Research Article

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Some Remarks on Profile Decomposition Theorems

DOI: 10.1515/ans-2015-5049
Received May 10, 2016; accepted May 15, 2016

Abstract: In this paper we present, in the case of Sobolev spaces, some concentration-compactness theorems, nowadays known as profile decomposition theorems, which imply the most known results in literature, clarifying the connections between the different versions.

Keywords: Concentration-Compactness, Profile Decompositions, Multiscale Analysis

MSC 2010: Primary 46E35, 46E30; secondary 35B33

Communicated by: Shair Ahmad
Dedicated to Marco Degiovanni on the occasion of his 60th birthday

1 Introduction

Several years ago, in [8] one of the authors has proved some compactness results for bounded sets of a Sobolev space with respect to the Lebesgue norm corresponding to the critical embedding. These results are due to the nonoptimality of such embedding in the wider category of Lorentz spaces. In particular, [8, Theorem 2] gives for bounded sequences in $H^{1,p}(\mathbb{R}^N)$ an analogous result to a theorem of M. Struwe [11, Proposition 2.1], established for Palais–Smale sequences of suitable functionals, which has been reproduced in the years in many different versions for various classes of functionals which do not satisfy the Palais–Smale condition. Many of these proofs do not use the fact that most of the thesis is known for general bounded sequences, even if [8, Theorem 2] has been followed from several other results of the same type (see [4, 5, 13]). Such statements are nowadays known as *profile decomposition theorems*. In particular, some of the subsequent versions of this kind of result, such as [4, Theorem 1.1], show that some sequences can be approximated by means of finite sums of singularities, while [8, Theorem 2] uses a sum which is potentially infinite and we have sometimes realized that these different versions generate some confusion even among the experts of the field and partially justify the limited use of such general results for particular Palais–Smale sequences. Indeed, the choice of the scalings in the statements of [8, Theorem 2] and [4, Theorem 1.1] is not exactly the same. Recently, we have read in a referee report a purported counterexample to [8, Theorem 2], which actually is a counterexample to the necessity of an assumption required in one of the results in this paper and which is a particular case of a more general example discussed in Corollary 5.5. The difference between the two statements has been underlined also in [12], in which the finite sum version (in another setting) has been defined “more convenient” while it is clearly a weaker result which could have been stated under weaker assumptions (compare the statements of Corollaries 6.3 and 6.4). The landscape has now even more variants since recently in [9] and [10], to get analogous results in suitable Banach spaces, a new technique has been introduced which makes use of polar convergence (or Δ-convergence, see [6] and [2]) instead of

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weak convergence. Following the approach introduced in this paper, even an extension to metric spaces has been recently proposed in [3].

The aim of this paper is to put some order among the different variants and to clarify their connections by showing the following facts:

- [8, Theorem 2] is an immediate consequence of [8, Theorem 1] at the light of some results concerning L^p spaces which can be seen as a multiscale version of the Banach–Alaoglu theorem (see Corollary 5.3).
- Results of the type of [8, Theorem 2] are contained in Theorem 6.2 below which makes use of suitable families of scalings, whose existence will be shown in Section 5, and do not hold for more general families of scalings, see Corollary 5.5 below. In this way we shall also supply some details missed and left to the reader in [8].
- Results of the type of [4, Theorem 1.1], on the contrary, do hold for the more general families of scalings excluded by Corollary 5.5, see Corollary 6.4 below.

The results will be presented in the simplest possible case (Sobolev spaces of integer order) even if a big part of the existing literature deals with more general spaces (for instance [8, Theorem 2] deals with Lorentz spaces and [4, Theorem 1.1] deals with Sobolev spaces of fractional order) since the main purpose of the paper is to show the connections between the various types of statements and, in our opinion, this can be done in the best way by minimizing technicalities.

The paper is organized as follows: In Section 2 we give some basic notions and, among them, that of profile, scale transitions sequence, multiplicity of a profile and the basic energy bound (see Lemma 2.12). In Section 3 we prove a multiscale weak compactness result (see Theorem 3.1), which generalizes the Banach–Alaoglu theorem, showing that any bounded sequence admits a subsequence which is, roughly speaking, weakly converging in all possible scales. In Section 4 we approach the "inverse problem" (actually solved in Section 5) by looking for bounded sequences which admit a given complete profile system with a related system of scale transitions sequences. In Section 5 we show that a fundamental assumption used in the previous section can be forced starting from an arbitrary family of scale transitions sequences. Finally, in Section 6 we show how the results obtained for the L^p spaces apply to Sobolev space $H^{1,p}$ allowing to deduce [8, Theorem 2] as a direct consequence of [8, Theorem 1] which, in turn, is an easy corollary of Sobolev embedding in Lorentz spaces. In Section 7 we briefly discuss the case of polar profile decomposition.

