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A micro-mechanical model for the Biot theory of acoustic waves in a fully saturated granular material

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In the context of the classical Biot theory for acoustic waves in a fully-saturated granular material, we improve upon the constitutive relation of the solid phase by means of micro-mechanical modeling. This is needed to explain discrepancies on the dependency of the frequency with the sound speed attenuation and dispersion between present models and experiment. As first step, we provide a more detailed description of the interaction between the particles and water. A Standard Linear Solid model is proposed to represent the behavior at contact level between particles immersed in a compressible fluid.

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1. INTRODUCTION

Our goal is to develop a micro-mechanical model to predict compressional and shear waves in a fully-saturated aggregate of elastic particles in the context of the Biot's theory.[1] Important ingredients of the theory are the effective moduli, their value and how they vary depending on the degree of consolidation and the frequency of the waves that propagate in the aggregate.[2] We attempt to predict the effective moduli on the basis of a micro-mechanical approach. The derivation is characterized by two main steps: in the first, we focus on a typical pair and its contact interaction, in which particle deformation and fluid compressibility are taken in account; in the second, we adopt an averaging process that permits scaling up from the local force interaction between particles to the macroscopic stress of the aggregate, from which the effective moduli are derived.[3] Here, we restrict our attention to the first step and defer to a sequel[4] the completion of the model with a comparison between theory and laboratory results.

2. THEORY

We consider a fully-saturated aggregate of identical, elastic particles with radius R. The sample is slightly consolidated; so that when waves propagate in it, deletion between particles does not occur. The compressibility of the water, with viscosity μ, and the deformability of the particles, with Young modulus E, are taken in account in a lubrication model that describes the interaction between a typical pair of particles. As a result, an interaction force is derived that is represented by a visco-elastic network.

A. THE LOCAL MODEL

In Fig. 1 we show a pair of particles of the aggregate and the deformation model, based upon single springs in the solid and in the water. The relative displacement between particle centers is given by

$$\delta = \varepsilon - \varepsilon \sin\left(\omega t + \frac{\pi}{2}\right) \tag{1}$$

where ε is half of the maximum displacement while the distance between the surfaces of the particles, h, is

$$h\left(r,t\right) = 2R - \delta\left(r,t\right) - 2\left(R^2 - r^2\right)^{1/2} + 2\eta\left(r,t\right)\left[1 - \left(\frac{r}{R}\right)^2\right]^{1/2} - 2u\left(r,t\right). \tag{2}$$

We assume that there is always a film of water between particles and that both the deformation of the particles and the water are idealized by springs whose displacements are, respectively, η and u (positive in compression). The distance of the spring from the initial contact point is r. The particle deformation is idealized through springs that deform according to

$$\eta = 2Rp/E, \tag{3}$$

where p is the pressure exerted by the fluid on the solid particle. The vertical component of the pressure is

$$p_v = \frac{E}{2R}\eta\left[1 - \left(\frac{r}{R}\right)^2\right]^{1/2}, \tag{4}$$

which is equal to the pressure exerted by the particles on the compressible fluid:

$$p_v = \frac{2\kappa}{h}u, \tag{5}$$

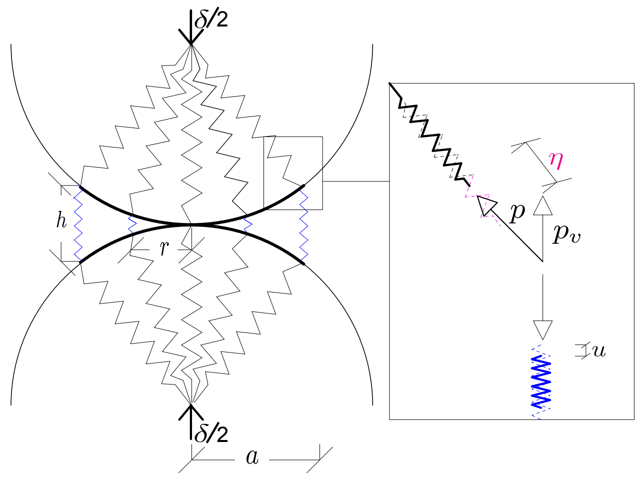

Figure 1: The solid and water springs model.

