Sobolev–Morrey regularity of solutions to general quasilinear elliptic equations

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Abstract

We deal with the Dirichlet problem for general quasilinear elliptic equations over Reifenberg flat domains. The principal part of the operator supports natural gradient growth and its x-discontinuity is of small-BMO type, while the lower order terms satisfy controlled growth conditions with x-behaviour modeled by Morrey spaces. We obtain a Calderón–Zygmund type result for the gradient of the weak solution by proving that the solution gains Sobolev–Morrey regularity from the data of the problem.

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1. Introduction

The regularity problem is one of the central topics in the general theory of PDEs. Its main goal is to establish how the smoothness of the data of a given differential problem influences the regularity of a solution, obtained under

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very general circumstances. Once having better smoothness, powerful tools of functional analysis apply to infer finer properties of the solution and the problem itself. The importance of these issues is even more evident if dealing with discontinuous differential operators over domains with non-smooth boundaries when many of the classical analysis techniques fail.

In the present paper we study the regularity problem in Sobolev–Morrey spaces for quasilinear divergence form elliptic equations. We are interested in obtaining an optimal Calderón-Zygmund type theory in such spaces under minimal assumptions to impose on the discontinuous nonlinearities and on the non-smooth underlying domain. We deal, precisely, with the Dirichlet problem

$$\begin{cases} \operatorname{div}\big(\mathbf{a}(x,u,Du)+\mathbf{b}(x,u)\big)=c(x,u,Du) & \text{in } \Omega \\ u=0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

over bounded n-dimensional domains Ω and where the nonlinear terms $\mathbf{a}$, $\mathbf{b}$ and c are given by suitable Carathéodory maps.

We suppose that the generally non-smooth boundary of Ω is sufficiently flat in the sense of Reifenberg that means, roughly speaking, $\partial\Omega$ is well approximated by hyperplanes at each point and at each scale. This is a sort of "minimal boundary regularity" ensuring the validity of the main geometric analysis results in Ω, and it has proved to be a natural assumption to be required on $\partial\Omega$ when dealing with regularity problems for divergence form PDEs. In particular, C^1-smooth and Lipschitz continuous boundaries (with small Lipschitz constant) belong to that class, but the category of Reifenberg flat domains extends beyond these common examples and contains sets with rough fractal boundaries such as the Helge von Koch snowflake (see [17]).

The principal part $\mathbf{a}(x,u,Du)$ of the differential operator is supposed to be elliptic, measurable in x and it supports natural gradient growth, that is, $\mathbf{a}(x,u,Du)$ behaves as $|Du|^{m-1}$ with $m>1$. The most notable, by now classical, example is given by the m-Laplacian $|Du|^{m-2}Du$, but our results apply also to $\mathbf{A}(x,u)|Du|^{m-2}Du$ with a suitable elliptic matrix $\mathbf{A}$ and to more general operators which do not possess necessarily a variational structure. As for the lower order terms $\mathbf{b}$ and c, these are subject to controlled growth conditions

$$\begin{aligned} |\mathbf{b}(x,u)| &\le \mathcal{O}\left(\varphi(x)+|u|^{\frac{m^*(m-1)}{m}}\right), \\ |c(x,u,Du)| &\le \mathcal{O}\left(\psi(x)+|u|^{m^*-1}+|Du|^{\frac{m(m^*-1)}{m^*}}\right), \end{aligned}$$

with the Sobolev conjugate m^* of m, and suitable Lebesgue integrable functions φ and ψ. It is worth noting that the growth requirements on the nonlinearities in (1.1) are indispensable in order to give sense of the concept of a weak solution $u\in W_0^{1,m}(\Omega)$ to (1.1), but these are very far from being sufficient to ensure better integrability of the gradient than that in $L^m(\Omega)$.

The importance of studying discontinuous problems of the type (1.1) over rough domains is justified by the fact that these arise naturally in mathematical models of real-world systems over media with fractal geometry such as blood vessels, composite materials and semiconductor devices, the internal structure of lungs, clouds, bacteria growth, optimal control of stock markets and the economic applications of the non-smooth variational analysis (see [11, 14]). The x-discontinuity of $\mathbf{a}(x,u,Du)$, instead, could be regarded to crack ruptures of the media, such as small multipliers of the Heaviside step function for instance.

We are interested here of the case when φ and ψ control the x-behaviour of $\mathbf{b}$ and c in terms of the Morrey spaces $L^{p,\lambda}(\Omega)$ and $L^{q,\mu}(\Omega)$, respectively, with exponents satisfying $(m-1)p+\lambda>n$ and $mq+\mu>n$. Assuming that the discontinuity with respect to x in $\mathbf{a}(x,u,Du)$ is measured in terms of smallness of the bounded mean oscillation (BMO) seminorm, and that $\partial\Omega$ is Reifenberg flat, we prove that the problem (1.1) supports the Calderón–Zygmund property in the Sobolev–Morrey functional scales. In other words, the gradient of each $W_0^{1,m}(\Omega)$ weak solution to (1.1) gains better Lebesgue and Morrey integrability from φ and ψ, both strictly defined by n,m and the exponents p,λ,q and μ. As a consequence, Hölder continuity up to the boundary follows for the weak solution with optimal exponent expressed in terms of n,m,p,λ,q and μ.

This article is a natural outgrowth of the papers [5] and [10] where weak solutions to the equation $\operatorname{div}\mathbf{a}(x,Du)=\operatorname{div}\big(|\mathbf{F}|^{m-2}\mathbf{F}\big)$ were considered. In particular, [5] provides gradient estimates in weighted Lebesgue spaces, while in [10] weighted Lorentz bounds have been proved for the gradient. In the both cases, gradient estimates in Morrey spaces do follow. Here we extend the results from [5, 10] to more general class of differential operators by analysing the exact influence of the lower order terms on the resulting Morrey regularity of the gradient. In that sense, our results generalize also these from [14] and [2], where the $W_0^{1,2}(\Omega)$-regularity problem was studied for *semilinear* elliptic operators, $\mathbf{a}(x,u,Du)=a^{ij}(x,u)D_ju$, with φ and ψ taken in Lebesgue or in Morrey spaces, respectively.

