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VENTTSEL BOUNDARY VALUE PROBLEMS WITH DISCONTINUOUS DATA

DARYA E. APUSHKINSKAYA, ALEXANDER I. NAZAROV, DIAN K. PALAGACHEV,
AND LUBOMIRA G. SOFTOVA

ABSTRACT. We study linear and quasilinear Venttsel boundary value problems involving elliptic operators with discontinuous coefficients. On the base of the *a priori* estimates obtained, maximal regularity and strong solvability in Sobolev spaces are proved.

Dedicated to Nina N. Uraltseva with admiration

1. INTRODUCTION

The paper deals with *discontinuous* Venttsel boundary value problems for linear and quasilinear second-order elliptic equations. The discontinuity regards the coefficients of the differential operators acting inside and on the boundary of the underlying domain, and is expressed in terms of appurtenance of the principal coefficients to the class of functions with *vanishing mean oscillation (VMO),* while optimal Lebesgue integrability requirements are imposed on the lower-order coefficients. We consider *strong solutions* belonging to Sobolev spaces with optimal exponent ranges and develop L^p-regularity and solvability theory for such problems.

The history of the Venttsel BVPs goes back to the pioneering work [59] where, given an elliptic operator $\mathcal{L}$ in a bounded domain $\Omega \subset \mathbb{R}^n$, A.D. Venttsel found general boundary conditions, given in terms of a second-order integro-differential operator, which restrict $\mathcal{L}$ to an infinitesimal generator of a Markov process in Ω. From a probabilistic point of view, the Venttsel conditions[1] are the *most general admissible boundary conditions*, where the differential terms describe the phenomena of diffusion along the boundary, absorption, reflection and viscosity, while the non-local integral terms represent jumps of the process along $\partial\Omega$ and inward jumps from $\partial\Omega$ into Ω (see also [27] and [60]).

We consider here the pure local case when the Venttsel conditions are given by a combination of second-order differential operator *along the boundary* and a full gradient term. Even in such settings, the Venttsel conditions include as particular cases the Dirichlet, Neumann, oblique derivative and mixed (Robin) boundary conditions.

[1]Notice that Venttsel conditions are known in the literature also as *Wentzell* or *Ventcel'* conditions. We mention also that Venttsel type conditions occur for all types of second-order equations – elliptic, parabolic and hyperbolic; we discuss here only the elliptic case.

The Venttsel problems describe many physical processes in media surrounded by a thin film, and appear in various branches of science, technology and industry: e.g. in water wave theory ([56]), electromagnetic and phase-transition phenomena ([55] and [26]), elasticity theory problems ([41]), engineering problems of hydraulic fracturing ([13]), models of fluid diffusion ([45]), as well as in some climate models or non-isothermal phase separation in a confined container ([21, 22]), and in various aspects of financial mathematics ([57, Chapter 8]). Some simple physical models leading to problems with Venttsel boundary conditions can be also found in [6], [36] and [25]. Moreover, the Venttsel problem is the simplest case of systems connected on manifolds of different dimensions (cf. [45]), and it also provides an example of problems on stratified sets (see, for instance, [52]).

There is a vast literature devoted to linear and semilinear Venttsel problems in both local and non-local settings, see e.g. [38], [10], [23], [33], [34], [16], [44], [17] and the references therein. The study of *quasilinear* problems with Venttsel boundary conditions was initiated by Y. Luo in [37] and continued later in a series of publications by D.E. Apushkinskaya and A.I. Nazarov. A detailed survey on the "quasilinear" results obtained up to 1999 can be found in [8]. We mention also a series of papers dealing with *two-phase quasilinear* Venttsel problems ([9, 11]) where the Venttsel condition is given on an interface separating the domain in two parts, and *degenerate quasilinear* Venttsel problems ([6, 7, 42]) where the thickness of the surrounding film can become zero on a subset of the boundary.

Notice that all the results mentioned above concern equations and boundary conditions with leading terms that depend at least *continuously* on the independent variable x.

The first relevant W_p^2-theory of linear elliptic operators $a^{ij}(x)D_iD_j$ with *discontinuous* coefficients is due to F. Chiarenza, M. Frasca and P. Longo. In their pioneer works [14, 15] the authors allowed discontinuity of a^{ij}s, taking these in the Sarason class VMO, that contains only as a proper subset the space of uniformly continuous functions (see Section 2 for more details). It is proved in [14, 15] that $a^{ij} \in VMO$ ensures the validity of the Calderón–Zygmund property for *any* $p > 1$. Namely, if $u \in W_q^2(\Omega)$ with $q \in (1, p)$ and $u = 0$ on $\partial\Omega$ in the sense of traces, then $a^{ij}(x)D_iD_ju \in L^p(\Omega)$ implies $u \in W_p^2(\Omega)$. This is a crucial point in the L^p-theory of PDEs allowing to extend the results for operators with continuous coefficients (cf. [31, 24]) to discontinuous ones. Moreover, since the space $W_n^1(\Omega)$ and the fractional Sobolev space $W_{n/\vartheta}^\vartheta(\Omega)$, $\vartheta \in (0, 1)$, are both contained in VMO, the VMO-discontinuity of the coefficients makes these results more general then those obtained before (see [15] and [40, Chapter 1] for more references). The technique in [14, 15] is based on an explicit representation formula for the derivatives D_iD_ju via Calderón–Zygmund singular integrals $\mathcal{K}f$ and their commutators $\mathcal{C}[a^{ij}, D_iD_j] = a^{ij}\mathcal{K}D_iD_j - \mathcal{K}(a^{ij}D_iD_j)$, and the vanishing property of the VMO-moduli of a^{ij}s permits to make the commutator norm small enough and hence to obtain an *a priori* estimate for the strong solutions. Combining this estimate with a fixed point theorem arguments, the authors of [14, 15] obtain unique strong solvability of the Dirichlet problem

$$a^{ij}(x)D_iD_ju = f(x) \quad \text{a.e. in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega$$

in $W_p^2(\Omega)$ for every $f \in L^p(\Omega)$ provided $p \in (1, \infty)$.

Similar regularity and strong solvability theory have been developed in [19] and [39] for linear oblique derivative problem for uniformly elliptic operators with *VMO* principal coefficients.

Combining the results of [14, 15] with the Aleksandrov–Bakel'man maximum principle, suitable *a priori* gradient estimates have been obtained in [46] for the strong solutions to *quasilinear* elliptic equations with *VMO* principal coefficients. As consequence, W_n^2-solvability of the *quasilinear* Dirichlet problem was proved in [46]. In [19] similar results were obtained for the oblique derivative problem.

In the present paper we develop a strong solvability theory for linear and quasilinear elliptic Venttsel problems with *discontinuous* coefficients. We deal with *strong* solutions lying in the space $V_{p,q}(\Omega) = W_p^2(\Omega) \cap W_q^2(\partial\Omega)$ that satisfy the interior and the boundary equations almost everywhere.

For the linear case, the exponents p and q vary in the full range $(1, \infty)$, restricted only by a natural requirement (cf. (3.1)) ensuring trace compatibility. The principal coefficients of both the domain and boundary operators are taken in *VMO*, while *optimal* integrability in Lebesgue or Orlicz spaces is assumed for the lower-order coefficients. The analytic core here is Theorem 3.1 that provides a coercive *a priori* estimate in $V_{p,q}(\Omega)$ for any solution to the linear Venttsel problem. The proof relies on the results of [14, 15], fine interpolation between various Sobolev spaces, depending on the admissible combinations of p and q, and W_p^2-bounds for suitable extension operators (see Theorem 2.2). The coercive estimate implies the *Fredholm property*, which, in turn, provides the *improving of integrability* property for the linear Venttsel problem (Theorem 3.5). Finally, a comparison principle of *Aleksandrov–Bakel'man* type (Theorem 3.6) allows to derive unique strong solvability for all admissible values of p and q.

The natural functional framework for the solutions of the *quasilinear* Venttsel problem is the space $V_{n,n-1}(\Omega)$. Due to technical difficulties, we restrict ourselves to the case when the principal coefficients both of the equation and the boundary condition are *independent* of the gradient of solution. However, these depend on the solution itself and exhibit *discontinuities* in the independent variable x, measured in terms of *VMO*. The lower-order terms support quadratic gradient growth and may have unbounded singularities in x with a proper control of the Lebesgue or Orlicz integrability. The existence approach relies on the Leray–Schauder fixed point theorem that reduces the solvability to suitable *a priori* estimates for all possible solutions to a family of Venttsel problems. In our case these are bounds for the $L^{2n}(\Omega)$-norm of the *full* gradient and the $L^{2(n-1)}(\partial\Omega)$-norm of the *tangential* gradient (Theorem 4.2). To get such estimates, we adapt to the specific Venttsel situation a homotopy-type machinery of Amann and Crandall [3] which reveals very useful when dealing with discontinuous quasilinear operators. This approach requires also estimates for the L^∞ and the Hölder norms of the solution. The first is obtained in Lemma 4.8, while the second one follows for free from [4]. As a result, strong solvability of the quasilinear Venttsel problem (Theorem 4.6) does follow.

The paper is organized as follows. In Section 2 we provide the necessary notation, collect the basic facts about *VMO* spaces, and prove an auxiliary result about extension operators in Sobolev spaces. In Sections 3 and 4 we deal with linear and quasilinear Venttsel problems, respectively, and we assume here that the dimension of the underlying domain Ω is at least 3. The 2D case is more simple, and we briefly discuss it in Section 5. Some remarks about possible generalizations of our

results are also given there, together with several open problems and indications for further research.

2. Auxiliary results

2.1. Notation and conventions. Throughout the paper we use the following notation:

$x = (x', x_n) = (x_1, \ldots, x_{n-1}, x_n)$ is a vector in $\mathbb{R}^n$ with Euclidean norm $|x|$;

$\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_n > 0\}$;

Ω is a bounded domain in $\mathbb{R}^n$ with compact closure $\overline{\Omega}$ and $(n-1)$-dimensional boundary $\partial\Omega$;

$\Gamma(\Omega)$ is the part of $\partial\Omega$ lying on the hyperplane $\{x_n = 0\}$;

$\mathbf{n} = \mathbf{n}(x)$ is the unit vector of the outward normal to $\partial\Omega$ at the point x;

$B_r(x^0)$ is the open ball in $\mathbb{R}^n$ with center x^0 and radius r; $B_r = B_r(0)$; $B'_r(x^0) = B_r(x^0) \cap \{x_n = 0\}$; $B'_r = B'_r(0)$;

The indices i and j run from 1 to n and we adopt the standard convention regarding summation with respect to repeated indices;

D_i denotes the operator of (weak) differentiation with respect to x_i;

$Du = (D'u, D_n u) = (D_1 u, \ldots, D_{n-1} u, D_n u)$ is the gradient of u;

d_i denotes the tangential differential operator on $\partial\Omega$, i.e.,

$$d_i = D_i - \mathbf{n}_i \mathbf{n}_j D_j;$$

$du = (d_i u)$ is the tangential gradient of u on $\partial\Omega$; in particular, we have $du = (D'u, 0)$ on $\Gamma(\Omega)$;

$\| \cdot \|_{p,\Omega}$ denotes the norm in $L^p(\Omega)$;

$W_p^1(\Omega)$ and $W_p^2(\Omega)$ are the Sobolev space with norms

$$\|u\|_{W_p^1(\Omega)} = \|Du\|_{p,\Omega} + \|u\|_{p,\Omega} \quad \text{and} \quad \|u\|_{W_p^2(\Omega)} = \|D(Du)\|_{p,\Omega} + \|u\|_{p,\Omega}$$

respectively; similarly, the symbols $W_p^1(\partial\Omega)$ and $W_p^2(\partial\Omega)$ stand for the Sobolev spaces of functions defined on $\partial\Omega$ and equipped with the corresponding norms

$$\|u\|_{W_p^1(\partial\Omega)} = \|du\|_{p,\partial\Omega} + \|u\|_{p,\partial\Omega}, \qquad \|u\|_{W_p^2(\partial\Omega)} = \|d(du)\|_{p,\partial\Omega} + \|u\|_{p,\partial\Omega};$$

We also define $V_{p,q}(\Omega)$ as the subspace of $W_p^2(\Omega)$ consisting of all functions u that have traces in $W_q^2(\partial\Omega)$, with the norm

$$\|u\|_{V_{p,q}(\Omega)} = \|u\|_{W_p^2(\Omega)} + \|u\|_{W_q^2(\partial\Omega)}.$$

We denote by $\mathcal{C}(\overline{\Omega})$ and $\mathcal{C}^1(\overline{\Omega})$ the spaces of continuous and continuously differentiable functions, respectively. In a similar way, we introduce the spaces $\mathcal{C}(\partial\Omega)$ and $\mathcal{C}^1(\partial\Omega)$.

$\mathcal{C}^{0,\lambda}(\overline{\Omega})$ is the space of Hölder continuous functions with exponent $\lambda \in (0,1]$, and with norm

$$\|u\|_{\mathcal{C}^{0,\lambda}(\overline{\Omega})} = \sup_{\Omega} |u| + \sup_{x,y \in \Omega} \frac{|u(x) - u(y)|}{|x - y|^\lambda}.$$

$\mathcal{C}^{1,\lambda}(\overline{\Omega})$ is the space of functions with first-order derivatives belonging to $\mathcal{C}^{0,\lambda}(\overline{\Omega})$.

For a functional space $\mathcal{X}$, we understand the notation $\partial\Omega \in \mathcal{X}$ as follows. There is a positive R_0 such that for every $x^0 \in \partial\Omega$ the set $\partial\Omega \cap B_{R_0}(x^0)$ is (in a suitable Cartesian coordinate system) a graph $y_n = f(y')$ of a function $f \in \mathcal{X}$. When saying that a given constant depends on "*the properties of $\partial\Omega$*" we simply mean

dependence on the $\mathcal{X}$-norms of the diffeomorphisms that flatten locally $\partial\Omega$, and on the area of $\partial\Omega$.

By p^* we denote the Sobolev conjugate of the exponent p, that is,

$$p^* := \begin{cases} \dfrac{np}{n-p} & \text{if } p < n, \\ \infty & \text{if } p \geq n. \end{cases}$$

We use the letters C and N (with or without indices) to denote various constants. To indicate that C depends on some parameters, we list these in parentheses: $C(\dots)$. Finally, we set $\frac{0}{0} = 0$, if such an uncertainty occurs.

2.2. *VMO* functions. We will deal here with differential operators with discontinuous principal coefficients belonging to the Sarason class of functions with mean oscillation that vanishes over shrinking balls. In [28] John and Nirenberg introduced the space of functions with *bounded mean oscillation (BMO)*. Their paper has been followed by various works exhibiting the importance of the *BMO* functions in the harmonic analysis (see [54] and the references therein). Later, Sarason [54] attracted the attention to a natural subspace of *BMO*, called *VMO,* consisting of the functions with *vanishing mean oscillation.* Let us give a precise definition of these spaces.

Definition 2.1. ([28, 54]) A locally integrable function f defined on $\mathbb{R}^n$ lies in *BMO* if its integral oscillation is bounded, that is, if

$$\|f\|_* := \sup_B \fint_B |f(x) - f_B| \, dx < \infty,$$

where B varies in the class of *all* balls in $\mathbb{R}^n$ and f_B stands for the integral average $|B|^{-1} \int_B f(x) \, dx$. Modulo constant functions, *BMO* becomes a Banach space under the norm $\|f\|_*$.