where κ is the compressibility of the water. If we define effective stiffnesses for the solid spring and the water, respectively $E/(2R)$ and $2\kappa/h$, we note that the former is constant while the latter is non linear through h. Equating Eqs.(4) and (5), we obtain the relation between the displacements u and η :

$$u = \frac{Eh}{2R\kappa}\eta\left[1-\left(\frac{r}{R}\right)^2\right]^{1/2}. \tag{6}$$

The particle-fluid interaction is governed by the lubrication equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(rh^3\frac{\partial p_v}{\partial r}\right) = 12\mu\frac{\partial h}{\partial t}. \tag{7}$$

This can be formulate as function of $\eta = \eta\,(r,t)$, so the boundary conditions are

$$\frac{\partial \eta(0,t)}{\partial r} = 0 \text{ and } \eta(1,t) = 0 \tag{8}$$

and the initial condition is

$$\eta\,(r,0) = 0. \tag{9}$$

We make the following normalizations

$$\hat{h} = h/(2R) \quad \hat{\delta} = \delta/(2R) \quad \hat{r} = r/R \quad \hat{\eta} = \eta/R \quad \hat{u} = u/R;$$

so, with Eqs.(2) and (6), the dimensionless surface distance $\hat{h}$ becomes

$$\hat{h} = \frac{1-\hat{\delta}-\left(1-\hat{r}^2\right)^{1/2}+\hat{\eta}\left(1-\hat{r}^2\right)^{1/2}}{1+\hat{\eta}\left(1-\hat{r}^2\right)^{1/2}E/\left(2\kappa\right)}, \tag{10}$$

and Eq. (7) becomes

$$\frac{1}{\hat{r}}\frac{\partial}{\partial \hat{r}}\left\{\hat{r}\hat{h}^3\frac{E}{2}\frac{\partial}{\partial \hat{r}}\left[\hat{\eta}\left(1-\hat{r}^2\right)^{1/2}\right]\right\} = 12\mu\frac{\partial \hat{h}}{\partial t}. \tag{11}$$

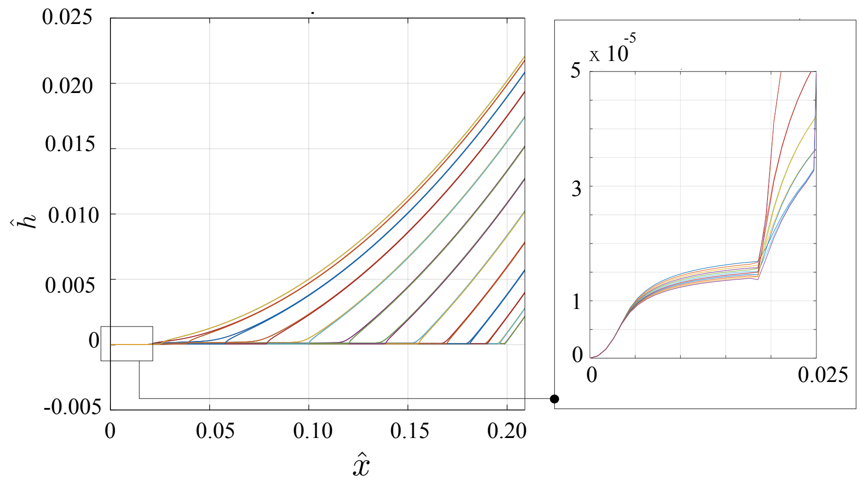

Figure 2: The dimensionless gap profile at different times.

