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Series solution of beams with variable cross-section

Valerio De Biagi[1*], Bernardino Chiaia[1], Giuseppe Carlo Marano[1], Alessandra Fiore[2], Rita Greco[3], Laura Sardone[2], Raffaele Cucuzza[1], Giuseppe L. Cascella[4], M. Spinelli[4], Nikos D. Lagaros[5]

[1]*Politecnico di Torino, Corso Duca degli Abruzzi, 24, 10129 Torino (Italy)*
[*]*corresponding author: valerio.debiagi@polito.it*
[2]*Politecnico di Bari, DICAR, Via Edoardo Orabona, 4, 70126 Bari (Italy)*
[3]*Politecnico di Bari, DicaTech, Via Edoardo Orabona, 4, 70126 Bari (Italy)*
[4]*IDEA75 s.r.l., Via Brigata e Divisione Bari, 122 – 70123 Bari (Italy)*
[5]*National Technical University of Athens, Patission Complex , 42, Patission str 10682 Athens*

Abstract

In structural engineering beams with non-constant cross-section or beams with variable cross-section represent a class of slender bodies, aim of practitioners' interest due to the possibility of optimizing their geometry with respect to specific needs. Despite the advantages that engineers can obtain from their applications, non-trivial difficulties occurring in the non-prismatic beam modeling often lead to inaccurate predictions that vanish the gain of the optimization process. As a consequence, an effective non-prismatic beam modeling still represents a branch of the structural engineering of interest for the community, especially for advanced design applications in large spans elements.

A straight beam of length *l* with variable inertia *J(z)* is provided in figure, subject to a generic live load condition *q(z)*. The vertical displacement y(z) can be obtained from the solution of the differential equation of the elastic line, i.e., taking into consideration the inertia variability and neglecting, as first approximation, any shear contribution. Even if this solution is an approximate one, it is able to deal with the problem in its basic formulation.

In this paper a solution for the problem stated is formulated using a series expansion of solutions, in a general load and cross section variability condition. Solution is thus obtained for a generic rectangular cross section beam with a variable height. Analytical solution is presented and evaluated using numerical evaluation of some cases of practical interest.

Keywords: Structural Design, Structural Optimization, Parametric Design, Architectural Design

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1. Introduction

In structural engineering beams with non-constant cross-section or beams of variable cross-section are represents a class of slender bodies, object of practitioners'< interest due to the possibility of optimizing their geometry with respect to specific needs. Beams with variable cross-section are widely adopted in many engineering fields, such as large span structures (bridges)[1], protective structures [2] or mechanical components [3]. Interests in such type of structures recently emerged in other disciplines, such as energy harvesting [4]. Evidences of variable cross-section largely emerge in damaged structures [5],[6],[7]. The modeling of such elements is nontrivial since the variable properties of the cross-section do not allow for straightforward solutions, event for simple geometries, loads and boundary conditions [8],[9]. Therefore, numerical tools need to be considered for solving such problems. Recently, a novel theory based on a simple definition of kinematics, equilibrium and constitutive equations was proposed for non-prismatic beams by Balduzzi et al. [10]. They showed that in Timoshenko beams the stress distribution of stresses due to the geometry play a relevant role in the response of the element. In addition, axial and shear-bending problems are strictly coupled. The solution of their model requires advanced numerical techniques, as reported in [1]. Other approaches can be found in the literature to solve the variable cross-section beams. Attarnejad et al. adopted displacement functions coupled with structural matrices [11], Beltempo et al. derived the displacements though the Hellinger-Reissner functional theory [12], Shooshtari and Khajavi proposed a procedure based on shape functions and stiffness matrices [13].

Despite the real advantages that practitioners can have from the use of beams with variable cross-section (for example, tapered beams), the nontrivial procedures and difficulties emerging in the solution vanish the gain of the optimization process [14]. In this conference paper we propose a simple solution scheme for an Euler-Bernoulli beam with variable cross section. Despite simple, the model offers interesting possibilities for future optimization procedures.

The paper is organized as follows: Section 2 details the basic theoretical aspects, Section 3 proposes the solution scheme based on power series. Section 4 illustrates the problem of applying boundary conditions to the solution and Section 5 proposes an example.