For a function $f \in BMO$, define

$$(2.1) \qquad \omega_f(r) = \sup_{\rho \leq r} \fint_{B_\rho} |f(x) - f_{B_\rho}| \, dx,$$

where B_ρ varies now in the class of *all* balls of radius ρ. Then $f \in VMO$ if

$$\lim_{r \to 0} \omega_f(r) = 0$$

and we refer to $\omega_f(r)$ as *VMO*-modulus of f.

For a bounded domain $\Omega \subset \mathbb{R}^n$, the spaces $BMO(\Omega)$ and $VMO(\Omega)$ are defined in the same manner, replacing B and B_ρ above by the respective intersections with Ω. Similarly, if $\partial\Omega$ is smooth, the spaces $BMO(\partial\Omega)$ and $VMO(\partial\Omega)$ are defined in a natural way by surface integral oscillations over $B \cap \partial\Omega$ and $B_\rho \cap \partial\Omega$ with balls centered at points of $\partial\Omega$.

Having a function $f \in VMO(\Omega)$ given on a Lipschitz domain, it is possible to extend it to the whole $\mathbb{R}^n$ by preserving the *VMO*-modulus as follows by results of Acquistapace [1].

It is worth to mention some examples that illustrate the embeddings between *VMO/BMO* and some classical function spaces. Note that *BMO* functions are not necessarily bounded but the space of the bounded uniformly continuous functions belongs to *VMO* and we can take as *VMO*-modulus the corresponding modulus of continuity. Further on, $W_n^1(\mathbb{R}^n)$ is a proper subset of *VMO* as it follows by the

Poincaré inequality. However, if $f \in W_1^1(\mathbb{R}^n)$ with a gradient belonging to the Marcinkiewicz space L_{weak}^n, then f belongs to *BMO* but not necessarily to *VMO*. This can be seen also by the following examples (cf. [28], [40, Section 2.1]).

Let $f(x) = \log|x|$ and set $f_\alpha(x) = |\log|x||^\alpha$ with $\alpha > 0$. Then

- $f \in BMO \setminus VMO$, $\sin(f) \in BMO \cap L^\infty$;
- $f_\alpha \in VMO$ for each $\alpha \in (0,1)$, but $f_\alpha \in W_n^1(\mathbb{R}^n)$ *only* if $\alpha \in (0, 1 - \frac{1}{n})$.

In [54] Sarason gave alternative descriptions of *VMO* which explain the wide application of this function space not only in the harmonic analysis but also in the theory of PDEs, stochastic theory, etc.

Proposition 2.1. ([54]) *For $f \in BMO$, the following conditions are equivalent:*

(1) *$f \in VMO$;*
(2) *f is in the BMO-closure of bounded uniformly continuous functions;*
(3) *$\lim_{y \to 0} \|f(\cdot - y) - f(\cdot)\|_* = 0$.*

Let us note that the last circumstance in Proposition 2.1 guarantees the good behavior of the mollifiers of *VMO* functions. Moreover, we are able to approximate the *VMO* functions with $\mathcal{C}_0^\infty$ functions.

2.3. An extension result. The next statement is a modification of Theorem 6.1 in [5] and allows to extend Sobolev functions defined on $\partial\Omega$ to Sobolev functions in the whole Ω.

Theorem 2.2. *Let the exponents q and p be chosen such that*

$$1 \le p \le \frac{nq}{n-1} \quad and \quad q > 1,$$

and let $\partial\Omega \in \mathcal{C}^{1,1}$.

Then there exists an extension operator

$$\Pi\colon \ W_q^2(\partial\Omega) \to W_p^2(\Omega)$$

such that

$$\text{(2.2)} \qquad \|\Pi u\|_{W_p^2(\Omega)} \le N_0 \|u\|_{W_q^2(\partial\Omega)},$$

where N_0 depends only on p, q and the properties of $\partial\Omega$.

Proof. It is easy to see that it suffices to prove the theorem when $p = \frac{nq}{n-1}$.

Step 1. We start with a procedure constructing an extension operator from a flat boundary surface to a boundary strip that acts continuously from the space $\overset{\circ}{W}_q^2(B_R')$ into the space $W_p^2(B_R' \times (0, R))$. Here $\overset{\circ}{W}_q^2(B_R')$ stands for the closure of $\mathcal{C}_0^\infty(B_R')$ with respect to the norm in W_q^2, and we assume that the function u is extended as zero to the whole $\mathbb{R}^{n-1}$.

Now successive application of some statements from [58] yields the following

[58, formula (2.8.1/18)]: $\quad W_q^2(\mathbb{R}^{n-1}) \to \mathbf{B}_{p,p}^{2-1/p}(\mathbb{R}^{n-1}) \qquad$ (embedding),

[58, Theorem 2.9.3 (a)]: $\quad \mathbf{B}_{p,p}^{2-1/p}(\mathbb{R}^{n-1}) \to W_p^2(\mathbb{R}^n) \qquad$ (extension).

(Here the notation $\mathbf{B}$ of the Besov spaces corresponds to the book [58]).

Note that multiplying by a suitable cut-off function, the following properties can be ensured

(1) the extended function is equal to 0 for $|x_n| > R/2$;

(2) if the initial function is equal to 0 for $|x'| > R/2$, then the extended one is equal to 0 for $|x'| > 3R/4$.

Moreover, the norm of the extension operator is bounded in terms of n, q, p and R.

Step 2. The condition $\partial\Omega \in \mathcal{C}^{1,1}$ implies that for each point $x^0 \in \partial\Omega$ there exists a Cartesian coordinate system with origin at x^0 satisfying the following conditions

(1) the surface $\partial\Omega$ is tangent to the hyperplane $\{x_n = 0\}$ at the point x^0;
(2) the intersection of $\partial\Omega$ with the neighborhood $U_R = \{(x', x_n): |x'| < R, |x_n| < R\}$ can be given by an equation $x_n = \omega(x')$ with $\omega \in W^2_\infty(B'_R)$.

Moreover, the radius R of the above neighborhood can be chosen one and the same for all points $x^0 \in \partial\Omega$. The change of variables $y' = x'$, $y_n = x_n - \omega(x')$ then maps $\partial\Omega \cap U_R$ into the ball of radius R lying on the hyperplane $\{y_n = 0\}$.

This change of the variables induces the "transplantation" operator $u(x) \to u(y)$ acting continuously from $W^2_q(\partial\Omega \cap U_R)$ to $W^2_q(B'_R)$.

Step 3. Using the results of Steps 1 and 2 above, one can construct local extension operators that map $W^2_q(\partial\Omega)$-functions with sufficiently small x-support into $W^2_p(\mathbb{R}^n)$-functions vanishing for $|x_n| > R/2$, $|x'| > 3R/4$. Finally, the desired operator Π can be glued from the local operators via appropriate partition of unity. $\quad\square$

3. The linear Venttsel problem

In the sequel we suppose that $n \geq 3$. Assume $\partial\Omega \in \mathcal{C}^{1,1}$ and let the exponents q and p be chosen such that (see Fig. 1)

$$(3.1) \qquad 1 < p \leq \frac{nq}{n-1} < p^* \quad \text{and} \quad q > 1.$$
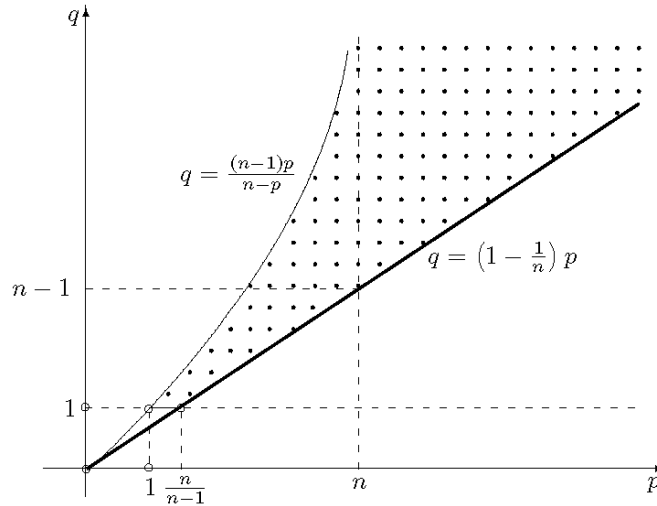

FIGURE 1. The exponents p and q

We introduce a linear elliptic operator $\mathcal{L}$,

$$(3.2) \qquad \mathcal{L} \equiv -a^{ij}(x)D_iD_j + b^i(x)D_i + c(x),$$

$$(L1) \qquad a^{ij}(x) = a^{ji}(x) \quad x \in \Omega, \qquad a^{ij} \in VMO(\Omega),$$

$$(L2) \qquad \nu|\xi|^2 \leq a^{ij}\xi_i\xi_j \leq \nu^{-1}|\xi|^2 \quad \forall \xi \in \mathbb{R}^n, \quad \nu = \mathrm{const} > 0,$$

and a linear boundary operator $\mathcal{B}$,

$$(3.3) \qquad \mathcal{B} \equiv -\alpha^{ij}(x)d_id_j + \beta^i(x)D_i + \gamma(x),$$

$$(B1) \qquad \alpha^{ij}(x) = \alpha^{ji}(x) \quad x \in \partial\Omega, \qquad \alpha^{ij} \in VMO(\partial\Omega),$$

$$(B2) \qquad \nu|\xi^*|^2 \leq \alpha^{ij}\xi_i^*\xi_j^* \leq \nu^{-1}|\xi^*|^2, \quad \forall \xi^* \in \mathbb{R}^n, \quad \xi^* \perp \mathbf{n}(x).$$

Set $\mathbf{b}(x) = \big(b^1(x), \ldots, b^n(x)\big)$ and assume that the lower-order coefficients of the operator $\mathcal{L}$ satisfy the following integrability conditions

$$(L3) \qquad \begin{aligned} &|\mathbf{b}| \in L^{\max\{p,n\}}(\Omega), &&\text{if } p \neq n, \\ &|\mathbf{b}|\left(\log(1+|\mathbf{b}|)\right)^{1-1/n} \in L^n(\Omega), &&\text{if } p = n, \end{aligned}$$

and

$$(L4) \qquad \begin{aligned} &c \in L^{\max\{p,n/2\}}(\Omega), &&\text{if } p \neq n/2, \\ &c\left(\log(1+|c|)\right)^{1-1/n} \in L^{n/2}(\Omega), &&\text{if } p = n/2. \end{aligned}$$

The assumptions on the lower-order coefficients of the operator $\mathcal{B}$ are as follows. Most of the results obtained require the vector field $\boldsymbol{\beta}(x) = \big(\beta^1(x), \ldots, \beta^n(x)\big)$ to be an *exterior field* on $\partial\Omega$, that is,

$$(B) \qquad \beta_0(x) := \beta^i(x)\mathbf{n}_i(x) \geq 0, \qquad x \in \partial\Omega.$$

We denote by $\boldsymbol{\beta}^*(x) := \boldsymbol{\beta}(x) - \beta_0(x)\mathbf{n}(x)$ the tangential component of $\boldsymbol{\beta}(x)$ and assume

$$(B3) \qquad \begin{aligned} &|\boldsymbol{\beta}^*| \in L^{\max\{q,n-1\}}(\partial\Omega), &&\text{if } q \neq n-1, \\ &|\boldsymbol{\beta}^*|\left(\log(1+|\boldsymbol{\beta}^*|)\right)^{1-1/(n-1)} \in L^{n-1}(\partial\Omega), &&\text{if } q = n-1, \end{aligned}$$

together with

$$(B4) \qquad \begin{aligned} &\gamma \in L^{\max\{q,(n-1)/2\}}(\partial\Omega), &&\text{if } q \neq (n-1)/2, \\ &\gamma\left(\log(1+|\gamma|)\right)^{1-1/(n-1)} \in L^{(n-1)/2}(\partial\Omega), &&\text{if } q = (n-1)/2. \end{aligned}$$

Finally, the normal component β_0 of the field $\boldsymbol{\beta}$ is supposed to satisfy

$$(B5) \qquad \begin{aligned} &\beta_0 \in L^q(\partial\Omega), &&\text{if } p > n, \\ &\beta_0 \in L^{qp^*/(p^* - qn/(n-1))}(\partial\Omega), &&\text{if } p < n, \\ &\beta_0\left(\log(1+|\beta_0|)\right)^{1-1/n} \in L^q(\partial\Omega), &&\text{if } p = n. \end{aligned}$$

Our first result provides an *a priori* estimate for any strong solution to the linear Venttsel problem in terms of the data of the problem.

Theorem 3.1. *Let the exponents p and q be chosen in accordance with (3.1), $\partial\Omega \in \mathcal{C}^{1,1}$ and assume that conditions (L1)–(L4) and (B1)–(B5) are verified.*
If a function $u \in V_{p,q}(\Omega)$ satisfies the equation

$$(3.4) \qquad \mathcal{L}u(x) = f(x) \quad a.e. \text{ in } \Omega$$

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and the boundary condition

$$(3.5) \qquad\qquad \mathcal{B}u(x) = g(x) \quad \text{a.e. on } \partial\Omega$$

with $f \in L^p(\Omega)$ and $g \in L^q(\partial\Omega)$, then

$$(3.6) \qquad \|u\|_{V_{p,q}(\Omega)} \leq C_1\Big(\|f\|_{p,\Omega} + \|g\|_{q,\partial\Omega} + \|u\|_{p,\Omega} + \|u\|_{q,\partial\Omega}\Big)$$

with a constant C_1 depending on n, ν, p, q, $\operatorname{diam}\Omega$, the properties of $\partial\Omega$, on the VMO-moduli of the coefficients $a^{ij}(x)$ and $\alpha^{ij}(x)$, and on the moduli of continuity of the functions $|\mathbf{b}|$, c, $|\boldsymbol{\beta}^*|$, γ, and β_0 in the corresponding functional spaces defined by conditions (L3)–(L4) *and* (B3)–(B5), *respectively.*[2]

Proof. If $\beta_0 \equiv 0$ then $\beta^i(x)D_i u = \beta^{*i}(x)d_i u$ on $\partial\Omega$ and (3.5) can be considered as autonomous equation on $\partial\Omega$:

$$-\alpha^{ij}(x)d_i d_j u + \beta^{*i}(x)d_i u + \gamma(x)u = g(x) \quad \text{a.e. on } \partial\Omega.$$

Using the standard procedure of finite covering of $\partial\Omega$ by balls, local flattening of $\partial\Omega$ and employing there the coercive estimates from [14, 15], and putting finally these together with the aid of partition of unity, we get

$$\|u\|_{W_q^2(\partial\Omega)} \leq C\Big(\|u\|_{q,\partial\Omega} + \|g\|_{q,\partial\Omega}\Big).$$

The function $u \in W_p^2(\Omega)$ solves the equation

$$-a^{ij}(x)D_i D_j u + b^i(x)D_i u + c(x)u = f(x) \quad \text{a.e. in } \Omega$$

and, according to Theorem 2.2, it assumes the boundary value $u|_{\partial\Omega}$ in the sense of W_p^1. Then we apply once again the global L^p-theory of [15] in order to conclude

$$\|u\|_{W_p^2(\Omega)} \leq C\Big(\|u\|_{p,\Omega} + \|f\|_{p,\Omega} + \|u\|_{W_q^2(\partial\Omega)}\Big)$$

that gives the claim (3.6). Actually, it is to be noted that the estimates in [15] regard second-order operators without lower-order terms, but appropriate use of interpolation inequalities leads to the same result for general second-order operators.

