \leq t \leq T$. We begin by observing that, by (28), we can consider the following function:

$$F_\varepsilon(t,x) = \int_{-\infty}^x P_\varepsilon(t,y)dy. \tag{33}$$

Integrating the second equation of (16) on $(-\infty, x)$, we have

$$P_\varepsilon(t,x) = \int_{-\infty}^x u_\varepsilon(t,y)dy. \tag{34}$$

Differentiating (34) with respect to t, we get

$$\partial_t P_\varepsilon(t,x) = \frac{d}{dt}\int_{-\infty}^x u_\varepsilon(t,y)dy = \int_{-\infty}^x \partial_t u_\varepsilon(t,x)dx. \tag{35}$$

Equation (33), (35) and an integration on $(-\infty, x)$ of the first equation of (16) give

$$\partial_t P_\varepsilon = bF_\varepsilon - \varepsilon\partial_x^3 u_\varepsilon - g u_\varepsilon^3 + a\partial_x^2 u_\varepsilon - q v_\varepsilon u_\varepsilon. \tag{36}$$

Multiplying (36) by $-2P_\varepsilon$, an integration on $\mathbb{R}$ gives

$$\begin{aligned}\frac{d}{dt}\|P_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} =&2b\int_{\mathbb{R}} F_\varepsilon P_\varepsilon dx - 2\varepsilon\int_{\mathbb{R}} P_\varepsilon\partial_x^3 u_\varepsilon dx - 2g\int_{\mathbb{R}} P_\varepsilon u_\varepsilon^3 dx\\ &+2a\int_{\mathbb{R}} P_\varepsilon\partial_x^2 u_\varepsilon dx - 2q\int_{\mathbb{R}} P_\varepsilon v_\varepsilon u_\varepsilon dx.\end{aligned} \tag{37}$$

Observe that, by (18), (30), (33) and the second equation of (16),

$$\begin{aligned}2b\int_{\mathbb{R}} F_\varepsilon P_\varepsilon dx =&2b\int_{\mathbb{R}} F_\varepsilon \partial_x F_\varepsilon dx = bF_\varepsilon^2(t,\infty) = b\left(\int_{\mathbb{R}} P_\varepsilon(t,x)dx\right)^2 dx = 0,\\ -2\varepsilon\int_{\mathbb{R}} P_\varepsilon\partial_x^3 u_\varepsilon dx =&2\varepsilon\int_{\mathbb{R}}\partial_x P_\varepsilon\partial_x^2 u_\varepsilon dx = 2\varepsilon\int_{\mathbb{R}} u_\varepsilon\partial_x^2 u_\varepsilon dx = -2\varepsilon\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})},\\ 2a\int_{\mathbb{R}} P_\varepsilon\partial_x^2 u_\varepsilon dx =&-2a\int_{\mathbb{R}}\partial_x P_\varepsilon\partial_x u_\varepsilon dx = -2a\int_{\mathbb{R}} u_\varepsilon\partial_x u_\varepsilon = 0,\\ -2q\int_{\mathbb{R}} P_\varepsilon v_\varepsilon u_\varepsilon dx =&-2q\int_{\mathbb{R}} P_\varepsilon v_\varepsilon\partial_x P_\varepsilon dx = 2q\int_{\mathbb{R}}\partial_x v_\varepsilon P_\varepsilon^2 dx.\end{aligned}$$

Consequently, by (24) and (37),

$$\begin{aligned}\frac{d}{dt}&\|P_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\varepsilon\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\\ &=2q\int_{\mathbb{R}}\partial_x v_\varepsilon P_\varepsilon^2 dx - 2g\int_{\mathbb{R}} P_\varepsilon u_\varepsilon^3 dx\\ &\le 2|q|\int_{\mathbb{R}}|\partial_x v_\varepsilon|P_\varepsilon^2 dx + 2|g|\int_{\mathbb{R}}|P_\varepsilon||u_\varepsilon|^3 dx\\ &\le 2|q|\|\partial_x v_\varepsilon(t,\cdot)\|_{L^\infty(\mathbb{R})}\|P_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2|g|\int_{\mathbb{R}}|P_\varepsilon||u_\varepsilon|^3 dx\\ &\le C_0\|P_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2|g|\int_{\mathbb{R}}|P_\varepsilon||u_\varepsilon|^3 dx.\end{aligned} \tag{38}$$

Due to Lemma 3 and the Young inequality,

$$\begin{aligned}2|g|\int_{\mathbb{R}}|P_\varepsilon||u_\varepsilon|^3 dx =& \int_{\mathbb{R}}|2P_\varepsilon u_\varepsilon||g u_\varepsilon^2| dx \le 2\int_{\mathbb{R}} P_\varepsilon^2 u_\varepsilon^2 dx + \frac{g^2}{2}\int_{\mathbb{R}} u_\varepsilon^4 dx\\ \le& 2\|P_\varepsilon(t,\cdot)\|^2_{L^\infty(\mathbb{R})}\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\\ &+\frac{g^2}{2}\|u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})}\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\\ \le& C_0\|P_\varepsilon(t,\cdot)\|^4_{L^\infty(\mathbb{R})} + C_0\|u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})}\\ \le& \|P_\varepsilon(t,\cdot)\|^4_{L^\infty(\mathbb{R})} + C_0 + C_0\|u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})}.\end{aligned}$$

It follows from (38) that

$$\begin{aligned}\frac{d}{dt}&\|P_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\varepsilon\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\\ &\le C_0\|P_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \|P_\varepsilon(t,\cdot)\|^4_{L^\infty(\mathbb{R})} + C_0\left(1+\|u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})}\right).\end{aligned} \tag{39}$$

Thanks to Lemma 3 and the Hölder inequality,

$$\begin{aligned}P_\varepsilon^2(t,x) =& 2\int_{-\infty}^x P_\varepsilon u_\varepsilon dy \le 2\int_{\mathbb{R}}|P_\varepsilon||u_\varepsilon| dx\\ &\le \|P_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}\|u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})} \le C_0\|P_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}.\end{aligned}$$

Hence,

$$\|P_\varepsilon(t,\cdot)\|^4_{L^\infty(\mathbb{R})} \le C_0 \|P_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}. \tag{40}$$

It follows from (39) and (40) that

$$\begin{aligned}&\frac{d}{dt}\|P_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\varepsilon\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\\ &\qquad\le C_0\|P_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + C_0\left(1+\|u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})}\right).\end{aligned}$$

The Gronwall Lemma and (17) give

$$\begin{aligned}&\|P_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\varepsilon e^{C_0t}\int_0^t e^{-C_0s}\|u_\varepsilon(s,\cdot)\|^2_{L^2(\mathbb{R})}ds\\ &\qquad\le C_0e^{C_0t} + C_0\left(1+\|u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})}\right)e^{C_0t}\int_0^t e^{-C_0s}ds\\ &\qquad\le C(T)\left(1+\|u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})}\right),\end{aligned}$$

which gives (31).

Finally, (32) follows from (31) and (40). □

Following ([6], Lemma 2.5), we prove the following result.

Lemma 7. *Assume (5). Fix $T>0$. There exists a constant $C(T)>0$, independent on ε, such that*

$$\|u_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})} \le C(T). \tag{41}$$

In particular, we have

$$\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\varepsilon e^{C_0t}\int_0^t e^{-C_0s}\left\|\partial_x^3 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds \le C(T), \tag{42}$$

for every $0\le t\le T$. Moreover,

$$\begin{aligned}\left\|\partial_x^2 v_\varepsilon\right\|_{L^\infty((0,T)\times\mathbb{R})} &\le C(T),\\ \|P_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})} &\le C(T),\\ \|P_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})} &\le C(T),\end{aligned} \tag{43}$$

for every $0\le t\le T$.

Proof. Let $0\le t\le T$. Multiplying the first equation of (1) by $-2\partial_x^2 u_\varepsilon + \frac{2g}{a}u_\varepsilon^3$, we have that

$$\begin{aligned}&\left(-2\partial_x^2u_\varepsilon + \frac{2g}{a}u_\varepsilon^3\right)\partial_t u_\varepsilon + 3g\left(-2\partial_x^2u_\varepsilon + \frac{2g}{a}u_\varepsilon^3\right)u_\varepsilon^2\partial_x u_\varepsilon\\ &\qquad - a\left(-2\partial_x^2u_\varepsilon + \frac{2g}{a}u_\varepsilon^3\right)\partial_x^3 u_\varepsilon + q\left(-2\partial_x^2u_\varepsilon + \frac{2g}{a}u_\varepsilon^3\right)\partial_x(v_\varepsilon u_\varepsilon)\\ &\qquad = b\left(-2\partial_x^2u_\varepsilon + \frac{2g}{a}u_\varepsilon^3\right)P_\varepsilon - \varepsilon\left(-2\partial_x^2u_\varepsilon + \frac{2g}{a}u_\varepsilon^3\right)\partial_x^4 u_\varepsilon.\end{aligned} \tag{44}$$

Observe that, by (18) and the second equation of (16),

$$-2b\int_{\mathbb{R}} P_\varepsilon\partial_x^2 u_\varepsilon dx = 2b\int_{\mathbb{R}}\partial_x P_\varepsilon\partial_x u_\varepsilon dx = 2b\int_{\mathbb{R}} u_\varepsilon\partial_x u_\varepsilon dx = 0. \tag{45}$$

Moreover,

$$
\begin{aligned}
\int_{\mathbb{R}} \left(-2\partial_x^2 u_\varepsilon + \frac{2g}{a}u_\varepsilon^3\right)\partial_t u_\varepsilon dx =& \frac{d}{dt}\left(\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{g}{2a}\int_{\mathbb{R}} u_\varepsilon^4 dx\right),\\
3g\int_{\mathbb{R}} \left(-2\partial_x^2 u_\varepsilon + \frac{2g}{a}u_\varepsilon^3\right)u_\varepsilon^2\partial_x u_\varepsilon dx =& -6g\int_{\mathbb{R}} u_\varepsilon^2\partial_x u_\varepsilon\partial_x^2 u_\varepsilon dx,\\
-a\int_{\mathbb{R}} \left(-2\partial_x^2 u_\varepsilon + \frac{2g}{a}u_\varepsilon^3\right)\partial_x^3 u_\varepsilon dx =& 6g\int_{\mathbb{R}} u_\varepsilon^2\partial_x u_\varepsilon\partial_x^2 u_\varepsilon dx,\\
-\varepsilon\int_{\mathbb{R}} \left(-2\partial_x^2 u_\varepsilon + \frac{2g}{a}u_\varepsilon^3\right)\partial_x^4 u_\varepsilon dx =& -2\varepsilon\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&+\frac{6g\varepsilon}{a}\int_{\mathbb{R}} u_\varepsilon^2\partial_x u_\varepsilon\partial_x^3 u_\varepsilon dx.
\end{aligned} \tag{46}
$$

Defined

$$
G(t) := \|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{g}{2a}\int_{\mathbb{R}} u_\varepsilon^4 dx, \tag{47}
$$

it follows from (45), (46) and an integration on $\mathbb{R}$ of (44) that

$$
\begin{aligned}
\frac{dG(t)}{dt} + 2\varepsilon\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} =& \frac{2bg}{a}\int_{\mathbb{R}} P_\varepsilon u_\varepsilon^3 dx + \frac{6g\varepsilon}{a}\int_{\mathbb{R}} u_\varepsilon^2\partial_x u_\varepsilon\partial_x^3 u_\varepsilon dx\\
&+2q\int_{\mathbb{R}} \partial_x(v_\varepsilon u_\varepsilon)\partial_x^2 u_\varepsilon dx - 2qga\int_{\mathbb{R}} \partial_x(v_\varepsilon u_\varepsilon)u_\varepsilon^3 dx.
\end{aligned} \tag{48}
$$

Observe that

$$
\begin{aligned}
2q\int_{\mathbb{R}} \partial_x(v_\varepsilon u_\varepsilon)\partial_x^2 u_\varepsilon dx =& 2q\int_{\mathbb{R}} u_\varepsilon\partial_x v_\varepsilon\partial_x^2 u_\varepsilon dx + 2q\int_{\mathbb{R}} v_\varepsilon\partial_x u_\varepsilon\partial_x^2 u_\varepsilon dx\\
=& -2q\int_{\mathbb{R}} \partial_x^2 v_\varepsilon u_\varepsilon\partial_x u_\varepsilon dx - 3q\int_{\mathbb{R}} \partial_x v_\varepsilon(\partial_x u_\varepsilon)^2 dx,\\
-2qga\int_{\mathbb{R}} \partial_x(v_\varepsilon u_\varepsilon)u_\varepsilon^3 dx =& -2qga\int_{\mathbb{R}} \partial_x v_\varepsilon u_\varepsilon^4 dx - 2qga\int_{\mathbb{R}} v_\varepsilon\partial_x u_\varepsilon u_\varepsilon^3 dx\\
&-\frac{3qg}{2a}\int_{\mathbb{R}} \partial_x v_\varepsilon u_\varepsilon^4 dx.
\end{aligned}
$$

Consequently, by (48),

$$
\begin{aligned}
\frac{dG(t)}{dt} + 2\varepsilon\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} =& \frac{2bg}{a}\int_{\mathbb{R}} P_\varepsilon u_\varepsilon^3 dx + \frac{6g\varepsilon}{a}\int_{\mathbb{R}} u_\varepsilon^2\partial_x u_\varepsilon\partial_x^3 u_\varepsilon dx\\
&-3q\int_{\mathbb{R}} \partial_x v_\varepsilon(\partial_x u_\varepsilon)^2 dx - 2q\int_{\mathbb{R}} \partial_x^2 v_\varepsilon u_\varepsilon\partial_x u_\varepsilon dx\\
&-\frac{3qg}{2a}\int_{\mathbb{R}} \partial_x v_\varepsilon u_\varepsilon^4 dx.
\end{aligned} \tag{49}
$$

Due to (26), (32), Lemma 3 and the Young inequality,

$$\begin{aligned}
\left|\frac{2bg}{a}\right|\int_{\mathbb{R}} P_\varepsilon u_\varepsilon^3 dx =& \int_{\mathbb{R}} |2P_\varepsilon u_\varepsilon| \left|\frac{bgu_\varepsilon^2}{a}\right| dx\\
\le& 2\int_{\mathbb{R}} P_\varepsilon^2 u_\varepsilon^2 dx + \frac{b^2g^2}{2a^2}\int_{\mathbb{R}} u_\varepsilon^4 dx\\
\le& 2\left\|P_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\left\|u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&+\frac{b^2g^2}{2a^2}\left\|u_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\left\|u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
\le& 2C_0\left\|P_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbb{R})} + C_0\left\|u_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\\
\le& \left\|P_\varepsilon\right\|^4_{L^\infty((0,T)\times\mathbb{R})} + C_0\left\|u_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbb{R})} + C_0\\
\le& C(T)\left(1+\left\|u_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\right),\\
3|q|\int_{\mathbb{R}} |\partial_x v_\varepsilon|(\partial_x u_\varepsilon)^2 dx \le& 3|q|\left\|\partial_x v_\varepsilon(t,\cdot)\right\|_{L^\infty(\mathbb{R})}\left\|\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
\le& C_0\left\|\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\
2|q|\int_{\mathbb{R}} \partial_x^2 v_\varepsilon u_\varepsilon \partial_x u_\varepsilon dx =& 2\int_{\mathbb{R}} |q\partial_x^2 v_\varepsilon u_\varepsilon| \partial_x u_\varepsilon| dx\\
\le& q^2\int_{\mathbb{R}} (\partial_x^2 v_\varepsilon)^2 u_\varepsilon^2 dx + \left\|\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
\le& q^2\left\|\partial_x^2 v_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\left\|u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \left\|\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
\le& C_0\left\|\partial_x^2 v_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbb{R})} + \left\|\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
\le& C(T)\left(1+\left\|u_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\right) + \left\|\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\
\left|\frac{3qg}{2a}\right|\int_{\mathbb{R}} |\partial_x v_\varepsilon| u_\varepsilon^4 dx \le& \left|\frac{3qg}{2a}\right|\left\|\partial_x v_\varepsilon(t,\cdot)\right\|_{L^\infty(\mathbb{R})}\int_{\mathbb{R}} u_\varepsilon^4 dx\\
\le& C_0\int_{\mathbb{R}} u_\varepsilon^4 dx \le C_0\left\|u_\varepsilon^2\right\|_{L^\infty((0,T)\times\mathbb{R})}\left\|u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
\le& C_0\left\|u_\varepsilon^2\right\|_{L^\infty((0,T)\times\mathbb{R})},\\
\left|\frac{6g\varepsilon}{a}\right|\int_{\mathbb{R}} |u_\varepsilon^2\partial_x u_\varepsilon||\partial_x^3 u_\varepsilon| dx =& 2\varepsilon\int_{\mathbb{R}}\left|\frac{3g}{a}u_\varepsilon^2\partial_x u_\varepsilon\right|\left|\partial_x^3 u_\varepsilon\right| dx\\
\le& \frac{9g^2\varepsilon}{a^2}\int_{\mathbb{R}} u_\varepsilon^4(\partial_x u_\varepsilon)^2 dx + \varepsilon\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.
\end{aligned}$$