With Eq.(10) , the solution of Eq.(11) permits the determination of $\hat{\eta}\left(\hat{r},t\right)$ for a given $\hat{\delta}$. In the final form, the differential equation that governs the problem is

$$\frac{1}{\hat{r}}\frac{\partial}{\partial\hat{r}}\left[\hat{r}\hat{h}^3\left(1-\hat{r}^2\right)^{1/2}\frac{\partial\hat{\eta}}{\partial\hat{r}}\right]-\frac{1}{\hat{r}}\frac{\partial}{\partial\hat{r}}\left[\frac{\hat{r}^2\hat{h}^3}{\left(1-\hat{r}^2\right)^{1/2}}\hat{\eta}\right] \tag{12}$$

$$=\quad\frac{24\mu/E}{1+\hat{\eta}\left(1-\hat{r}^2\right)^{1/2}E/\left(2\kappa\right)}\left[-\frac{\partial\hat{\delta}}{\partial t}+\left(1-\hat{r}^2\right)^{1/2}\left(1-\hat{h}\right)\frac{E}{2\kappa}\frac{\partial\hat{\eta}}{\partial t}\right]$$

with the initial condition

$$\hat{\eta}\left(\hat{r},0\right)=0, \tag{13}$$

and the boundary conditions

$$\frac{\partial\hat{\eta}(0,t)}{\partial\hat{r}}=0,\ \ \text{and}\ \hat{\eta}(1,t)=0. \tag{14}$$

B. NUMERICAL SOLUTIONS

The solution of the differential equation depends on R, E and κ. As an example , we take $E=1$ GPa and $\kappa=0.1$ GPa and $R=0.1$ mm. From the initial contact point, the two particles centers move according to Eq. (1), where we take $\varepsilon=10^{-2}R$. Both solid and water springs will oscillate, but no deletion between the two particles occurs. In Figs. 2, 3 and 4, we show the profiles of the gap h, the solid and water spring displacements at different times, and it is clear that most of the activity occurs at $\hat{x}<0.2$. This length is indicated with a and it is defined as the greatest value of the radius r for which there would be an overlap of rigid spheres for a given ε. In this small region, the water will be squeezed out by a very high pressure, which induces a deformation of the particles' surface. We distinguish two regions: the first, limited by a, where h goes almost to zero and the highest pressure of the fluid over the solid springs occurs; the second, where particles deformation is negligible and the fluid is simply compressed. The radius a seems to be independent of the frequency. Near the origin (inset of Fig. 2), only the elasticity and not the flux of the water plays a role.

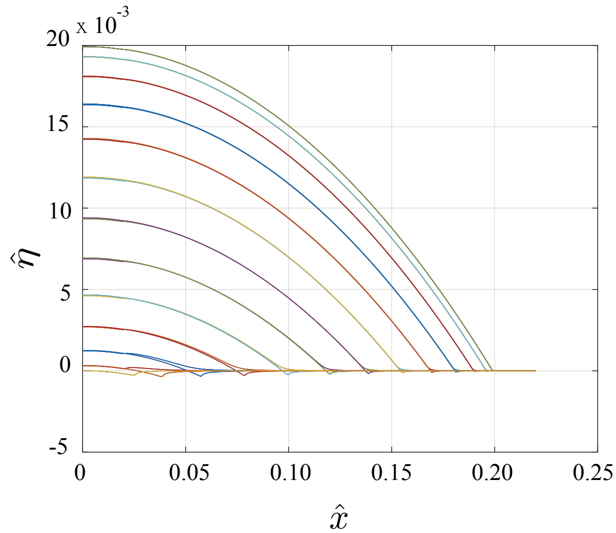

Figure 3: The dimensionless solid spring displacement profiles at different times.