2 Profiles

In this paper we shall make use of the notion of scaling as introduced in [8]. Given $1 \le p \le +\infty$, $x_0 \in \mathbb{R}^N$ and $\lambda > 0$, we shall denote by ρ an L^p-invariant scaling which maps every function $u \in L^p(\mathbb{R}^N)$ into the function defined by setting

$$\rho(u)(x) = \lambda^{\frac{N}{p}} u(x_0 + \lambda(x - x_0)) \quad \text{for all } x \in \mathbb{R}^N.$$

We shall refer to x_0 and λ respectively as to the *center* and the *modulus* of the scaling ρ, while we shall denote by G the group generated by the scalings. Note that G includes scalings and translations but for simplicity we shall keep to call all the elements of G scalings. Finally, we shall denote by $\mathcal{G}$ the space of sequences $\boldsymbol{\rho} = (\rho_n)_{n\in\mathbb{N}} \subset G$. Note that for any scaling ρ, which is not the identity function i_d, $\rho(u) = u$ if and only if $u = 0$. When we shall work with both $L^p(\mathbb{R}^N)$ and (its dual) $L^{p'}(\mathbb{R}^N)$ we shall denote by ρ' the scaling $\rho^{-\top}$, i.e. the $L^{p'}$-invariant scaling which has the same center x_0 and modulus λ of ρ, defined by

$$\rho'(u)(x) = \lambda^{\frac{N}{p'}} u(x_0 + \lambda(x - x_0)) \quad \text{for all } x \in \mathbb{R}^N.$$

If $\rho \in G$, $u \in L^p(\mathbb{R}^N)$ and $v \in L^{p'}(\mathbb{R}^N)$ then the following "duality" relation holds:

$$\int_{\mathbb{R}^N} \rho^{-1}(u)(x)v(x)\,dx = \int_{\mathbb{R}^N} u(x)\rho'(v)(x)\,dx. \tag{2.1}$$

Remark 2.1. Given any sequence of scalings $\boldsymbol{\rho} = (\rho_n)_{n\in\mathbb{N}} \in \mathcal{G}$, one of the following alternatives holds true:
- $(\rho_n)_{n\in\mathbb{N}}$ is diverging, i.e. $\rho_n \rightharpoonup 0$ in $L^p(\mathbb{R}^N)$ weakly pointwise,
- there exists a scaling ρ such that, modulo subsequences, $\lim_{n\to+\infty} \rho_n = \rho$ in $L^p(\mathbb{R}^N)$ strongly pointwise.

Definition 2.2 (Scale Equivalence). Let $\boldsymbol{\rho} = (\rho_n)_{n\in\mathbb{N}}$, $\boldsymbol{\sigma} = (\sigma_n)_{n\in\mathbb{N}} \in \mathcal{G}$ be two sequences of scalings. We shall say that $\boldsymbol{\rho}$ and $\boldsymbol{\sigma}$ are *scale equivalent* if the sequence $(\sigma_n^{-1} \circ \rho_n)_{n\in\mathbb{N}}$ converges strongly pointwise to the identity function i_d.

Note that the already defined relation is an equivalence relation on the set $\mathcal{G}$ of the sequences of scalings and we denote by $[\boldsymbol{\rho}]_S$ the *scale equivalence class* containing $\boldsymbol{\rho}$.

Definition 2.3 (Profiles and s.t.s.). Let $(u_n)_{n\in\mathbb{N}} \subset L^p(\mathbb{R}^N)$ be a given bounded sequence, we shall say that $\varphi \in L^p(\mathbb{R}^N) \setminus \{0\}$ is a profile of the sequence $(u_n)_{n\in\mathbb{N}}$ if there exists $\boldsymbol{\rho} = (\rho_n)_{n\in\mathbb{N}} \in \mathcal{G}$ such that

$$\rho_n^{-1}(u_n) \rightharpoonup \varphi. \tag{2.2}$$

In such a case we shall call $\boldsymbol{\rho} = (\rho_n)_{n\in\mathbb{N}}$ a *scale transitions sequence* (s.t.s. for short) of the profile φ.

Remark 2.4. Note that if φ is a profile of the sequence $(u_n)_{n\in\mathbb{N}}$ and $\boldsymbol{\rho} = (\rho_n)_{n\in\mathbb{N}}$ is an s.t.s. of φ, then any $\boldsymbol{\sigma} \in [\boldsymbol{\rho}]_S$ is still an s.t.s. of φ, while for all $g \in G$, $g(\varphi)$ is still a profile of the sequence $(u_n)_{n\in\mathbb{N}}$ and $(\rho_n \circ g^{-1})_{n\in\mathbb{N}}$ is an s.t.s. of the profile $g(\varphi)$. Therefore we shall say that two profiles φ and ψ of a sequence $(u_n)_{n\in\mathbb{N}}$ are *distinct* if $\psi \neq g(\varphi)$ for all $g \in G$ while they are *copies* if there exists a $g \in G$ such that $\psi = g(\varphi)$. So any profile can be thought as a whole orbit of copies $(g(\varphi))_{g\in G}$. Finally, by taking into account Remark 2.1, we deduce that if $(\rho_n)_{n\in\mathbb{N}}$ and $(\sigma_n)_{n\in\mathbb{N}}$ are s.t.s. related to distinct profiles they must be mutually diverging or quasi orthogonal (i.e. $(\sigma_n^{-1} \circ \rho_n)_{n\in\mathbb{N}}$ is diverging).