Consequently, by (49),

$$\begin{aligned}
\frac{dG(t)}{dt} + \varepsilon\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \le& C_0\left\|\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{9g^2\varepsilon}{a^2}\int_{\mathbb{R}} u_\varepsilon^4(\partial_x u_\varepsilon)^2 dx\\
&+ C(T)\left(1+\left\|u_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\right).
\end{aligned} \tag{50}$$

Lemma 2.6 of [6] says that

$$\int_{\mathbb{R}} u_\varepsilon^4(\partial_x u_\varepsilon)^2 dx \le 4\left\|u_\varepsilon(t,\cdot)\right\|^4_{L^2(\mathbb{R})}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}. \tag{51}$$

Hence, by Lemma 3, we have that

$$\begin{aligned}\frac{9g^2\varepsilon}{a^2}\int_{\mathbb{R}} u_\varepsilon^4(\partial_x u_\varepsilon)^2 dx \leq & \frac{36g^2\varepsilon}{a^2}\left\|u_\varepsilon(t,\cdot)\right\|^4_{L^2(\mathbb{R})}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ \leq & C_0\varepsilon\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.\end{aligned}$$

Therefore, by (50),

$$\begin{aligned}\frac{dG(t)}{dt}+\varepsilon\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \leq & C_0\left\|\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+C_0\varepsilon\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ &+C(T)\left(1+\left\|u_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\right).\end{aligned} \tag{52}$$

Observe that, by (47) and Lemma 3,

$$\begin{aligned}C_0\left\|\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} = & C_0G(t)-\frac{gC_0}{2a}\int_{\mathbb{R}} u_\varepsilon^4 dx\\ \leq & C_0G(t)+\left|\frac{gC_0}{2a}\right|\left\|u_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\left\|u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ \leq & C_0G(t)+C_0\left\|u_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbb{R})}.\end{aligned} \tag{53}$$

It follows from (52) and (53) that

$$\begin{aligned}\frac{dG(t)}{dt}+\varepsilon\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \leq & C_0G(t)+C_0\varepsilon\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ &+C(T)\left(1+\left\|u_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\right).\end{aligned}$$

The Gronwall Lemma, (17), (47) and Lemma 3 that

$$\begin{aligned}&\left\|\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\frac{g}{2a}\int_{\mathbb{R}} u_\varepsilon^4 dx+\varepsilon e^{C_0t}\int_0^t e^{-C_0s}\left\|\partial_x^3 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds\\ &\leq C_0e^{C_0t}+C(T)\left(1+\left\|u_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\right)e^{C_0t}\int_0^t e^{-C_0s} ds\\ &\quad +C_0\varepsilon e^{C_0t}\int_0^t e^{-C_0s}\left\|\partial_x^2 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds\\ &\leq C(T)\left(1+\left\|u_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\right)+C(T)\varepsilon\int_0^t\left\|\partial_x^2 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds\\ &\leq C(T)\left(1+\left\|u_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\right).\end{aligned}$$

Consequently, by Lemma 3,

$$\begin{aligned}&\left\|\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+2\varepsilon e^{C_0t}\int_0^t e^{-C_0s}\left\|\partial_x^3 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds\\ &\quad \leq C(T)\left(1+\left\|u_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\right)-\frac{g}{2a}\left\|u_\varepsilon(t,\cdot)\right\|^4_{L^4(\mathbb{R})}\\ &\quad \leq C(T)\left(1+\left\|u_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\right)+\left|\frac{g}{2a}\right|\int_{\mathbb{R}} u_\varepsilon^4 dx\\ &\quad \leq C(T)\left(1+\left\|u_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\right)+\left|\frac{g}{2a}\right|\left\|u_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\left\|u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ &\quad \leq C(T)\left(1+\left\|u_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\right)+C_0\left\|u_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\\ &\quad \leq C(T)\left(1+\left\|u_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\right).\end{aligned} \tag{54}$$

We prove (41). Thanks to (54), Lemma 3 and the Hölder inequality,

$$\begin{aligned} u_\varepsilon^2(t,x) =& 2\int_{-\infty}^{x} u_\varepsilon \partial_x u_\varepsilon dx \le \int_{\mathbb{R}} |u_\varepsilon||\partial_x u_\varepsilon| dx \\ & \le \|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} \|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} \le C(T)\sqrt{\left(1+\|u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})}\right)}. \end{aligned}$$

Hence,

$$\|u_\varepsilon\|^4_{L^\infty((0,T)\times\mathbb{R})} - C(T)\,\|u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})} - C(T) \le 0,$$

which gives (41).

Finally, (42) follows from (41) and (54), while (26), (31), (32) and (41) give (43). □

Arguing as in ([15], Lemmas 2.8 and 2.9), we have the following result.

Lemma 8. *Assume (5). Fix $T > 0$. There exists a constant $C(T) > 0$, independent on ε, such that*

$$\left\|\partial_x^2 v_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})}, \left\|\partial_x^3 v_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})} \le C(T), \tag{55}$$

for every $0 \le t \le T$.

Lemma 9. *Assume (5). Fix $T > 0$. There exists a constant $C(T) > 0$, independent on ε, such that*

$$\|\partial_x u_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})} \le C(T), \tag{56}$$

In particular, we have that

$$\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\varepsilon e^{C_0 t}\int_0^t e^{-C_0 s}\left\|\partial_x^4 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \le C(T), \tag{57}$$

for every $0 \le t \le T$.

Proof. Let $0 \le t \le T$. Consider two real constants D, E which will be specified later Multiplying the first equation of (16) by

$$2a^2\partial_x^4 u_\varepsilon + Dagu_\varepsilon(\partial_x u_\varepsilon)^2 + Eagu_\varepsilon^2\partial_x^2 u_\varepsilon,$$

we have that

$$\begin{aligned} &\left(2a^2\partial_x^4 u_\varepsilon + Dagu_\varepsilon(\partial_x u_\varepsilon)^2 + Eagu_\varepsilon^2\partial_x^2 u_\varepsilon\right)\partial_t u_\varepsilon \\ &\quad + 3g\left(2a^2\partial_x^4 u_\varepsilon + Dagu_\varepsilon(\partial_x u_\varepsilon)^2 + Eagu_\varepsilon^2\partial_x^2 u_\varepsilon\right)u_\varepsilon^2\partial_x u_\varepsilon \\ &\quad - a\left(2a^2\partial_x^4 u_\varepsilon + Dagu_\varepsilon(\partial_x u_\varepsilon)^2 + Eagu_\varepsilon^2\partial_x^2 u_\varepsilon\right)\partial_x^3 u_\varepsilon \\ &\quad + q\left(2a^2\partial_x^4 u_\varepsilon + Dagu_\varepsilon(\partial_x u_\varepsilon)^2 + Eagu_\varepsilon^2\partial_x^2 u_\varepsilon\right)u_\varepsilon\partial_x v_\varepsilon \\ &\quad + q\left(2a^2\partial_x^4 u_\varepsilon + Dagu_\varepsilon(\partial_x u_\varepsilon)^2 + Eagu_\varepsilon^2\partial_x^2 u_\varepsilon\right)v_\varepsilon\partial_x u_\varepsilon \\ &= b\left(2a^2\partial_x^4 u_\varepsilon + Dagu_\varepsilon(\partial_x u_\varepsilon)^2 + Eagu_\varepsilon^2\partial_x^2 u_\varepsilon\right)P_\varepsilon \\ &\quad - \varepsilon\left(2a^2\partial_x^4 u_\varepsilon + Dagu_\varepsilon(\partial_x u_\varepsilon)^2 + Eagu_\varepsilon^2\partial_x^2 u_\varepsilon\right)\partial_x^4 u_\varepsilon. \end{aligned} \tag{58}$$

Observe that

$$\begin{aligned}
&\int_{\mathbb{R}} \left(2a^2\partial_x^4 u_\varepsilon + Dagu_\varepsilon(\partial_x u_\varepsilon)^2 + Eagu_\varepsilon^2\partial_x^2 u_\varepsilon\right)\partial_t u_\varepsilon dx\\
&\quad = a^2\frac{d}{dt}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + Dag\int_{\mathbb{R}} u_\varepsilon(\partial_x u_\varepsilon)^2\partial_t u_\varepsilon dx + Eag\int_{\mathbb{R}} u_\varepsilon^2\partial_x^2 u_\varepsilon\partial_t u_\varepsilon dx,\\
&3g\int_{\mathbb{R}} \left(2a^2\partial_x^4 u_\varepsilon + Dagu_\varepsilon(\partial_x u_\varepsilon)^2 + Eagu_\varepsilon^2\partial_x^2 u_\varepsilon\right)u_\varepsilon^2\partial_x u_\varepsilon dx\\
&\quad = -12a^2g\int_{\mathbb{R}} u_\varepsilon(\partial_x u_\varepsilon)^2\partial_x^3 u_\varepsilon dx - 6a^2g\int_{\mathbb{R}} u_\varepsilon^2\partial_x^2 u_\varepsilon\partial_x^3 u_\varepsilon dx\\
&\qquad + (3D-6E)\,ag^2\int_{\mathbb{R}} u_\varepsilon^3(\partial_x u_\varepsilon)^3 dx\\
&\quad = 30a^2g\int_{\mathbb{R}} u_\varepsilon\partial_x u_\varepsilon(\partial_x^2 u_\varepsilon)^2 dx + (3D-6E)\,ag^2\int_{\mathbb{R}} u_\varepsilon^3(\partial_x u_\varepsilon)^3 dx,\\
&-a\int_{\mathbb{R}} \left(2a^2\partial_x^4 u_\varepsilon + Dagu_\varepsilon(\partial_x u_\varepsilon)^2 + Eagu_\varepsilon^2\partial_x^2 u_\varepsilon\right)\partial_x^3 u_\varepsilon dx\\
&\quad = -(2D+E)\,a^2g\int_{\mathbb{R}} u_\varepsilon\partial_x u_\varepsilon(\partial_x^2 u_\varepsilon)^2 dx,\\
&q\int_{\mathbb{R}} \left(2a^2\partial_x^4 u_\varepsilon + Dagu_\varepsilon(\partial_x u_\varepsilon)^2 + Eagu_\varepsilon^2\partial_x^2 u_\varepsilon\right)u_\varepsilon\partial_x v_\varepsilon dx\\
&\quad = -2a^2q\int_{\mathbb{R}} \partial_x u_\varepsilon\partial_x v_\varepsilon\partial_x^3 u_\varepsilon dx - 2a^2q\int_{\mathbb{R}} u_\varepsilon\partial_x^2 v_\varepsilon\partial_x^3 u_\varepsilon dx\\
&\qquad + (D-3E)\,agq\int_{\mathbb{R}} u_\varepsilon^2(\partial_x u_\varepsilon)^2\partial_x v_\varepsilon dx - agqE\int_{\mathbb{R}} u_\varepsilon^3\partial_x u_\varepsilon\partial_x^2 v_\varepsilon dx\\
&\quad = 2a^2q\int_{\mathbb{R}} \partial_x v_\varepsilon(\partial_x^2 u_\varepsilon)^2 dx + 4a^2q\int_{\mathbb{R}} \partial_x u_\varepsilon\partial_x^2 v_\varepsilon\partial_x^2 u_\varepsilon dx\\
&\qquad + 2a^2q\int_{\mathbb{R}} u_\varepsilon\partial_x^3 v_\varepsilon\partial_x^2 u_\varepsilon dx + aq\,(D-3E)\int_{\mathbb{R}} u_\varepsilon^2(\partial_x u_\varepsilon)^2\partial_x v_\varepsilon dx,\\
&q\int_{\mathbb{R}} \left(2a^2\partial_x^4 u_\varepsilon + Dagu_\varepsilon(\partial_x u_\varepsilon)^2 + Eagu_\varepsilon^2\partial_x^2 u_\varepsilon\right)v_\varepsilon\partial_x u_\varepsilon dx\\
&\quad = -2a^2q\int_{\mathbb{R}} \partial_x v_\varepsilon\partial_x u_\varepsilon\partial_x^3 u_\varepsilon dx - 2a^2q\int_{\mathbb{R}} v_\varepsilon\partial_x^2 u_\varepsilon\partial_x^3 u_\varepsilon dx\\
&\qquad + (D-E)\,agq\int_{\mathbb{R}} u_\varepsilon v_\varepsilon(\partial_x u_\varepsilon)^3 dx - \frac{Eagq}{2}\int_{\mathbb{R}} u_\varepsilon^2(\partial_x u_\varepsilon)^3\partial_x^2 v_\varepsilon dx\\
&\quad = 2a^2q\int_{\mathbb{R}} \partial_x^2 v_\varepsilon\partial_x u_\varepsilon\partial_x^2 u_\varepsilon dx + 3a^2q\int_{\mathbb{R}} \partial_x v_\varepsilon(\partial_x^2 u_\varepsilon)^2 dx\\
&\qquad + (D-E)\,agq\int_{\mathbb{R}} u_\varepsilon v_\varepsilon(\partial_x u_\varepsilon)^3 dx - \frac{Eagq}{2}\int_{\mathbb{R}} u_\varepsilon^2(\partial_x u_\varepsilon)^3\partial_x^2 v_\varepsilon dx,\\
&-\varepsilon\int_{\mathbb{R}} \left(2a^2\partial_x^4 u_\varepsilon + Dagu_\varepsilon(\partial_x u_\varepsilon)^2 + Eagu_\varepsilon^2\partial_x^2 u_\varepsilon\right)\partial_x^4 u_\varepsilon dx\\
&\quad = -2a^2\varepsilon\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + Dag\varepsilon\int_{\mathbb{R}} u_\varepsilon(\partial_x u_\varepsilon)^2\partial_x^4 u_\varepsilon dx\\
&\qquad + Eag\varepsilon\int_{\mathbb{R}} u_\varepsilon^2\partial_x^2 u_\varepsilon\partial_x^4 u_\varepsilon dx.
\end{aligned}$$

Consequently, an integration on $\mathbb{R}$ of (58) gives

$$\begin{aligned}
&a^2\frac{d}{dt}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + Dag\int_{\mathbb{R}} u_\varepsilon(\partial_x u_\varepsilon)^2\partial_t u_\varepsilon dx + Eag\int_{\mathbb{R}} u_\varepsilon^2\partial_x^2 u_\varepsilon\partial_t u_\varepsilon dx\\
&\quad + 2a^2\varepsilon\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&= -a^2 g\,(30+2D+E)\int_{\mathbb{R}} u_\varepsilon\partial_x u_\varepsilon(\partial_x^2 u_\varepsilon)^2 dx - (3D-6E)\,ag^2\int_{\mathbb{R}} u_\varepsilon^3(\partial_x u_\varepsilon)^3 dx\\
&\quad - 5a^2 q\int_{\mathbb{R}}\partial_x v_\varepsilon(\partial_x^2 u_\varepsilon)^2 dx - 6a^2 q\int_{\mathbb{R}}\partial_x^2 v_\varepsilon\partial_x u_\varepsilon\partial_x^2 u_\varepsilon dx\\
&\quad - 2a^2 q\int_{\mathbb{R}} u_\varepsilon\partial_x^3 v_\varepsilon\partial_x^2 u_\varepsilon dx - aq\,(D-3E)\int_{\mathbb{R}} u_\varepsilon^2(\partial_x u_\varepsilon)^2\partial_x v_\varepsilon dx\\
&\quad - (D-E)\,agq\int_{\mathbb{R}} u_\varepsilon v_\varepsilon(\partial_x u_\varepsilon)^3 dx + \frac{Eagq}{2}\int_{\mathbb{R}} u_\varepsilon^2(\partial_x u_\varepsilon)^3\partial_x^2 v_\varepsilon dx\\
&\quad + 2a^2 b\int_{\mathbb{R}} P_\varepsilon\partial_x^4 u_\varepsilon dx + Dagb\int_{\mathbb{R}} P_\varepsilon u_\varepsilon(\partial_x u_\varepsilon)^2 dx + Eagb\int_{\mathbb{R}} P_\varepsilon u_\varepsilon^2\partial_x^2 u_\varepsilon dx\\
&\quad - Dag\varepsilon\int_{\mathbb{R}} u_\varepsilon(\partial_x u_\varepsilon)^2\partial_x^4 u_\varepsilon dx + Eag\varepsilon\int_{\mathbb{R}} u_\varepsilon^2\partial_x^2 u_\varepsilon\partial_x^4 u_\varepsilon dx.
\end{aligned}\tag{59}$$