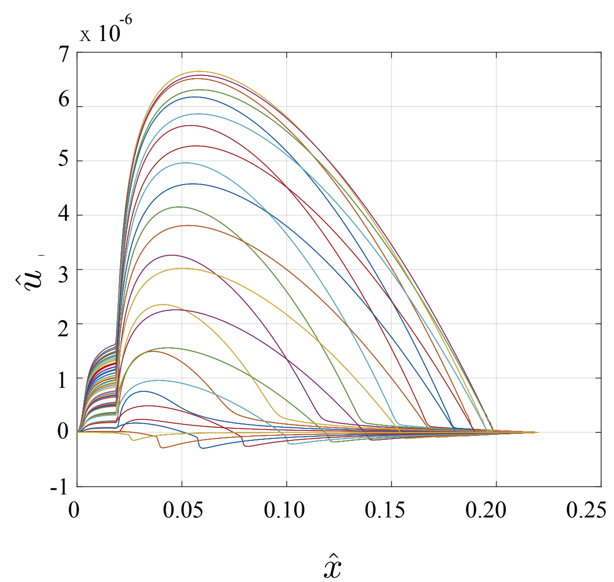

Figure 4: The dimensionless water spring displacement profiles at different times.

Given $\eta(r,t)$, solution of the lubrication equation, we determine the total compressive force as an integration of the springs that act over a circular area of radius a

$$Q(t) = \int_0^{2\pi} \int_0^a r p_v(r,t) dr d\theta. \tag{15}$$

In Fig. 5, we plot the oscillating displacement ε, given by Eq.(1), and the output force $Q(t)$ given by Eq. (15). We note that $Q(t)$ is not in phase with ε. We represent the interaction between the particles through a Standar Linear System (SLS) model, see Fig. 6, where we have to determine the costant k_1, k_2 and μ_2. When the SLS model deforms according to Eq.(1) , the output force is

$$F(t) = \varepsilon k_1 \left[1 - \cos(\omega t)\right] - \varepsilon k_2 \frac{(\omega/\tilde{\omega})^2}{1 + (\omega/\tilde{\omega})^2} \cos(\omega t) + \varepsilon k_2 \frac{\omega/\tilde{\omega}}{1 + (\omega/\tilde{\omega})^2} \sin(\omega t), \tag{16}$$

where $\tilde{\omega} = k_2/\mu_2$ is the characteristic frequency of the Maxwell arm and also a measure of the out-of-phase response of the SLS device with respect to the input vibration δ. In order to determine the parameters of the model, k_1, k_2 and μ_2, we solve the differential equation for three different frequencies, $0.5, 5$ and 10 KHz and compare the corresponding $Q(t)$ with $F(t)$ given by Eq. (16). An example of best fit comparison is given in Fig. 5, for 500 Hz. We note that when δ reaches its maximum value (particles are compressed), the force $Q(t)$ is in phase; when δ is almost zero there is an out-of-phase response which increases with the frequency. Close to the minimum value of δ, $Q(t)$ becomes slightly negative and then increases again.

In general we define a low and high frequency regime when we compare the given ω with $\tilde{\omega}$. So, at low frequency, $\omega << \tilde{\omega}$, $\omega/\tilde{\omega} \to 0$ and the response is

$$F(t) = \varepsilon k_1 \left[1 - \cos(\omega t)\right] \tag{17}$$

while at high frequency $\omega >> \tilde{\omega}$, then $\omega/\tilde{\omega} \to \infty$ and

$$F(t) = \varepsilon k_1 \left[1 - \cos(\omega t)\right] - \varepsilon k_2 \cos(\omega t). \tag{18}$$

When $\omega/\tilde{\omega}$ is finite, the response is out of phase. The numerical comparison produces the following result: $k_1 = 1.0$ KN/m , $k_2 = 0.7$ KN/m and $\mu_2 = 0.03$ N s/m, while the characteristic frequency $\tilde{\omega} \approx 3.7$ KHz. With these values the interaction force in Eq. (16) is a candidate to represent the interaction force between a typical pair. The next step is to employ a proper average to derive the effective moduli of the aggregate and test the influence in the Biot's theory.[4]