2. Elastic line for beams with variable cross-section inertia

The elastic line formula for the transversal displacement of beams is a well-known mathematical expression that originates from Euler-Bernoulli beam theory [15]. Consider a straight beam with variable cross-section inertia (Fig.1). The length of the beam is denoted as ℓ and z direction coincides with beam axis. Beam cross-section inertia is denoted as $J(z)$. The beam is loaded with a variable load, denoted as $q(z)$. The displacement v of the beam along the transversal y-axis is related to the other variables as:

$$\frac{d^2}{dz^2}\left[EJ(z)\frac{d^2v}{dz^2}\right] = q(z),$$

where E is the Young's modulus of elasticity. Considering that the function representing the inertia belongs, at least, to C^2 differentiability class, the previous expression can be rewritten as:

$$\frac{d^2J}{dz^2}\frac{d^2v}{dz^2} + 2\frac{dJ}{dz}\frac{d^3v}{dz^3} + J\frac{d^4v}{dz^4} = \frac{q(z)}{E}.$$

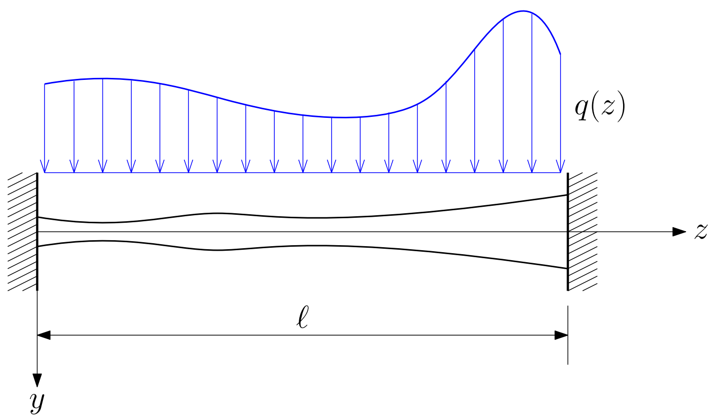

Fig. 1. Example of a beam with variable height, thus, variable stiffness. The distributed load is not constant along the length of the beam. For sake of simplicity, the beam has fixed supports at both ends.

3. Power series solution

The partial differential equation describing the transversal displacement at each position z along beam axis can be solved using various mathematical techniques. As described in the introduction, the solution of the equation can be performed through numerical integration.

To overcome such problem, we propose a solution based on power series. The idea at the base of the method is that the lateral displacement can be expressed as

$$v = \sum_{n=0}^{\infty} a_n z^n,$$

where a_i, with $i = 1..\infty$, is a set of constant terms to be determined on the bases of the boundary conditions. The length ℓ of the beam is finite, thus a normalization of the position in the range [0;1] can be done. The power series expansion of the lateral displacement, thus, becomes

$$v = \sum_{n=0}^{\infty} a_n \left(\frac{z}{\ell}\right)^n.$$

Similarly, the variable inertia and the applied loads are expressed as

$$J = \sum_{n=0}^{\infty} j_n \left(\frac{z}{\ell}\right)^n$$

and

$$q = \sum_{n=0}^{\infty} q_n \left(\frac{z}{\ell}\right)^n.$$

In case of constant load, q_0 is the only nonnull term of the series. The solution scheme consists in deriving the previous terms and inserting them in a unique expression with the same summation index. Details on how to deal with power series solutions of differential equations are provided in [16].

The elastic line differential equation can be rewritten as

$$\sum_{n=0}^{\infty}\left\{\sum_{p=0}^{n+2}\left[\mathcal{N}^{\mathrm{II}}_{(n-p+2)}\mathcal{J}^{\mathrm{II}}_{(p-2)}+2\mathcal{N}^{\mathrm{III}}_{(n-p+1)}\mathcal{J}^{\mathrm{I}}_{(p-1)}+\mathcal{N}^{\mathrm{IV}}_{(n-p)}\right]a_{(n-p+4)}j_p-\frac{q_n\ell^4}{E}\right\}\left(\frac{z}{\ell}\right)^n=0$$

where

$$\mathcal{N}^{\mathrm{II}}_t=(t+1)(t+2)$$
$$\mathcal{N}^{\mathrm{III}}_t=(t+1)(t+2)(t+3)$$
$$\mathcal{N}^{\mathrm{IV}}_t=(t+1)(t+2)(t+3)(t+4)$$

and

$$\mathcal{J}^{\mathrm{I}}_{\mathrm{t}}=(t+1)$$
$$\mathcal{J}^{\mathrm{II}}_{\mathrm{t}}=(t+1)(t+2)$$