In the general case $\beta_0 \not\equiv 0$, we apply the so-called *Munchhausen trick* ([53], [17, Theorem 2.1], see also [5, Sect. 2]) and estimate the directional derivative $\partial_{\mathbf{n}} u$ in terms of itself. The procedure consists of three steps.

Step 1. Making use of

$$\beta^i(x)D_i u = \beta^{*i}(x)d_i u + \beta_0(x)\partial_{\mathbf{n}} u,$$

we rewrite the boundary condition (3.5) in the form

$$(3.7) \qquad -\alpha^{ij}(x)d_i d_j u = g(x) - \beta_0(x)\partial_{\mathbf{n}} u - \beta^{*i}(x)d_i u - \gamma(x)u$$
$$=: g_1(x) \quad \text{a.e. on } \partial\Omega.$$

Now we consider (3.7) as an elliptic equation on $\partial\Omega$. As above, the standard procedure of finite covering of $\partial\Omega$ by balls, local flattening of $\partial\Omega$ and employing there the coercive estimates from [14, 15], putting these together via a partition of unity, implies that any solution u of (3.7) satisfies the bound

$$\|u\|_{W_q^2(\partial\Omega)} \leq N_1\Big(\|g_1\|_{q,\partial\Omega} + \|u\|_{q,\partial\Omega}\Big).$$

---

[2]It is to be noted that the vector field $\boldsymbol{\beta}$ is **not** required to be exterior to $\partial\Omega$ for the validity of Theorem 3.1!

Here N_1 is determined by n, q, ν, the properties of $\partial\Omega$ and by the *VMO*-moduli of the coefficients α^{ij}.

Employing (3.7) in the last inequality, it takes on the form

$$(3.8) \qquad \|u\|_{W^2_q(\partial\Omega)} \leq N_1 \Big(\|g\|_{q,\partial\Omega} + \|u\|_{q,\partial\Omega} + \|\beta_0 \partial_\mathbf{n} u\|_{q,\partial\Omega}$$
$$+ \|\beta^{*i} d_i u\|_{q,\partial\Omega} + \|\gamma u\|_{q,\partial\Omega} \Big).$$

We proceed now to estimate the last two terms in the right-hand of (3.8). Take an arbitrary $\varepsilon > 0$ and consider $\|\beta^{*i} d_i u\|_{q,\partial\Omega}$. The argument falls naturally into three possible cases.

Case 1a. Let $q > n - 1$. Since $W^2_q(\partial\Omega)$ is compactly embedded into $\mathcal{C}^1(\partial\Omega)$, we have the estimate

$$(3.9) \qquad \|\beta^{*i} d_i u\|_{q,\partial\Omega} \leq \|\boldsymbol{\beta}^*\|_{q,\partial\Omega} \|du\|_{\infty,\partial\Omega}$$
$$\leq \varepsilon \|\boldsymbol{\beta}^*\|_{q,\partial\Omega} \|u\|_{W^2_q(\partial\Omega)} + N_2(\varepsilon) \|\boldsymbol{\beta}^*\|_{q,\partial\Omega} \|u\|_{q,\partial\Omega},$$

where $N_2(\varepsilon)$ depends also on n, q, $\mathrm{diam}\,\Omega$ and the properties of $\partial\Omega$.

Case 1b. Let $q < n - 1$. Now $|\boldsymbol{\beta}^*| \in L^{n-1}(\partial\Omega)$ and we use the well-known idea (see, for example, [31, Ch. III, §8, Remark 8.2]) to decompose $|\boldsymbol{\beta}^*|$ into the sum

$$|\boldsymbol{\beta}^*(x)| = \varphi_1(x) + \varphi_2(x),$$

where $\|\varphi_1\|_{n-1,\partial\Omega} \leq \delta$ with a small positive δ to be chosen later, and $\varphi_2 \in L^\infty(\partial\Omega)$. Notice that $\|\varphi_2\|_{\infty,\partial\Omega}$ is also determined by δ and by the moduli of continuity of $|\boldsymbol{\beta}^*|$ in $L^{n-1}(\partial\Omega)$. An application of the Hölder inequality yields

$$(3.10) \qquad \|\beta^{*i} d_i u\|_{q,\partial\Omega} \leq \|\varphi_1\|_{n-1,\partial\Omega} \|du\|_{q^*,\partial\Omega} + \|\varphi_2\|_{\infty,\partial\Omega} \|du\|_{q,\partial\Omega},$$

where $q^* = q(n-1)/(n-1-q)$. The first term in (3.10) is estimated from above with the help of the Sobolev embedding on $\partial\Omega$, while the upper bound for the second term follows from the compact embedding of $W^2_q(\partial\Omega)$ into $W^1_q(\partial\Omega)$. Thus, choosing δ small enough, we obtain

$$(3.11) \qquad \|\beta^{*i} d_i u\|_{q,\partial\Omega} \leq \varepsilon \left(1 + \|\varphi_2\|_{\infty,\partial\Omega}\right) \|u\|_{W^2_q(\partial\Omega)} + N_3(\varepsilon) \|\varphi_2\|_{\infty,\partial\Omega} \|u\|_{q,\partial\Omega},$$

where $N_3(\varepsilon)$ depends on the same parameters as $N_2(\varepsilon)$.

Case 1c. If $q = n - 1$ we argue in the same manner as in Case 1b. What is the difference now is that we use the Yudovich–Pohozhaev embedding theorem into the Orlicz space

$$W^1_{n-1}(\partial\Omega) \hookrightarrow L_\psi(\partial\Omega) \quad \text{with} \quad \psi(t) = e^{|t|^{(n-1)/(n-2)}} - 1$$

(see, e.g., [12, Sec. 10.5-10.6]). Therefore,

$$(3.12) \qquad |du|^{n-1} \in L_\Psi(\partial\Omega) \quad \text{with} \quad \Psi(t) = e^{|t|^{1/(n-2)}} - 1,$$

and we observe that in the considered case the assumption (B3) ensures that $|\boldsymbol{\beta}^*|^{n-1}$ belongs to the Orlicz space $L_{\Psi^*}(\partial\Omega)$ dual to $L_\Psi(\partial\Omega)$, see [29, Sec. 14]. As a result we get again the estimate (3.11), but now $\|\varphi_2\|_{\infty,\partial\Omega}$ is determined by the moduli of continuity of $|\boldsymbol{\beta}^*|$ in the Orlicz space related to (B3).[3]

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[3]Actually, the estimate (3.11) shows that $u \mapsto \beta^{*i} d_i u$ is a compact operator acting from $W^2_q(\partial\Omega)$ into $L^q(\partial\Omega)$, and further estimates of the lower-order terms can be interpreted in a similar way.

Summarizing, for sufficiently small ε, we have

$$(3.13) \qquad \|\beta^{*i} d_i u\|_{q,\partial\Omega} \le \frac{1}{4N_1} \|u\|_{W_q^2(\partial\Omega)} + \widehat{N}_1 \|u\|_{q,\partial\Omega}$$

in the all three cases, where N_1 is the constant from (3.8), while $\widehat{N}_1$ is determined by n, q, $\operatorname{diam}\Omega$, the properties of $\partial\Omega$ and on the moduli of continuity of $|\boldsymbol{\beta}^*|$ in corresponding functional spaces defined by conditions (B3).

Arguing in the same way as in deriving of (3.13), we conclude that

$$(3.14) \qquad \|\gamma u\|_{q,\partial\Omega} \le \frac{1}{4N_1} \|u\|_{W_q^2(\partial\Omega)} + \widehat{N}_2 \|u\|_{q,\partial\Omega},$$

where $\widehat{N}_2$ depends on n, q, $\operatorname{diam}\Omega$, the properties of $\partial\Omega$ and on the moduli of continuity of γ in corresponding functional spaces defined by conditions (B4).

Substituting (3.13) and (3.14) into the right-hand side of (3.8), we obtain

$$(3.15) \qquad \|u\|_{W_q^2(\partial\Omega)} \le N_4 \left(\|g\|_{q,\partial\Omega} + \|u\|_{q,\partial\Omega} + \|\beta_0 \partial_{\mathbf{n}} u\|_{q,\partial\Omega} \right),$$

where $N_4 = 2N_1 \left(1 + \widehat{N}_1 + \widehat{N}_2 \right)$.

Step 2. Consider in Ω the function

$$(3.16) \qquad v(x) = u(x) - \widetilde{u}(x)$$

with $\widetilde{u} = \Pi(u\big|_{\partial\Omega})$, where Π is the extension operator constructed in Theorem 2.2. It is evident that v solves the boundary value problem

$$(3.17) \qquad \begin{cases} -a^{ij}(x) D_i D_j v = f_1(x) & \text{a.e. in } \Omega, \\ v = 0 & \text{a.e. on } \partial\Omega, \end{cases}$$

where $f_1(x) := f(x) - b^i(x) D_i u - c(x) u + a^{ij}(x) D_i D_j \widetilde{u}$.

It follows from Theorem 4.2 in [15] that the solution of (3.17) satisfies the bound

$$\|v\|_{W_p^2(\Omega)} \le N_5 \left(\|f_1\|_{p,\Omega} + \|v\|_{p,\Omega} \right),$$

where N_5 depends only on n, p, ν, on the properties of $\partial\Omega$ and on the *VMO*-moduli of the coefficients a^{ij}. In view of (3.16), (2.2) and the definition of f_1, one can transform the last inequality into

$$(3.18) \qquad \|u\|_{W_p^2(\Omega)} \le N_6 \Big(\|f\|_{p,\Omega} + \|u\|_{p,\Omega} + \|u\|_{W_q^2(\partial\Omega)}$$
$$+ \|b^i D_i u\|_{p,\Omega} + \|cu\|_{p,\Omega} \Big)$$

and N_6 depends on the same quantities as N_5. Repeating the arguments used in deriving (3.15), we estimate the last two terms on the right-hand side of (3.18) and arrive finally at

$$(3.19) \qquad \|u\|_{W_p^2(\Omega)} \le N_7 \Big(\|f\|_{p,\Omega} + \|u\|_{p,\Omega} + \|u\|_{W_q^2(\partial\Omega)} \Big).$$

Here N_7 is determined by the same parameters as N_5 and, in addition, it depends also on the moduli of continuity of $|\mathbf{b}|$ and c in the corresponding functional spaces given by (L3) and (L4), respectively

Combining (3.15) with (3.19), we get

$$(3.20) \qquad \|u\|_{W_p^2(\Omega)} \le N_8 \Big(\|f\|_{p,\Omega} + \|g\|_{q,\partial\Omega} + \|u\|_{p,\Omega}$$
$$+ \|u\|_{q,\partial\Omega} + \|\beta_0 \partial_{\mathbf{n}} u\|_{q,\partial\Omega} \Big),$$

where $N_8 = N_7 (1 + N_4)$.

Step 3. We are in a position now to estimate the term $\|\beta_0 \partial_{\mathbf{n}} u\|_{q,\partial\Omega}$. We argue along the same lines as above when derived (3.15) and (3.19). For $p > n$ we use the embedding $W_p^2(\Omega) \hookrightarrow \mathcal{C}^1(\overline{\Omega})$, for $p < n$ and $p = n$ we use the trace embeddings of $W_p^1(\Omega)$ into $L^{p^*(n-1)/n}(\partial\Omega)$ and into the Orlicz space, respectively (see [12, Sec. 10.5-10.6]). Thus, in the all three cases we have

$$(3.21) \qquad \|\beta_0 \partial_{\mathbf{n}} u\|_{q,\partial\Omega} \le \frac{1}{2N_8} \|u\|_{W_p^2(\Omega)} + \widehat{N}_3 \|u\|_{p,\Omega},$$

where N_8 is the constant from (3.20), and $\widehat{N}_3$ is determined by n, p, $\mathrm{diam}\,\Omega$, the properties of $\partial\Omega$ and on the moduli of continuity of $|\beta_0|$ in corresponding functional spaces defined by conditions (B5).

Substituting (3.20) into (3.21), we finalize the Munchhausen trick and arrive at

$$\|\beta_0 \partial_{\mathbf{n}} u\|_{q,\partial\Omega} \le (1 + 2\widehat{N}_3) \Big(\|f\|_{p,\Omega} + \|g\|_{q,\partial\Omega} + \|u\|_{p,\Omega} + \|u\|_{q,\partial\Omega} \Big).$$

Finally, inserting the last inequality into the right-hand sides of (3.15) and (3.20) gives the claim (3.6) and this completes the proof. □

For the sake of further application of Theorem 3.1 to the study of quasilinear Venttsel problems, we need its variant concerning sequences of differential operators.

Remark 3.2. Let the sequence of operators

$$\mathcal{L}_k \equiv -a_k^{ij}(x)D_iD_j + b_k^i(x)D_i + c_k(x), \qquad \mathcal{B}_k \equiv -\alpha_k^{ij}(x)d_id_j + \beta_k^i(x)D_i + \gamma_k(x),$$

satisfy the assumptions (L1)–(L2), (B1)–(B2). Suppose that the principal coefficients a_k^{ij} and α_k^{ij} are *VMO* functions uniformly in k, that is,

$$\lim_{r \to 0} \sup_k \omega_{a_k^{ij}}(r) = \lim_{r \to 0} \sup_k \omega_{\alpha_k^{ij}}(r) = 0.$$

Define further $\beta_{0,k}(x) := \beta_k^i(x)\mathbf{n}_i(x)$ and assume that

$$|\mathbf{b}_k| \le |\mathbf{b}|, \qquad |c_k| \le |c|, \qquad |\boldsymbol{\beta}_k^*| \le |\boldsymbol{\beta}^*|, \qquad |\gamma_k| \le |\gamma|, \qquad |\beta_{0,k}| \le |\beta_0|,$$

where the functions $|\mathbf{b}|, c, |\boldsymbol{\beta}^*|, \gamma$ and β_0 satisfy the assumptions (L3), (L4), (B3), (B4) and (B5), respectively. Then the solutions of the boundary value problems

$$\mathcal{L}_k u_k(x) = f_k(x) \quad \text{a.e. in } \Omega, \qquad \mathcal{B}_k u_k(x) = g_k(x) \quad \text{a.e. on } \partial\Omega$$

satisfy the estimate

$$\|u_k\|_{V_{p,q}(\Omega)} \le C_1 \Big(\|f_k\|_{p,\Omega} + \|g_k\|_{q,\partial\Omega} + \|u_k\|_{p,\Omega} + \|u_k\|_{q,\partial\Omega} \Big)$$

with a constant C_1 which is *independent* of k.

In the case when uniqueness theorem holds for the linear Venttsel problem (3.4)–(3.5), the lower-order terms $\|u\|_{p,\Omega}$ and $\|u\|_{q,\partial\Omega}$ can be dropped from the right-hand side of (3.6).

Corollary 3.3. *Let the domain Ω and the operators $\mathcal{L}$ and $\mathcal{B}$ in (3.4)–(3.5) satisfy all the assumptions of Theorem 3.1. Assume also that the homogeneous problem*

$$(3.22) \qquad \mathcal{L}u = 0 \quad a.e. \ in \ \Omega, \qquad \mathcal{B}u = 0 \quad a.e. \ on \ \partial\Omega$$

admits only *the trivial solution in $V_{p,q}(\Omega)$.*
Then a solution $u \in V_{p,q}(\Omega)$ of (3.4)–(3.5) satisfies the estimate

$$(3.23) \qquad \|u\|_{V_{p,q}(\Omega)} \leq \widetilde{C}_1 \Big(\|f\|_{p,\Omega} + \|g\|_{q,\partial\Omega} \Big)$$

with a constant $\widetilde{C}_1$ independent of u[4].