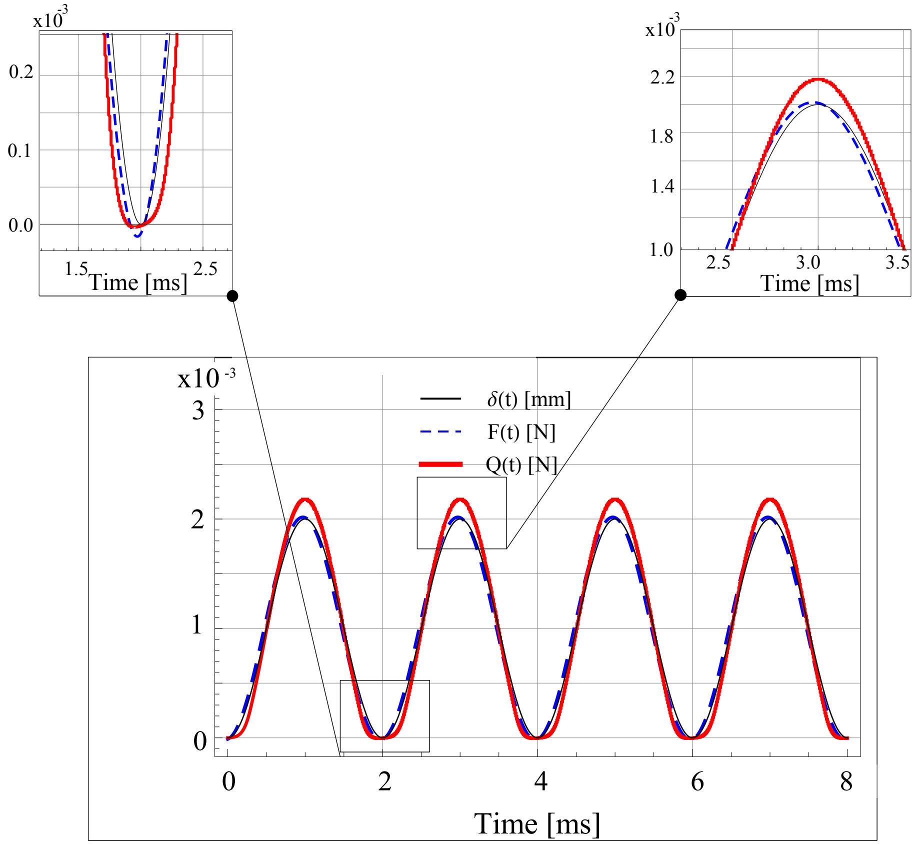

Figure 5: The comparison between the oscillating displacement, δ, the resultant force $Q(t)$, and the SLS model output force $F(t)$.

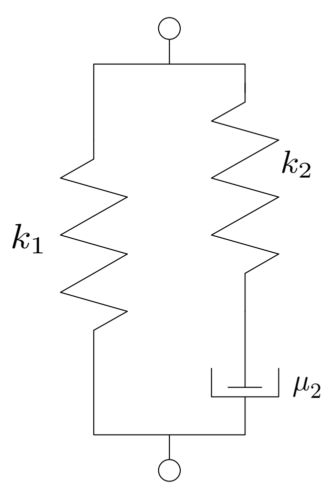

Figure 6: The Standar Linear System (SLS) adopted to simulate the particle-particle interaction with an interstitial fluid.

3. CONCLUSION

A particle-particle interaction has been studied, including the influence of the water between the grains. It is possible to identify the particle interaction behavior when a wave propagates in the aggregate through an SLS model that introduces a frequency dependency. This may have some effect on the effective moduli over the entire aggregate. In this respect, Chotiros and Isakson[5] have employed a SLS model to understand sound speed dispersion with a modification of the Biot theory through a squirt flow mechanism.[6] Our characteristic frequency prediction, $\tilde{\omega}$, is quantitative close to what is estimated by Chotiros and Isakson.[5]

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