From the previous equation, a general expression for the coefficient $a_{(n+4)}$ can be found:

$$a_{(n+4)}=\frac{1}{\mathcal{N}^{\mathrm{IV}}_n j_0}\left[\frac{q_n\ell^4}{E}-\sum_{p=1}^{n+2}\left(\mathcal{N}^{\mathrm{II}}_{(n-p+2)}\mathcal{J}^{\mathrm{II}}_{(p-2)}+2\mathcal{N}^{\mathrm{III}}_{(n-p+1)}\mathcal{J}^{\mathrm{I}}_{(p-1)}+\mathcal{N}^{\mathrm{IV}}_{(n-p)}\right)\ a_{(n-p+4)}j_p\right].$$

The previous equation represents the general expression for computing the coefficients of the power series of the lateral displacement of a straight beam with variable cross-section inertia and variable distributed load.

The coefficient $a_{(n+4)}$ can be evaluated starting from the previously computed terms and from the known values of j_p and q_n. The values a_0, a_1, a_2 and a_3 have to be determined from the boundary conditions.

4. Boundary conditions and coefficients a_0, a_1, a_2 and a_3

The boundary conditions allow to determine the values a_0, a_1, a_2 and a_3. For a fourth-order differential equation, 4 boundary conditions have to be determined. In power series solutions adopted in physics, the initial conditions, i.e. at $z=0$, are provided. This allows to directly evaluate the unknown four coefficients. In structural mechanics, the boundary conditions (b.c.) are usually provided for both ends of the beams. The conditions are directly related to the supports of the beams. For example, the fixed end at $z=0$ can be described through two b.c.:

$$v(0)=0 \qquad \varphi(0)=\left.\frac{dv}{dz}\right|_{z=0}=0.$$

A simply supported end at $z=\ell$ can be mathematically described as

$$v(\ell)=0 \qquad \aleph(\ell)=-\left.\frac{d^2v}{dz^2}\right|_{z=\ell}=0.$$

Focusing the attention on beams with fixed-fixed boundary conditions, it clearly results that $a_0=0$ and $a_1=0$. The terms a_2 and a_3 must be determined from the solution of the following system of equations

$$\begin{cases} a_2 + a_3 + \sum_{n=4}^{\infty} a_n = 0 \\ 2a_2 + 3a_3 + \sum_{n=4}^{\infty} n a_n = 0 \end{cases}$$

where $a_4, a_5, \ldots$ are computed through the expression previously mentioned. To solve this problem an optimization scheme is proposed in which dimensionless kinematic quantities expressing the displacement and the rotations are minimized.

5. Example

As an example of the approach herein presented, a straight doubly fixed beam is proposed, as sketched in Fig. 2. The beam has variable cross-section depth defined as a function of beam length

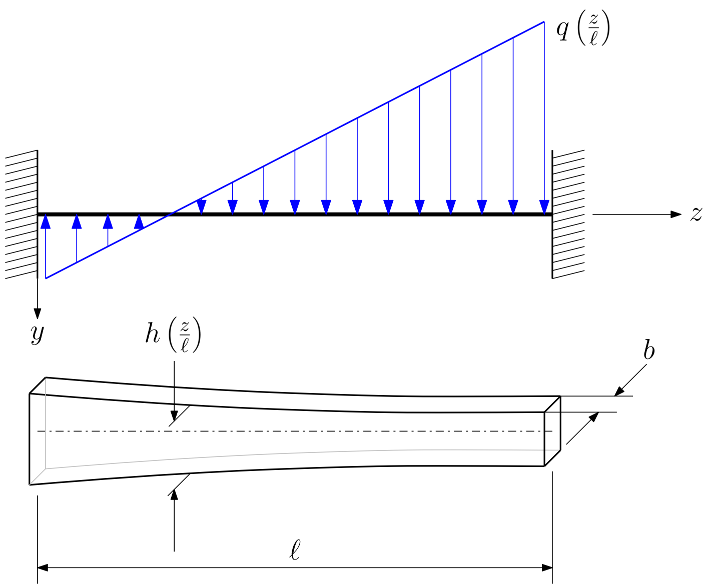

Fig.2. Example of a beam with variable depth and loading.