Thanks to the second equation of (16) and (18), we have that

$$\begin{aligned}
2a^2 b\int_{\mathbb{R}} P_\varepsilon\partial_x^4 u_\varepsilon dx &= -2a^2 b\int_{\mathbb{R}}\partial_x P_\varepsilon\partial_x^3 u_\varepsilon = -2a^2 b\int_{\mathbb{R}} u_\varepsilon\partial_x^3 u_\varepsilon dx\\
&= 2a^2 b\int_{\mathbb{R}}\partial_x u_\varepsilon\partial_x^2 u_\varepsilon dx = 0,\\
Eagb\int_{\mathbb{R}} P_\varepsilon u_\varepsilon^2\partial_x^2 u_\varepsilon dx &= -Eagb\int_{\mathbb{R}}\partial_x P_\varepsilon u_\varepsilon^2\partial_x u_\varepsilon dx - 2Eagb\int_{\mathbb{R}} P_\varepsilon u_\varepsilon(\partial_x u_\varepsilon)^2 dx\\
&= -2Eagb\int_{\mathbb{R}} P_\varepsilon u_\varepsilon(\partial_x u_\varepsilon)^2 dx - Eagb\int_{\mathbb{R}} u_\varepsilon^3\partial_x u_\varepsilon dx\\
&= -2Eagb\int_{\mathbb{R}} P_\varepsilon u_\varepsilon(\partial_x u_\varepsilon)^2 dx.
\end{aligned}$$

Therefore, by (59),

$$\begin{aligned}
&a^2\frac{d}{dt}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + Dag\int_{\mathbb{R}} u_\varepsilon(\partial_x u_\varepsilon)^2\partial_t u_\varepsilon dx + Eag\int_{\mathbb{R}} u_\varepsilon^2\partial_x^2 u_\varepsilon\partial_t u_\varepsilon dx\\
&\quad + 2a^2\varepsilon\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&= -a^2 g\,(30+2D+E)\int_{\mathbb{R}} u_\varepsilon\partial_x u_\varepsilon(\partial_x^2 u_\varepsilon)^2 dx - (3D-6E)\,ag^2\int_{\mathbb{R}} u_\varepsilon^3(\partial_x u_\varepsilon)^3 dx\\
&\quad - 5a^2 q\int_{\mathbb{R}}\partial_x v_\varepsilon(\partial_x^2 u_\varepsilon)^2 dx - 6a^2 q\int_{\mathbb{R}}\partial_x^2 v_\varepsilon\partial_x u_\varepsilon\partial_x^2 u_\varepsilon dx\\
&\quad - 2a^2 q\int_{\mathbb{R}} u_\varepsilon\partial_x^3 v_\varepsilon\partial_x^2 u_\varepsilon dx - aq\,(D-3E)\int_{\mathbb{R}} u_\varepsilon^2(\partial_x u_\varepsilon)^2\partial_x v_\varepsilon dx\\
&\quad - (D-E)\,agq\int_{\mathbb{R}} u_\varepsilon v_\varepsilon(\partial_x u_\varepsilon)^3 dx + \frac{Eagq}{2}\int_{\mathbb{R}} u_\varepsilon^2(\partial_x u_\varepsilon)^3\partial_x^2 v_\varepsilon dx\\
&\quad + (D-2E)\,agb\int_{\mathbb{R}} P_\varepsilon u_\varepsilon(\partial_x u_\varepsilon)^2 dx + Dag\varepsilon\int_{\mathbb{R}} u_\varepsilon(\partial_x u_\varepsilon)^2\partial_x^4 u_\varepsilon dx\\
&\quad + Eag\varepsilon\int_{\mathbb{R}} u_\varepsilon^2\partial_x^2 u_\varepsilon\partial_x^4 u_\varepsilon dx.
\end{aligned}\tag{60}$$

Observe that

$$
\begin{aligned}
&Dag\int_{\mathbb{R}} u_\varepsilon(\partial_x u_\varepsilon)^2\partial_t u_\varepsilon dx + Eag\int_{\mathbb{R}} u_\varepsilon^2\partial_x^2 u_\varepsilon\partial_t u_\varepsilon dx\\
&\quad= \frac{Dag}{2}\int_{\mathbb{R}}\partial_t(u_\varepsilon^2)(\partial_x u_\varepsilon)^2 dx - Eag\int_{\mathbb{R}}\partial_x u_\varepsilon\partial_x(u_\varepsilon^2\partial_t u_\varepsilon)dx\\
&\quad= \frac{Dag}{2}\int_{\mathbb{R}}\partial_t(u_\varepsilon^2)(\partial_x u_\varepsilon)^2 dx - 2Eag\int_{\mathbb{R}} u_\varepsilon(\partial_x u_\varepsilon)^2\partial_t u_\varepsilon dx - Ea\int_{\mathbb{R}} u_\varepsilon^2\partial_x u_\varepsilon\partial_{tx}^2 u_\varepsilon dx\\
&\quad ag\left(\frac{D}{2}-E\right)\int_{\mathbb{R}}\partial_t(u_\varepsilon^2)(\partial_x u_\varepsilon)^2 dx - \frac{Eag}{2}\int_{\mathbb{R}} u_\varepsilon^2\partial_t((\partial_x u_\varepsilon)^2)dx.
\end{aligned}
$$

Consequently, by (60),

$$
\begin{aligned}
&a^2\frac{d}{dt}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + ag\left(\frac{D}{2}-E\right)\int_{\mathbb{R}}\partial_t(u_\varepsilon^2)(\partial_x u_\varepsilon)^2 dx\\
&\qquad - \frac{Eag}{2}\int_{\mathbb{R}} u_\varepsilon^2\partial_t((\partial_x u_\varepsilon)^2)dx + 2a^2\varepsilon\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad= -a^2 g\,(30+2D+E)\int_{\mathbb{R}} u_\varepsilon\partial_x u_\varepsilon(\partial_x^2 u_\varepsilon)^2 dx - (3D-6E)\,ag^2\int_{\mathbb{R}} u_\varepsilon^3(\partial_x u_\varepsilon)^3 dx\\
&\qquad - 5a^2 q\int_{\mathbb{R}}\partial_x v_\varepsilon(\partial_x^2 u_\varepsilon)^2 dx - 6a^2 q\int_{\mathbb{R}}\partial_x^2 v_\varepsilon\partial_x u_\varepsilon\partial_x^2 u_\varepsilon dx\\
&\qquad - 2a^2 q\int_{\mathbb{R}} u_\varepsilon\partial_x^3 v_\varepsilon\partial_x^2 u_\varepsilon dx - aq\,(D-3E)\int_{\mathbb{R}} u_\varepsilon^2(\partial_x u_\varepsilon)^2\partial_x v_\varepsilon dx\\
&\qquad - (D-E)\,agq\int_{\mathbb{R}} u_\varepsilon v_\varepsilon(\partial_x u_\varepsilon)^3 dx + \frac{Eagq}{2}\int_{\mathbb{R}} u_\varepsilon^2(\partial_x u_\varepsilon)^3\partial_x^2 v_\varepsilon dx\\
&\qquad + (D-2E)\,agb\int_{\mathbb{R}} P_\varepsilon u_\varepsilon(\partial_x u_\varepsilon)^2 dx + Dag\varepsilon\int_{\mathbb{R}} u_\varepsilon(\partial_x u_\varepsilon)^2\partial_x^4 u_\varepsilon dx\\
&\qquad + Eag\varepsilon\int_{\mathbb{R}} u_\varepsilon^2\partial_x^2 u_\varepsilon\partial_x^4 u_\varepsilon dx.
\end{aligned}
\tag{61}
$$

We search $D,\ E$ such that

$$
\frac{D}{2} - E = -\frac{E}{2}, \qquad 30 + 2D + E = 0,
$$

that is

$$
D = E, \qquad 30 + 2D + E = 0. \tag{62}
$$

Since $D = E - 10$ is the unique solution of (62), it follows from (61) that

$$
\begin{aligned}
&a^2\frac{d}{dt}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 5ag\int_{\mathbb{R}}\partial_t(u_\varepsilon^2)(\partial_x u_\varepsilon)^2 dx + 5ag\int_{\mathbb{R}} u_\varepsilon^2\partial_t((\partial_x u_\varepsilon)^2)dx\\
&\qquad + 2a^2\varepsilon\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad= -30ag^2\int_{\mathbb{R}} u_\varepsilon^3(\partial_x u_\varepsilon)^3 dx - 5a^2 q\int_{\mathbb{R}}\partial_x v_\varepsilon(\partial_x^2 u_\varepsilon)^2 dx - 6a^2 q\int_{\mathbb{R}}\partial_x^2 v_\varepsilon\partial_x u_\varepsilon\partial_x^2 u_\varepsilon dx\\
&\qquad - 2a^2 q\int_{\mathbb{R}} u_\varepsilon\partial_x^3 v_\varepsilon\partial_x^2 u_\varepsilon dx - 20aq\int_{\mathbb{R}} u_\varepsilon^2(\partial_x u_\varepsilon)^2\partial_x v_\varepsilon dx - 5agq\int_{\mathbb{R}} u_\varepsilon^2(\partial_x u_\varepsilon)^3\partial_x^2 v_\varepsilon dx\\
&\qquad + 10agb\int_{\mathbb{R}} P_\varepsilon u_\varepsilon(\partial_x u_\varepsilon)^2 dx - 10ag\varepsilon\int_{\mathbb{R}} u_\varepsilon(\partial_x u_\varepsilon)^2\partial_x^4 u_\varepsilon dx\\
&\qquad - 10ag\varepsilon\int_{\mathbb{R}} u_\varepsilon^2\partial_x^2 u_\varepsilon\partial_x^4 u_\varepsilon dx,
\end{aligned}
$$

that is

$$
\begin{aligned}
&\frac{d}{dt}\left(a^2\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+5ag\int_{\mathbb{R}}u_\varepsilon^2(\partial_x u_\varepsilon)^2dx\right)+2a^2\varepsilon\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad=-30ag^2\int_{\mathbb{R}}u_\varepsilon^3(\partial_x u_\varepsilon)^3dx-5a^2q\int_{\mathbb{R}}\partial_x v_\varepsilon(\partial_x^2 u_\varepsilon)^2dx\\
&\qquad-6a^2q\int_{\mathbb{R}}\partial_x^2 v_\varepsilon\partial_x u_\varepsilon\partial_x^2 u_\varepsilon dx-2a^2q\int_{\mathbb{R}}u_\varepsilon\partial_x^3 v_\varepsilon\partial_x^2 u_\varepsilon dx\\
&\qquad-20aq\int_{\mathbb{R}}u_\varepsilon^2(\partial_x u_\varepsilon)^2\partial_x v_\varepsilon dx-5agq\int_{\mathbb{R}}u_\varepsilon^2(\partial_x u_\varepsilon)^3\partial_x^2 v_\varepsilon dx\\
&\qquad+10agb\int_{\mathbb{R}}P_\varepsilon u_\varepsilon(\partial_x u_\varepsilon)^2dx-10ag\varepsilon\int_{\mathbb{R}}u_\varepsilon(\partial_x u_\varepsilon)^2\partial_x^4 u_\varepsilon dx\\
&\qquad-10ag\varepsilon\int_{\mathbb{R}}u_\varepsilon^2\partial_x^2 u_\varepsilon\partial_x^4 u_\varepsilon dx.
\end{aligned}
\tag{63}
$$

Due to (41), (42), (43), (55), Lemma 3 and the Young inequality,

$$
\begin{aligned}
|30ag^2|\int_{\mathbb{R}}|u_\varepsilon|^3|\partial_x u_\varepsilon|^3dx\le&|30ag^2|\left\|u_\varepsilon\right\|^3_{L^\infty((0,T)\times\mathbb{R})}\int_{\mathbb{R}}|\partial_x u_\varepsilon|^3dx\\
\le&C(T)\int_{\mathbb{R}}|\partial_x u_\varepsilon|^3dx\\
\le&C(T)\left\|\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+C(T)\int_{\mathbb{R}}(\partial_x u_\varepsilon)^4dx\\
\le&C(T)+C(T)\left\|\partial_x u_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\left\|\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
\le&C(T)\left(1+\left\|\partial_x u_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\right),\\
|5a^2q|\int_{\mathbb{R}}|\partial_x v_\varepsilon|(\partial_x^2 u_\varepsilon)^2dx\le&|5a^2q|\left\|\partial_x v_\varepsilon(t,\cdot)\right\|_{L^\infty(\mathbb{R})}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
\le&C_0\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\
|6a^2q|\int_{\mathbb{R}}|\partial_x^2 v_\varepsilon\partial_x u_\varepsilon||\partial_x^2 u_\varepsilon|dx\le&3a^4q^2\int_{\mathbb{R}}(\partial_x^2 v_\varepsilon)^2(\partial_x u_\varepsilon)^2dx+3\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
\le&3a^4q^2\left\|\partial_x^2 v_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\left\|\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+3\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
\le&C(T)+3\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\
|2a^2q|\int_{\mathbb{R}}|u_\varepsilon\partial_x^3 v_\varepsilon||\partial_x^2 u_\varepsilon|dx\le&a^4q^2\int_{\mathbb{R}}u_\varepsilon^2(\partial_x^3 v_\varepsilon)^2dx+\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
\le&a^4q^2\left\|u_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\left\|\partial_x^3 v_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
\le&C(T)+\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\
|20aq|\int_{\mathbb{R}}u_\varepsilon^2(\partial_x u_\varepsilon)^2|\partial_x v_\varepsilon|dx\le&|20aq|\left\|\partial_x v_\varepsilon(t,\cdot)\right\|_{L^\infty(\mathbb{R})}\left\|u_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\left\|\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
\le&C(T)\left\|\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\le C(T),
\end{aligned}
$$

$$
\begin{aligned}
|5agq|\int_{\mathbb{R}} u_\varepsilon^2|\partial_x u_\varepsilon|^3|\partial_x^2 v_\varepsilon|dx \le& |5agq|\,\|u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})}\left\|\partial_x^2 v_\varepsilon\right\|_{L^\infty((0,T)\times\mathbb{R})}\int_{\mathbb{R}}|\partial_x u_\varepsilon|^3dx\\
\le& C(T)\int_{\mathbb{R}}|\partial_x u_\varepsilon|^3dx\\
\le& C(T)\,\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}+C(T)\int_{\mathbb{R}}(\partial_x u_\varepsilon)^4dx\\
\le& C(T)+C(T)\,\|\partial_x u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})}\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
\le& C(T)\left(1+\|\partial_x u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})}\right),\\
|10agb|\int_{\mathbb{R}}|P_\varepsilon u_\varepsilon|(\partial_x u_\varepsilon)^2dx \le& |10agb|\,\|P_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})}\|u_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})}\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
\le& C(T)\,\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\le C(T),\\
|10ag|\varepsilon\int_{\mathbb{R}}|u_\varepsilon(\partial_x u_\varepsilon)^2||\partial_x^4 u_\varepsilon|dx =& \varepsilon\int_{\mathbb{R}}|10gu_\varepsilon(\partial_x u_\varepsilon)^2||a\partial_x^4 u_\varepsilon|dx\\
\le& 50g^2\varepsilon\int_{\mathbb{R}}u_\varepsilon^2(\partial_x u_\varepsilon)^4dx+\frac{a^2\varepsilon}{2}\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
\le& 50g^2\varepsilon\,\|u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})}\int_{\mathbb{R}}(\partial_x u_\varepsilon)^4dx+\frac{a^2\varepsilon}{2}\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
\le& C(T)\varepsilon\,\|\partial_x u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})}\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&+\frac{a^2\varepsilon}{2}\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\
|10ag|\varepsilon\int_{\mathbb{R}}|u_\varepsilon^2\partial_x^2 u_\varepsilon||\partial_x^4 u_\varepsilon|dx =& \varepsilon\int_{\mathbb{R}}|10gu_\varepsilon^2\partial_x^2 u_\varepsilon||a\partial_x^4 u_\varepsilon|dx\\
\le& 50g^2\varepsilon\int_{\mathbb{R}}u_\varepsilon^4(\partial_x^2 u_\varepsilon)^2dx+\frac{a^2\varepsilon}{2}\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
\le& 50g^2\varepsilon\,\|u_\varepsilon\|^4_{L^\infty((0,T)\times\mathbb{R})}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\frac{a^2\varepsilon}{2}\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
\le& C(T)\varepsilon\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\frac{a^2\varepsilon}{2}\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.
\end{aligned}
$$