Proof. The proof is quite standard (see [24, Lemma 9.17] and the proof of Lemma 5.1 [35] for a more general result). We argue by contradiction. If (3.23) is false, then there is a sequence u_k such that

$$f_k := \mathcal{L}u_k \to 0 \ \text{ in } \ L^p(\Omega), \quad g_k := \mathcal{B}u_k \to 0 \ \text{ in } \ L^q(\partial\Omega), \quad \|u_k\|_{V_{p,q}(\Omega)} \equiv 1.$$

Without loss of generality we may assume that $u_k \to u$ in $L^p(\Omega)$ and $u_k \to v$ in $L^q(\partial\Omega)$. The estimate (3.6) applied to pairwise differences $u_k - u_m$ yields

$$\|u_k - u_m\|_{V_{p,q}(\Omega)} \to 0 \quad \text{as} \quad k, m \to \infty,$$

and therefore $u_k \to u$ in $V_{p,q}(\Omega)$, $v = u|_{\partial\Omega}$. This, in turn, implies $\mathcal{L}u = 0$ and $\mathcal{B}u = 0$, whence $u = 0$ by the uniqueness, which is a contradiction. $\qquad \square$

Standard arguments, based on the parameter continuation and the coercive estimate (3.6), lead to the following existence theorem.

Theorem 3.4. *Under the hypotheses of Corollary 3.3, the non-homogeneous problem (3.4)–(3.5) admits a unique solution $u \in V_{p,q}(\Omega)$ for all $f \in L^p(\Omega)$ and $g \in L^q(\partial\Omega)$.*

Using this result we can prove that the couple $(\mathcal{L}, \mathcal{B})$ supports the *classical elliptic regularization property:* if the data of (3.4)–(3.5) allow the solution to have better integrability, then the solution does gain it indeed.

Theorem 3.5. *Let the domain Ω and the operators $\mathcal{L}$ and $\mathcal{B}$ satisfy all the assumptions of Theorem 3.1. Suppose that the exponents $\tilde{p} < p$, $\tilde{q} < q$ verify (3.1).*
If $u \in V_{\tilde{p},\tilde{q}}(\Omega)$ is a solution of (3.4)–(3.5) with $f \in L^p(\Omega)$ and $g \in L^q(\partial\Omega)$ then $u \in V_{p,q}(\Omega)$.

Proof. In case the homogeneous problem (3.22) admits only the trivial solution in $V_{\tilde{p},\tilde{q}}(\Omega)$, then uniqueness in $V_{\tilde{p},\tilde{q}}(\Omega)$ implies uniqueness in $V_{\hat{p},\hat{q}}(\Omega)$ for all $\hat{p} \in [\tilde{p}, p]$ and $\hat{q} \in [\tilde{q}, q]$ that satisfy (3.1). Therefore, Theorem 3.4 gives the claim $u \in V_{p,q}(\Omega)$.

Otherwise, if the kernel of the couple $(\mathcal{L}, \mathcal{B})$ in $V_{\tilde{p},\tilde{q}}(\Omega)$ is non-trivial, then we hope to get at least the Fredholm property in $V_{\tilde{p},\tilde{q}}(\Omega)$. Unfortunately, the hypotheses of Theorem 3.1 ensure it in $V_{p,q}(\Omega)$, but generally not in $V_{\tilde{p},\tilde{q}}(\Omega)$. The reason is that the requirement (B5), in contrast to the other assumptions on the lower-order terms in (3.4)–(3.5), becomes stronger when p decreases. To bypass that obstacle,

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[4]In Lemma 4.7 below we give a more general result which regards sequences of differential operators.

we transfer the "bad" term from the left-hand side into the right one and rewrite the problem (3.4)–(3.5) as follows

$$\widetilde{\mathcal{L}}u := (\mathcal{L} + \lambda)u = f + \lambda u =: \tilde{f} \qquad\qquad \text{a.e. in } \Omega;$$

$$\widetilde{\mathcal{B}}u := -\alpha^{ij}d_i d_j u + \beta^{*i}d_i u + \gamma u = g - \beta_0 \partial_{\mathbf{n}}u =: \tilde{g} \qquad \text{a.e. on } \partial\Omega.$$

The operator $\widetilde{\mathcal{B}}$ has zero normal derivative component, and our hypotheses ensure the Fredholm property in $V_{\tilde{p},\tilde{q}}(\Omega)$. Choosing λ in a way to avoid the discrete spectrum of the couple $(\widetilde{\mathcal{L}},\widetilde{\mathcal{B}})$, the above problem results uniquely solvable in $V_{\tilde{p},\tilde{q}}(\Omega)$ for any $\tilde{f} \in L^{\tilde{p}}(\Omega)$ and $\tilde{g} \in L^{\tilde{q}}(\partial\Omega)$. Again, uniqueness in $V_{\tilde{p},\tilde{q}}(\Omega)$ implies uniqueness in $V_{\hat{p},\hat{q}}(\Omega)$ for all $\hat{p} \in [\tilde{p}, p]$ and $\hat{q} \in [\tilde{q}, q]$ that satisfy (3.1).
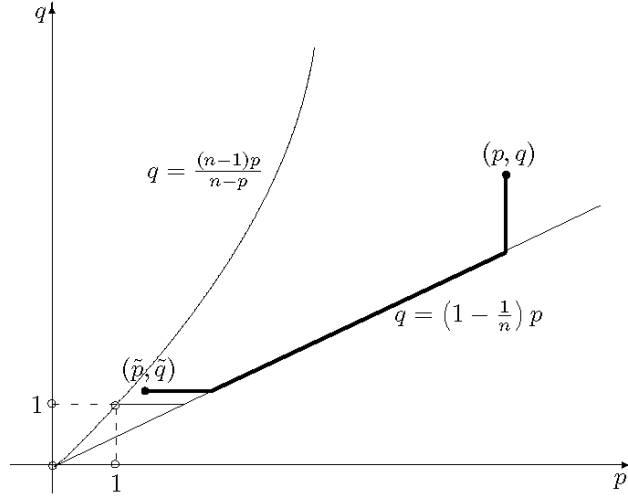

FIGURE 2. The passage from $(\tilde{p}, \tilde{q})$ to (p, q)

In general, the assumption (B5) does not guarantee $\beta_0 \partial_{\mathbf{n}}u \in L^{\tilde{q}}(\partial\Omega)$ for arbitrary $u \in V_{\tilde{p},\tilde{q}}(\Omega)$. Nevertheless, since all terms of $\widetilde{\mathcal{B}}u$ belong to $L^{\tilde{q}}(\partial\Omega)$, we conclude $\tilde{g} \in L^{\tilde{q}}(\partial\Omega)$. Moreover, $\tilde{f} \in L^{\hat{p}}(\Omega)$ with $\hat{p} = \min\{p, (\tilde{p}^*)^*\}$, and since $\tilde{p}$ and $\tilde{q}$ satisfy (3.1), we have $(\tilde{p}^*)^* > \frac{n\tilde{q}}{n-1}$. At that point Theorem 3.4 yields $u \in V_{\mathfrak{p},\tilde{q}}(\Omega)$ with $\mathfrak{p} = \min\{p, \frac{n\tilde{q}}{n-1}\}$ (the horizontal tract of the thick line on Fig. 2).

Now we move back the normal derivative term in the left-hand side and rewrite (3.4)–(3.5) as

$$\widetilde{\mathcal{L}}u = \tilde{f} \quad \text{a.e. in } \Omega; \qquad \mathcal{B}u = g \quad \text{a.e. on } \partial\Omega.$$

For this problem, the assumptions ensure the Fredholm property in $V_{\hat{p},\hat{q}}(\Omega)$ for any $\hat{p} \in [\mathfrak{p}, p]$, $\hat{q} = (1 - \frac{1}{n})\hat{p}$, and in $V_{p,\hat{q}}(\Omega)$ for any $\hat{q} \in \left[(1 - \frac{1}{n})p, q\right]$. So, we can choose again λ in a way that this problem is uniquely solvable in $V_{\mathfrak{p},\tilde{q}}(\Omega)$ for any $\tilde{f} \in L^{\mathfrak{p}}(\Omega)$ and $\tilde{g} \in L^{\tilde{q}}(\partial\Omega)$. Repeating the previous arguments, we get successively $u \in V_{p,(1-\frac{1}{n})p}(\Omega)$ and $u \in V_{p,q}(\Omega)$ (the oblique and vertical tracts on Fig. 2). $\qquad\square$

Theorem 3.4 shows that uniqueness is a sufficient condition guaranteeing existence of strong solutions to the linear Venttsel problem (3.4)–(3.5). There are various types of additional requirements to impose on the coefficients of $\mathcal{L}$ and $\mathcal{B}$ that ensure the validity of global maximum principle, and hence triviality of the kernel of (3.22). For instance, the following statement holds true.

Theorem 3.6. *Let $\partial\Omega \in W_n^2$ and let the operators $\mathcal{L}$ and $\mathcal{B}$ be defined by the formulas (3.2) and (3.3), respectively. Suppose that $\{a^{ij}\}$ and $\{\alpha^{ij}\}$ are symmetric matrices and that hypotheses (L2), (B2) and (B) are fulfilled.*
Assume also that

$$|\mathbf{b}| \in L^n(\Omega), \qquad |\boldsymbol{\beta}^*| \in L^{n-1}(\partial\Omega)$$

and

(3.24) $\qquad c \geq 0 \quad a.e.\ in\ \Omega, \qquad \gamma \geq \gamma_0 \quad a.e.\ on\ \partial\Omega, \quad \gamma_0 = \mathrm{const} > 0.$

If $u \in V_{n,n-1}(\Omega)$ satisfies

$$\mathcal{L}u \leq 0 \quad a.e.\ in\ \Omega, \qquad \mathcal{B}u \leq 0 \quad a.e.\ on\ \partial\Omega,$$

then $u \leq 0$ in $\overline{\Omega}$. In particular, the problem (3.22) admits only the trivial solution in the space $V_{n,n-1}(\Omega)$.

Proof. We argue by contradiction. Note that $u \in V_{n,n-1}(\Omega)$ implies $u \in \mathcal{C}(\overline{\Omega})$ and let $M := \max_{\overline{\Omega}} u > 0$. By the Aleksandrov–Bakel'man maximum principle ([2], see also Theorem 1.5 in the survey paper [43]), the maximum of u is achieved at a point $x^0 \in \partial\Omega$. We take a coordinate system centered at the point x^0 and flatten $\partial\Omega$ in a neighborhood of x^0, so that $\Omega \cap B_R \subset \mathbb{R}^n_+$ for some $R > 0$. It is worth noting that all the assumptions of Theorem 3.6 are invariant with respect to this coordinate transformation.

Further, we put $C = \nu^{-1}\sqrt{n-1}$ and introduce the set

$$\mathcal{O}_\rho = \left\{ x \in \mathbb{R}^n\colon \quad |x'| < \rho,\ 0 < x_n < C^{-1}\rho \right\}$$

and the function

$$u_\varepsilon(x) = u(x) - M + \varepsilon\left(1 - \frac{|x'|^2}{\rho^2} + \frac{C^2 x_n^2}{\rho^2} - \frac{2C x_n}{\rho}\right),$$

where $\rho < R$ and $\varepsilon < M$ are positive parameters to be chosen later.

It is evident that $u_\varepsilon|_{\partial\mathcal{O}_\rho \setminus \Gamma(\mathcal{O}_\rho)} \leq 0$ and $u_\varepsilon(0) = \varepsilon > 0$. Applying the local Aleksandrov-type estimate for the Venttsel problem [4, Theorem 3] (see also [43, Theorem 3.1]) to the function u_ε in $\mathcal{O}_\rho$, we obtain

$$\varepsilon \leq \widetilde{C}\left(n, \nu, \|\mathbf{b}\|_{n,\mathcal{O}_\rho}, \|\boldsymbol{\beta}^*\|_{n-1,\Gamma(\mathcal{O}_\rho)}\right) \times$$

$$\times \rho \left[\frac{2\varepsilon C}{\rho}\|\mathbf{b}\|_{n,\mathcal{O}_\rho} + \left\|\left(-\gamma_0(M - \varepsilon) + \frac{2\varepsilon(n-1)}{\nu\rho^2} + \frac{2\varepsilon}{\rho}|\boldsymbol{\beta}^*|\right)_+\right\|_{n-1,\Gamma(\mathcal{O}_\rho)}\right].$$

Thus, for $\varepsilon < \frac{M}{2}\left(1 + \frac{2(n-1)}{\gamma_0\rho^2\nu}\right)^{-1}$, we have

(3.25) $\qquad 1 \leq 2\widetilde{C}\left[C\|\mathbf{b}\|_{n,\mathcal{O}_\rho} + \left\|\left(|\boldsymbol{\beta}^*| - \frac{\gamma_0 M\rho}{4\varepsilon}\right)_+\right\|_{n-1,\Gamma(\mathcal{O}_\rho)}\right].$

The first term in the square brackets in (3.25) tends to zero as $\rho \to 0$. Therefore, there exists a value of ρ such that

$$\frac{1}{4\widetilde{C}} \leq \left\|\left(|\boldsymbol{\beta}^*| - \frac{\gamma_0 M\rho}{4\varepsilon}\right)_+\right\|_{n-1,\Gamma(\mathcal{O}_\rho)}.$$

However, the right-hand side of the last inequality tends to zero as $\varepsilon \to 0$ and the contradiction obtained gives the claim. $\qquad\square$

Corollary 3.7. *Let the exponents p and q satisfy (3.1), $\partial\Omega \in \mathcal{C}^{1,1}$, and assume that conditions* (L1)–(L2)*,* (B1)–(B2)*,* (B) *and* (3.24) *are satisfied.*
Suppose also that

$$c, |\mathbf{b}| \in L^p(\Omega), \qquad\qquad\qquad\quad \text{if} \quad p > n,$$

$$c, |\mathbf{b}| \left(\log\left(1 + |\mathbf{b}|\right)\right)^{1-1/n} \in L^n(\Omega), \quad \text{if} \quad p \le n;$$

$$\gamma, |\boldsymbol{\beta}^*| \in L^q(\partial\Omega), \qquad\qquad\qquad\qquad \text{if} \quad q > n - 1,$$

$$\gamma, |\boldsymbol{\beta}^*| \left(\log\left(1 + |\boldsymbol{\beta}^*|\right)\right)^{1-1/(n-1)} \in L^{n-1}(\partial\Omega), \quad \text{if} \quad q \le n - 1;$$

$$\beta_0 \in L^q(\partial\Omega), \qquad\qquad\qquad \text{if} \quad p > n,$$

$$\beta_0 \in L^{qp^*/(p^* - qn/(n-1))}(\partial\Omega), \qquad \text{if} \quad p \le n, \quad q > \left(1 - \frac{1}{n}\right)p,$$

$$\beta_0 \left(\log\left(1 + |\beta_0|\right)\right)^{1-1/n} \in L^{n-1}(\partial\Omega), \quad \text{if} \quad \frac{n}{n-1} < p \le n, \quad q = \left(1 - \frac{1}{n}\right)p.$$

Then the problem (3.4)–(3.5) *is uniquely solvable in $V_{p,q}(\Omega)$ for any $f \in L^p(\Omega)$ and $g \in L^q(\partial\Omega)$.*

Proof. If $p \ge n$ and $q \ge n - 1$, then Theorem 3.6 ensures triviality of the solution of (3.22) in $V_{n,n-1}(\Omega)$ and thus in $V_{p,q}(\Omega)$ as well, and the claim follows from Theorem 3.4.