$$\frac{h}{\ell} = 0.1 - 0.1\left(\frac{z}{\ell}\right) + 0.08\left(\frac{z}{\ell}\right)^2,$$

and a variable distributed load defined as

$$\frac{q}{E\ell^2} = \left[-1 + 4\left(\frac{z}{\ell}\right)\right] \times 10^{-7}.$$

Dimensionless quantities are adopted following Buckingham's Π-theorem. The length of the beam and material Young's modulus are chosen as repeating terms. Considering a dimensionless beam width $b/\ell = 0.5$, the dimensionless inertia is

$$\frac{J}{\ell^4} \approx 0.0026 - 0.0078\left(\frac{z}{\ell}\right) + 0.0141\left(\frac{z}{\ell}\right)^2 - 0.0151\left(\frac{z}{\ell}\right)^3 + 0.0113\left(\frac{z}{\ell}\right)^4 - 0.0050\left(\frac{z}{\ell}\right)^5 + 0.0013\left(\frac{z}{\ell}\right)^6.$$

In order to determine the coefficients a_0, a_1, a_2 and a_3, the optimization procedure previously described was implemented on Matlab considering a maximum summation index equal to 2000. An unconstrained nonlinear minimization scheme was adopted to solve the optimization problem. The following parameter vector results

$$[a_0 \quad a_1 \quad a_2 \quad a_3] = [0 \quad 0 \quad 0.004006 \quad 0.001676].$$

The stem plot of Fig. 3 reports the values of the coefficients a_n for $n = 0..500$. Double colors are adopted for marking positive and negative coefficients. Since the y-axis of the plot is in logarithmic scale, the values of the coefficients decreases with increasing coefficient index, n. As an example, the coefficient a_{2000}, referring to the maximum computed index, is equal to 3.39 x 10-99. Fig. 4 shows the displacement of the beam under variable distributed load. It is observed that the boundary conditions are satisfied since the displacement and the curvature at $z = 0$ and $z = \ell$ are null. The forces diagrams, i.e., bending moment and shear force, are reported in Fig. 5.

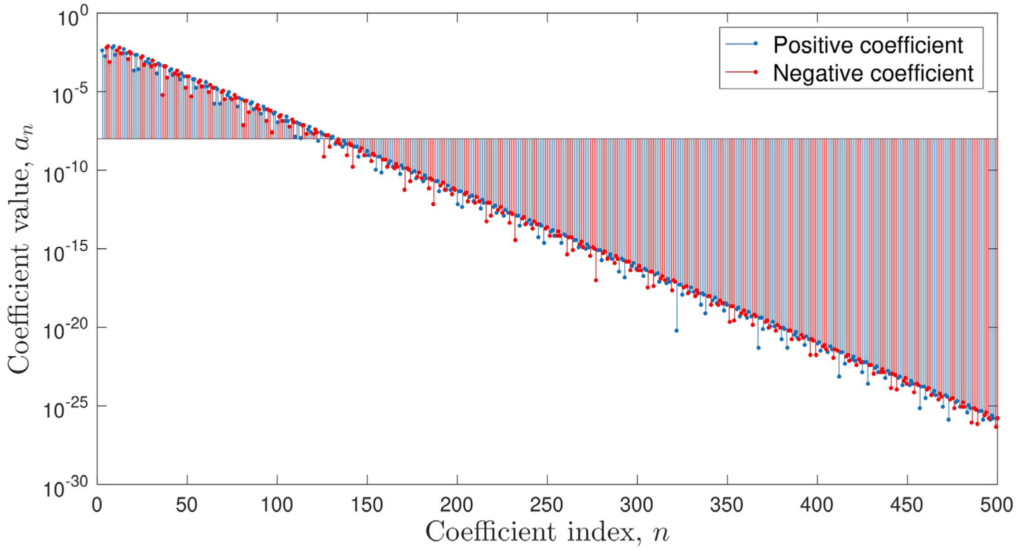

Fig. 3. Stem plot of the coefficients a_n for $n = 0 \ldots 500$. Negative values are marked in red and logarithmic y-axis is adopted.