The first step to prove the main result (Theorem 2.2) is ensured by our recent paper [4] (see the announcement in [3] also), where (1.1) has been studied in very rough domains satisfying a sort of variational capacity thickness condition and where only *measurability* with respect to x is required in $\mathbf{a}(x,u,Du)$. Since the Reifenberg flat domains satisfy the thickness condition, the restrictions imposed on the exponents p,λ,q and μ, and the results from [4] guarantee that each $W_0^{1,m}(\Omega)$ weak solution to (1.1) supports the Gehring–Giaquinta–Modica property, that is, $u\in W^{1,m_0}(\Omega)$ with $m_0\in(m,m+\varepsilon)$. Moreover, a De Giorgi type result holds in the sense that u is globally bounded and Hölder continuous up to $\partial\Omega$ with *some* exponent $\alpha\in(0,1)$. At this point the composition $\mathbf{A}(x,\xi)=\mathbf{a}(x,u(x),\xi)$ turns out to have *small BMO seminorm* in x that allows to apply the recent regularity results from [5] to the problem (1.1). Making use of the mapping properties of Riesz potential in Morrey spaces, we apply a bootstrapping procedure to improve the rate of the Lebesgue integrability of Du, after that a similar approach improves also the Morrey exponent of the gradient. A central role in this approach is played by the growth assumptions imposed on the nonlinearities as well as by the restrictions on the exponents p,λ,q and μ, which turn out to be also optimal as shown in [4]. In the particular case when $\lambda=\mu=0$, Theorem 2.2 implies Calderón–Zygmund property for (1.1) also in the framework of the classical Sobolev scales.

2. Hypotheses and Main Results

Throughout the paper, we will use standard notations and will assume that the functions and sets considered are measurable.

We denote by $B_\rho(x)$ (or simply B_ρ if there is no ambiguity) the n-dimensional open ball with center $x \in \mathbb{R}^n$ and radius ρ. The Lebesgue measure of a measurable set $E \subset \mathbb{R}^n$ will be denoted by $|E|$ while, for any integrable function u defined on a set A, its integral average is given by

$$\overline{u}_A := \fint_A u(x)\, dx = \frac{1}{|A|}\int_A u(x)\, dx.$$

We will denote by $C_0^\infty(\Omega)$ the space of infinitely differentiable functions over a bounded domain $\Omega \subset \mathbb{R}^n$ with compact support contained in that domain, and $L^p(\Omega)$ stands for the standard Lebesgue space with a given $p \in [1,\infty]$. The Sobolev space $W_0^{1,p}(\Omega)$ is defined, as usual, by the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{W^{1,p}(\Omega)} := \|u\|_{L^p(\Omega)} + \|Du\|_{L^p(\Omega)}$$

for $p \in [1,\infty)$.

Given $s \in [1,\infty)$ and $\theta \in [0,n]$, the Morrey space $L^{s,\theta}(\Omega)$ is the collection of all functions $u \in L^s(\Omega)$ such that

$$\|u\|_{L^{s,\theta}(\Omega)} := \sup_{x_0\in\Omega,\ \rho>0} \left(\rho^{-\theta}\int_{B_\rho(x_0)\cap\Omega} |u(x)|^s\, dx\right)^{1/s} < \infty.$$

The space $L^{s,\theta}(\Omega)$ equipped with the norm $\|\cdot\|_{L^{s,\theta}(\Omega)}$ is a Banach space and the limit cases $\theta = 0$ and $\theta = n$ give rise, respectively, to $L^s(\Omega)$ and $L^\infty(\Omega)$.

For $0 < \alpha < n$, the Riesz potential $I_\alpha f$ of a locally integrable function f on $\mathbb{R}^n$ is defined by

$$(I_\alpha f)(x) := \frac{1}{c_\alpha}\int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy$$

with a constant c_α, expressed in terms of the Euler Γ-function

$$c_\alpha = \pi^{n/2}2^\alpha \frac{\Gamma(\alpha/2)}{\Gamma((n-\alpha)/2)}.$$

In what follows we will consider a bounded domain $\Omega \subset \mathbb{R}^n$ with $n \geq 2$, the boundary $\partial\Omega$ of which is *Reifenberg flat* in the sense of the following definition.

Definition 2.1. *The domain Ω is said to be (δ, R)-Reifenberg flat if there exist positive constants δ and R with the property that for each $x_0 \in \partial\Omega$ and each $\rho \in (0,R)$ there is a local coordinate system $\{x_1,\cdots,x_n\}$ with origin at the point x_0 such that*

$$B_\rho(x_0) \cap \{x : x_n > \rho\delta\} \subset B_\rho(x_0)\cap\Omega \subset B_\rho(x_0)\cap\{x : x_n > -\rho\delta\}. \qquad (2.1)$$

Let us note that in the above definition R could be taken 1 by the scaling invariance, while (2.1) makes sense for $0<\delta<2^{-n-1}$ (see [17]). The Reifenberg flatness means that the boundary $\partial\Omega$ is well approximated by hyperplanes at every point and at every scale and it is a sort of a "minimal regularity requirement" to impose on $\partial\Omega$ to ensure the validity in Ω of the main natural properties of the geometric analysis. In particular, (2.1) holds in the cases of C^1-smooth, or Lipschitz continuous boundaries with small Lipschitz constant, and domains with Reifenberg flat boundaries satisfy the known (A)-property of Ladyzhenskaya and Ural'tseva (cf. [14, 2]).

Turning back to the Dirichlet problem (1.1), the nonlinearities considered are given by the Carathéodory maps $\mathbf{a}\colon \Omega\times\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}^n$, $\mathbf{b}\colon\Omega\times\mathbb{R}\to\mathbb{R}^n$ and $c\colon\Omega\times\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$, where $\mathbf{a}(x,z,\xi)=\big(a^1(x,z,\xi),\cdots,a^n(x,z,\xi)\big)$ and $\mathbf{b}(x,z)=\big(b^1(x,z),\cdots,b^n(x,z)\big)$. We suppose moreover that $\mathbf{a}(x,z,\xi)$ is differentiable with respect to ξ and $D_\xi\mathbf{a}$ is a Carathéodory map.

Throughout the paper, $m>1$ and the following structure and regularity conditions on the data will be assumed:

• *Uniform ellipticity:* There exists a constant $\gamma>0$ such that

$$\begin{cases}\gamma|\xi|^{m-2}|\eta|^2\leq\langle D_\xi\mathbf{a}(x,z,\xi)\eta,\eta\rangle,\\ |\mathbf{a}(x,z,\xi)|+|\xi||D_\xi\mathbf{a}(x,z,\xi)|\leq\gamma^{-1}|\xi|^{m-1}\end{cases}\tag{2.2}$$

for a.a. $x\in\Omega$, all $(z,\xi)\in\mathbb{R}\times\mathbb{R}^n$ and all $\eta\in\mathbb{R}^n$.