Therefore, defining

$$G_1(t)=a^2\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+5ag\int_{\mathbb{R}}u_\varepsilon^2(\partial_x u_\varepsilon)^2dx, \tag{64}$$

by (63) and (64), we have

$$
\begin{aligned}
\frac{dG_1(t)}{dt}+\varepsilon\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \le& C_0\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+C(T)\left(1+\|\partial_x u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})}\right)\\
&+C(T)\varepsilon\,\|\partial_x u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})}\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&+C(T)\varepsilon\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.
\end{aligned} \tag{65}
$$

Observe that by (41), (42) and (64),

$$\begin{aligned} C_0\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} =& \frac{C_0a^2}{a^2}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ =&\frac{C_0}{a^2}G_1(t)-\frac{5C_0g}{a}\int_{\mathbb{R}} u_\varepsilon^2(\partial_x u_\varepsilon)^2dx\\ \le& C_0G_1(t)+\left|\frac{5C_0g}{a}\right|\int_{\mathbb{R}} u_\varepsilon^2(\partial_x u_\varepsilon)^2dx\\ \le& C_0G_1(t)+\left|\frac{5C_0g}{a}\right|\|u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})}\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\\ \le& C_0G_1(t)+C(T). \end{aligned} \tag{66}$$

It follows from (65) and (66) that

$$\begin{aligned} \frac{dG_1(t)}{dt}+\varepsilon\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \le& C_0G_1(t)+C(T)\left(1+\|\partial_x u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})}\right)\\ &+C(T)\varepsilon\|\partial_x u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})}\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\\ &+C(T)\varepsilon\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}. \end{aligned}$$

The Gronwall Lemma, (17) and Lemma 3 give

$$\begin{aligned} &\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+5ag\int_{\mathbb{R}} u_\varepsilon^2(\partial_x u_\varepsilon)^2dx+2\varepsilon e^{C_0t}\int_0^t e^{-C_0s}\left\|\partial_x^4 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds\\ &\le C_0e^{C_0t}+C(T)\left(1+\|\partial_x u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})}\right)e^{C_0t}\int_0^t e^{-C_0s}ds\\ &\quad+C(T)\varepsilon\|\partial_x u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})}e^{C_0t}\int_0^t e^{-C_0s}\|\partial_x u_\varepsilon(s,\cdot)\|^2_{L^2(\mathbb{R})}ds\\ &\quad+C(T))\varepsilon e^{C_0t}\int_0^t e^{-C_0s}\left\|\partial_x^2 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds\\ &\le C(T)\left(1+\|\partial_x u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})}\right)\\ &\quad+C(T)\varepsilon\|\partial_x u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})}\int_0^t\|\partial_x u_\varepsilon(s,\cdot)\|^2_{L^2(\mathbb{R})}ds\\ &\quad+C(T))\varepsilon\int_0^t\left\|\partial_x^2 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds\\ &\le C(T)\left(1+\|\partial_x u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})}\right). \end{aligned}$$

Therefore, thanks to (41) and (42),

$$\begin{aligned} &\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+2\varepsilon e^{C_0t}\int_0^t e^{-C_0s}\left\|\partial_x^4 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds\\ &=C(T)\left(1+\|\partial_x u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})}\right)-5ag\int_{\mathbb{R}} u_\varepsilon^2(\partial_x u_\varepsilon)^2dx\\ &\le C(T)\left(1+\|\partial_x u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})}\right)+|5ag|\int_{\mathbb{R}} u_\varepsilon^2(\partial_x u_\varepsilon)^2dx\\ &\le C(T)\left(1+\|\partial_x u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})}\right)+|5ag|\|u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})}\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\\ &\le C(T)\left(1+\|\partial_x u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})}\right). \end{aligned} \tag{67}$$

We prove (56). Due to (42), (67) and the Hölder inequality,

$$\begin{aligned}(\partial_x u_\varepsilon(t,x))^2 =&2\int_{-\infty}^{x}\partial_x u_\varepsilon \partial_x^2 u_\varepsilon dx \le 2\int_{\mathbb{R}}|\partial_x u_\varepsilon||\partial_x^2 u_\varepsilon|dx\\ &\le \|\partial_x u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})} \le C(T)\sqrt{\left(1+\|\partial_x u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})}\right)}.\end{aligned}$$

Therefore,

$$\|\partial_x u_\varepsilon\|^4_{L^\infty((0,T)\times\mathbb{R})} - C(T)\,\|\partial_x u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})} - C(T) \le 0,$$

which gives (56).

Finally, (57) follows from (56) and (67). □

Lemma 10. *Assume (5). Fix $T>0$. There exists a constant $C(T)>0$, independent on ε, such that*

$$\left\|\partial_x^4 v_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \le C(T), \tag{68}$$

for every $0\le t\le T$. In particular, we have that

$$\left\|\partial_x^3 v_\varepsilon\right\|_{L^\infty((0,T)\times\mathbb{R})} \le C(T), \tag{69}$$

Proof. Let $0\le t\le T$. Differentiating the third equation of (16) twice with respect to x, we have

$$\alpha\partial_x^4 v_\varepsilon = 2\kappa(\partial_x u_\varepsilon)^2 + 2\kappa u_\varepsilon \partial_x^2 u_\varepsilon - \beta\partial_x^3 v_\varepsilon - \gamma\partial_x^2 v_\varepsilon. \tag{70}$$

Since

$$u_\varepsilon(t,\pm\infty) = \partial_x u_\varepsilon(t,\pm\infty) = \partial_x^2 u_\varepsilon(t,\pm\infty) = 0, \tag{71}$$

it follows from (24) and (55) that

$$\partial_x^4 v_\varepsilon(t,\pm\infty) = 0. \tag{72}$$

Multiplying (70) by $2\alpha\partial_x^4 v_\varepsilon$, an integration on $\mathbb{R}$ gives

$$\begin{aligned}2\alpha^2\left\|\partial_x^4 v_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} =&2\kappa\alpha\int_{\mathbb{R}}(\partial_x u_\varepsilon)^2\partial_x^4 v_\varepsilon dx + 2\kappa\alpha\int_{\mathbb{R}} u_\varepsilon\partial_x^2 u_\varepsilon \partial_x^4 v_\varepsilon dx\\ &-2\beta\alpha\int_{\mathbb{R}}\partial_x^3 v_\varepsilon\partial_x^4 v_\varepsilon dx - 2\gamma\alpha\int_{\mathbb{R}}\partial_x^2 v_\varepsilon \partial_x^4 v_\varepsilon dx.\end{aligned} \tag{73}$$

Observe that, thanks to (24), (55) and (72),

$$\begin{aligned}-2\beta\alpha\int_{\mathbb{R}}\partial_x^3 v_\varepsilon \partial_x^4 v_\varepsilon dx =&0,\\ -2\gamma\alpha\int_{\mathbb{R}}\partial_x^2 v_\varepsilon\partial_x^4 v_\varepsilon dx =&2\gamma\alpha\left\|\partial_x^3 v_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.\end{aligned}$$

Therefore, by (55) and (73),

$$\begin{aligned}2\alpha^2\left\|\partial_x^4 v_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \le&2|\kappa\alpha|\int_{\mathbb{R}}(\partial_x u_\varepsilon)^2|\partial_x^4 v_\varepsilon|dx + 2|\kappa\alpha|\int_{\mathbb{R}}|u_\varepsilon\partial_x^2 u_\varepsilon||\partial_x^4 v_\varepsilon|dx\\ &+2|\gamma\alpha|\left\|\partial_x^3 v_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ \le&2|\kappa\alpha|\int_{\mathbb{R}}(\partial_x u_\varepsilon)^2|\partial_x^4 v_\varepsilon|dx + 2|\kappa\alpha|\int_{\mathbb{R}}|u_\varepsilon\partial_x^2 u_\varepsilon||\partial_x^4 v_\varepsilon|dx + C(T).\end{aligned} \tag{74}$$

Due to (41), (42), (56), (57) and the Young inequality,

$$\begin{aligned}
2|\kappa\alpha| \int_{\mathbb{R}} (\partial_x u_\varepsilon)^2 |\partial_x^4 v_\varepsilon| dx \leq& \kappa^2 \int_{\mathbb{R}} (\partial_x u_\varepsilon)^4 dx + \alpha^2 \left\| \partial_x^4 v_\varepsilon(t,\cdot) \right\|^2_{L^2(\mathbb{R})} \\
\leq& \kappa^2 \left\| \partial_x u_\varepsilon \right\|^2_{L^\infty((0,T)\times\mathbb{R})} \left\| \partial_x u_\varepsilon(t,\cdot) \right\|^2_{L^2(\mathbb{R})} + \alpha^2 \left\| \partial_x^4 v_\varepsilon(t,\cdot) \right\|^2_{L^2(\mathbb{R})} \\
\leq& C(T) + \alpha^2 \left\| \partial_x^4 v_\varepsilon(t,\cdot) \right\|^2_{L^2(\mathbb{R})}, \\
2|\kappa\alpha| \int_{\mathbb{R}} |u_\varepsilon \partial_x^2 u_\varepsilon| |\partial_x^4 v_\varepsilon| =& \int_{\mathbb{R}} |2\kappa u_\varepsilon \partial_x^2 u_\varepsilon| |\alpha \partial_x^4 v_\varepsilon| dx \\
\leq& 2\kappa^2 \int_{\mathbb{R}} u_\varepsilon^2 (\partial_x^2 u_\varepsilon)^2 dx + \frac{\alpha^2}{2} \left\| \partial_x^4 v_\varepsilon(t,\cdot) \right\|^2_{L^2(\mathbb{R})} \\
\leq& 2\kappa^2 \left\| u_\varepsilon \right\|^2_{L^\infty((0,T)\times\mathbb{R})} \left\| \partial_x^2 u_\varepsilon(t,\cdot) \right\|^2_{L^2(\mathbb{R})} + \frac{\alpha^2}{2} \left\| \partial_x^4 v_\varepsilon(t,\cdot) \right\|^2_{L^2(\mathbb{R})} \\
\leq& C(T) + \frac{\alpha^2}{2} \left\| \partial_x^4 v_\varepsilon(t,\cdot) \right\|^2_{L^2(\mathbb{R})}.
\end{aligned}$$

Consequently, by (74),

$$\frac{\alpha^2}{2} \left\| \partial_x^4 v_\varepsilon(t,\cdot) \right\|^2_{L^2(\mathbb{R})} \leq C(T),$$

which gives (68).

Finally, we prove (69). Due to (55), (68) and the Hölder inequality,

$$\begin{aligned}
(\partial_x^3 v_\varepsilon(t,x))^2 =& 2 \int_{-\infty}^{x} \partial_x^3 v_\varepsilon \partial_x^4 v_\varepsilon dx \leq 2 \int_{\mathbb{R}} |\partial_x^3 v_\varepsilon| |\partial_x^4 v_\varepsilon| dx \\
\leq& \left\| \partial_x^3 v_\varepsilon(t,\cdot) \right\|_{L^2(\mathbb{R})} \left\| \partial_x^4 v_\varepsilon(t,\cdot) \right\|_{L^2(\mathbb{R})} \leq C(T).
\end{aligned}$$

Hence,

$$\left\| \partial_x^3 v_\varepsilon \right\|^2_{L^\infty((0,T)\times\mathbb{R})} \leq C(T),$$

which gives (69). □

3. Proof of Theorem 1

This section is devoted to the proof of Theorem 1.

We begin by proving the following lemma.

Lemma 11. *Fix $T > 0$. Then,*

$$\text{the sequence } \{u_\varepsilon\}_{\varepsilon>0} \text{ is compact in } L^2_{loc}((0,\infty)\times\mathbb{R}). \tag{75}$$

Consequently, there exists a subsequence $\{u_{\varepsilon_k}\}_{k\in\mathbb{N}}$ of $\{u_\varepsilon\}_{\varepsilon>0}$ and $u \in L^2_{loc}((0,\infty)\times\mathbb{R})$ such that, for each compact subset K of $(0,\infty)\times\mathbb{R})$,

$$u_{\varepsilon_k} \to u \text{ in } L^2(K) \text{ and a.e.}, \tag{76}$$

$$v_{\varepsilon_k} \rightharpoonup v \text{ in } H^1((0,T)\times\mathbb{R}), \tag{77}$$

$$P_{\varepsilon_k} \rightharpoonup P \text{ in } L^2((0,T)\times\mathbb{R}). \tag{78}$$

Moreover, (u, v, P) is a solution of (1) satisfying (11) and (12).

Proof. We begin by proving (75). To prove (75), we rely on the Aubin–Lions Lemma (see [58–60]). We recall that

$$H^1_{loc}(\mathbb{R}) \hookrightarrow\hookrightarrow L^2_{loc}(\mathbb{R}) \hookrightarrow H^{-1}_{loc}(\mathbb{R}),$$

where the first inclusion is compact and the second is continuous. Owing to the Aubin–Lions Lemma [60], to prove (75), it suffices to show that

$$\{u_\varepsilon\}_{\varepsilon>0} \text{ is uniformly bounded in } L^2(0,T;H^1_{loc}(\mathbb{R})), \tag{79}$$

$$\{\partial_t u_\varepsilon\}_{\varepsilon>0} \text{ is uniformly bounded in } L^2(0,T;H^{-1}_{loc}(\mathbb{R})). \tag{80}$$

We prove (79). Thanks to (42), (57) and Lemma 3,

$$\|u_\varepsilon(t,\cdot)\|^2_{H^2(\mathbb{R})} = \|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \le C(T).$$

Therefore,

$$\{u_\varepsilon\}_{\varepsilon>0} \text{ is uniformly bounded in } L^\infty(0,T;H^2(\mathbb{R})),$$

which gives (79).

We prove (80). By the first equation of (16),

$$\partial_t u_\varepsilon = \partial_x\left(-gu_\varepsilon^3 + a\partial_x^2 u_\varepsilon - qv_\varepsilon u_\varepsilon - \varepsilon\partial_x^3 u_\varepsilon\right) + bP_\varepsilon. \tag{81}$$

We have that

$$\|u_\varepsilon\|_{L^6((0,T)\times\mathbb{R})} \le C(T). \tag{82}$$

Indeed, thanks to (41) and Lemma 3,

$$\begin{aligned} g^2\int_0^T\int_{\mathbb{R}} u_\varepsilon^6 dtdx \le& g^2\|u_\varepsilon\|^4_{L^\infty((0,T)\times\mathbb{R})}\int_0^T\int_{\mathbb{R}} u_\varepsilon^2 dtdx\\ \le& C(T)\int_0^T\int_{\mathbb{R}} u_\varepsilon^2 dtdx \le C(T). \end{aligned}$$

We prove that

$$q^2\|v_\varepsilon u_\varepsilon\|^2_{L^2((0,T)\times\mathbb{R})} \le C(T). \tag{83}$$

Due to Lemma 3,

$$\begin{aligned} q^2\int_0^T\int_{\mathbb{R}} v_\varepsilon^2 u_\varepsilon^2 dtdx \le& q^2\|v_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})}\int_0^T\int_{\mathbb{R}} u_\varepsilon^2 dtdx\\ \le& C(T)\int_0^T\int_{\mathbb{R}} u_\varepsilon^2 dtdx \le C(T). \end{aligned}$$

Observe that, since $0<\varepsilon<1$, thanks to (42) and (57),

$$\varepsilon\left\|\partial_x^3 u_\varepsilon\right\|^2_{L^2((0,T)\times\mathbb{R})},\ \beta^2\left\|\partial_x^2 u_\varepsilon\right\|^2_{L^2((0,T)\times\mathbb{R})} \le C(T). \tag{84}$$

Therefore, by (82), (83) and (84),

$$\left\{\partial_x\left(-gu_\varepsilon^3 + a\partial_x^2 u_\varepsilon - qv_\varepsilon u_\varepsilon - \varepsilon\partial_x^3 u_\varepsilon\right)\right\}_{\varepsilon>0} \text{ is bounded in } H^1((0,T)\times\mathbb{R}). \tag{85}$$

Moreover, by (43), we have that

$$b^2\|P_\varepsilon\|^2_{L^2((0,T)\times\mathbb{R})} \le C(T). \tag{86}$$

Equation (80) follows from (85) and (86).