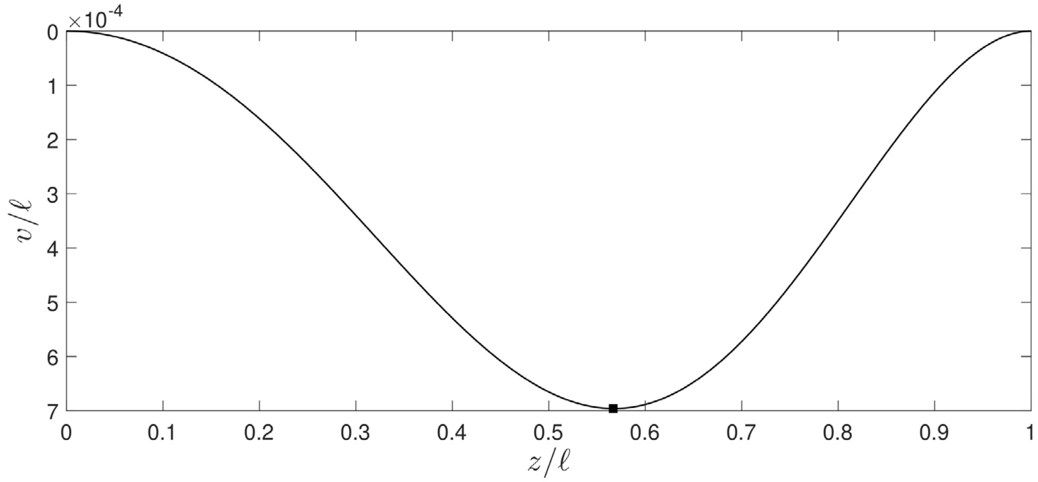

Fig. 4. Plot of the vertical displacements of the beam under the variable distributed load. The maximum displacement v/ℓ = 6.59 × 10^{-4} occurs at z/ℓ = 0.5675.

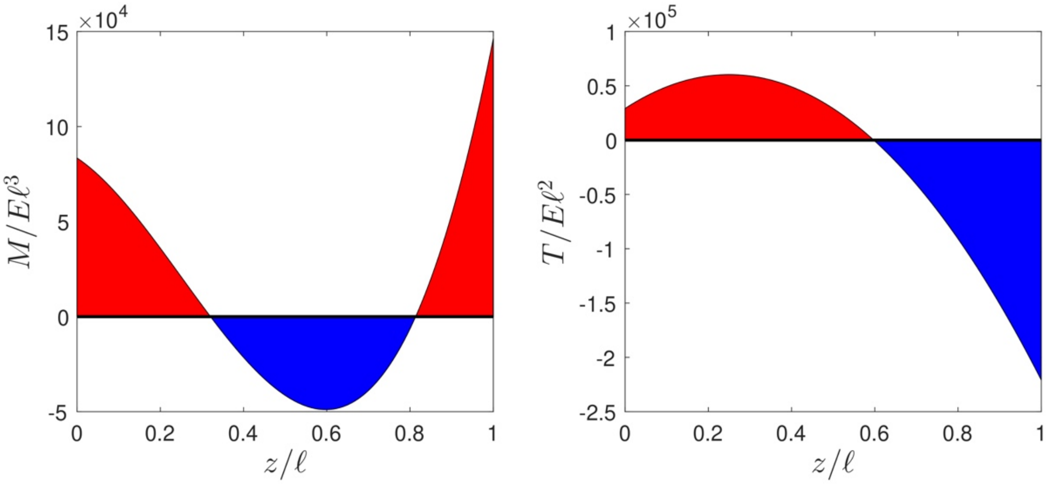

Fig. 5. Plot of the bending moment and shear force diagrams. The plotted forces are dimensionless.

6. Conclusions

Structures with variable cross-section elements are common in civil and mechanical engineering. The solution of such elements, i.e., the evaluation of the end reactions, the displacement and the internal forces, is usually obtained through numerical integration schemes or FEM modeling. We proposed a novel solution scheme based on power series solution for estimating the end reactions and the displacement of such type of beams. The method integrates the fourth-order ordinary elastic line differential equation considering the effects of variable inertia. The idea at the base of the method is that the lateral displacement can be expressed as a summation of terms. Substituting the terms into the original differential equations, a recurrence formula that allows to compute the coefficients of the power series can be found. An optimization algorithm is proposed for estimating the initial terms, which depend on the boundary conditions.

The proposed example, despite simple, clearly shows the capabilities of the method, with reference to the possible implementations for studying the behavior of structures with known variable inertia [17], components subjected to corrosion with variable resisting area [18], or for the optimization of the structural behavior by varying cross-section depth.

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