• *Controlled growth conditions:* There exist a constant $\Lambda>0$ and nonnegative functions $\varphi\in L^{p,\lambda}(\Omega)$ with $p>\frac{m}{m-1}$, $\lambda\in[0,n)$ and $(m-1)p+\lambda>n$, and $\psi\in L^{q,\mu}(\Omega)$ with $q>\max\left\{1,\frac{mn}{mn+m-n}\right\}$, $\mu\in[0,n)$ and $mq+\mu>n$, such that

$$\begin{cases}|\mathbf{b}(x,z)|\leq\Lambda\left(\varphi(x)+|z|^{\frac{m^*(m-1)}{m}}\right),\\ |c(x,z,\xi)|\leq\Lambda\left(\psi(x)+|z|^{m^*-1}+|\xi|^{\frac{m(m^*-1)}{m^*}}\right)\end{cases}\tag{2.3}$$

for a.a. $x\in\Omega$ and all $(z,\xi)\in\mathbb{R}\times\mathbb{R}^n$. Here, m^* is the Sobolev conjugate of m that is given by

$$m^*=\begin{cases}\frac{nm}{n-m} & \text{if } m<n,\\ \text{arbitrary large number}>m & \text{if } m\geq n.\end{cases}\tag{2.4}$$

• *Local uniform continuity:* For each $M>0$ there is a nondecreasing function $\sigma_M:\mathbb{R}^+\to\mathbb{R}^+$ with $\lim_{t\to0^+}\sigma_M(t)=0$ such that

$$|\mathbf{a}(x,z_1,\xi)-\mathbf{a}(x,z_2,\xi)|\leq\sigma_M(|z_1-z_2|)|\xi|^{m-1}\tag{2.5}$$

for a.a. $x\in\Omega$, all $z_1,z_2\in[-M,M]$ and all $\xi\in\mathbb{R}^n$.

• (δ,R)*-vanishing property:* For each constant $M>0$ there exist $R_M>0$ and $\delta_M>0$ such that

$$\sup_{z\in[-M,M]}\ \sup_{0<\rho\leq R_M}\ \sup_{y\in\mathbb{R}^n}\fint_{B_\rho(y)}\Theta\big(\mathbf{a};B_\rho(y)\big)(x,z)\,dx\leq\delta_M.\tag{2.6}$$

Here the function Θ is defined by

$$\Theta\big(\mathbf{a}; B_\rho(y)\big)(x,z) := \sup_{\xi\in\mathbb{R}^n\setminus\{0\}} \frac{|\mathbf{a}(x,z,\xi)-\overline{\mathbf{a}}_{B_\rho(y)}(z,\xi)|}{|\xi|^{m-1}}, \tag{2.7}$$

where $\overline{\mathbf{a}}_{B_\rho(y)}(z,\xi)$ is the integral average of $\mathbf{a}(x,z,\xi)$ in the variables x for the fixed $(z,\xi)\in\mathbb{R}\times\mathbb{R}^n$, that is,

$$\overline{\mathbf{a}}_{B_\rho(y)}(z,\xi) = \fint_{B_\rho(y)} \mathbf{a}(x,z,\xi)\,dx.$$

It is clear that (2.6) requires a sort of small-BMO behaviour of $\mathbf{a}(x,z,\xi)$ with respect to x and it is automatically satisfied when $\mathbf{a}(\cdot,z,\xi)$ is continuous or VMO in Ω.

A typical example of a nonlinear differential operator satisfying the above hypotheses is that given by $\mathbf{a}(x,z,\xi)=A(x,z)|\xi|^{m-2}\xi$ where $A(x,z)\in L^\infty(\Omega\times\mathbb{R})$, $A(x,z)\geq\gamma>0$ and $A(\cdot,z)$ is of small-BMO for all z.

Recall that a function $u\in W_0^{1,m}(\Omega)$ is called *weak solution* to the Dirichlet problem (1.1) if

$$\int_\Omega \Big(\big(\mathbf{a}(x,u(x),Du(x))+\mathbf{b}(x,u(x))\big)\cdot Dv(x) \\ +c(x,u(x),Du(x))v(x)\Big)\,dx = 0 \tag{2.8}$$

for each test function $v\in W_0^{1,m}(\Omega)$. It is worth noting that the convergence of the integrals involved in (2.8) for all admissible u and v is ensured by (2.2) and (2.3) under the *sole* assumptions $p\geq\frac{m}{m-1}$ and $q\geq\frac{mn}{mn+m-n}$ when $m<n$, $q>1$ if $m=n$, $q\geq 1$ if $m>n$.

In what follows, given any two exponents $s>1$ and $\theta\in(0,n)$, we set

$$s^*_\theta = \begin{cases} \frac{(n-\theta)s}{n-\theta-s} & \text{if } s+\theta<n,\\ \text{arbitrary large number} & \text{if } s+\theta\geq n \end{cases} \tag{2.9}$$

for the *Sobolev–Morrey* conjugate of s.

We will use in the sequel the omnibus phrase "*known quantities*" which means that a given constant depends on the data in hypotheses (2.2)–(2.6), which include n, m, m^*, γ, Λ, p, λ, q, μ, $\|\varphi\|_{L^{p,\lambda}(\Omega)}$, $\|\psi\|_{L^{q,\mu}(\Omega)}$, $\operatorname{diam}\Omega$, δ, R, σ_M, R_M and δ_M, and the letter C will stand for a generic constant, depending on known quantities, which may vary within the same formula.

The main results of the paper is the following

Theorem 2.2. *Suppose* (2.2)*,* (2.3)*,* (2.5) *and* (2.6)*, and let $u\in W_0^{1,m}(\Omega)$ be a weak solution to the Dirichlet problem* (1.1)*.*

There exists a small $\delta_0>0$ such that if Ω is (δ,R)-Reifenberg flat and $\mathbf{a}(x,z,\xi)$ is (δ,R)-vanishing in the sense of (2.6) *with $\delta<\delta_0$, then*

$$u\in L^\infty(\Omega)\cap W^{1,(m-1)r}(\Omega) \quad \textit{with} \quad r=\min\{p,q^*_\mu\}.$$

Moreover, the gradient Du belongs to an appropriate Morrey space,

$$Du \in L^{(m-1)r,\nu}(\Omega) \quad \textit{with} \quad \nu = \min\left\{\frac{r(\lambda-n)}{p}+n, \frac{r(\mu-n)}{q^*_\mu}+n\right\}.$$

Indeed, in the particular case $\lambda = \mu = 0$ when $\varphi \in L^p(\Omega)$ with $(m-1)p > n$ and $\psi \in L^q(\Omega)$ with $mq > n$, we get $Du \in L^{(m-1)r}(\Omega)$ where $r = \min\{p, q^*\}$. Thus Theorem 2.2 provides a Calderón–Zygmund property in Sobolev spaces for the weak solutions to (1.1), extending this way the semilinear result from [14] to more general quasilinear elliptic equations.