Otherwise, the integrability requirements on the lower-order coefficients of $\mathcal{L}$ and $\mathcal{B}$ guarantee that any solution $u \in V_{p,q}(\Omega)$ of (3.22) in fact belongs to $V_{n,n-1}(\Omega)$ through Theorem 3.5. Then the desired unique solvability follows once again from (3.24) and Theorems 3.6 and 3.4. □

4. The quasilinear Venttsel problem

We aim now to the study of quasilinear elliptic equation

$$(4.1) \qquad\qquad -a^{ij}(x, u)D_i D_j u + a(x, u, Du) = 0 \quad \text{a.e. in } \Omega$$

coupled with the quasilinear Venttsel boundary condition

$$(4.2) \qquad\qquad -\alpha^{ij}(x, u)d_i d_i u + \alpha(x, u, Du) = 0 \quad \text{a.e. on } \partial\Omega$$

over domains Ω with $\mathcal{C}^{1,1}$-smooth boundaries. As in the previous Section, we suppose that $n \ge 3$.

Notice that (4.2) is not an autonomous equation on $\partial\Omega$ because it involves not only tangential derivatives but also the normal component of the gradient Du.

We suppose that the functions $a^{ij}(x, z)$ and $a(x, z, p)$ are Carathéodory functions, i.e., these are measurable in $x \in \Omega$ for all $(z, p) \in \mathbb{R} \times \mathbb{R}^n$ and continuous with respect to z and p for almost all $x \in \Omega$. The equation (4.1) will be assumed to be uniformly elliptic, that is, for almost all $x \in \Omega$ and for all $z \in \mathbb{R}$ we have

$$(A1) \qquad \nu|\xi|^2 \le a^{ij}(x, z)\xi_i\xi_j \le \nu^{-1}|\xi|^2 \quad \forall \xi \in \mathbb{R}^n, \; \nu = \text{const} > 0, \quad a^{ij} = a^{ji}.$$

Regarding the regularity conditions of the coefficients a^{ij}, we suppose that

$$a^{ij}(\cdot, z) \in \textit{VMO locally uniformly in } z, \textit{that is,}$$
$$(A2) \qquad\qquad\qquad \lim_{r \to 0} \sup_{z \in [-M, M]} \omega_{a^{ij}(\cdot, z)}(r) = 0,$$

where $\omega_{a^{ij}(\cdot,z)}$ is the *VMO*-modulus of $a^{ij}(\cdot,z)$ defined by (2.1) with B_ρ replaced by $B_\rho(x) \cap \Omega$, $x \in \Omega$. Moreover, we need a^{ij} to be locally uniformly continuous of with respect to z, uniformly in x :

(A3)
$$|a^{ij}(x,z) - a^{ij}(x,\widehat{z})| \leq \tau_M(|z - \widehat{z}|) \quad a.e. \ in \ \Omega, \quad \forall z, \widehat{z} \in [-M, M]$$
$$with \ a \ non\text{-}decreasing \ function \ \tau_M(t), \ \lim_{t \to 0} \tau_M(t) = 0.$$

The function $a(x, z, p)$ is assumed to grow quadratically with respect to the gradient, i.e., for almost all $x \in \Omega$ and for all $(z, p) \in \mathbb{R} \times \mathbb{R}^n$

(A4)
$$|a(x,z,p)| \leq \eta(|z|)\Big(\mu|p|^2 + b(x)|p| + \Phi(x)\Big)$$

with a constant $\mu \geq 0$ and a non-decreasing function $\eta \in \mathcal{C}(\mathbb{R}_+)$, and where

(A5)
$$b \left(\log\left(1 + |b|\right)\right)^{1-1/n} \in L^n(\Omega), \quad \Phi \in L^n(\Omega).$$

Further on, we assume that the boundary condition (4.2) is a uniformly elliptic Venttsel condition in the sense that for almost all $x \in \partial\Omega$ and for all $(z, p) \in \mathbb{R} \times \mathbb{R}^n$ we have:

(V)
$$the \ function \ \alpha(x,z,p) \ is \ weakly \ differentiable \ with \ respect \ to \ p_i, \ and$$
$$0 \leq \alpha_{p_i}(x,z,p)\mathbf{n}_i(x) \leq \eta(|z|)\beta_0(x),$$

with η as above and

(V0)
$$\beta_0 \left(\log\left(1 + |\beta_0|\right)\right)^{1-1/n} \in L^{n-1}(\partial\Omega)$$

and

(V1)
$$\nu|\xi^*|^2 \leq \alpha^{ij}(x,z)\xi_i^*\xi_j^* \leq \nu^{-1}|\xi^*|^2 \quad \forall \xi^* \in \mathbb{R}^n, \quad \xi^* \perp \mathbf{n}(x),$$
$$\nu = \text{const} > 0, \quad \alpha^{ij} = \alpha^{ji},$$

In addition, we impose regularity conditions on the coefficients α^{ij} similar to these required for a^{ij}. Precisely,

(V2)
$$\alpha^{ij}(\cdot,z) \in VMO \ locally \ uniformly \ in \ z, \ that \ is,$$
$$\lim_{r \to 0} \sup_{z \in [-M,M]} \omega_{\alpha^{ij}(\cdot,z)}(r) = 0,$$

where $\omega_{\alpha^{ij}(\cdot,z)}$ is the *VMO*-modulus of $\alpha^{ij}(\cdot,z)$ defined by (2.1) with $\partial\Omega \cap B_\rho(x)$, $x \in \partial\Omega$, in the place of B_ρ, and

(V3)
$$|\alpha^{ij}(x,z) - \alpha^{ij}(x,\widehat{z})| \leq \tau_M(|z - \widehat{z}|) \quad a.e. \ on \ \partial\Omega, \quad \forall z, \widehat{z} \in [-M, M],$$
$$with \ \tau_M(t) \ as \ in \ (A3).$$

The quasilinear term $\alpha(x, z, p)$ of (4.2) is required to support quadratic growth with respect to the tangential gradient, i.e., for almost all $x \in \partial\Omega$ and for all $z \in \mathbb{R}$

(V4)
$$|\alpha(x,z,p^*)| \leq \eta(|z|)\Big(\mu|p^*|^2 + \beta(x)|p^*| + \Theta(x)\Big) \quad \forall p^* \in \mathbb{R}^n, \quad p^* \perp \mathbf{n}(x)$$

with μ and η as in (A4) and where

(V5)
$$\beta \left(\log\left(1 + |\beta|\right)\right)^{1-1/(n-1)} \in L^{n-1}(\partial\Omega), \quad \Theta \in L^{n-1}(\partial\Omega).$$

It is well known (see [46, Lemma 2.1]) that conditions (A2)–(A3) and (V2)–(V3) provide for $u(x) \in \mathcal{C}(\overline{\Omega})$ the inclusions $a^{ij}(x, u(x)) \in VMO \cap L^\infty(\Omega)$ and $\alpha^{ij}(x, u(x)) \in VMO \cap L^\infty(\partial\Omega)$ with *VMO*-moduli bounded in terms of the continuity modulus of $u(x)$ and $\sup_\Omega |u|$.

The strong solvability of the problem (4.1)–(4.2) will be proved in $V_{n,n-1}(\Omega)$ by the aid of the Leray–Schauder fixed point theorem. To apply it, we have to derive *a priori* estimates in a suitable functional space for any solution to a family of quasi-linear Venttsel problems. Following the classical approach of O.A. Ladyzhenskaya and N.N. Ural'tseva [32], we obtain these estimates assuming we already dispose of a bound for the supremum norm $\sup_\Omega |u|$ of a solution $u \in V_{n,n-1}(\Omega)$.

We recall, first of all, the *a priori* estimate for the Hölder norm of a solution.

Proposition 4.1 (Theorem 1′ in [4])**.** *Let $\partial\Omega \in W_n^2$. Suppose that the function $u \in V_{n,n-1}(\Omega)$ is a solution of* (4.1)–(4.2).
Assume also that conditions (A1), (A4), (V), (V1) *and* (V4) *are satisfied with*

$$b, \Phi \in L^n(\Omega); \qquad \beta_0, \beta, \Theta \in L^{n-1}(\partial\Omega).$$

Then there exists a constant $\lambda > 0$ depending only on n, ν and the properties of $\partial\Omega$, such that

$$\|u\|_{\mathcal{C}^{0,\lambda}(\overline{\Omega})} \leq M_\lambda,$$

where M_λ depends only on n, ν, μ, the properties of $\partial\Omega$, $M_0 = \sup_\Omega |u|$, $\eta(M_0)$, $\|\Phi\|_{n,\Omega}$, $\|\Theta\|_{n-1,\partial\Omega}$ and on the moduli of continuity of the functions b, β_0 and β in the corresponding Lebesgue spaces.

The key *a priori* estimate of our approach is the gradient one.

Theorem 4.2. *Let $\partial\Omega \in \mathcal{C}^{1,1}$ and assume that conditions* (A1)–(A5), (V), (V0)–(V5) *are satisfied.*
Then any solution $u \in V_{n,n-1}(\Omega)$ of the problem (4.1)–(4.2) *fulfills the estimate*

$$(4.3) \qquad \|Du\|_{2n,\Omega} + \|du\|_{2(n-1),\partial\Omega} \leq M_1$$

with a constant M_1 depending on:

- n, ν, $\mathrm{diam}\,\Omega$ *and the properties of $\partial\Omega$;*
- $M_0 = \sup_\Omega |u|$ *and $\eta(M_0)$;*
- *the norms $\|\Phi\|_{n,\Omega}$ and $\|\Theta\|_{n-1,\partial\Omega}$;*
- *the constant μ and the moduli of continuity of the functions b, β_0 and β in the Orlicz spaces defined by conditions* (A5), (V0) *and* (V5)*, respectively;*
- *the VMO-moduli w.r.t. x and on the moduli of continuity w.r.t. z of the Carathéodory functions $a^{ij}(x,z)$ and $\alpha^{ij}(x,z)$, see conditions* (A2)–(A3), (V2)–(V3).

Proof. The *a priori* estimate (4.3) will be derived with the aid of a homotopy technique which goes back to Amann and Crandall [3] and has been used in [46] and [20] in the study of the Dirichlet and oblique derivative problems for quasilinear elliptic operators with discontinuous coefficients.

For the sake of brevity, we define hereafter

$$(4.4) \qquad \tilde{a}^{ij}(x) := a^{ij}(x, u(x)),$$

$$(4.5) \qquad \tilde{a}(x) := \frac{a(x, u(x), Du(x))}{\mu|Du(x)|^2 + b(x)|Du(x)| + \Phi(x)},$$

$$(4.6) \qquad \tilde{b}^i(x) := b(x)\tilde{a}(x)\frac{D_i u(x)}{|Du(x)|}$$

(recall that we set $\frac{0}{0} = 0$, if such an uncertainty occurs) and note that the assumption (A1) implies immediately the uniform ellipticity condition (L2) for $\tilde{a}^{ij}$.

Further, we make use of the hypotheses (A2), (A3) and employ [46, Lemma 2.1] in order to get that the VMO-moduli of $\tilde{a}^{ij}(x)$ and $\tilde{\alpha}^{ij}(x)$ are controlled in terms of $M_0 = \sup_\Omega |u|$ and the modulus of continuity of u. Further, Proposition 4.1 provides estimates the continuity modulus of u in terms of M_0 and therefore $\tilde{a}^{ij}$ satisfy (L1). Moreover, $|\tilde{a}| \leq \eta(M_0)$ in view of (A4) while $\tilde{b}^i$ verify (L3) as consequence of (A5).

This way, the equation (4.1) can be rewritten as

$$(4.7) \qquad -\tilde{a}^{ij}(x)D_iD_ju + \tilde{b}^i(x)D_iu + \mu\tilde{a}(x)|Du|^2 + \Phi(x)u = \tilde{f}(x) \quad \text{a.e. in } \Omega$$

with

$$\tilde{f}(x) := \Phi(x)\big(u(x) - \tilde{a}(x)\big) \in L^n(\Omega).$$

Similarly, we define

$$(4.8) \qquad \tilde{\alpha}^{ij}(x) := \alpha^{ij}(x, u(x)),$$

$$(4.9) \qquad \tilde{\alpha}(x) := \frac{\alpha(x, u(x), du(x))}{\mu|du(x)|^2 + \beta(x)|du(x)| + \Theta(x)},$$

$$(4.10) \qquad \tilde{\beta}^{*i}(x) := \beta(x)\tilde{\alpha}(x)\frac{d_iu(x)}{|du(x)|},$$

$$(4.11) \qquad \tilde{\beta}_0(x) := \int_0^1 \alpha_{p_i}\big(x, u(x), du(x) + t\mathbf{n}\partial_\mathbf{n}u(x)\big)\mathbf{n}_i(x)\, dt$$

and note that the assumptions (V1)–(V3) and [46, Lemma 2.1] imply the conditions (B1)–(B2) for $\tilde{\alpha}^{ij}$. Further on, $|\tilde{\alpha}| \leq \eta(M_0)$ by (V4), $\tilde{\beta}^{*i}$ satisfy (B3) because of (V5), while (V) implies $0 \leq \tilde{\beta}_0(x) \leq \eta(M_0)\beta_0(x)$ for a.a. $x \in \partial\Omega$. Moreover,

$$\alpha(x, u, Du) = \alpha(x, u, du) + \tilde{\beta}_0(x)\partial_\mathbf{n}u$$

and the boundary equation (4.2) takes on the form

$$(4.12) \qquad -\tilde{\alpha}^{ij}(x)d_id_ju + \tilde{\beta}^{*i}(x)d_iu + \tilde{\beta}_0(x)\partial_\mathbf{n}u$$
$$+ \mu\tilde{\alpha}(x)|du(x)|^2 + \Theta(x)u = \tilde{g}(x) \quad \text{a.e. on } \partial\Omega,$$

where $\tilde{g}(x) := \Theta(x)\big(u(x) - \tilde{\alpha}(x)\big) \in L^{n-1}(\partial\Omega)$. Having in mind (V4) and without loss of generality we may suppose hereafter that $\Theta \geq 1$.

Consider now the one-parameter family of Venttsel problems

$$(4.13) \qquad -\tilde{a}^{ij}D_iD_jv_\delta + \tilde{b}^iD_iv_\delta + \mu\tilde{a}|Dv_\delta|^2 + \Phi v_\delta = \delta\tilde{f} \qquad \text{a.e. in } \Omega,$$

$$(4.14) \qquad -\tilde{\alpha}^{ij}d_id_jv_\delta + \tilde{\beta}^{*i}d_iv_\delta + \tilde{\beta}_0\partial_\mathbf{n}v_\delta + \mu\tilde{\alpha}|dv_\delta|^2 + \Theta v_\delta = \delta\tilde{g} \quad \text{a.e. on } \partial\Omega.$$

Following the strategy of [3], we will prove unique solvability of (4.13)–(4.14) in $V_{n,n-1}(\Omega)$ for each $\delta \in [0,1]$ and will estimate the gradient of v_{δ_2} in terms of the gradient of v_{δ_1} for small enough $\delta_2 - \delta_1 > 0$. Then we will easily have $v_0 \equiv 0$, while the coincidence of the problem (4.13)–(4.14) with (4.7)–(4.12) for $\delta = 1$ would give $v_1 = u$, and finite iteration in δ will give the desired bound (4.3).