Thanks to the Aubin–Lions Lemma, (75) and (76) hold.

Observe that, (77) follows from Lemma 3, while, by (43), we have (78). Consequently, (u, v, P) solves (1).

Observe again that, thanks to Lemmas 3, 7, 8, 9, (10) and the second equation of (16), we obtain (11).

Finally, we prove (12). Thanks to Lemmas 3 and 7, we have

$$u_{\varepsilon_k} \rightharpoonup u \text{ in } H^1((0,T)\times\mathbb{R}). \tag{87}$$

Therefore, (12) follows from (19) and (87). □

We are ready for the proof of Theorem 1.

Proof of Theorem 1. Lemma 11 gives the existence of a solution of (1) satisfying (11) and (12). Let (u_1, P_1) and (u_2, P_2) be two solutions of (1) satisfying (11) and (12), namely

$$\begin{cases} \partial_t u_1 + 3gu_1^2\partial_x u_1 - a\partial_x^3 u_1 + q\partial_x(u_1 v_1) = bP_1, & t>0,\ x\in\mathbb{R},\\ \partial_x P_1 = u_1, & t>0,\ x\in\mathbb{R},\\ \alpha\partial_x^2 v_1 + \beta\partial_x v_1 + \gamma v_1 = \kappa u_1^2, & t>0,\ x\in\mathbb{R},\\ P_1(t,-\infty)=0, & t>0,\\ u_1(0,x) = u_{1,0}(x), & x\in\mathbb{R}, \end{cases}$$

$$\begin{cases} \partial_t u_2 + 3gu_2^2\partial_x u_2 - a\partial_x^3 u_2 + q\partial_x(u_2 v_2) = bP_2, & t>0,\ x\in\mathbb{R},\\ \partial_x P_2 = u_2, & t>0,\ x\in\mathbb{R},\\ \alpha\partial_x^2 v_2 + \beta\partial_x v_2 + \gamma v_2 = \kappa u_2^2, & t>0,\ x\in\mathbb{R},\\ P_2(t,-\infty)=0, & t>0,\\ u_2(0,x) = u_{2,0}(x), & x\in\mathbb{R}. \end{cases}$$

Then, the triad (ω, V, Ω) defined by

$$\begin{aligned} &\omega(t,x) = u_1(t,x) - u_2(t,x), \quad V(t,x) = v_1(t,x) - v_2(t,x),\\ &\Omega(t,x) = \int_{-\infty}^x \omega(t,y)dy = \int_{-\infty}^x u_1(t,y)dy - \int_{-\infty}^x u_2(t,y)dy,\\ &\Omega(0,x) = \int_{-\infty}^x \omega(0,y)dy = \int_{-\infty}^x u_1(0,y)dy - \int_{-\infty}^x u_2(0,y)dy. \end{aligned} \tag{88}$$

is solution of the following Cauchy problem:

$$\begin{cases} \partial_t\omega + 3g\left(u_1^2\partial_x u_1 - u_2^2\partial_x u_2\right) & \\ \qquad -a\partial_x^3\omega + q\partial_x(u_1v_1 - u_2v_2) = b\Omega, & t>0,\ x\in\mathbb{R},\\ \partial_x\Omega = \omega, & t>0,\ x\in\mathbb{R},\\ \alpha\partial_x^2 V + \beta\partial_x V + \gamma V = \kappa(u_1^2 - u_2^2), & t>0,\ x\in\mathbb{R},\\ \Omega(t,-\infty) = 0, & t>0,\\ \omega(0,x) = u_{1,0} - u_{2,0}(x), & x\in\mathbb{R}. \end{cases} \tag{89}$$

Arguing as in ([15], Theorem 1.1), we have that

$$\begin{aligned} \|V(t,\cdot)\|^2_{H^2(\mathbb{R})} \leq & C(T)\,\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})}, && (90)\\ \|V(t,\cdot)\|^2_{L^\infty(\mathbb{R})} \leq & C(T)\,\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})}, && (91)\\ \|\partial_x V(t,\cdot)\|^2_{L^\infty(\mathbb{R})} \leq & C(T)\,\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})}. && (92) \end{aligned}$$

Moreover, by (12) and (88),

$$\Omega(t,\infty) = \int_{\mathbb{R}} \omega(t,x)dx = \int_{\mathbb{R}} u_1(t,y)dy - \int_{\mathbb{R}} u_2(t,x)dx = 0. \tag{93}$$

Observe that, by (88)

$$\begin{aligned} 3g\left(u_1^2\partial_x u_1 - u_2^2\partial_x u_2\right) =& 3g\left(u_1^2\partial_x u_1 - u_2^2\partial_x u_1 + u_2^2\partial_x u_1 - u_2^2\partial_x u_2\right)\\ =& 3g\left(\partial_x u_1\left(u_1^2 - u_2^2\right) + u_2^2\partial_x\omega\right)\\ =& 3g\left(\partial_x u_1\left(u_1+u_2\right)\omega + u_2^2\partial_x\omega\right). \end{aligned} \tag{94}$$

Moreover, arguing as in ([15], Theorem 1.1),

$$q\partial_x(u_1v_1 - u_2v_2) = q\partial_x(u_1v_1 - u_2v_1 + u_2v_1 - u_2v_2) = q\partial_x(v_1\omega) + q\partial_x(u_2V). \tag{95}$$

Therefore, thanks to (94) and (95), the first equation of (89) is equivalent to the following one:

$$\partial_t\omega = b\Omega - 3g\partial_x u_1\left(u_1+u_2\right)\omega - 3gu_2^2\partial_x\omega + a\partial_x^3\omega - q\partial_x(v_1\omega) - q\partial_x(u_2V). \tag{96}$$

Multiplying (96) by 2ω, an integration on $\mathbb{R}$ gives

$$\begin{aligned} \frac{d}{dt}\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} =& 2b\int_{\mathbb{R}}\Omega\partial_x\omega dx - 6g\int_{\mathbb{R}}\partial_x u_1\left(u_1+u_2\right)\omega^2 dx - 2a\int_{\mathbb{R}}\omega\partial_x^3\omega dx\\ &- 6g\int_{\mathbb{R}} u_2^2\omega\partial_x\omega dx - 2q\int_{\mathbb{R}}\partial_x(v_1\omega)\omega dx - 2q\int_{\mathbb{R}}\partial_x(u_2V)\omega dx. \end{aligned} \tag{97}$$

Observe that, by (88) and (93),

$$\begin{aligned} 2b\int_{\mathbb{R}}\Omega\partial_x\omega dx =& 2b\int_{\mathbb{R}}\Omega\partial_x\Omega dx = b\Omega^2(t,\infty) = 0,\\ -6g\int_{\mathbb{R}} u_2^2\omega\partial_x\omega dx =& 6g\int_{\mathbb{R}} u_2\partial_x u_2\omega^2 dx,\\ -2a\int_{\mathbb{R}}\omega\partial_x^3\omega dx =& 2a\int_{\mathbb{R}}\partial_x\omega\partial_x^2\omega = 0,\\ -2q\int_{\mathbb{R}}\partial_x(v_1\omega)\omega dx =& 2q\int_{\mathbb{R}} v_1\omega\partial_x\omega dx = -q\int_{\mathbb{R}}\partial_x v_1\omega^2 dx,\\ -2q\int_{\mathbb{R}}\partial_x(u_2V)\omega dx =& -2q\int_{\mathbb{R}}\partial_x u_2 V\omega dx - 2q\int_{\mathbb{R}} u_2\partial_x V\omega dx. \end{aligned} \tag{98}$$

It follows from (97) and (98) that

$$\begin{aligned} \frac{d}{dt}\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} =& -6g\int_{\mathbb{R}}\partial_x u_1\left(u_1+u_2\right)\omega^2 dx + 6g\int_{\mathbb{R}} u_2\partial_x u_2\omega^2 dx\\ &- q\int_{\mathbb{R}}\partial_x v_1\omega^2 dx - 2q\int_{\mathbb{R}}\partial_x u_2 V\omega dx - 2q\int_{\mathbb{R}} u_2\partial_x V\omega dx. \end{aligned} \tag{99}$$

Since (11) holds, we have that

$$\begin{aligned}\|u_1\|_{L^\infty((0,T)\times\mathbb{R})}, \|u_2\|_{L^\infty((0,T)\times\mathbb{R})} \le& C(T),\\ \|\partial_x u_1\|_{L^\infty((0,T)\times\mathbb{R})}, \|\partial_x u_2\|_{L^\infty((0,T)\times\mathbb{R})} \le& C(T),\\ \|\partial_x v_1\|_{L^\infty((0,T)\times\mathbb{R})}, \|u_2(t,\cdot)\|_{L^2(\mathbb{R})}, \|\partial_x u_2(t,\cdot)\|_{L^2(\mathbb{R})} \le& C(T),\end{aligned} \tag{100}$$

for evert $0 \le t \le T$. Consequently, by (91), (100) and the Hölder inequality,

$$\begin{aligned}|6g| \int_{\mathbb{R}} |\partial_x u_1||u_1+u_2|\omega^2 dx \le& |6g| \|\partial_x u_1\|_{L^\infty((0,T)\times\mathbb{R})} \int_{\mathbb{R}} |u_1+u_2|\omega^2 dx\\ \le& C(T)\left(\|u_1\|_{L^\infty((0,T)\times\mathbb{R})} + \|u_2\|_{L^\infty((0,T)\times\mathbb{R})}\right)\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})}\\ \le& C(T)\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})},\\ |6g| \int_{\mathbb{R}} |u_2||\partial_x u_2|\omega^2 dx \le& |6g| \|u_1\|_{L^\infty((0,T)\times\mathbb{R})} \int_{\mathbb{R}} |\partial_x u_2|\omega^2 dx\\ \le& C(T)\|\partial_x u_2\|_{L^\infty((0,T)\times\mathbb{R})}\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})}\\ \le& C(T)\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})},\\ |q| \int_{\mathbb{R}} |\partial_x v_1|\omega^2 dx \le& |q| \|\partial_x v_1\|_{L^\infty((0,T)\times\mathbb{R})}\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})}\\ \le& C(T)\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})},\\ |2q| \int_{\mathbb{R}} |\partial_x u_2||V||\omega| dx \le& |2q| \|V(t,\cdot)\|_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} |\partial_x u_2||\omega| dx\\ \le& C(T)\|\partial_x u_2(t,\cdot)\|_{L^2(\mathbb{R})}\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})}\\ \le& C(T)\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})},\\ |2q| \int_{\mathbb{R}} |u_2||\partial_x V||\omega| dx \le& |2q| \|\partial_x V(t,\cdot)\|_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} |u_2|||\omega| dx\\ \le& C(T)\|u_2(t,\cdot)\|_{L^2(\mathbb{R})}\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})}\\ \le& C(T)\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})}.\end{aligned}$$

It follows from (99) that

$$\frac{d}{dt}\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} \le C(T)\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})}. \tag{101}$$

The Gronwall Lemma and (89) give

$$\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} \le e^{C(T)t}\|\omega(0,x)\|^2_{L^2(\mathbb{R})}. \tag{102}$$

Since (11) holds, by (88), arguing as in Lemma 5, $\Omega(t,\cdot)$ is integrable at $\pm\infty$. Moreover, thanks to (93) and Lemma 5, we have that

$$\int_{\mathbb{R}} \Omega(t,x)dx = 0. \tag{103}$$

Consider the following function:

$$\Omega_1(t,x) = \int_{-\infty}^{x} \Omega(t,y)dy, \tag{104}$$

since, by the second equation of (89),

$$\partial_t\Omega = \frac{d}{dt}\int_{-\infty}^{x}\omega(t,y)dy = \int_{-\infty}^{x}\partial_t\omega(t,y)dy, \tag{105}$$

integrating the first equation of (89) on $(-\infty, x)$, by (104) and (105), we have that

$$\partial_t \Omega = b\Omega_1 - g\left(u_1^3 - u_2^3\right) + a\partial_x^2\omega - q\left(u_1 v_1 - u_2 v_2\right). \tag{106}$$

Observe that, by (88),

$$\begin{aligned} u_1^3 - u_2^3 &= \left(u_1^2 + u_2^2 + u_1 u_2\right)\omega, \\ u_1 v_1 - u_2 v_2 &= v_1\omega + u_2 V. \end{aligned}$$

Consequently, by (106),

$$\partial_t \Omega = b\Omega_1 - g\left(u_1^2 + u_2^2 + u_1 u_2\right)\omega + a\partial_x^2\omega - qv_1\omega - qu_2 V. \tag{107}$$

It follows from (88), (93), (103) and (104) that

$$\begin{aligned} 2b\int_{\mathbb{R}} \Omega_1 \Omega dx &= 2b\int_{\mathbb{R}} \Omega_1 \partial_x \Omega_1 dx = b\Omega_1^2(t,\infty) = b\left(\int_{\mathbb{R}} \Omega(t,x)dx\right)^2 = 0, \\ 2a\int_{\mathbb{R}} \Omega\partial_x^2\omega dx &= -2a\int_{\mathbb{R}} \partial_x\Omega\partial_x\omega dx = -2a\int_{\mathbb{R}} \omega\partial_x\omega dx = 0. \end{aligned}$$

Therefore, multiplying (107) by 2Ω, an integration on $\mathbb{R}$ gives

$$\begin{aligned} \frac{d}{dt}\|\Omega(t,\cdot)\|^2_{L^2(\mathbb{R})} = &-2g\int_{\mathbb{R}}\left(u_1^2 + u_2^2 + u_1 u_2\right)\omega\Omega dx \\ &- 2q\int_{\mathbb{R}} v_1\omega\Omega dx - 2q\int_{\mathbb{R}} u_2 V\Omega dx. \end{aligned} \tag{108}$$

Due to (91), (100) and the Young inequality,

$$\begin{aligned} &|2g|\int_{\mathbb{R}}\left|u_1^2 + u_2^2 + u_1 u_2\right||\omega||\Omega|dx \\ &\quad\le g^2\int_{\mathbb{R}}\left(u_1^2 + u_2^2 + u_1 u_2\right)^2\omega^2 dx + \|\Omega(t,\cdot)\|^2_{L^2(\mathbb{R})} \\ &\quad\le C(T)\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} + \|\Omega(t,\cdot)\|^2_{L^2(\mathbb{R})}, \\ &|2q|\int_{\mathbb{R}}|v_1\omega||\Omega|dx \\ &\quad\le q^2\int_{\mathbb{R}} v_1^2\omega^2 dx + \|\Omega(t,\cdot)\|^2_{L^2(\mathbb{R})} \\ &\quad\le q^2\|v_1\|^2_{L^\infty((0,T)\times\mathbb{R})}\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} + \|\Omega(t,\cdot)\|^2_{L^2(\mathbb{R})} \\ &\quad\le C(T)\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} + \|\Omega(t,\cdot)\|^2_{L^2(\mathbb{R})}, \\ &|2q|\int_{\mathbb{R}}|u_2 V|\Omega dx \\ &\quad\le q^2\int_{\mathbb{R}} V^2 u_2^2 dx + \|\Omega(t,\cdot)\|^2_{L^2(\mathbb{R})} \\ &\quad\le q^2\|V(t,\cdot)\|^2_{L^\infty(\mathbb{R})}\|u_2(t,\cdot)\|^2_{L^2(\mathbb{R})} + \|\Omega(t,\cdot)\|^2_{L^2(\mathbb{R})} \\ &\quad\le C(T)\|V(t,\cdot)\|^2_{L^\infty(\mathbb{R})} + \|\Omega(t,\cdot)\|^2_{L^2(\mathbb{R})} \\ &\quad\le C(T)\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} + \|\Omega(t,\cdot)\|^2_{L^2(\mathbb{R})}. \end{aligned}$$

Therefore, by (108),

$$\frac{d}{dt}\|\Omega(t,\cdot)\|^2_{L^2(\mathbb{R})} \le C(T)\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} + 3\|\Omega(t,\cdot)\|^2_{L^2(\mathbb{R})}. \tag{109}$$

Adding (101) and (109), by (88) and the second equation of (89), we have that

$$\frac{d}{dt}\|\Omega(t,\cdot)\|^2_{H^1(\mathbb{R})} \leq C(T)\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} + 3\|\Omega(t,\cdot)\|^2_{L^2(\mathbb{R})} \leq C(T)\|\Omega(t,\cdot)\|^2_{H^1(\mathbb{R})}$$

and

$$\|\Omega(t,\cdot)\|^2_{H^1(\mathbb{R})} \leq e^{C(T)t}\|\Omega(0,\cdot)\|^2_{H^1(\mathbb{R})}. \tag{110}$$

Therefore, (13) follows (14), (88), (89), (90), (102) and (110). □

4. Conclusions

In this paper we studied the Cauchy problem for the Spectrum Pulse equation. It is a third order nonlocal nonlinear evolutive equation related to the dynamics of the electrical field of linearly polarized continuum spectrum pulses in optical waveguides. Our existence analysis is based on on passing to the limit in a fourth order perturbation of the equation. If the initial datum belongs to $H^2(\mathbb{R}) \cap L^1(\mathbb{R})$ and has zero mean we use the Aubin–Lions Lemma while if it belongs to $H^3(\mathbb{R}) \cap L^1(\mathbb{R})$ and has zero mean we use the Sobolev Immersion Theorem. Finally, we directly prove a stability estimate that implies the uniqueness of the solution.