An immediate consequence of Theorem 2.2, Proposition 3.3, $(m-1)p+\lambda > n$ and $mq + \mu > n$ is the following global Hölder continuity of the weak solutions to (1.1).

Corollary 2.3. *Under the assumptions of Theorem* 2.2*, each* $W^{1,m}_0(\Omega)$*-weak solution to* (1.1) *is Hölder continuous in* $\overline{\Omega}$*,*

$$u \in C^{0,\min\left\{1-\frac{n-\lambda}{(m-1)p}, 1-\frac{n-\mu-q}{(m-1)q}\right\}}(\overline{\Omega}).$$

As shown in Lemma 3.8 below, thanks to (2.2) and (2.3), the weak solutions of (1.1) own always some Hölder continuity. What Corollary 2.3 gives is the *exact* Hölder exponent defined by the hypotheses imposed on the data φ and ψ. In particular, when $m > n$, we have automatically $u \in W^{1,m}_0(\Omega) \subset C^{1-n/m}(\overline{\Omega})$ by the Sobolev imbedding and the Morrey lemma, while

$$\min\left\{1-\frac{n-\lambda}{(m-1)p}, 1-\frac{n-\mu-q}{(m-1)q}\right\} > 1-\frac{n}{m}$$

now and thus Corollary 2.3 ensures better Hölder continuity for the solution.

At this end, it is worth noting that, even if we are mainly dealing with regularity issues here, the $W^{1,m}_0(\Omega)$-weak solvability of (1.1) can be obtained by standard techniques, such as Galerkin or Minty–Browder methods (see [9, 8]), under the above hypotheses, and imposing additional assumptions on the lower order terms to ensure the desired coercivity of the left-hand side of (2.8) (see also [12, 13] for some particular existence results).

3. Auxiliary results

For the sake of completeness, we present here some auxiliary assertions to be used in the proof of Theorem 2.2.

3.1. Basic facts about Morrey spaces

Proposition 3.1. (Embeddings between Morrey spaces, [15]) *For arbitrary* $s', s'' \in [1, \infty)$ *and* $\theta', \theta'' \in [0, n)$*, one has*

$$L^{s',\theta'}(\Omega) \subseteq L^{s'',\theta''}(\Omega)$$

if and only if

$$s' \geq s'' \geq 1 \quad \textit{and} \quad \frac{s'}{n-\theta'} \geq \frac{s''}{n-\theta''}.$$

Proposition 3.2. (Mapping properties of Riesz potentials, [1, Theorem 3.1]) *Let $s \in (1,\infty)$ and $\theta \in [0,n)$. Then the Riesz potential I_1 is bounded from $L^{s,\theta}(\Omega)$ into $L^{s^*_\theta,\theta}(\Omega)$, that is,*

$$\|I_1 f\|_{L^{s^*_\theta,\theta}(\Omega)} \le C(n,s,\theta,\Omega)\|f\|_{L^{s,\theta}(\Omega)}.$$

As already mentioned above, a Reifenberg flat domain supports the (A)-property of Ladyzhenskaya and Ural'tseva (cf. [14, 2]), whence [6, Lemma 3.III, Lemma 3.IV] give the next result.

Proposition 3.3. (Regularity of functions with gradients in Morrey spaces) *Assume that Ω is a Reifenberg flat domain and let $u \in W^{1,s}(\Omega)$ with $Du \in L^{s,\theta}(\Omega)$, $\theta \in [0,n)$. Then*
(1) *If $s+\theta < n$ then $u \in L^{\frac{ns}{n-s},\frac{n\theta}{n-s}}(\Omega) \subset L^{s,\theta+s}(\Omega)$ with*

$$\|u\|_{L^{s,\theta+s}(\Omega)} \le \|u\|_{L^{\frac{ns}{n-s},\frac{n\theta}{n-s}}(\Omega)} \le C(n,s,\theta,\partial\Omega)\left(\|u\|_{L^s(\Omega)} + \|Du\|_{L^{s,\theta}(\Omega)}\right).$$

(2) *If $s+\theta = n$ then $u \in L^{s',\theta'}(\Omega)$ for any $s' < \infty$ and any $\theta' < n$, and*

$$\|u\|_{L^{s',\theta'}(\Omega)} \le C(n,s,\theta,\partial\Omega)\left(\|u\|_{L^s(\Omega)} + \|Du\|_{L^{s,\theta}(\Omega)}\right).$$

(3) *If $s+\theta > n$ then $u \in C^{0,\alpha}(\overline{\Omega})$ with $\alpha = 1 - \frac{n-\theta}{s}$ and*

$$\sup_{\overline{\Omega}} |u(x)| + \sup_{x,y\in\overline{\Omega},\ x\neq y} \frac{|u(x)-u(y)|}{|x-y|^\alpha} \le C(n,s,\theta,\partial\Omega)\left(\|u\|_{L^s(\Omega)} + \|Du\|_{L^{s,\theta}(\Omega)}\right).$$

3.2. Nonlinear elliptic equations

Consider the following Dirichlet problem

$$\begin{cases} \operatorname{div}\big(\mathbf{A}(x,Du(x))\big) = \operatorname{div}\big(|\mathbf{F}(x)|^{m-2}\mathbf{F}(x)\big) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where $\mathbf{F} = (f^1,\cdots,f^n) \in L^m(\Omega,\mathbb{R}^n)$ is a given vector-valued function and the nonlinearity $\mathbf{A}(x,\xi)$ together with $D_\xi\mathbf{A}(x,\xi)$ are Carathéodory maps satisfying the following conditions

$$\begin{cases} \gamma|\xi|^{m-2}|\eta|^2 \le \langle D_\xi\mathbf{A}(x,\xi)\eta,\eta\rangle, \\ |\mathbf{A}(x,\xi)| + |\xi||D_\xi\mathbf{A}(x,\xi)| \le \gamma^{-1}|\xi|^{m-1}. \end{cases} \tag{3.2}$$

According to the *Minty–Browder method*, (3.1) possesses a unique weak solution $u \in W_0^{1,m}(\Omega)$ and

$$\|Du\|_{L^m(\Omega)} \le C\|\mathbf{F}\|_{L^m(\Omega)}$$

with a constant C depending only on γ, n, m and Ω ([16, Chapter 2]).