To realize that plan, we need two lemmata.

Lemma 4.3. *Suppose that $v_{\delta_1}, v_{\delta_2} \in V_{n,n-1}(\Omega)$ solve (4.13)–(4.14) with $\delta_1 \leq \delta_2$. Then*

$$(4.15) \qquad \|v_{\delta_1} - v_{\delta_2}\|_{\infty,\Omega} \leq (\delta_2 - \delta_1)\big(M_0 + \eta(M_0)\big).$$

Proof. Setting $w = v_{\delta_1} - v_{\delta_2}$ we obtain the following *linear* Venttsel problem

$$\tilde{\mathcal{L}}w := -\tilde{a}^{ij}D_iD_jw + (\tilde{a}^i + \tilde{b}^i)D_iw + \Phi w = (\delta_1 - \delta_2)\tilde{f} \qquad \text{a.e. in } \Omega,$$

$$\tilde{\mathcal{B}}w := -\tilde{\alpha}^{ij}d_id_jw + (\tilde{\alpha}^i + \tilde{\beta}^{*i})d_iw + \tilde{\beta}_0\partial_{\mathbf{n}}w + \Theta w = (\delta_1 - \delta_2)\tilde{g} \quad \text{a.e. on } \partial\Omega,$$

with

$$\tilde{a}^i(x) := 2\mu\tilde{a}(x)\int_0^1 \big(tD_iw(x) + D_iv_{\delta_2}(x)\big)dt,$$

$$\tilde{\alpha}^i(x) := 2\mu\tilde{\alpha}(x)\int_0^1 \big(td_iw(x) + d_iv_{\delta_2}(x)\big)dt.$$

We recall that $|\tilde{a}| \leq \eta(M_0)$, $w, v_{\delta_2} \in W_n^2(\Omega)$ and therefore $\tilde{a}^i \in L^n(\Omega)$. Similarly, $\tilde{\alpha}^i \in L^{n-1}(\partial\Omega)$.

Using $\tilde{f}(x) \geq -\Phi(x)\big(M_0 + \eta(M_0)\big)$ we get

$$\tilde{\mathcal{L}}w \leq \tilde{\mathcal{L}}\big((\delta_2 - \delta_1)(M_0 + \eta(M_0))\big) \qquad \text{a.e. in } \Omega$$

and similarly

$$\tilde{\mathcal{B}}w \leq \tilde{\mathcal{B}}\big((\delta_2 - \delta_1)(M_0 + \eta(M_0))\big) \qquad \text{a.e. on } \partial\Omega.$$

It follows from Theorem 3.6 that

$$w(x) \leq (\delta_2 - \delta_1)\big(M_0 + \eta(M_0)\big) \qquad \text{in } \Omega.$$

In the same manner the lower estimate $w(x) \geq -(\delta_2 - \delta_1)\big(M_0 + \eta(M_0)\big)$ follows and this gives the claim (4.15). $\qquad\square$

It is worth noting that setting $\delta_1 = \delta_2$ in (4.15), we get immediately $v_{\delta_1} \equiv v_{\delta_2}$ and thus uniqueness of solutions to (4.13)–(4.14). Precisely,

Corollary 4.4. *The problem* (4.13)–(4.14) *cannot have more than one solution in* $V_{n,n-1}(\Omega)$ *for any* $\delta \in [0,1]$.

Lemma 4.5. *Under the hypotheses of Lemma 4.3, there is a* $\varkappa > 0$ *such that the inequality* $\delta_2 - \delta_1 \leq \varkappa$ *implies*

$$(4.16) \quad \|Dv_{\delta_2} - Dv_{\delta_1}\|_{2n,\Omega} + \|dv_{\delta_2} - dv_{\delta_1}\|_{2(n-1),\partial\Omega}$$

$$\leq C_2(\delta_2 - \delta_1)\Big(1 + \|Dv_{\delta_1}\|_{2n,\Omega} + \|dv_{\delta_1}\|_{2(n-1),\partial\Omega}\Big).$$

The constants $\varkappa$ *and* C_2 *depend on the same quantities as* M_1 *in the statement of Theorem 4.2.*

Proof. We rewrite the problem for $w = v_{\delta_1} - v_{\delta_2}$ as follows:

$$(4.17) \qquad \widehat{\mathcal{L}}w := -\tilde{a}^{ij}D_iD_jw + \tilde{b}^iD_iw + \Phi w = \widehat{f} \qquad\qquad \text{a.e. in } \Omega,$$

$$(4.18) \qquad \widehat{\mathcal{B}}w := -\tilde{\alpha}^{ij}d_id_jw + \tilde{\beta}^{*i}d_iw + \tilde{\beta}_0\partial_{\mathbf{n}}w + \Theta w = \widehat{g} \qquad \text{a.e. on } \partial\Omega,$$

with

$$\widehat{f} = (\delta_1 - \delta_2)\tilde{f} - \mu\tilde{a}\big(|Dv_{\delta_1}|^2 - |Dv_{\delta_2}|^2\big) \in L^n(\Omega),$$

$$\widehat{g} = (\delta_1 - \delta_2)\tilde{g} - \mu\tilde{\alpha}\big(|dv_{\delta_1}|^2 - |dv_{\delta_2}|^2\big) \in L^{n-1}(\partial\Omega).$$

Theorem 3.1 and Lemma 4.3 yield

$$(4.19) \qquad \|w\|_{V_{n,n-1}(\Omega)} \leq N_9\left(\big\|\widehat{f}\big\|_{n,\Omega} + \big\|\widehat{g}\big\|_{n-1,\partial\Omega} + (\delta_2 - \delta_1)\big(M_0 + \eta(M_0)\big)\right),$$

where N_9 depends only on n, ν, $\operatorname{diam}\Omega$, the properties of $\partial\Omega$, M_0, the norms $\|\Phi\|_{n,\Omega}$ and $\|\Theta\|_{n-1,\partial\Omega}$, the moduli of continuity of the functions b, β_0 and β in the corresponding Orlicz spaces defined by conditions (A5), (V0) and (V5), respectively, and on the *VMO*-moduli of the coefficients $\tilde{a}^{ij}(x)$ and $\tilde{\alpha}^{ij}(x)$.

However, as explained before, the *VMO*-moduli of $\tilde{a}^{ij}(x)$ and $\tilde{\alpha}^{ij}(x)$ are controlled in terms of M_0 through [46, Lemma 2.1] and Proposition 4.1. Thus, the constant N_9 in (4.19) depends only on data listed in the statement of Lemma 4.5.

Taking advantage of the bounds

$$\|\tilde{f}\|_{n,\Omega} \le \|\Phi\|_{n,\Omega}\big(M_0 + \eta(M_0)\big), \qquad \|\tilde{g}\|_{n-1,\partial\Omega} \le \|\Theta\|_{n-1,\partial\Omega}\big(M_0 + \eta(M_0)\big),$$

and of the evident inequalities

$$\big\||Dv_{\delta_1}|^2 - |Dv_{\delta_2}|^2\big\|_{n,\Omega} \le \|Dw\|_{2n,\Omega}^2 + 2\|Dw\|_{2n,\Omega}\|Dv_{\delta_1}\|_{2n,\Omega},$$

$$\big\||dv_{\delta_1}|^2 - |dv_{\delta_2}|^2\big\|_{n-1,\partial\Omega} \le \|dw\|_{2(n-1),\partial\Omega}^2 + 2\|dw\|_{2(n-1),\partial\Omega}\|dv_{\delta_1}\|_{2(n-1),\partial\Omega},$$

we rewrite (4.19) as follows

$$(4.20) \quad \|w\|_{V_{n,n-1}(\Omega)} \le N_{10}\Big((\delta_2 - \delta_1) + \|Dw\|_{2n,\Omega}^2 + \|dw\|_{2(n-1),\partial\Omega}^2$$
$$+ \|Dw\|_{2n,\Omega}\|Dv_{\delta_1}\|_{2n,\Omega} + \|dw\|_{2(n-1),\partial\Omega}\|dv_{\delta_1}\|_{2(n-1),\partial\Omega}\Big),$$

where N_{10} depends on the same quantities as N_9.

We infer now the Gagliardo–Nirenberg interpolation inequality [12, Theorem 15.1] and the estimate (4.15) to get

$$(4.21) \qquad \|Dw\|_{2n,\Omega}^2 \le C(n,\Omega)\big(\|D^2w\|_{n,\Omega} + \|w\|_{\infty,\Omega}\big)\|w\|_{\infty,\Omega}$$
$$\le C(n,\Omega)(\delta_2 - \delta_1)\big(M_0 + \eta(M_0)\big)\|w\|_{V_{n,n-1}(\Omega)},$$

and similarly

$$(4.22) \qquad \|dw\|_{2(n-1),\partial\Omega}^2 \le C(n,\Omega)(\delta_2 - \delta_1)\big(M_0 + \eta(M_0)\big)\|w\|_{V_{n,n-1}(\Omega)}.$$

We substitute these inequalities into (4.20) and estimate the last two terms by the Cauchy inequality. This gives

$$\|w\|_{V_{n,n-1}(\Omega)} \le N_{11}\Big((\delta_2 - \delta_1) + (\delta_2 - \delta_1 + \varkappa)\|w\|_{V_{n,n-1}(\Omega)}$$
$$+ \frac{\delta_2 - \delta_1}{\varkappa}\big(\|Dv_{\delta_1}\|_{2n,\Omega}^2 + \|dv_{\delta_1}\|_{2(n-1),\partial\Omega}^2\big)\Big)$$

with arbitrary $\varkappa \in (0,1)$ and where N_{11} depends on the same quantities as N_9. Choosing $\varkappa = \frac{1}{4N_{11}}$ we obtain (4.16) for $\delta_2 - \delta_1 \le \varkappa$, in view of (4.21) and (4.22). $\square$

Turning back to the proof of Theorem 4.2, we fix $\delta_1 = 0$ and $\delta_2 = \varkappa$ in (4.16) and remember that $v_0 \equiv 0$ by Corollary 4.4. This gives the *a priori* estimate

$$(4.23) \qquad \|Dv_\varkappa\|_{2n,\Omega} + \|dv_\varkappa\|_{2(n-1),\partial\Omega} \le C_2\varkappa.$$

The solvability of (4.13)–(4.14) with $\delta = \varkappa$ is a consequence of the Leray–Schauder fixed point theorem. Indeed, define the nonlinear operator

$$\mathcal{F}:\ W_{2n}^1(\Omega) \cap W_{2(n-1)}^1(\partial\Omega) \mapsto V_{n,n-1}(\Omega)$$

which associates to any $w \in W^1_{2n}(\Omega) \cap W^1_{2(n-1)}(\partial\Omega)$ the unique solution $v = \mathcal{F}(w)$ of the *linear* Venttsel problem

$$\begin{cases} \widehat{\mathcal{L}}v = \varkappa \tilde{f}(x) - \tilde{a}(x)|Dw|^2 & \text{a.e. in } \Omega, \\ \widehat{\mathcal{B}}v = \varkappa \tilde{g}(x) - \tilde{\alpha}(x)|dw|^2 & \text{a.e. on } \partial\Omega, \end{cases}$$

with operators $\widehat{\mathcal{L}}$ and $\widehat{\mathcal{B}}$ given by (4.17) and (4.18), respectively.

The unique solvability of that problem follows from Theorems 3.4 and 3.6 due to assumptions of Theorem 4.2 and $w \in W^1_{2n}(\Omega) \cap W^1_{2(n-1)}(\partial\Omega)$ (recall that $\Theta \geq 1$). Therefore, the nonlinear operator $\mathcal{F}$ is well defined. Moreover, the problem (4.13)–(4.14) with $\delta = \varkappa$ is equivalent to the equation $u = \mathcal{F}(u)$.

The estimate (3.23) yields the continuity of $\mathcal{F}$, while the compactness of the embedding $V_{n,n-1}(\Omega) \hookrightarrow W^1_{2n}(\Omega) \cap W^1_{2(n-1)}(\partial\Omega)$ guarantees the compactness of $\mathcal{F}$ considered as a mapping from $W^1_{2n}(\Omega) \cap W^1_{2(n-1)}(\partial\Omega)$ into itself. Finally, any solution of the equation $v = \sigma\mathcal{F}(v)$, $0 \leq \sigma \leq 1$, that is,

$$\begin{cases} \widehat{\mathcal{L}}v = \sigma\big(\varkappa\tilde{f}(x) - \tilde{a}(x)|Dv|^2\big) & \text{a.e. in } \Omega, \\ \widehat{\mathcal{B}}v = \sigma\big(\varkappa\tilde{g}(x) - \tilde{\alpha}(x)|dv|^2\big) & \text{a.e. on } \partial\Omega, \end{cases}$$

satisfies, by (4.23), the *a priori* estimate

$$\|Dv\|_{2n,\Omega} + \|dv\|_{2(n-1),\partial\Omega} \leq C_3$$

with $C_3 = C_2\varkappa$ independent of σ. This suffices to combine the Leray–Schauder theorem (see, e.g., [24, Theorem 11.6]) with Corollary 4.4 in order to get unique solvability of (4.13)–(4.14) with $\delta = \varkappa$.

To complete the proof of Theorem 4.2, we take successively $\delta_1 = k\varkappa$, $\delta_2 = (k+1)\varkappa$, $k \in \mathbb{N}$, and repeat the above procedure. Finitely many iterations of (4.16) lead to (4.3) since v_1 is nothing else than the solution u of the problem (4.7), (4.12). □

Based on the *a priori* gradient estimate derived in Theorem 4.2, we can get solvability of the quasilinear Venttsel problem (4.1)–(4.2) under the hypotheses listed at the beginning of Section 4.

Theorem 4.6. *Let $\partial\Omega \in \mathcal{C}^{1,1}$ and let the functions involved in (4.1)–(4.2) satisfy the conditions* (A1)–(A5), (V), (V0)–(V5).

If any solution to the one-parameter family of Venttsel problems

$$(4.24) \qquad -a^{ij}(x,u)D_iD_ju + \sigma a(x,u,Du) = 0 \qquad\qquad \textit{a.e. in } \Omega,$$

$$(4.25) \qquad -\alpha^{ij}(x,u)d_id_ju + (1-\sigma)u + \sigma\alpha(x,u,Du) = 0 \qquad \textit{a.e. on } \partial\Omega$$

satisfies the a priori estimate

$$(4.26) \qquad\qquad \sup_{\Omega}|u| \leq M_0$$

with M_0 independent of u and $\sigma \in [0,1]$, then the quasilinear Venttsel problem (4.1)–(4.2) is solvable in the space $V_{n,n-1}(\Omega)$.