Author Contributions: Writing–original draft, Giuseppe Maria Coclite and Lorenzo di Ruvo

Funding: This research received no external funding.

Acknowledgments: The authors are members of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).

Conflicts of Interest: The authors declare no conflict of interest.

Appendix A. $u_0 \in H^3(\mathbb{R}) \cap L^1(\mathbb{R})$

In this appendix, we consider the Cauchy problem (1), where, on the initial datum, we assume

$$u_0(x) \in H^3(\mathbb{R}) \cap L^1(\mathbb{R}), \quad \int_{\mathbb{R}} u_0(x)dx = 0, \tag{A1}$$

while on the function $P(x)$, defined in (3), we assume (4). Moreover, we assume (5). The main result of this appendix is the following theorem.

Theorem A1. *Assume* (3), (4), (5) *and* (A1). *Fix* $T > 0$, *there exists an unique solution* (u, v, P) *of* (1) *such that*

$$\begin{aligned}
&u \in H^1((0,T)\times\mathbb{R}) \cap L^\infty(0,T;H^3(\mathbb{R})),\\
&v \in H^1((0,T)\times\mathbb{R}) \cap L^\infty(0,T;H^5(\mathbb{R})) \cap W^{1,\infty}((0,T)\times\mathbb{R}),\\
&\partial^2_{tx} v \in L^\infty((0,T)\times\mathbb{R}) \cap L^\infty(0,T;L^2(\mathbb{R})),\\
&P \in L^\infty(0,T;H^4(\mathbb{R})).
\end{aligned} \tag{A2}$$

Moreover, (12) *and* (13) *hold.*

To prove Theorem A1, we consider the approximation (16), where $u_{\varepsilon,0}$ is a C^∞ approximation of u_0 such that

$$\begin{aligned}
&\|u_{\varepsilon,0}\|_{H^3(\mathbb{R})} \leq \|u_0\|_{H^3(\mathbb{R})}, \quad \int_{\mathbb{R}} u_{\varepsilon,0}dx = 0,\\
&\|P_{\varepsilon,0}\|_{L^2(\mathbb{R})} \leq \|P_0\|_{L^2(\mathbb{R})}, \quad \int_{\mathbb{R}} P_{\varepsilon,0}dx = 0,\\
&\varepsilon\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \leq C_0,
\end{aligned} \tag{A3}$$

where C_0 is a positive constant, independent on ε.

Let us prove some a priori estimates on u_ε, v_ε and P_ε.

Since $H^2(\mathbb{R}) \subset H^3(\mathbb{R})$, then Lemmas 1, 2, 3, 4, 5, 6, 7, 8, 9 and 10 are still valid.

We prove the following result.

Lemma A1. *Assume (5). Fix $T > 0$. There exists a constant $C(T) > 0$, independent on ε, such that,*

$$\left\|\partial_x^3 u_\varepsilon\right\|_{L^\infty((0,T)\times\mathbb{R})} \le C(T). \tag{A4}$$

In particular, we have that

$$\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\varepsilon e^{C(T)t}\int_0^t e^{-C(T)s}\left\|\partial_x^5 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \le C(T), \tag{A5}$$

for every $0 \le t \le T$.

Proof. Let $0 \le t \le T$. Multiplying the first equation of (16) by $-2\partial_x^6 u_\varepsilon$, we have that

$$\begin{aligned}-2\partial_x^6 u_\varepsilon \partial_t u_\varepsilon =& -2bP_\varepsilon\partial_x^6 u_\varepsilon + 2\varepsilon\partial_x^6 u_\varepsilon\partial_x^4 u_\varepsilon + 6gu_\varepsilon^2\partial_x u_\varepsilon\partial_x^6 u_\varepsilon\\ &- 2a\partial_x^3 u_\varepsilon\partial_x^6 u_\varepsilon + 2qu_\varepsilon\partial_x v_\varepsilon\partial_x^6 u_\varepsilon + 2qv_\varepsilon\partial_x u_\varepsilon\partial_x^6 u_\varepsilon.\end{aligned} \tag{A6}$$

Observe that by (18) and the second equation of (16),

$$\begin{aligned}-2b\int_{\mathbb{R}} P_\varepsilon\partial_x^6 u_\varepsilon dx =& 2b\int_{\mathbb{R}}\partial_x P_\varepsilon\partial_x^5 u_\varepsilon dx = 2b\int_{\mathbb{R}} u_\varepsilon\partial_x^5 u_\varepsilon dx\\ =& -2b\int_{\mathbb{R}}\partial_x u_\varepsilon\partial_x^4 u_\varepsilon dx = 2b\int_{\mathbb{R}}\partial_x^2 u_\varepsilon\partial_x^3 u_\varepsilon dx = 0.\end{aligned} \tag{A7}$$

Moreover,

$$\begin{aligned}-2\int_{\mathbb{R}}\partial_x^6 u_\varepsilon\partial_t u_\varepsilon =& \frac{d}{dt}\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\ 2\varepsilon\int_{\mathbb{R}}\partial_x^6 u_\varepsilon\partial_x^4 u_\varepsilon dx =& -2\varepsilon\left\|\partial_x^5 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\ -2a\int_{\mathbb{R}}\partial_x^3 u_\varepsilon\partial_x^6 u_\varepsilon dx =& 2a\int_{\mathbb{R}}\partial_x^4 u_\varepsilon\partial_x^5 u_\varepsilon dx = 0.\end{aligned} \tag{A8}$$

It follows from (A7), (A8) and an integration of (A6) on $\mathbb{R}$ that

$$\begin{aligned}&\frac{d}{dt}\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\varepsilon\left\|\partial_x^5 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ &\quad= 6g\int_{\mathbb{R}} u_\varepsilon^2\partial_x u_\varepsilon\partial_x^6 u_\varepsilon dx + 2q\int_{\mathbb{R}} u_\varepsilon\partial_x v_\varepsilon\partial_x^6 u_\varepsilon dx\\ &\qquad+ 2q\int_{\mathbb{R}} v_\varepsilon\partial_x u_\varepsilon\partial_x^6 u_\varepsilon dx.\end{aligned} \tag{A9}$$

Observe that

$$
\begin{aligned}
6g\int_{\mathbb{R}} u_\varepsilon^2\partial_x u_\varepsilon \partial_x^6 u_\varepsilon dx =& -12g\int_{\mathbb{R}} u_\varepsilon(\partial_x u_\varepsilon)^2\partial_x^5 u_\varepsilon dx - 6g\int_{\mathbb{R}} u_\varepsilon^2\partial_x^2 u_\varepsilon\partial_x^5 u_\varepsilon dx\\
=&12g\int_{\mathbb{R}}(\partial_x u_\varepsilon)^3\partial_x^4 u_\varepsilon dx + 36g\int_{\mathbb{R}} u_\varepsilon\partial_x u_\varepsilon\partial_x^2 u_\varepsilon\partial_x^4 u_\varepsilon dx\\
&+6g\int_{\mathbb{R}} u_\varepsilon^2\partial_x^3 u_\varepsilon\partial_x^4 u_\varepsilon dx\\
=&-72g\int_{\mathbb{R}}(\partial_x u_\varepsilon)^2\partial_x^2 u_\varepsilon\partial_x^3 u_\varepsilon dx - 36g\int_{\mathbb{R}} u_\varepsilon(\partial_x^2 u_\varepsilon)^2\partial_x^3 u_\varepsilon dx\\
&-42g\int_{\mathbb{R}} u_\varepsilon\partial_x u_\varepsilon(\partial_x^3 u_\varepsilon)^2 dx,\\
2q\int_{\mathbb{R}} u_\varepsilon\partial_x v_\varepsilon\partial_x^6 u_\varepsilon dx =& -2q\int_{\mathbb{R}}\partial_x u_\varepsilon\partial_x v_\varepsilon\partial_x^5 u_\varepsilon dx - 2q\int_{\mathbb{R}} u_\varepsilon\partial_x^2 v_\varepsilon\partial_x^5 u_\varepsilon dx\\
=&2q\int_{\mathbb{R}}\partial_x^2 u_\varepsilon\partial_x v_\varepsilon\partial_x^4 u_\varepsilon dx + 4q\int_{\mathbb{R}}\partial_x u_\varepsilon\partial_x^2 v_\varepsilon\partial_x^4 u_\varepsilon dx\\
&+2q\int_{\mathbb{R}} u_\varepsilon\partial_x^3 v_\varepsilon\partial_x^4 u_\varepsilon dx\\
=&-2q\int_{\mathbb{R}}\partial_x v_\varepsilon(\partial_x^3 u_\varepsilon)^2 dx - 6q\int_{\mathbb{R}}\partial_x^2 u_\varepsilon\partial_x^2 v_\varepsilon\partial_x^3 u_\varepsilon dx\\
&-6q\int_{\mathbb{R}}\partial_x u_\varepsilon\partial_x^3 v_\varepsilon\partial_x^3 u_\varepsilon dx - 2q\int_{\mathbb{R}} u_\varepsilon\partial_x^4 v_\varepsilon\partial_x^3 u_\varepsilon dx\\
=&-2q\int_{\mathbb{R}}\partial_x v_\varepsilon(\partial_x^3 u_\varepsilon)^2 dx + 3q\int_{\mathbb{R}}\partial_x^3 v_\varepsilon(\partial_x^2 u_\varepsilon)^2 dx\\
&-6q\int_{\mathbb{R}}\partial_x u_\varepsilon\partial_x^3 v_\varepsilon\partial_x^3 u_\varepsilon dx - 2q\int_{\mathbb{R}} u_\varepsilon\partial_x^4 v_\varepsilon\partial_x^3 u_\varepsilon dx,\\
2q\int_{\mathbb{R}} v_\varepsilon\partial_x u_\varepsilon\partial_x^6 u_\varepsilon dx =& -2q\int_{\mathbb{R}}\partial_x v_\varepsilon\partial_x u_\varepsilon\partial_x^5 u_\varepsilon dx - 2q\int_{\mathbb{R}} v_\varepsilon\partial_x^2 u_\varepsilon\partial_x^5 u_\varepsilon dx\\
=&2q\int_{\mathbb{R}}\partial_x^2 v_\varepsilon\partial_x u_\varepsilon\partial_x^4 u_\varepsilon dx + 4q\int_{\mathbb{R}}\partial_x v_\varepsilon\partial_x^2 u_\varepsilon\partial_x^4 u_\varepsilon dx\\
&+2q\int_{\mathbb{R}} v_\varepsilon\partial_x^3 u_\varepsilon\partial_x^4 u_\varepsilon dx\\
=&-2q\int_{\mathbb{R}}\partial_x^3 v_\varepsilon\partial_x u_\varepsilon\partial_x^3 u_\varepsilon dx - 6q\int_{\mathbb{R}}\partial_x^2 v_\varepsilon\partial_x^2 u_\varepsilon\partial_x^3 u_\varepsilon dx\\
&-5q\int_{\mathbb{R}}\partial_x v_\varepsilon(\partial_x^3 u_\varepsilon)^2 dx\\
=&-2q\int_{\mathbb{R}}\partial_x^3 v_\varepsilon\partial_x u_\varepsilon\partial_x^3 u_\varepsilon dx + 3q\int_{\mathbb{R}}\partial_x^3 v_\varepsilon(\partial_x^2 u_\varepsilon)^2 dx\\
&-5q\int_{\mathbb{R}}\partial_x v_\varepsilon(\partial_x^3 u_\varepsilon)^2 dx.
\end{aligned}
$$

Consequently, by (A9),

$$
\begin{aligned}
&\frac{d}{dt}\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\varepsilon\left\|\partial_x^5 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\qquad= -72g\int_{\mathbb{R}}(\partial_x u_\varepsilon)^2\partial_x^2 u_\varepsilon\partial_x^3 u_\varepsilon dx - 36g\int_{\mathbb{R}} u_\varepsilon(\partial_x^2 u_\varepsilon)^2\partial_x^3 u_\varepsilon dx\\
&\qquad\quad -42g\int_{\mathbb{R}} u_\varepsilon\partial_x u_\varepsilon(\partial_x^3 u_\varepsilon)^2 dx - 7q\int_{\mathbb{R}}\partial_x v_\varepsilon(\partial_x^3 u_\varepsilon)^2 dx\\
&\qquad\quad +6q\int_{\mathbb{R}}\partial_x^3 v_\varepsilon(\partial_x^2 u_\varepsilon)^2 dx - 8q\int_{\mathbb{R}}\partial_x^3 v_\varepsilon\partial_x u_\varepsilon\partial_x^3 u_\varepsilon dx\\
&\qquad\quad -2q\int_{\mathbb{R}} u_\varepsilon\partial_x^4 v_\varepsilon\partial_x^3 u_\varepsilon dx.
\end{aligned}
\tag{A10}
$$

Due to (24), (41), (42), (56), (57), (68), (69) and the Young inequality,

$$
\begin{aligned}
&|72g|\int_{\mathbb{R}}|(\partial_x u_\varepsilon)^2\partial_x^2 u_\varepsilon||\partial_x^3 u_\varepsilon|dx\\
&\quad\le 36g^2\int_{\mathbb{R}}(\partial_x u_\varepsilon)^4(\partial_x^2 u_\varepsilon)^2dx+36\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad\le 36g^2\left\|\partial_x u_\varepsilon\right\|^4_{L^\infty((0,T)\times\mathbb{R})}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+36\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad\le C(T)+36\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\
&|36g|\int_{\mathbb{R}}|u_\varepsilon(\partial_x^2 u_\varepsilon)^2||\partial_x^3 u_\varepsilon|dx\\
&\quad\le 18g^2\int_{\mathbb{R}}u_\varepsilon^2(\partial_x^2 u_\varepsilon)^4dx+18\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad\le 18g^2\left\|u_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\left\|\partial_x^2 u_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+18\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad\le C(T)\left\|\partial_x^2 u_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbb{R})}+18\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\
&|7q|\int_{\mathbb{R}}|\partial_x v_\varepsilon|(\partial_x^3 u_\varepsilon)^2dx\le|7q|\left\|\partial_x v_\varepsilon(t,\cdot)\right\|_{L^\infty(\mathbb{R})}\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad\le C_0\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\
&|6q|\int_{\mathbb{R}}|\partial_x^3 v_\varepsilon|(\partial_x^2 u_\varepsilon)^2dx\\
&\quad\le|6q|\left\|\partial_x^3 v_\varepsilon\right\|_{L^\infty((0,T)\times\mathbb{R})}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\le C(T),\\
&|8q|\int_{\mathbb{R}}|\partial_x^3 v_\varepsilon\partial_x u_\varepsilon||\partial_x^3 u_\varepsilon|dx\\
&\quad\le 4q^2\int_{\mathbb{R}}(\partial_x^3 v_\varepsilon)^2(\partial_x u_\varepsilon)^2dx+4\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad\le 4q^2\left\|\partial_x^3 v_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\left\|\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+4\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad\le C(T)+4\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\
&|2q|\int_{\mathbb{R}}|u_\varepsilon\partial_x^4 v_\varepsilon||\partial_x^3 u_\varepsilon|dx\\
&\quad\le q^2\int_{\mathbb{R}}u_\varepsilon^2(\partial_x^4 v_\varepsilon)^2dx+\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\
&\quad\le q^2\left\|u_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\left\|\partial_x^4 v_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.
\end{aligned}
$$