Definition 3.4. *We say that the vector field* $\mathbf{A}(x,\xi)$ *is* (δ,R)*-vanishing if there exist* $\delta, R>0$ *such that*

$$\sup_{0<\rho\le R}\sup_{y\in\mathbb{R}^n}\fint_{B_\rho(y)}\Theta\big(\mathbf{A};B_\rho(y)\big)(x)\,dx\le\delta,$$

where

$$\Theta\big(\mathbf{A};B_\rho(y)\big)(x)=\sup_{\xi\in\mathbb{R}^n\setminus\{0\}}\frac{|\mathbf{A}(x,\xi)-\overline{\mathbf{A}}_{B_\rho(y)}(\xi)|}{|\xi|^{m-1}}.$$

As consequence of the weighted $L^p(\Omega)$-theory, we have the following result regarding gradient Morrey regularity of the weak solution to (3.1).

Lemma 3.5. (see [5]) *For each* $s\in(m,\infty)$ *and each* $\theta\in[0,n)$*, there exist a small positive constant* δ *and a constant* C*, depending on* γ*,* n*,* m*,* s*,* θ *and* Ω*, such that if* $\mathbf{A}$ *is* (δ,R)*-vanishing,* Ω *is* (δ,R)*-Reifenberg flat and* $\mathbf{F}\in L^{s,\theta}(\Omega,\mathbb{R}^n)$*, then the unique weak solution* $u\in W_0^{1,m}(\Omega)$ *of the problem* (3.1) *satisfies* $Du\in L^{s,\theta}(\Omega)$ *and*

$$\|Du\|_{L^{s,\theta}(\Omega)}\le C\|\mathbf{F}\|_{L^{s,\theta}(\Omega)}.$$

3.3. (δ,R)*-vanishing properties of superposition operators*

Lemma 3.6. *Under the assumptions* (2.5) *and* (2.6)*, for each* $u\in C^0(\overline{\Omega})$ *the composition* $\mathbf{A}(x,\xi):=\mathbf{a}(x,u(x),\xi)$ *is* (δ,R)*-vanishing with* $\delta=\delta_M+2\sigma_M(\omega_u(R_M))$*,* $R=R_M$*,* $M=\|u\|_{L^\infty(\Omega)}$ *and* $\omega_u(\cdot)$ *is the modulus of continuity of the function* u*.*

PROOF. Set $M=\|u\|_{L^\infty(\Omega)}$ and take $R=R_M$ as given by (2.6). Let $x,y\in\mathbb{R}^n$, $0<\rho\le R$, $|x-y|<\rho$ and $\xi\in\mathbb{R}^n\setminus\{0\}$. In view of triangle inequality, we have

$$\begin{aligned}\big|\mathbf{A}(x,\xi)-\overline{\mathbf{A}}_{B_\rho(y)}(\xi)\big|\le\ &|\mathbf{a}(x,u(x),\xi)-\mathbf{a}(x,u(y),\xi)|\\&+\big|\overline{\mathbf{a}}_{B_\rho(y)}(u(y),\xi))-\overline{\mathbf{A}}_{B_\rho(y)}(\xi)\big|\\&+\big|\mathbf{a}(x,u(y),\xi)-\overline{\mathbf{a}}_{B_\rho(y)}(u(y),\xi))\big|\\=:\ &J_1+J_2+J_3.\end{aligned}$$

Since $|u|\le M$ in Ω, the local uniform continuity (2.5) gives

$$J_1=|\mathbf{a}(x,u(x),\xi)-\mathbf{a}(x,u(y),\xi)|\le\sigma_M(|u(x)-u(y)|)|\xi|^{m-1},$$

whence

$$J_1\le\sigma_M(|u(x)-u(y)|)|\xi|^{m-1}\le\sigma_M(\omega_u(R))|\xi|^{m-1}$$

by employing the modulus of continuity of u, and the properties of the function σ_M.

In a similar manner one has

$$
\begin{aligned}
J_2 &\le \fint_{B_\rho(y)} |\mathbf{a}(\tilde{x}, u(y), \xi) - \mathbf{a}(\tilde{x}, u(\tilde{x}), \xi)|\, d\tilde{x} \\
&\le \fint_{B_\rho(y)} \sigma_M(|u(y) - u(\tilde{x})|)|\xi|^{m-1} d\tilde{x} \\
&\le \sigma_M(\omega_u(R))|\xi|^{m-1},
\end{aligned}
$$

while

$$J_3 \le \Theta\big(\mathbf{a}; B_\rho(y)\big)(x, u(y))|\xi|^{m-1}$$

with Θ given in (2.7).

Therefore we have

$$\Theta\big(\mathbf{A}; B_\rho(y)\big)(x) \le \Theta\big(\mathbf{a}; B_\rho(y)\big)(x, u(y)) + 2\sigma_M(\omega_u(R))$$

and

$$
\begin{aligned}
\fint_{B_\rho(y)} \Theta\big(\mathbf{A}; B_\rho(y)\big)(x)\, dx &\le \fint_{B_\rho(y)} \Theta\big(\mathbf{a}; B_\rho(y)\big)(x, u(y))\, dx + 2\sigma_M(\omega_u(R)) \\
&\le \delta_M + 2\sigma_M(\omega_u(R))
\end{aligned}
$$

as consequence of (2.6). □

3.4. Hölder continuity of solutions to general nonlinear equations

The following two results regard the weak solutions to general quasilinear elliptic equations without *any* regularity requirements on the principal term with respect to the variable x. These are classical when φ and ψ are only Lebesgue integrable functions, that is, when $\lambda = \mu = 0$, and are due to Gehring, Giaquinta and Modica ([7, Chapter V]) and Ladyzhenskaya and Uralt'seva ([8, Chapter IV]), respectively. In the case when φ and ψ belong to Morrey spaces, Lemmas 3.7 and 3.8 have been proved in [4] and announced in [3] for very general domains Ω with *thick* enough complements. The last means that $\partial\Omega$ satisfies a sort of variational *capacity density condition* which is surely fulfilled when $\partial\Omega$ supports the (A)-property of Ladyzhenskaya and Uralt'seva, that is, it holds automatically also for Reifenberg flat domains.