Proof. We again proceed by using the Leray–Schauder theorem. Introduce the linear space

$$U_{n,n-1}(\Omega) := \Big\{v \in W^1_{2n}(\Omega) \cap W^1_{2(n-1)}(\partial\Omega)\colon \quad \partial_{\mathbf{n}}v \in L^1(\partial\Omega)\Big\},$$

equipped with the natural norm

$$\|v\|_{U_{n,n-1}(\Omega)} = \|v\|_{W^1_{2n}(\Omega)} + \|v\|_{W^1_{2(n-1)}(\partial\Omega)} + \|\partial_{\mathbf{n}}v\|_{L^1(\partial\Omega)}.$$

and define the nonlinear operator

$$\mathcal{F}_1 \colon \quad U_{n,n-1}(\Omega) \mapsto V_{n,n-1}(\Omega)$$

which associates to any $u \in U_{n,n-1}(\Omega)$ the solution $v = \mathcal{F}_1(u)$ of the *linear* Venttsel problem

$$(4.27) \qquad \mathbb{L}v := -\tilde{a}^{ij}D_iD_jv + \tilde{b}^iD_iv = -\tilde{a}(\mu|Du|^2 + \Phi) \qquad \text{a.e. in } \Omega,$$

$$(4.28) \qquad \mathbb{B}v := -\tilde{\alpha}^{ij}d_id_jv + \tilde{\beta}^{*i}d_iv + \tilde{\beta}_0\partial_{\mathbf{n}}v + v$$
$$= -\tilde{\alpha}(\mu|du|^2 + \Theta) + u \qquad \text{a.e. on } \partial\Omega,$$

where $\tilde{a}^{ij}$, $\tilde{a}$ and $\tilde{b}^i$ are defined by (4.4)–(4.6), while $\tilde{\alpha}^{ij}$, $\tilde{\alpha}$, $\tilde{\beta}^{*i}$ and $\tilde{\beta}_0$ are given by (4.8)–(4.11), respectively.

By the Morrey embedding theorem, see [24, Theorem 7.17], we have $u \in \mathcal{C}^{0,1/2}(\overline{\Omega})$. Similarly to the proof of Theorem 4.2, we establish that the problem (4.27)–(4.28) satisfies all the hypotheses of Theorems 3.1 and 3.6. On the base of Theorems 3.4 and 3.6 we conclude that the problem (4.27)–(4.28) is uniquely solvable in $V_{n,n-1}(\Omega)$ and therefore the nonlinear operator $\mathcal{F}_1$ is well defined. Moreover, it is easy to see that the original quasilinear Venttsel problem (4.1)–(4.2) is equivalent to the equation $u = \mathcal{F}_1(u)$.

Since $u \in V_{n,n-1}(\Omega)$ implies $\partial_{\mathbf{n}}u \in W^1_n(\Omega) \subset L^r(\partial\Omega)$ for all $r < \infty$, see, e.g., [12, Sec. 10.5], the space $V_{n,n-1}(\Omega)$ is embedded into $U_{n,n-1}(\Omega)$ and the embedding is compact. This guarantees the compactness of $\mathcal{F}_1$ considered as a mapping from $U_{n,n-1}(\Omega)$ into itself.

To prove the continuity of $\mathcal{F}_1$ in the space $U_{n,n-1}(\Omega)$, we consider a sequence $u_h \in U_{n,n-1}(\Omega)$ such that $u_h \to u$ in $U_{n,n-1}(\Omega)$ as $h \to \infty$, and set $v_h = \mathcal{F}_1(u_h)$, $v = \mathcal{F}_1(u)$. Thus, v is the solution of (4.27)–(4.28) while v_h solves the problem

$$(4.29) \qquad \mathbb{L}_hv_h := -\tilde{a}^{ij}_hD_iD_jv_h + \tilde{b}^i_hD_iv_h = -\tilde{a}_h(\mu|Du_h|^2 + \Phi) \qquad \text{a.e. in } \Omega,$$

$$(4.30) \qquad \mathbb{B}_hv_h := -\tilde{\alpha}^{ij}_hd_id_jv_h + \tilde{\beta}^{*i}_hd_iv_h + \tilde{\beta}_{0,h}\partial_{\mathbf{n}}v_h + v_h$$
$$= -\tilde{\alpha}_h(\mu|du_h|^2 + \Theta) + u_h \qquad \text{a.e. on } \partial\Omega,$$

where $\tilde{a}^{ij}_h$, $\tilde{a}_h$, $\tilde{b}^i_h$, $\tilde{\alpha}^{ij}_h$, $\tilde{\alpha}_h$, $\tilde{\beta}^{*i}_h$ and $\tilde{\beta}_{0,h}$ are defined similarly to (4.4)–(4.6) and (4.8)–(4.11) with u replaced by u_h.

Notice that $\|u_h\|_{\mathcal{C}^{0,1/2}(\overline{\Omega})}$ is uniformly bounded by the Morrey theorem. So, by [46, Lemma 2.1] the *VMO*-moduli of $\tilde{a}^{ij}_h(x)$ and $\tilde{\alpha}^{ij}_h(x)$ are uniformly bounded. Further, by definition we have

$$|\tilde{b}^i_h(x)| \le \eta(M)b(x), \quad |\tilde{\beta}^{*i}_h(x)| \le \eta(M)\beta(x), \quad 0 \le \tilde{\beta}_{0,h}(x) \le \eta(M)\beta_0(x),$$

where $M = \sup_h \|u_h\|_{\mathcal{C}(\overline{\Omega})}$ while the functions b, β and β_0 verify the assumptions (L3), (B3) and (B5), respectively. It follows from Remark 3.2 that any solution of the problem

$$(4.31) \qquad \mathbb{L}_hw_h = \mathbb{F}_h \quad \text{a.e. in } \Omega, \qquad \mathbb{B}_hw_h = \mathbb{G}_h \quad \text{a.e. on } \partial\Omega$$

satisfies the estimate

$$(4.32) \quad \|w_h\|_{V_{n,n-1}(\Omega)} \le C_4\Big(\|\mathbb{F}_h\|_{n,\Omega} + \|\mathbb{G}_h\|_{n-1,\partial\Omega} + \|w_h\|_{n,\Omega} + \|w_h\|_{n-1,\partial\Omega}\Big),$$

where the constant C_4 is independent of h.

As in Corollary 3.3, the lower-order terms can be dropped from the right-hand side of (4.32).

Lemma 4.7. *For any solution of the problem* (4.31), *the following estimate holds*

$$(4.33) \qquad \|w_h\|_{V_{n,n-1}(\Omega)} \leq \widetilde{C}_4 \big(\|\mathbb{F}_h\|_{n,\Omega} + \|\mathbb{G}_h\|_{n-1,\partial\Omega} \big)$$

with a constant $\widetilde{C}_4$ *independent of* h *and* w_h.

Proof. As in the proof of Corollary 3.3, we proceed by contradiction. Assume that the statement is false. Then there is a sequence

$$\mathbb{F}_h \to 0 \quad \text{in } L^n(\Omega), \qquad \mathbb{G}_h \to 0 \quad \text{in } L^{n-1}(\partial\Omega),$$

such that the corresponding solutions of (4.31) are bounded away from zero in $V_{n,n-1}(\Omega)$. Without loss of generality we may suppose that $\|w_h\|_{V_{n,n-1}(\Omega)} = 1$. Then, passing if necessary to a subsequence, one has

$$w_h \rightharpoonup w \quad \text{in } V_{n,n-1}(\Omega), \qquad w_h \to w \quad \text{in } L^n(\Omega), \qquad w_h \to w \quad \text{in } L^{n-1}(\partial\Omega).$$

Since the difference $w_{h,k} := w_h - w_k$ solves the problem

$$\mathbb{L}_h w_{h,k} = \big(\tilde{a}_h^{ij} - \tilde{a}_k^{ij}\big) D_i D_j w_k + \big(\tilde{b}_k^i - \tilde{b}_h^i\big) D_i w_k + \mathbb{F}_h - \mathbb{F}_k \quad \text{a.e. in } \Omega,$$
$$\mathbb{B}_h w_{h,k} = \big(\tilde{\alpha}_h^{ij} - \tilde{\alpha}_k^{ij}\big) d_i d_j w_k + \big(\tilde{\beta}_k^{*i} - \tilde{\beta}_h^{*i}\big) d_i w_k$$
$$+ \big(\tilde{\beta}_{0,k} - \tilde{\beta}_{0,h}\big) \partial_{\mathbf{n}} w_k + \mathbb{G}_h - \mathbb{G}_k \quad \text{a.e. on } \partial\Omega,$$

the estimate (4.32) yields

$$(4.34) \qquad \|w_{h,k}\|_{V_{n,n-1}(\Omega)} \leq C_4 \Big(\big\| \big(\tilde{a}_h^{ij} - \tilde{a}_k^{ij}\big) D_i D_j w_k \big\|_{n,\Omega}$$
$$+ \big\| \big(\tilde{\alpha}_h^{ij} - \tilde{\alpha}_k^{ij}\big) d_i d_j w_k \big\|_{n-1,\partial\Omega} + \big\| \big(\tilde{b}_k^i - \tilde{b}_h^i\big) d_i w_k \big\|_{n,\Omega}$$
$$+ \big\| \big(\tilde{\beta}_k^{*i} - \tilde{\beta}_h^{*i}\big) d_i w_k \big\|_{n-1,\partial\Omega} + \big\| \big(\tilde{\beta}_{0,k} - \tilde{\beta}_{0,h}\big) \partial_{\mathbf{n}} w_k \big\|_{n-1,\partial\Omega}$$
$$+ \|\mathbb{F}_h - \mathbb{F}_k\|_{n,\Omega} + \|\mathbb{G}_h - \mathbb{G}_k\|_{n-1,\partial\Omega}$$
$$+ \|w_{h,k}\|_{n,\Omega} + \|w_{h,k}\|_{n-1,\partial\Omega} \Big).$$

We know that $\|D_i D_j w_k\|_{n,\Omega}$ are bounded, while

$$\big\| \tilde{a}_h^{ij} - \tilde{a}_k^{ij} \big\|_{\infty,\Omega} = \big\| a^{ij}(\cdot, u_h) - a^{ij}(\cdot, u_k) \big\|_{\infty,\Omega} \to 0$$

as $h, k \to \infty$, as consequence of $u_h \to u$ in $\mathcal{C}^{0,1/2}(\overline{\Omega})$ and the hypothesis (A3).

Therefore, the first term on the right-hand side of (4.34) tends to 0 as $h, k \to \infty$. The second term is managed in the same way.

Further on,

$$\tilde{\beta}_k^{*i} - \tilde{\beta}_h^{*i} = \beta \left(\frac{\alpha(\cdot, u_k, du_k)}{\mu |du_k|^2 + \beta |du_k| + \Theta} \frac{d_i u_k}{|du_k|} - \frac{\alpha(\cdot, u_h, du_h)}{\mu |du_h|^2 + \beta |du_h| + \Theta} \frac{d_i u_h}{|du_h|} \right)$$

tends to zero a.e. on $\partial\Omega$ as $h, k \to \infty$. Since $\big| \tilde{\beta}_k^{*i}(x) - \tilde{\beta}_h^{*i}(x) \big| \leq 2\eta(M)\beta(x)$, the hypothesis (V5) together with the Lebesgue dominated convergence theorem ensure that

$$\int_{\partial\Omega} \big| \tilde{\beta}_k^{*i}(x) - \tilde{\beta}_h^{*i}(x) \big|^{n-1} \left(\log \left(1 + \big| \tilde{\beta}_k^{*i}(x) - \tilde{\beta}_h^{*i}(x) \big|^{n-1} \right) \right)^{1-1/(n-1)} dx \to 0$$

as $h, k \to \infty$. This gives

$$\left\|\left(\tilde{\beta}_k^{*i} - \tilde{\beta}_h^{*i}\right)d_i w_k\right\|_{n-1,\partial\Omega} \leq \left\||\tilde{\boldsymbol{\beta}}_k^* - \tilde{\boldsymbol{\beta}}_h^*|^{n-1}\right\|_{L_{\Psi^*}(\partial\Omega)}\left\||dw_k|^{n-1}\right\|_{L_{\Psi}(\partial\Omega)} \to 0$$

as $h, k \to \infty$ (recall that $L_{\Psi^*}(\partial\Omega)$ is the Orlicz space dual to $L_{\Psi}(\partial\Omega)$ introduced in (3.12)). The third and the fifth terms on the right-hand side of (4.34) are managed in the same manner. The last four terms evidently tend to zero and we conclude that $w_h \to w$ in $V_{n,n-1}(\Omega)$. In particular, $\|w\|_{V_{n,n-1}(\Omega)} = 1$.

Finally, passing to the limit in (4.31) yields

$$\mathbb{L}w = 0 \quad \text{a.e. in } \Omega, \qquad \mathbb{B}w = 0 \quad \text{a.e. on } \partial\Omega,$$

whence $w = 0$ by Theorem 3.6. The contradiction obtained completes the proof. $\square$

Turning back to the proof of Theorem 4.6, we have

$$\|v_h\|_{V_{n,n-1}(\Omega)} \leq \widetilde{C}_4 \eta(M)\Big(\big\|\mu|Du_h|^2 + \Phi\big\|_{n,\Omega} + \big\|\mu|du_h|^2 + \Theta + |u_h|\big\|_{n-1,\partial\Omega}\Big)$$

as consequence of Lemma 4.7 (recall that $|\tilde{a}_h(x)| \leq \eta(M)$ by (A4) and $|\tilde{\alpha}_h(x)| \leq \eta(M)$ by (V4)). The terms on the right-hand side above are uniformly bounded because of the boundedness of u_h in $U_{n,n-1}(\Omega)$. This means that the sequence v_h is bounded in $V_{n,n-1}(\Omega)$.

The difference $v - v_h$ solves the problem

$$\begin{aligned}
\mathbb{L}(v - v_h) &= \big(\tilde{a}^{ij} - \tilde{a}_h^{ij}\big)D_i D_j v_h + \big(\tilde{b}_h^i - \tilde{b}^i\big)D_i v_h \\
&\quad + \tilde{a}_h\big(\mu|Du_h|^2 + \Phi\big) - \tilde{a}\big(\mu|Du|^2 + \Phi\big) \quad \text{a.e. in } \Omega, \\
\mathbb{B}(v - v_h) &= \big(\tilde{\alpha}^{ij} - \tilde{\alpha}_h^{ij}\big)d_i d_j v_h + \big(\tilde{\beta}_h^{*i} - \tilde{\beta}^{*i}\big)d_i v_h \\
&\quad + \big(\tilde{\beta}_{0,h} - \tilde{\beta}_0\big)\partial_{\mathbf{n}} v_h + (u - u_h) \\
&\quad + \tilde{\alpha}_h\big(\mu|du_h|^2 + \Theta\big) - \tilde{\alpha}\big(\mu|du|^2 + \Theta\big) \quad \text{a.e. on } \partial\Omega
\end{aligned}$$

and the estimate (3.23) yields

$$\begin{aligned}
\|v - v_h\|_{V_{n,n-1}(\Omega)} \leq \widetilde{C}_1 \Big(&\left\|\big(\tilde{a}^{ij} - \tilde{a}_h^{ij}\big)D_i D_j v_h\right\|_{n,\Omega} + \left\|\big(\tilde{b}_h^i - \tilde{b}^i\big)d_i v_h\right\|_{n,\Omega} \\
&+ \left\|\big(\tilde{\alpha}^{ij} - \tilde{\alpha}_h^{ij}\big)d_i d_j v_h\right\|_{n-1,\partial\Omega} + \left\|\big(\tilde{\beta}_h^{*i} - \tilde{\beta}^{*i}\big)d_i v_h\right\|_{n-1,\partial\Omega} \\
&+ \left\|\big(\tilde{\beta}_{0,h} - \tilde{\beta}_0\big)\partial_{\mathbf{n}} v_h\right\|_{n-1,\partial\Omega} + \|u - u_h\|_{n-1,\partial\Omega} \\
&+ \mu\left\|\tilde{a}_h|Du_h|^2 - \tilde{a}|Du|^2\right\|_{n,\Omega} + \mu\left\|\tilde{\alpha}_h|du_h|^2 - \tilde{\alpha}|du|^2\right\|_{n-1,\partial\Omega} \\
&+ \left\|\big(\tilde{a}_h - \tilde{a}\big)\Phi\right\|_{n,\Omega} + \left\|\big(\tilde{\alpha}_h - \tilde{\alpha}\big)\Theta\right\|_{n-1,\partial\Omega}\Big).
\end{aligned}$$

We estimate the first five terms on the right-hand side above in the same way as done for the corresponding terms in (4.34). Further on,

$$\begin{aligned}
\left\|\tilde{a}_h|Du_h|^2 - \tilde{a}|Du|^2\right\|_{n,\Omega} \leq \|\tilde{a}_h\|_{\infty,\Omega}\left\||Du_h|^2 - |Du|^2\right\|_{n,\Omega} \\
+ \left\|\big(\tilde{a}_h - \tilde{a}\big)|Du|^2\right\|_{n,\Omega}.
\end{aligned}$$

The first term tends to zero since $u_h \to u$ in $U_{n,n-1}(\Omega)$, while the second one is infinitesimal by the Lebesgue theorem. All the remaining terms are estimated in a similar way and we obtain finally $v_h \to v$ in $V_{n,n-1}(\Omega)$ that proves the continuity of the operator $\mathcal{F}_1$.