It follows from (A10) that

$$
\begin{aligned}
&\frac{d}{dt}\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+2\varepsilon\left\|\partial_x^5 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\qquad\le C(T)\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+C(T)\left(1+\left\|\partial_x^2 u_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\right).
\end{aligned}
$$

The Gronwall Lemma and (A3) give

$$\begin{aligned}\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} &+ 2\varepsilon e^{C(T)t}\int_0^t e^{-C(T)s}\left\|\partial_x^5 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds\\ &\le C_0 e^{C(T)t} + C(T)\left(1+\left\|\partial_x^2 u_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\right) e^{C(T)t}\int_0^t e^{-C(T)s} ds \\ &\le C(T)\left(1+\left\|\partial_x^2 u_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\right).\end{aligned} \tag{A11}$$

We prove (A4). Thanks to (57), (A11) and the Hölder inequality,

$$\begin{aligned}(\partial_x^2 u_\varepsilon(t,x))^2 =& 2\int_{-\infty}^x \partial_x^2 u_\varepsilon \partial_x^3 u_\varepsilon dx \le 2\int_{\mathbb{R}} |\partial_x^2 u_\varepsilon||\partial_x^3 u_\varepsilon| dx\\ &\le \left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})}\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \le C(T)\sqrt{\left(1+\|\partial_x^2 u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})}\right)}.\end{aligned}$$

Hence,

$$\left\|\partial_x^2 u_\varepsilon\right\|^4_{L^\infty((0,T)\times\mathbb{R})} - C(T)\left\|\partial_x^2 u_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbb{R})} - C(T) \le 0,$$

which gives (A4).

Finally, (A5) follows from (A4) and (A11). □

Lemma A2. *Assume* (5)*. Fix* $T > 0$*. There exists a constant* $C(T) > 0$*, independent on* ε*, such that*

$$\left\|\partial_x^5 v_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \le C(T), \tag{A12}$$

for every $0 \le t \le T$*. In particular, we have that*

$$\left\|\partial_x^4 v_\varepsilon\right\|_{L^\infty((0,T)\times\mathbb{R})} \le C(T). \tag{A13}$$

Proof. Let $0 \le t \le T$. Differentiating (70) with respect to x, we have

$$\alpha\partial_x^5 v_\varepsilon = 6\kappa\partial_x u_\varepsilon \partial_x^2 u_\varepsilon + 2\kappa u_\varepsilon \partial_x^3 u_\varepsilon - \beta\partial_x^4 v_\varepsilon - \gamma\partial_x^3 v_\varepsilon. \tag{A14}$$

Since $\partial_x^3 u_\varepsilon(t,\pm\infty) = 0$, by (55), (71) and (72), we have that

$$\partial_x^5 v_\varepsilon(t,\pm\infty) = 0. \tag{A15}$$

Observe that

$$\begin{aligned}-2\beta\alpha\int_{\mathbb{R}} \partial_x^4 v_\varepsilon \partial_x^5 v_\varepsilon dx =& 0,\\ -2\alpha\gamma\int_{\mathbb{R}} \partial_x^3 v_\varepsilon \partial_x^5 v_\varepsilon dx =& 2\alpha\gamma\left\|\partial_x^4 v_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.\end{aligned}$$

Consequently, multiplying (A14) by $2\alpha\partial_x^5 v_\varepsilon$, an integration on $\mathbb{R}$ gives

$$\begin{aligned}2\alpha^2\left\|\partial_x^5 v_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} =& 12\alpha\kappa\int_{\mathbb{R}} \partial_x u_\varepsilon \partial_x^2 u_\varepsilon \partial_x^5 v_\varepsilon dx + 4\alpha\kappa\int_{\mathbb{R}} u_\varepsilon \partial_x^3 u_\varepsilon \partial_x^5 v_\varepsilon dx\\ &+ 2\alpha\gamma\left\|\partial_x^4 v_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.\end{aligned} \tag{A16}$$

Due to (41), (56), (57), (A4) and the Young inequality,

$$
\begin{aligned}
&12|\alpha\kappa|\int_{\mathbb{R}}|\partial_x u_\varepsilon \partial_x^2 u_\varepsilon||\partial_x^5 v_\varepsilon|dx\\
&\quad=\int_{\mathbb{R}}|12\kappa\partial_x u_\varepsilon\partial_x^2 u_\varepsilon||\alpha\partial_x^5 v_\varepsilon|dx\\
&\quad\le 72\kappa^2\int_{\mathbb{R}}(\partial_x u_\varepsilon)^2(\partial_x^2 u_\varepsilon)^2dx+\frac{\alpha^2}{2}\left\|\partial_x^5 v_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad\le 72\kappa^2\left\|\partial_x u_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\frac{\alpha^2}{2}\left\|\partial_x^5 v_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad\le C(T)+\frac{\alpha^2}{2}\left\|\partial_x^5 v_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\
&|4\alpha\kappa|\int_{\mathbb{R}}|u_\varepsilon\partial_x^3 u_\varepsilon||\partial_x^5 v_\varepsilon|dx\\
&\quad=\int_{\mathbb{R}}|4\kappa u_\varepsilon\partial_x^3 u_\varepsilon|||\alpha\partial_x^5 v_\varepsilon|dx\\
&\quad\le 8\kappa^2\int_{\mathbb{R}}u_\varepsilon^2(\partial_x^3 u_\varepsilon)^2dx+\frac{\alpha^2}{2}\left\|\partial_x^5 v_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad\le 8\kappa^2\left\|u_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\frac{\alpha^2}{2}\left\|\partial_x^5 v_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad\le C(T)+\frac{\alpha^2}{2}\left\|\partial_x^5 v_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.
\end{aligned}
$$

It follows from (68) and (A16) that

$$\alpha^2\left\|\partial_x^5 v_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\le C(T)+|2\alpha\gamma|\left\|\partial_x^4 v_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\le C(T),$$

which gives (A12).

Finally, we prove (A13). Thanks to (68), (A12) and the Hölder inequality,

$$
\begin{aligned}
(\partial_x^4 v_\varepsilon(t,x))^2=&2\int_{-\infty}^x\partial_x^4 v_\varepsilon\partial_x^5 v_\varepsilon dx\le 2\int_{\mathbb{R}}|\partial_x^4 v_\varepsilon||\partial_x^5 v_\varepsilon|dx\\
&\le\left\|\partial_x^4 v_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})}\left\|\partial_x^5 v_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})}\le C(T).
\end{aligned}
$$

Hence,

$$\left\|\partial_x^4 v_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\le C(T),$$

which gives (A13). □

Lemma A3. *Assume (5). Fix $T>0$. There exists a constant $C(T)>0$, independent on ε, such that,*

$$\varepsilon\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\varepsilon^2\int_0^t\left\|\partial_x^6 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds\le C(T), \tag{A17}$$

for every $0\le t\le T$.

Proof. Let $0\le t\le T$. Multiplying the first equation of (16) by $2\varepsilon\partial_x^8 u_\varepsilon$, we have

$$
\begin{aligned}
2\varepsilon\partial_x^8 u_\varepsilon\partial_t u_\varepsilon=&2b\varepsilon P_\varepsilon\partial_x^8 u_\varepsilon-2\varepsilon^2\partial_x^4 u_\varepsilon\partial_x^8 u_\varepsilon-6g\varepsilon u_\varepsilon^2\partial_x u_\varepsilon\partial_x^8 u_\varepsilon\\
&+2a\varepsilon\partial_x^3 u_\varepsilon\partial_x^8 u_\varepsilon-2q\varepsilon u_\varepsilon\partial_x v_\varepsilon\partial_x^8 u_\varepsilon-2q\varepsilon v_\varepsilon\partial_x u_\varepsilon\partial_x^8 u_\varepsilon.
\end{aligned} \tag{A18}
$$

Observe that by the second equation of (16) and (18),

$$\begin{aligned} 2b\varepsilon\int_{\mathbb{R}} P_\varepsilon\partial_x^8 u_\varepsilon dx =& -2b\varepsilon\int_{\mathbb{R}}\partial_x P_\varepsilon\partial_x^7 u_\varepsilon dx = -2b\varepsilon\int_{\mathbb{R}} u_\varepsilon\partial_x^7 u_\varepsilon dx = 2b\varepsilon\int_{\mathbb{R}}\partial_x u_\varepsilon\partial_x^6 u_\varepsilon dx \\ =& -2b\varepsilon\int_{\mathbb{R}}\partial_x^2 u_\varepsilon\partial_x^5 u_\varepsilon dx = 2b\varepsilon\int_{\mathbb{R}}\partial_x^3 u_\varepsilon\partial_x^4 u_\varepsilon dx = 0. \end{aligned} \tag{A19}$$

Moreover,

$$\begin{aligned} 2\varepsilon\int_{\mathbb{R}}\partial_x^8 u_\varepsilon\partial_t u_\varepsilon dx =& \varepsilon\frac{d}{dt}\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}, \\ -2\varepsilon^2\int_{\mathbb{R}}\partial_x^4 u_\varepsilon\partial_x^8 u_\varepsilon dx =& -2\varepsilon^2\left\|\partial_x^6 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}, \\ 2a\varepsilon\int_{\mathbb{R}}\partial_x^3 u_\varepsilon\partial_x^8 u_\varepsilon dx =& 0. \end{aligned} \tag{A20}$$

Therefore, (A19), (A20) and an integration of (A18) on $\mathbb{R}$ give

$$\begin{aligned} \varepsilon\frac{d}{dt}\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} &+ 2\varepsilon^2\left\|\partial_x^6 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\ &= -6g\varepsilon\int_{\mathbb{R}} u_\varepsilon^2\partial_x u_\varepsilon\partial_x^8 u_\varepsilon dx - 2q\varepsilon\int_{\mathbb{R}} u_\varepsilon\partial_x v_\varepsilon\partial_x^8 u_\varepsilon dx \\ &\quad - 2q\varepsilon\int_{\mathbb{R}} v_\varepsilon\partial_x u_\varepsilon\partial_x^8 u_\varepsilon dx. \end{aligned} \tag{A21}$$

Observe that

$$\begin{aligned} -6g\varepsilon\int_{\mathbb{R}} u_\varepsilon^2\partial_x u_\varepsilon\partial_x^8 u_\varepsilon dx =& 12g\varepsilon\int u_\varepsilon(\partial_x u_\varepsilon)^2\partial_x^7 u_\varepsilon dx + 6g\varepsilon\int_{\mathbb{R}} u_\varepsilon^2\partial_x^2 u_\varepsilon\partial_x^7 u_\varepsilon dx \\ =& -12g\varepsilon\int_{\mathbb{R}}(\partial_x u_\varepsilon)^3\partial_x^6 u_\varepsilon dx - 36g\varepsilon\int_{\mathbb{R}} u_\varepsilon\partial_x u_\varepsilon\partial_x^2 u_\varepsilon\partial_x^6 u_\varepsilon dx \\ & -6g\varepsilon\int_{\mathbb{R}} u_\varepsilon^2\partial_x^3 u_\varepsilon\partial_x^6 u_\varepsilon dx, \\ -2q\varepsilon\int_{\mathbb{R}} u_\varepsilon\partial_x v_\varepsilon\partial_x^8 u_\varepsilon dx =& 2q\varepsilon\int_{\mathbb{R}}\partial_x u_\varepsilon\partial_x v_\varepsilon\partial_x^7 u_\varepsilon dx + 2q\varepsilon\int_{\mathbb{R}} u_\varepsilon\partial_x^2 v_\varepsilon\partial_x^7 u_\varepsilon dx \\ =& -2q\varepsilon\int_{\mathbb{R}}\partial_x^2 u_\varepsilon\partial_x v_\varepsilon\partial_x^6 u_\varepsilon dx - 4q\varepsilon\int_{\mathbb{R}}\partial_x u_\varepsilon\partial_x^2 v_\varepsilon\partial_x^6 u_\varepsilon dx \\ & -2q\varepsilon\int_{\mathbb{R}} u_\varepsilon\partial_x^3 v_\varepsilon\partial_x^6 u_\varepsilon dx, \\ -2q\varepsilon\int_{\mathbb{R}} v_\varepsilon\partial_x u_\varepsilon\partial_x^8 u_\varepsilon dx =& 2q\varepsilon\int_{\mathbb{R}}\partial_x v_\varepsilon\partial_x u_\varepsilon\partial_x^6 u_\varepsilon dx + 2q\varepsilon\int_{\mathbb{R}} v_\varepsilon\partial_x^2 u_\varepsilon\partial_x^7 u_\varepsilon dx \\ =& -2q\varepsilon\int_{\mathbb{R}}\partial_x^2 v_\varepsilon\partial_x u_\varepsilon\partial_x^6 u_\varepsilon dx - 4q\varepsilon\int_{\mathbb{R}}\partial_x v_\varepsilon\partial_x^2 u_\varepsilon\partial_x^6 u_\varepsilon dx \\ & -2q\varepsilon\int_{\mathbb{R}} v_\varepsilon\partial_x^3 u_\varepsilon\partial_x^6 u_\varepsilon dx. \end{aligned} \tag{A22}$$

Consequently, by (A21),

$$\begin{aligned}
\varepsilon\frac{d}{dt}\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}&+2\varepsilon^2\left\|\partial_x^6 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
=&-12g\varepsilon\int_{\mathbb{R}}(\partial_x u_\varepsilon)^3\partial_x^6 u_\varepsilon dx-36g\varepsilon\int_{\mathbb{R}}u_\varepsilon\partial_x u_\varepsilon\partial_x^2 u_\varepsilon\partial_x^6 u_\varepsilon dx\\
&-6g\varepsilon\int_{\mathbb{R}}u_\varepsilon^2\partial_x^3 u_\varepsilon\partial_x^6 u_\varepsilon dx-6q\varepsilon\int_{\mathbb{R}}\partial_x v_\varepsilon\partial_x^2 u_\varepsilon\partial_x^6 u_\varepsilon dx\\
&-6q\varepsilon\int_{\mathbb{R}}\partial_x^2 v_\varepsilon\partial_x u_\varepsilon\partial_x^6 u_\varepsilon dx-2q\varepsilon\int_{\mathbb{R}}u_\varepsilon\partial_x^3 v_\varepsilon\partial_x^6 u_\varepsilon dx\\
&-2q\varepsilon\int_{\mathbb{R}}v_\varepsilon\partial_x^3 u_\varepsilon\partial_x^6 u_\varepsilon dx.
\end{aligned}\tag{A23}$$