The first result claims gradient integrability improvement for the weak solutions to (1.1) in the spirit of Gehring–Giaquinta–Modica, whereas the second one gives their global boundedness and Hölder continuity.

Lemma 3.7. ([3, Lemma 2.2]) *Assume* (2.2), (2.3)*, let* Ω *be a* (δ, R)*-Reifenberg flat domain and* $u \in W_0^{1,m}(\Omega)$ *a weak solution to the Dirichlet problem* (1.1).

Then there exists an exponent $m_0 > m$ *such that* $Du \in L^{m_0}(\Omega)$ *and*

$$\|Du\|_{L^{m_0}(\Omega)} \le C$$

with a constant C *depending on known quantities and on* $\|Du\|_{L^m(\Omega)}$ *in addition.*

Actually, Lemma 3.7 has been proved in [4] in the case $m \le n$ but a careful look of the proof shows that it works also when $m > n$ under the assumption of (δ, R)-Reifenberg flatness of Ω.

Lemma 3.8. ([3, Theorems 2.1, 3.1]) *Assume* (2.2), (2.3) *and let* Ω *be a* (δ, R)-*Reifenberg flat domain.*

Then each $W_0^{1,m}(\Omega)$-*weak solution of the Dirichlet problem* (1.1) *is* essentially bounded *and* globally *Hölder continuous in* $\overline{\Omega}$. *Precisely,*

$$\sup_{x\in\overline{\Omega}} |u(x)| + \sup_{x,y\in\overline{\Omega},\ x\neq y} \frac{|u(x)-u(y)|}{|x-y|^{\alpha}} \le H,$$

with exponent $\alpha \in (0,1)$ *and constant* $H > 0$ *depending on known quantities and on* $\|Du\|_{L^m(\Omega)}$.

Indeed, if $m > n$ then Lemma 3.8 is a trivial consequence of the Morrey lemma and the $W^{1,p}$-extension properties of the Reifenberg flat domains.

4. Proof of the main results

Without loss of generality, we assume that the solution u and the data φ and ψ are extended as zero outside Ω.

The essential boundedness of the weak solutions to (1.1), $u \in L^\infty(\Omega)$, follows from Lemma 3.8.

Fix now the solution $u \in W_0^{1,m}(\Omega) \cap L^\infty(\Omega)$ into the nonlinear terms of (1.1) and define

$$\mathbf{A}(x,\xi) := \mathbf{a}(x,u(x),\xi), \quad \mathbf{B}(x) := -\mathbf{b}(x,u(x)), \quad f(x) := c(x,u(x),Du(x)).$$

Let $\Gamma(x-y)$ be the normalized fundamental solution of the Laplace operator and set

$$\mathcal{N}f(x) := \int_\Omega \Gamma(x-y) f(y)\, dy$$

for the Newtonian potential with density f. Since $f \in L^{\min\left\{q,\frac{m^*}{m^*-1}\right\}}(\Omega)$ by (2.3), the Newtonian potential is well-defined, $\mathcal{N}f \in W^{2,\min\left\{q,\frac{m^*}{m^*-1}\right\}}(\Omega)$ and $\Delta\big(\mathcal{N}f(x)\big) = f(x)$ for a.a. $x \in \Omega$ as it follows from the Caldeón–Zygmung theorem. Thus, defining

$$\overline{\mathbf{F}}(x) := D\big(\mathcal{N}f(x)\big) = \int_\Omega D_x\Gamma(x-y)f(y)\, dy = C(n)\int_\Omega \frac{x-y}{|x-y|^n} f(y)\, dy,$$

it is clear that $\operatorname{div} \overline{\mathbf{F}}(x) = f(x)$ for a.a. $x \in \Omega$.

With the setting

$$\mathbf{F}(x) := \begin{cases} |\mathbf{B}(x)+\overline{\mathbf{F}}(x)|^{\frac{2-m}{m-1}}\big(\mathbf{B}(x)+\overline{\mathbf{F}}(x)\big) & \text{if} \quad |\mathbf{B}(x)+\overline{\mathbf{F}}(x)| > 0, \\ 0 & \text{if} \quad |\mathbf{B}(x)+\overline{\mathbf{F}}(x)| = 0, \end{cases}$$

we have

$$|\mathbf{F}|^{m-2}\mathbf{F} = \mathbf{B} + \overline{\mathbf{F}}$$

and (1.1) can be rewritten into

$$\begin{cases} \operatorname{div}\big(\mathbf{A}(x, Du(x))\big) = \operatorname{div}\big(|\mathbf{F}(x)|^{m-2}\mathbf{F}(x)\big) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.1}$$

The term $\mathbf{A}(x,\xi)$ is (δ, R)-vanishing as consequence of Lemma 3.6, and it satisfies (3.2) because of (2.2) and Lemma 3.8. Further on, it follows from (2.3)

$$\begin{aligned} |\overline{\mathbf{F}}(x)| &\le C \int_\Omega \frac{|f(y)|}{|x-y|^{n-1}}\, dy \\ &\le C \int_\Omega \frac{1}{|x-y|^{n-1}} \left(1 + \psi(y) + |Du(y)|^{\frac{m(m^*-1)}{m^*}}\right) dy \\ &= C\left((I_1 1)(x) + (I_1\psi)(x) + \left(I_1 |Du|^{\frac{m(m^*-1)}{m^*}}\right)(x)\right) \end{aligned}$$

and therefore

$$|\mathbf{F}(x)|^{m-1} \le C\left(1 + \varphi(x) + (I_1\psi)(x) + \left(I_1|Du|^{\frac{m(m^*-1)}{m^*}}\right)(x)\right) \tag{4.2}$$

with C depending on known quantities and on $\|Du\|_{L^m(\Omega)}$ in addition.