In order to apply the Leray–Schauder theorem and to get existence of a fixed point of $\mathcal{F}_1$, we present a family of continuous, compact nonlinear operators $\mathcal{T}(\cdot, \sigma)$ continuously depending on the parameter $\sigma \in [0, 1]$ such that

$$\mathcal{T}(u, 0) \equiv 0; \qquad \mathcal{T}(u, 1) = \mathcal{F}_1(u), \qquad u \in U_{n,n-1}(\Omega).$$

Namely, given a $\sigma \in [0, 1]$, the operator $\mathcal{T}(\cdot, \sigma)$ associates to any $u \in U_{n,n-1}(\Omega)$ the unique solution v_σ of the linear Venttsel problem

$$- \tilde{a}^{ij} D_i D_j v_\sigma + \sigma \tilde{b}^i D_i v_\sigma = -\sigma \tilde{a}(\mu |Du|^2 + \Phi) \qquad \text{a.e. in } \Omega,$$

$$- \tilde{\alpha}^{ij} d_i d_j v_\sigma + \sigma \tilde{\beta}^{*i} d_i v_\sigma + \sigma \tilde{\beta}_0 \partial_{\mathbf{n}} v_\sigma + v_\sigma$$
$$= -\sigma \tilde{\alpha}(\mu |du|^2 + \Theta) + \sigma u \qquad \text{a.e. on } \partial\Omega.$$

All mentioned properties of the family $\mathcal{T}(\cdot, \sigma)$ follow from the previous arguments, and to apply the Leray–Schauder theorem it remains only to derive the *a priori* estimate

(4.35) $$\|u\|_{U_{n,n-1}(\Omega)} \leq C_5$$

for any solution of the equation $u = \mathcal{T}(u, \sigma)$ in $U_{n,n-1}(\Omega)$, with a constant C_5 independent of $\sigma \in [0, 1]$ and u.

To this end, we notice that the equation $u = \mathcal{T}(u, \sigma)$ is equivalent to the Venttsel problem (4.24)–(4.25). We apply Theorem 4.2 and take into account that under the assumption (4.26) the constant M_1 in (4.3) can be evidently chosen to be independent of σ.

Finally, we rewrite the problem (4.24)–(4.25) in the form

$$- \tilde{a}^{ij} D_i D_j u + \sigma \tilde{b}^i D_i u = -\sigma F \qquad \text{a.e. in } \Omega,$$

$$- \tilde{\alpha}^{ij} d_i d_j u + \sigma \tilde{\beta}^{*i} d_i u + \sigma \tilde{\beta}_0 \partial_{\mathbf{n}} u + u = -\sigma G \qquad \text{a.e. on } \partial\Omega,$$

with

$$F = \tilde{a}(\mu |Du|^2 + \Phi), \qquad G = \tilde{\alpha}(\mu |du|^2 + \Theta) - u,$$

and conclude by (4.3) that $\|F\|_{n,\Omega}$ and $\|G\|_{n-1,\partial\Omega}$ are bounded uniformly with respect to σ and u. Thus, (3.23) implies

$$\|u\|_{V_{n,n-1}(\Omega)} \leq C$$

with C independent of u and σ, which immediately implies the desired *a priori* estimate (4.35). Application of the Leray–Schauder fixed point theorem completes the proof of Theorem 4.6. □

Theorem 4.6 is of conditional type. It reduces the solvability of the quasilinear Venttsel problem (4.1)–(4.2) to the *a priori* estimate (4.26) for *any* solution $u \in V_{n,n-1}(\Omega)$ of the problem (4.24)–(4.25), with M_0 independent of u and $\sigma \in [0, 1]$. We provide now a simple sufficient condition ensuring the validity of (4.26).

Lemma 4.8. *Let $\partial\Omega \in W_n^2$. Assume that conditions* (A1), (V) *and* (V1) *are satisfied, together with* (A4) *and* (V4) *where*

$$b, \Phi \in L^n(\Omega); \quad \beta, \Theta \in L^{n-1}(\partial\Omega).$$

Finally, suppose that for $|z| \geq z_0$ the functions $a(x, z, p)$ and $\alpha(x, z, p^*)$ with $p^* \perp \mathbf{n}(x)$ are weakly differentiable with respect to z and

$$a_z(x, z, p) \geq \theta_0 \Phi(x); \qquad \alpha_z(x, z, p^*) \geq \theta_0 \max\{1, \Theta(x)\}, \quad \theta_0 = \text{const} > 0.$$

Then any solution $u \in V_{n,n-1}(\Omega)$ of (4.24)–(4.25) satisfies the bound (4.26) with $M_0 = z_0 + \theta_0^{-1}\eta(z_0)$.

Proof. We will get the estimate $u \leq M_0$, the proof of $u \geq -M_0$ is similar.

Suppose that the set $\Omega^+ = \{u > z_0\}$ is non-empty. Then, using (A4) and the lower bound for a_z, we can write a.e. in $\Omega \cap \Omega^+$ that

$$
\begin{aligned}
a(x, u, Du) &= a(x, z_0, Du) + \int_0^1 a_z\big(x, tu(x) + (1-t)z_0, Du(x)\big)(u - z_0)\, dt \\
&\geq -\eta(z_0)\Big(\mu|Du(x)|^2 + b(x)|Du(x)| + \Phi(x)\Big) + \theta_0 \Phi(x)(u - z_0).
\end{aligned}
$$

Therefore, (4.24) yields

$$
\begin{aligned}
(4.36) \qquad \mathcal{L}u(x) &:= -\tilde{a}^{ij}(x)D_iD_ju + \sigma\eta(z_0)b^i(x)D_iu + \sigma\theta_0\Phi(x)u \\
&\leq \big(\eta(z_0) + \theta_0 z_0\big)\sigma\Phi(x) \\
&= \mathcal{L}\big(\theta_0^{-1}\eta(z_0) + z_0\big),
\end{aligned}
$$

where

$$
\tilde{a}^{ij}(x) := a^{ij}\big(x, u(x)\big), \qquad b^i(x) := -D_iu(x)\Big(\mu + \frac{b(x)}{|Du(x)|}\Big) \in L^n(\Omega).
$$

Further on, assuming without loss of generality that $\Theta \geq 1$, we write $\alpha(x, u, Du) = \alpha(x, u, du) + \tilde{\beta}_0(x)\partial_{\mathbf{n}}u$ as in the proof of Theorem 4.2 with $\tilde{\beta}_0(x)$ given by (4.11). Using (V4) and the lower bound for α_z, we obtain a.e. on $\partial\Omega \cap \Omega^+$ that

$$
\begin{aligned}
\alpha(x, u, Du) &= \alpha(x, z_0, du) + \int_0^1 \alpha_z\big(x, tu(x) + (1-t)z_0, du(x)\big)(u - z_0)\, dt \\
&\quad + \tilde{\beta}_0(x)\partial_{\mathbf{n}}u \\
&\geq -\eta(z_0)\Big(\mu|du(x)|^2 + \beta(x)|du(x)| + \Theta(x)\Big) + \theta_0\Theta(x)(u - z_0) \\
&\quad + \tilde{\beta}_0(x)\partial_{\mathbf{n}}u.
\end{aligned}
$$

This way, (4.25) implies

$$
\begin{aligned}
(4.37) \qquad \mathcal{B}u(x) &:= -\tilde{\alpha}^{ij}(x)d_id_ju + \sigma\eta(z_0)\beta^{*i}(x)d_iu + \sigma\tilde{\beta}_0(x)\partial_{\mathbf{n}}u \\
&\quad + \big(1 - \sigma + \sigma\theta_0\Theta(x)\big)u \\
&\leq \big(\eta(z_0) + \theta_0 z_0\big)\sigma\Theta(x) \leq \mathcal{B}\big(\theta_0^{-1}\eta(z_0) + z_0\big)
\end{aligned}
$$

with

$$
\tilde{\alpha}^{ij}(x) := \alpha^{ij}\big(x, u(x)\big), \quad \beta^{*i}(x) := -d_iu(x)\Big(\mu + \frac{\beta(x)}{|du(x)|}\Big) \in L^{n-1}(\partial\Omega).
$$

It is to be noted that $\tilde{\beta}_0 \geq 0$ as it follows from (V), while $1 - \sigma + \sigma\theta_0\Theta(x) \geq \min\{1, \theta_0\} > 0$.

Setting $M_0 := \theta_0^{-1}\eta(z_0) + z_0$, we have from (4.36) and (4.37) that

$$
\mathcal{L}(u - M_0) \leq 0 \quad \text{a.e. in } \Omega \cap \Omega^+, \qquad \mathcal{B}(u - M_0) \leq 0 \quad \text{a.e. on } \partial\Omega \cap \Omega^+.
$$

To get the claim, we suppose the contrary, that is, let $\max_{\overline{\Omega}}(u - M_0) > 0$. Then the function $u(x) - M_0$ achieves its maximum at a point $x^0 \in \Omega^+$ and the Aleksandrov–Bakel'man maximum principle implies that $x^0 \in \partial\Omega \cap \Omega^+$. This leads to a contradiction as in Theorem 3.6 and that completes the proof. $\qquad\square$

It is to be noted that the monotonicity hypotheses on $a(x, z, p)$ and $\alpha(x, z, p^*)$ in Lemma 4.8 could be replaced with suitable sign-conditions of these terms with respect to z.

5. Concluding remarks and open problems

1. In Sections 3 and 4 we dealt with the multidimensional case $n \geq 3$. If $n = 2$, then the problem is simpler because the boundary equation is essentially an ordinary differential equation. So, in this case we can allow $q = 1$ in (3.1). Moreover, for both linear and quasilinear problems, the principal coefficient in the *boundary* operator can be merely measurable. All the results obtained in Sections 3 and 4 remain valid in the 2D case.

2. The machinery developed in Sections 3 and 4 runs without essential changes also for *two-phase* linear and quasilinear Venttsel problems with discontinuous coefficients (cf. [9, 10]).

3. Modulo technical details, the results of Section 3 can be extended to operators with *partially VMO* principal coefficients in the spirit of [30]. Moreover, since all statements in Section 4 use the *VMO*-hypothesis only via the linear theory, assumptions (A2) and (V2) can be weakened in a similar manner.

4. The integrability assumptions (L3), (L4), (B3)–(B5) on the lower-order coefficients of the linear operators $\mathcal{L}$ and $\mathcal{B}$ are *sharp* for the validity of the results obtained in Section 3. The technique employed in the proof of Theorem 3.1 can be successfully adopted to Dirichlet or oblique derivative problems for elliptic operator with *VMO* coefficients in order to generalize the results in [14, 15] and [19, 39].

5. The assumptions (A4) and (V4) allow not only quadratic gradient growth but also linear one with an *unbounded* coefficient. The approach used in the proof of Theorem 4.2, applied to quasilinear Dirichlet or regular oblique derivative problem, could give generalization of the results from [46] and [20].

Open problems and directions for further research. The results in Section 4 regard quasilinear elliptic operators and quasilinear Venttsel boundary conditions with principal coefficients depending on x and u only. In order to get statements at the same level of generality as in classical papers of O.A. Ladyzhenskaya and N.N. Ural'tseva (see the survey [32]), it is necessary to allow dependence also on Du and du, respectively, in the principal coefficients, and this is a fascinating open problem.

Another interesting problem is to extend the results of [11] to the case of discontinuous principle coefficients. If the support of the Venttsel condition in the two-phase problem is the interface meeting transversally the exterior boundary of the medium, then both parts of the medium are domains with smooth closed edges. This requires to study the problem in weighted Sobolev spaces where the weight is a power of the distance from a point to the edge (see [11] for more details).

Very difficult open problems appear in the case where the principal part of the boundary operator is *positively semidefinite.* There occur three different types of degeneracy:

1) Partial, when the matrix $\{\alpha^{ij}(x)\}$ is not vanishing but has zero as an eigenvalue. There are some results due to Luo and Trudinger [38] regarding the linear case, but the quasilinear one is completely open.

2) *Complete*, where the set $\mathcal{D} = \{x \in \partial\Omega\colon \alpha^{ij}(x) = 0\}$ is non-empty, but $\beta_0 \neq 0$ there. Now the boundary value problem is of oblique derivative type on $\mathcal{D}$, and some satisfactory results are available in [6, 7, 42]. We refer the reader also to [50, Chapter 6] where similar problems are treated with the machinery of pseudo-differential operators and Hörmander vector fields. It is a challenging task to overcome the continuity of the principal coefficients in that case.

3) *Over-degeneration*, when $\mathcal{D} \neq \emptyset$ and $\mathcal{D}_0 = \{x \in \mathcal{D}\colon \beta_0(x) = 0\}$ is *non-empty*. Now the Venttsel boundary condition is prescribed in terms only of *tangential* derivatives, $\beta^{*i}(x)d_i u + \gamma(x)u$, and the corresponding boudary value problem is no more regular (see [18, 51, 50, 47, 48, 49]). The properties of the problem, known as Poincaré problem, depend essentially on the behaviour of the vector field $\boldsymbol{\beta^*}$ near the set $\mathcal{D}_0$, and new effects appear such as loss of smoothness, loss of Fredholmness, etc. Nothing is known, instead, for the quasilinear Poincaré problem.

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DARYA E. APUSHKINSKAYA: SAARLAND UNIVERSITY, SAARBRÜCKEN, GERMANY; PEOPLES' FRIENDSHIP UNIVERSITY OF RUSSIA (RUDN UNIVERSITY), MOSCOW AND CHEBYSHEV LABORATORY, ST. PETERSBURG STATE UNIVERSITY, ST. PETERSBURG, RUSSIA
    Email address: `darya@math.uni-sb.de`

ALEXANDER I. NAZAROV: PDMI RAS AND ST. PETERSBURG STATE UNIVERSITY, ST. PETERSBURG, RUSSIA
    Email address: `al.il.nazarov@gmail.com`

DIAN K. PALAGACHEV: POLYTECHNIC UNIVERSITY OF BARI, DEPARTMENT OF MECHANICS, MATHEMATICS AND MANAGEMENT, 70125 BARI, ITALY
    Email address: `dian.palagachev@poliba.it`

LUBOMIRA G. SOFTOVA: UNIVERSITY OF SALERNO, DEPARTMENT OF MATHEMATICS, 84084 SALERNO, ITALY
    Email address: `lsoftova@unisa.it`