Due to (24), (41), (42), (43), (56), (57), (A5) and the Young inequality,

$$\begin{aligned}
12|g\varepsilon|&\int_{\mathbb{R}}|\partial_x u_\varepsilon|^3|\partial_x^6 u_\varepsilon|dx=12\int_{\mathbb{R}}\left|\frac{g(\partial_x u_\varepsilon)^3}{\sqrt{D_1}}\right|\left|\varepsilon\sqrt{D_1}\partial_x^6 u_\varepsilon\right|dx\\
&\le\frac{6g^2}{D_1}\int_{\mathbb{R}}(\partial_x u_\varepsilon)^6 dx+6D_1\varepsilon^2\left\|\partial_x^6 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\le\frac{6g^2}{D_1}\|\partial_x u_\varepsilon\|^4_{L^\infty((0,T)\times\mathbb{R})}\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}+6D_1\varepsilon^2\left\|\partial_x^6 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\le\frac{C(T)}{D_1}+6D_1\varepsilon^2\left\|\partial_x^6 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\
|36g\varepsilon|&\int_{\mathbb{R}}|u_\varepsilon\partial_x u_\varepsilon\partial_x^2 u_\varepsilon||\partial_x^6 u_\varepsilon|dx=36\int_{\mathbb{R}}\left|\frac{gu_\varepsilon\partial_x u_\varepsilon\partial_x^2 u_\varepsilon}{\sqrt{D_1}}\right|\left|\sqrt{D_1}\varepsilon\partial_x^6 u_\varepsilon\right|dx\\
&\le\frac{18g^2}{D_1}\int_{\mathbb{R}}u_\varepsilon^2(\partial_x u_\varepsilon)^2(\partial_x^2 u_\varepsilon)^2 dx+18D_1\varepsilon^2\left\|\partial_x^6 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\le\frac{18g^2}{D_1}\|u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})}\|\partial_x u_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+18D_1\varepsilon^2\left\|\partial_x^6 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\le\frac{C(T)}{D_1}+18D_1\varepsilon^2\left\|\partial_x^6 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\
|6g\varepsilon|&\int_{\mathbb{R}}|u_\varepsilon^2\partial_x^3 u_\varepsilon||\partial_x^6 u_\varepsilon|dx=6\int_{\mathbb{R}}\left|\frac{gu_\varepsilon^2\partial_x^3 u_\varepsilon}{\sqrt{D_1}}\right|\left|\sqrt{D_1}\varepsilon\partial_x^6 u_\varepsilon\right|dx\\
&\le\frac{3g^2}{D_1}\int_{\mathbb{R}}u_\varepsilon^4(\partial_x^3 u_\varepsilon)^2 dx+3D_1\varepsilon^2\left\|\partial_x^6 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\le\frac{3g^2}{D_1}\|u_\varepsilon\|^4_{L^\infty((0,T)\times\mathbb{R})}\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+3D_1\varepsilon^2\left\|\partial_x^6 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\le\frac{C(T)}{D_1}+3D_1\varepsilon^2\left\|\partial_x^6 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\
|6q\varepsilon|&\int_{\mathbb{R}}|\partial_x v_\varepsilon\partial_x^2 u_\varepsilon||\partial_x^6 u_\varepsilon|dx=6\int_{\mathbb{R}}\left|\frac{q\partial_x v_\varepsilon\partial_x^2 u_\varepsilon}{\sqrt{D_1}}\right|\left|\sqrt{D_1}\varepsilon\partial_x^6 u_\varepsilon\right|dx\\
&\le\frac{3q^2}{D_1}\int_{\mathbb{R}}(\partial_x v_\varepsilon)^2(\partial_x^2 u_\varepsilon)^2 dx+3D_1\varepsilon^2\left\|\partial_x^6 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\le\frac{3q^2}{D_1}\|\partial_x v_\varepsilon(t,\cdot)\|^2_{L^\infty(\mathbb{R})}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+3D_1\varepsilon^2\left\|\partial_x^6 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\le\frac{C(T)}{D_1}+3D_1\varepsilon^2\left\|\partial_x^6 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})},
\end{aligned}$$

$$
\begin{aligned}
|6q\varepsilon|\int_{\mathbb{R}}|\partial_x^2 v_\varepsilon\partial_x u_\varepsilon||\partial_x^6 u_\varepsilon|dx &= 6\int_{\mathbb{R}}\left|\frac{q\partial_x^2 v_\varepsilon\partial_x u_\varepsilon}{\sqrt{D_1}}\right|\left|\sqrt{D_1}\varepsilon\partial_x^6 u_\varepsilon\right|dx\\
&\le \frac{3q^2}{D_1}\int_{\mathbb{R}}(\partial_x^2 v_\varepsilon)^2(\partial_x u_\varepsilon)^2dx + 3D_1\varepsilon^2\left\|\partial_x^6 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\le \frac{3q^2}{D_1}\left\|\partial_x^2 v_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + 3D_1\varepsilon^2\left\|\partial_x^6 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\le \frac{C(T)}{D_1} + 3D_1\varepsilon^2\left\|\partial_x^6 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\
|2q\varepsilon|\int_{\mathbb{R}}|u_\varepsilon\partial_x^3 v_\varepsilon||\partial_x^6 u_\varepsilon|dx &= 2\int_{\mathbb{R}}\left|\frac{qu_\varepsilon\partial_x^3 v_\varepsilon}{\sqrt{D_1}}\right|\left|\sqrt{D_1}\varepsilon\partial_x^6 u_\varepsilon\right|dx\\
&\le \frac{q^2}{D_1}\int_{\mathbb{R}}u_\varepsilon^2(\partial_x^3 v_\varepsilon)^2dx + D_1\varepsilon^2\left\|\partial_x^6 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\le \frac{q^2}{D_1}\|u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})}\left\|\partial_x^3 v_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + D_1\varepsilon^2\left\|\partial_x^6 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\le \frac{C(T)}{D_1} + D_1\varepsilon^2\left\|\partial_x^6 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\
|2q\varepsilon|\int_{\mathbb{R}}|v_\varepsilon\partial_x^3 u_\varepsilon||\partial_x^6 u_\varepsilon|dx &= \int_{\mathbb{R}}\left|\frac{qv_\varepsilon\partial_x^3 u_\varepsilon}{\sqrt{D_1}}\right|\left|\sqrt{D_1}\varepsilon\partial_x^6 u_\varepsilon\right|dx\\
&\le \frac{q^2}{D_1}\int_{\mathbb{R}}v_\varepsilon^2(\partial_x^3 u_\varepsilon)^2dx + D_1\varepsilon^2\left\|\partial_x^6 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\le \frac{q^2}{D_1}\|v_\varepsilon(t,\cdot)\|^2_{L^\infty(\mathbb{R})}\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + D_1\varepsilon^2\left\|\partial_x^6 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\le \frac{C(T)}{D_1} + D_1\varepsilon^2\left\|\partial_x^6 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})},
\end{aligned}
$$

where D_1 is a positive constant, which will be specified later. Consequently, by (A23),

$$
\varepsilon\frac{d}{dt}\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + (2-35D_1)\,\varepsilon^2\left\|\partial_x^6 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \le \frac{C(T)}{D_1}.
$$

Taking $D_1 = \frac{1}{35}$, we have that

$$
\varepsilon\frac{d}{dt}\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \varepsilon^2\left\|\partial_x^6 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \le C(T).
$$

Equation (A3) and an integration on $(0,t)$ give

$$
\varepsilon\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \varepsilon^2\int_0^t\left\|\partial_x^6 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds \le C_0 + C(T)t \le C(T),
$$

that is (A17). □

Lemma A4. *Assume (5). Fix $T > 0$. There exists a constant $C(T) > 0$, independent on ε, such that,*

$$
\|\partial_t u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})} \le C(T), \tag{A24}
$$

for every $0 \le t \le T$.

Proof. Let $0 \leq t \leq T$. Multiplying the first equation of (16) by $2\partial_t u_\varepsilon$, an integration on $\mathbb{R}$ gives

$$\begin{aligned}2\left\|\partial_t u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} =& 2b\int_{\mathbb{R}} P_\varepsilon \partial_t u_\varepsilon dx - 2\varepsilon\int_{\mathbb{R}} \partial_x^4 u_\varepsilon \partial_t u_\varepsilon dx - 6g\int_{\mathbb{R}} u_\varepsilon^2 \partial_x u_\varepsilon \partial_t u_\varepsilon dx\\ &+ 2a\int_{\mathbb{R}} \partial_x^3 u_\varepsilon \partial_t u_\varepsilon dx - 2q\int_{\mathbb{R}} u_\varepsilon \partial_x v_\varepsilon \partial_t u_\varepsilon dx\\ &- 2q\int_{\mathbb{R}} v_\varepsilon \partial_x u_\varepsilon \partial_t u_\varepsilon dx.\end{aligned} \tag{A25}$$

Since $0 < \varepsilon < 1$, thanks to (24), (41), (42), (43), (A5), (A17) and the Young inequality,

$$\begin{aligned}|2b|\int_{\mathbb{R}} |P_\varepsilon||\partial_t u_\varepsilon| dx =& 2\int_{\mathbb{R}} \left|\frac{bP_\varepsilon}{\sqrt{D_2}}\right|\left|\sqrt{D_2}\partial_t u_\varepsilon\right| dx\\ \leq& \frac{b^2}{D_2}\left\|P_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + D_2\left\|\partial_t u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ \leq& \frac{C(T)}{D_2} + D_2\left\|\partial_t u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\ 2\varepsilon\int_{\mathbb{R}} |\partial_x^4 u_\varepsilon||\partial_t u_\varepsilon| dx =& 2\int_{\mathbb{R}} \left|\frac{\varepsilon\partial_x^4 u_\varepsilon}{\sqrt{D_2}}\right|\left|\sqrt{D_2}\partial_t u_\varepsilon\right| dx\\ \leq& \frac{\varepsilon^2}{D_2}\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + D_2\left\|\partial_t u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ \leq& \frac{\varepsilon}{D_2}\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + D_2\left\|\partial_t u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ \leq& \frac{C(T)}{D_2} + D_2\left\|\partial_t u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\ |6g|\int_{\mathbb{R}} |u_\varepsilon^2\partial_x u_\varepsilon||\partial_t u_\varepsilon| dx =& 6\int_{\mathbb{R}} \left|\frac{gu_\varepsilon^2\partial_x u_\varepsilon}{\sqrt{D_2}}\right|\left|\sqrt{D_2}\partial_t u_\varepsilon\right| dx\\ \leq& \frac{3g^2}{D_2}\int_{\mathbb{R}} u_\varepsilon^4(\partial_x u_\varepsilon)^2 dx + 3D_2\left\|\partial_t u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ \leq& \frac{3g^2}{D_2}\left\|u_\varepsilon\right\|^4_{L^\infty((0,T)\times\mathbb{R})}\left\|\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 3D_2\left\|\partial_t u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ \leq& \frac{C(T)}{D_2} + 3D_2\left\|\partial_t u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\ |2a|\int_{\mathbb{R}} |\partial_x^3 u_\varepsilon||\partial_t u_\varepsilon| dx =& 2\int_{\mathbb{R}} \left|\frac{a\partial_x^3 u_\varepsilon}{\sqrt{D_2}}\right|\left|\sqrt{D_2}\partial_t u_\varepsilon\right| dx\\ \leq& \frac{a^2}{D_2}\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + D_2\left\|\partial_t u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ \leq& \frac{C(T)}{D_2} + D_2\left\|\partial_t u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\ |2q|\int_{\mathbb{R}} |u_\varepsilon\partial_x v_\varepsilon||\partial_t u_\varepsilon| dx =& 2\int_{\mathbb{R}} \left|\frac{qu_\varepsilon\partial_x v_\varepsilon}{\sqrt{D_2}}\right|\left|\sqrt{D_2}\partial_t u_\varepsilon\right| dx\\ \leq& \frac{q^2}{D_2}\int_{\mathbb{R}} u_\varepsilon^2(\partial_x v_\varepsilon)^2 dx + D_2\left\|\partial_t u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ \leq& \frac{q^2}{D_2}\left\|u_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\left\|\partial_x v_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + D_2\left\|\partial_t u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ \leq& \frac{C(T)}{D_2} + D_2\left\|\partial_t u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\end{aligned}$$

$$
\begin{aligned}
|2q| \int_{\mathbb{R}} |v_\varepsilon \partial_x u_\varepsilon||\partial_t u_\varepsilon| dx &= 2\int_{\mathbb{R}} \left|\frac{q v_\varepsilon \partial_x u_\varepsilon}{\sqrt{D_2}}\right| \left|\sqrt{D_2}\partial_t u_\varepsilon\right| dx \\
&\le \frac{q^2}{D_2}\int_{\mathbb{R}} v_\varepsilon^2 (\partial_x u_\varepsilon)^2 dx + D_2 \|\partial_t u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} \\
&\le \frac{q^2}{D_2}\|v_\varepsilon(t,\cdot)\|^2_{L^\infty(\mathbb{R})}\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + D_2 \|\partial_t u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} \\
&\le \frac{C(T)}{D_2} + D_2 \|\partial_t u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})},
\end{aligned}
$$

where D_2 is a positive constant, which will be specified later. Therefore, by (A25),

$$2\left(1-4D_2\right)\|\partial_t u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} \le \frac{C(T)}{D_2}.$$

Choosing $D_2 = \frac{1}{8}$, we have that

$$\|\partial_t u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} \le C(T),$$

which gives (A24). □

Arguing as in ([15], Lemma 2.12), we have the following result.

Lemma A5. *Assume* (5). *Let $T > 0$. There exists a constant $C(T) > 0$, independent on ε, such that*

$$
\begin{aligned}
\left\|\partial^2_{tx} v_\varepsilon(t,\cdot)\right\|_{L^\infty(\mathbb{R})}, \left\|\partial^2_{tx} v_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})} &\le C(T), \\
\|\partial_t v_\varepsilon(t,\cdot)\|_{L^\infty(\mathbb{R})}, \|\partial_t v_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})} &\le C(T),
\end{aligned}
\tag{A26}
$$

for every $0 \le t \le T$.

Using the Sobolev Immersion Theorem, we begin by proving the following result.

Lemma A6. *Fix $T > 0$. There exist a subsequence $\{(u_{\varepsilon_k}, v_{\varepsilon_k}, P_{\varepsilon_k})\}_{k\in\mathbb{N}}$ of $\{(u_\varepsilon, v_\varepsilon, P_\varepsilon)\}_{\varepsilon>0}$ and an a limit triplet (u, v, P) which satisfies* (11) *such that*

$$
\begin{aligned}
& u_{\varepsilon_k} \to u \text{ a.e. and in } L^p_{loc}((0,T)\times\mathbb{R}), 1 \le p < \infty, \\
& u_{\varepsilon_k} \rightharpoonup u \text{ in } H^1((0,T)\times\mathbb{R}), \\
& v_{\varepsilon_k} \to v \text{ a.e. and in } L^p_{loc}((0,T)\times\mathbb{R}), 1 \le p < \infty, \\
& v_{\varepsilon_k} \rightharpoonup v \text{ in } H^1((0,T)\times\mathbb{R}), \\
& P_{\varepsilon_k} \rightharpoonup P \text{ in } L^2((0,T)\times\mathbb{R}).
\end{aligned}
\tag{A27}
$$

Moreover, (u, v, P) is solution of (1) *satisfying* (12).

Proof. Let $0 \le t \le T$. We begin by observing that, thanks to Lemmas 3, 7, 9, A1 and A4,

$$\{u_\varepsilon\}_{\varepsilon>0} \text{ is uniformly bounded in } H^1((0,T)\times\mathbb{R}). \tag{A28}$$

Lemmas 3 and A5 say that

$$\{v_\varepsilon\}_{\varepsilon>0} \text{ is uniformly bounded in } H^1((0,T)\times\mathbb{R}). \tag{A29}$$

Instead, by Lemma 7, we have that

$$\{P_\varepsilon\}_{\varepsilon>0} \text{ is uniformly bounded in } L^2((0,T)\times\mathbb{R}). \tag{A30}$$

Equation (A28), (A29) and (A30) give (A27).

Observe that, thanks to Lemmas 3, 7, 9, A1 and the second equation of (16), we have that

$$P \in L^\infty(0,T;H^4(\mathbb{R})).$$

Lemmas 3, 7, 9, A1 say that

$$u \in L^\infty(0,T;H^3(\mathbb{R})).$$

Instead, thanks to Lemmas 3, 8, 10, A2 and (A26), we get

$$v \in L^\infty(0,T;H^5(\mathbb{R})) \cap W^{1,\infty}((0,T)\times\mathbb{R}).$$

Moreover, Lemmas A5 says also that

$$\partial_{tx}^2 v \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R}),$$

for every $0 \le t \le T$. Therefore, (11) holds and $(u,\, v,\, P)$ is solution of (1).

Finally, (12) follows from (19) and (A27). □

Now, we prove Theorem A1.

Proof of Theorem A1. Lemma A6 gives the existence of a solution of (1) such that (12) and (A27) hold. Arguing as in Theorem 1, we have (13). □

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