Step 1: In this step, we will show that the weak solution $u \in W_0^{1,m}(\Omega)$ of (1.1) improves the gradient summability to $u \in W^{1,(m-1)r}(\Omega)$, that is,

$$|Du|^{m-1} \in L^{\min\{p, q_\mu^*\}}(\Omega).$$

Employing *the better gradient integrability* result from Lemma 3.7, we may assume

$$|Du|^{m-1} \in L^s(\Omega)$$

for some $s \in [\frac{m_0}{m-1}, \infty)$. This is equivalent to $|Du|^{\frac{m(m^*-1)}{m^*}} \in L^{\frac{(m-1)m^*}{m(m^*-1)}s}(\Omega)$ and Proposition 3.2 yields

$$I_1|Du|^{\frac{m(m^*-1)}{m^*}} \in L^{\left(\frac{(m-1)m^*}{m(m^*-1)}s\right)^*}(\Omega) \equiv L^{\left(\frac{(m-1)m^*}{m(m^*-1)}s\right)^*,0}(\Omega). \tag{4.3}$$

If $\frac{(m-1)m^*}{m(m^*-1)}s < n$, then

$$\left(\frac{(m-1)m^*}{m(m^*-1)}s\right)^* = \frac{(m-1)m^*ns}{mn(m^*-1) - (m-1)m^*s} \ge \frac{(m-1)m^*ns}{mn(m^*-1) - m_0m^*}$$

because of $(m-1)s \ge m_0$. Defining

$$\tau_0 := \frac{(m-1)m^*n}{mn(m^*-1) - m_0m^*},$$

we have

$$\tau_0 = \frac{(m-1)m^*n}{(m-1)m^*n + (n-m)m^* - nm - (m_0-m)m^*} > 1$$

since $m_0 > m$, $(n-m)m^* \leq mn$ and

$$\frac{m_0 m^*}{m(m^*-1)} \leq \frac{(m-1)m^*}{m(m^*-1)} s < n.$$

Therefore,

$$\left(\frac{(m-1)m^*}{m(m^*-1)} s\right)^* \geq \tau_0 s > s. \tag{4.4}$$

Further on, (2.3) and Proposition 3.2 yield

$$I_1\psi \in L^{q_\mu^*,\mu}(\Omega) \tag{4.5}$$

whence

$$|\mathbf{F}|^{m-1} \in L^{\min\{p,q_\mu^*,\tau_0 s\}}(\Omega)$$

as consequence of (2.3) and (4.2) − (4.4). Noting that

$$\min\{p, q_\mu^*, \tau_0 s\} > \frac{m}{m-1},$$

Lemma 3.5, applied to (4.1), implies

$$|Du|^{m-1} \in L^{\min\{p,q_\mu^*,\tau_0 s\}}(\Omega).$$

To proceed further, we take $s = s_k = \tau_0^k \frac{m_0}{m-1}$ for $k = 0, 1, 2, \ldots$.

Keeping in mind Lemma 3.7, we iterate the above procedure finite times until $s_k \geq \min\left\{p, q_\mu^*, \frac{(m^*-1)mn}{m^*(m-1)}\right\}$, getting this way

$$|Du|^{m-1} \in L^{\min\left\{p,q_\mu^*,\frac{(m^*-1)mn}{m^*(m-1)}\right\}}(\Omega).$$

In particular, if $\frac{(m^*-1)mn}{m^*(m-1)} \geq \min\{p, q_\mu^*\}$, then $|Du|^{m-1} \in L^{\min\{p,q_\mu^*\}}(\Omega)$ and we obtain the claim of Step 1.

If instead $\frac{(m^*-1)mn}{m^*(m-1)} < \min\{p, q_\mu^*\}$ then $|Du|^{m-1} \in L^{\frac{(m^*-1)mn}{m^*(m-1)}}(\Omega)$ and we take $s = \frac{(m^*-1)mn}{m^*(m-1)}$ that gives $\frac{(m-1)m^*}{m(m^*-1)} s = n$. Let us concentrate therefore to the case $\frac{(m-1)m^*}{m(m^*-1)} s \geq n$. Choosing the Sobolev conjugate as

$$\left(\frac{(m-1)m^*}{m(m^*-1)} s\right)^* = \min\{p, q_\mu^*\},$$

we get

$$|\mathbf{F}|^{m-1} \in L^{\min\{p,q_\mu^*\}}(\Omega)$$

as consequence of (2.3), (4.3) and (4.5), and thus we have

$$|Du|^{m-1} \in L^{\min\{p,q_\mu^*\}}(\Omega).$$

Setting hereafter $r := \min\{p, q_\mu^*\}$, we have $Du \in L^{(m-1)r}(\Omega)$ that is $u \in W^{1,(m-1)r}(\Omega)$.

Step 2: To obtain the Morrey regularity of Du as claimed in Theorem 2.2, we assume that $|Du|^{m-1} \in L^{r,\theta}(\Omega)$ for some $\theta \in [0, n)$. This is equivalent to $|Du|^{\frac{m(m^*-1)}{m^*}} \in L^{\frac{(m-1)m^*}{m(m^*-1)}r,\theta}(\Omega)$ and Proposition 3.2 yields

$$I_1|Du|^{\frac{m(m^*-1)}{m^*}} \in L^{\left(\frac{(m-1)m^*}{m(m^*-1)}r\right)^*_\theta,\theta}(\Omega). \tag{4.6}$$

If $\frac{(m-1)m^*}{m(m^*-1)}r < n - \theta$, we have

$$\left(\frac{(m-1)m^*}{m(m^*-1)}r\right)^*_\theta = \frac{(n-\theta)(m-1)m^*r}{(n-\theta)m(m^*-1) - (m-1)m^*r} > r$$

with the last inequality following by the hypotheses imposed on p and q. Thus, Proposition 3.1 gives

$$L^{\left(\frac{(m-1)m^*}{m(m^*-1)}r\right)^*_\theta,\theta}(\Omega) \subset L^{r,n+r-(n-\theta)\frac{m(m^*-1)}{(m-1)m^*}}(\Omega).$$

Further on, set

$$\tau_1 := \frac{n(m-m^*) + rm^*(m-1)}{m^*(m-1)}, \quad \tau_2 := \frac{m(m^*-1)}{(m-1)m^*}.$$

Keeping in mind $r > \frac{m}{m-1}$ and $(n-m)m^* \leq mn$, we get

$$\begin{aligned} \tau_1 = \frac{n(m-m^*) + rm^*(m-1)}{m^*(m-1)} &> \frac{n(m-m^*) + m^*m}{m^*(m-1)} \\ &= \frac{mn - m^*(n-m)}{m^*(m-1)} \geq 0. \end{aligned}$$

On the other hand, it is clear that

$$\tau_2 = \frac{m(m^*-1)}{(m-1)m^*} > 1$$

as consequence of $m^* > m$, and therefore straightforward calculations show

$$\begin{aligned} n + r - (n-\theta)\frac{m(m^*-1)}{(m-1)m^*} &= \frac{n(m-m^*) + rm^*(m-1)}{m^*(m-1)} + \theta\frac{m(m^*-1)}{(m-1)m^*} \\ &= \tau_1 + \tau_2\theta > \theta. \end{aligned}$$

Thus, we obtain

$$I_1|Du|^{\frac{m(m^*-1)}{m^*}} \in L^{r,\tau_1+\theta\tau_2}(\Omega).$$

Arguing in the same manner as in step 1 above, one has

$$|\mathbf{F}|^{m-1} \in L^{r,\min\left\{\tau_1+\theta\tau_2, \frac{\lambda-n}{p}r+n, \frac{\mu-n}{q_\mu^*}r+n\right\}}(\Omega)$$

whence, applying the Lemma 3.5 once again, we have

$$|Du|^{m-1} \in L^{r,\min\left\{\tau_1+\theta\tau_2, \frac{\lambda-n}{p}r+n, \frac{\mu-n}{q_\mu^*}r+n\right\}}(\Omega).$$

To proceed further, we first take $\theta = 0$ to get

$$|Du|^{m-1} \in L^{r,\min\left\{\tau_1, \frac{\lambda-n}{p}r+n, \frac{\mu-n}{q_\mu^*}r+n\right\}}(\Omega).$$

If $\tau_1 \geq \min\left\{\frac{\lambda-n}{p}r+n, \frac{\mu-n}{q_\mu^*}r+n\right\}$ then we are done. Otherwise, $|Du|^{m-1} \in L^{r,\tau_1}(\Omega)$ and we repeat the above procedure with $\theta = \tau_1$, obtaining thus

$$\begin{aligned}|Du|^{m-1} &\in L^{r,\min\left\{\tau_1+\tau_1\tau_2, \frac{\lambda-n}{p}r+n, \frac{\mu-n}{q_\mu^*}r+n\right\}}(\Omega)\\ &\subset L^{r,\min\left\{\tau_1\tau_2, \frac{\lambda-n}{p}r+n, \frac{\mu-n}{q_\mu^*}r+n\right\}}(\Omega).\end{aligned}$$

At this point we let $\theta = \theta_k = \tau_1\tau_2^k$ for $k = 1, 2, \ldots$ and iterate the above procedure. This gives

$$\begin{aligned}|Du|^{m-1} &\in L^{r,\min\left\{\tau_1+\tau_1\tau_2^k, \frac{\lambda-n}{p}r+n, \frac{\mu-n}{q_\mu^*}r+n\right\}}(\Omega)\\ &\subset L^{r,\min\left\{\tau_1\tau_2^k, \frac{\lambda-n}{p}r+n, \frac{\mu-n}{q_\mu^*}r+n\right\}}(\Omega).\end{aligned}$$

We choose now k so large to have

$$\tau_1\tau_2^k \geq \min\left\{\frac{\lambda-n}{p}r+n, \frac{\mu-n}{q_\mu^*}r+n, n-\frac{(m-1)m^*}{m(m^*-1)}r\right\},$$

getting this way

$$|Du|^{m-1} \in L^{r,\min\left\{\frac{\lambda-n}{p}r+n, \frac{\mu-n}{q_\mu^*}r+n, n-\frac{(m-1)m^*}{m(m^*-1)}r\right\}}(\Omega).$$

If

$$n-\frac{(m-1)m^*}{m(m^*-1)}r \geq \min\left\{\frac{\lambda-n}{p}r+n, \frac{\mu-n}{q_\mu^*}r+n\right\}$$

then

$$|Du|^{m-1} \in L^{r,\min\left\{\frac{\lambda-n}{p}r+n, \frac{\mu-n}{q_\mu^*}r+n\right\}}(\Omega)$$

and this completes the proof of Theorem 2.2.

If instead

$$n - \frac{(m-1)m^*}{m(m^*-1)} r < \min\left\{ \frac{\lambda - n}{p} r + n, \frac{\mu - n}{q^*_\mu} r + n \right\}$$

then

$$|Du|^{m-1} \in L^{r, n - \frac{(m-1)m^*}{m(m^*-1)} r}(\Omega)$$

and we take $\theta = n - \frac{(m-1)m^*}{m(m^*-1)} r$ that gives

$$\frac{(m-1)m^*}{m(m^*-1)} r = n - \theta.$$

So, consider now the case

$$\frac{(m-1)m^*}{m(m^*-1)} r \geq n - \theta$$

when

$$|Du|^{m-1} \in L^{r,\theta}(\Omega) \subset L^{r, n - \frac{(m-1)m^*}{m(m^*-1)} r}(\Omega).$$

Remembering (4.6), we choose the Sobolev–Morrey conjugate as

$$\left(\frac{(m-1)m^*}{m(m^*-1)} r \right)^*_{n - \frac{(m-1)m^*}{m(m^*-1)} r} = \frac{nr - \frac{(m-1)m^*}{m(m^*-1)} r^2}{n - \min\left\{ \frac{\lambda - n}{p} r + n, \frac{\mu - n}{q^*_\mu} r + n \right\}},$$

that gives

$$\begin{aligned} I_1 |Du|^{\frac{m(m^*-1)}{m^*}} &\in L^{\frac{nr - \frac{(m-1)m^*}{m(m^*-1)} r^2}{n - \min\left\{ \frac{\lambda - n}{p} r + n, \frac{\mu - n}{q^*_\mu} r + n \right\}}, n - \frac{(m-1)m^*}{m(m^*-1)} r}(\Omega) \\ &\subset L^{r, \min\left\{ \frac{\lambda - n}{p} r + n, \frac{\mu - n}{q^*_\mu} r + n \right\}}(\Omega). \end{aligned}$$

Moreover,

$$\varphi + I_1 \psi \in L^{r, \min\left\{ \frac{\lambda - n}{p} r + n, \frac{\mu - n}{q^*_\mu} r + n \right\}}(\Omega),$$

whence

$$|\mathbf{F}|^{m-1} \in L^{r, \min\left\{ \frac{\lambda - n}{p} r + n, \frac{\mu - n}{q^*_\mu} r + n \right\}}(\Omega)$$

as consequence of (4.2). Applying Lemma 3.5 once again, we have

$$|Du|^{m-1} \in L^{r, \min\left\{ \frac{\lambda - n}{p} r + n, \frac{\mu - n}{q^*_\mu} r + n \right\}}(\Omega)$$

which gives the claim of Theorem 2.2 also in this case. □

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