Definition 2.5 (Multiplicity). Let φ be a profile of a bounded sequence $(u_n)_{n\in\mathbb{N}}$. We shall define the *multiplicity* of the profile φ as the supremum $m(\varphi)$ of the cardinality of the sets of mutually diverging s.t.s. of φ. If $m(\varphi) = 1$ we shall say that φ is a *simple profile* while, if $m(\varphi) \geq 2$, we shall say that φ is a *multiple profile*.

We shall prove in the sequel, see Lemma 2.12 below, that in L^p spaces the multiplicity of a profile of a bounded sequence is always finite.

Remark 2.6. Every subsequence maintains any profile φ of the whole sequence, at least with the same multiplicity $m(\varphi)$. Indeed, if φ is a profile of a sequence $(u_n)_{n\in\mathbb{N}}$ and $\boldsymbol{\rho} = (\rho_n)_{n\in\mathbb{N}}$ is a related s.t.s., then for any extraction law $(k_n)_{n\in\mathbb{N}} \subset \mathbb{N}$, φ is a profile of the subsequence $(u_{k_n})_{n\in\mathbb{N}}$ and $(\rho_{k_n})_{n\in\mathbb{N}}$ is a related s.t.s. (i.e. $\rho_{k_n}^{-1}(u_{k_n}) \rightharpoonup \varphi$).

Definition 2.7 (Profile System). Let $(u_n)_{n\in\mathbb{N}} \subset L^p(\mathbb{R}^N)$ be a bounded sequence. A family $(\varphi_i)_{i\in I}$ of profiles of the sequence $(u_n)_{n\in\mathbb{N}}$ is said to be a *profile system* (in $L^p(\mathbb{R}^N)$) of the sequence $(u_n)_{n\in\mathbb{N}}$ if, for any profile φ, all elements φ_i which are copies of φ are equal and their number is (finite and) less or equal to $m(\varphi)$.

Taking into account Remark 2.6, we deduce that any profile system is also a profile system of every subsequence.

Definition 2.8 (s.t.s. Systems). Combining Remark 2.4 with Definition 2.5, we deduce that if $(\varphi_i)_{i\in I}$ is a profile system of the sequence $(u_n)_{n\in\mathbb{N}}$, then there exists a family $(\boldsymbol{\rho}_i)_{i\in I}$ such that
(1) for all $i \in I$, $\boldsymbol{\rho}_i = (\rho_n^i)_{n\in\mathbb{N}}$ is an s.t.s. of the profile φ_i,
(2) for all $i, j \in I$, $i \neq j$, $\boldsymbol{\rho}_i$ and $\boldsymbol{\rho}_j$ are mutually diverging.
In such a case we shall call the family $(\boldsymbol{\rho}_i)_{i\in I}$ an *s.t.s. system* related to the profile system $(\varphi_i)_{i\in I}$.

Remark 2.9. By Remark 2.4, if for all $i \in I$, $\boldsymbol{\sigma}_i \in [\boldsymbol{\rho}_i]_S$, then also the family $(\boldsymbol{\sigma}_i)_{i\in I}$ is an s.t.s. system of the profile system $(\varphi_i)_{i\in I}$.

The mutual divergence of the elements of an s.t.s. system allows to easily prove that, in the limit, $\rho_n^i(\varphi_i)$ and $\rho_n^j(\varphi_j)$ behave as two functions with disjoint supports. In particular, the following remark holds.

Remark 2.10. If $(\varphi_i)_{i\in I}$ is a finite profile system of $(u_n)_{n\in\mathbb{N}}$ and $(\boldsymbol{\rho}_i)_{i\in I}$ is a related s.t.s. system, then

$$\forall\varepsilon > 0\ \exists\nu \in \mathbb{N} \text{ such that } \forall n \geq \nu : \quad \Big|\Big\|\sum_{i\in I} \rho_n^i(\varphi_i)\Big\|_p^p - \sum_{i\in I} \|\varphi_i\|_p^p\Big| < \varepsilon. \tag{2.3}$$

In order to quantify how rich a profile system is, we define the function s_p on the set of the profile systems of a given sequence $(u_n)_{n\in\mathbb{N}}$, by setting for any profile system $(\varphi_i)_{i\in I}$,

$$s_p((\varphi_i)_{i\in I}) = \sum_{i\in I} \|\varphi_i\|_p^p. \tag{2.4}$$

The function s_p is increasing with respect to the richness of profile systems and allows us to evaluate also the *profile richness* of a sequence by setting

$$S_p((u_n)_{n\in\mathbb{N}}) = \sup\{s_p((\varphi_i)_{i\in I}) \mid (\varphi_i)_{i\in I} \text{ is a profile system of } (u_n)_{n\in\mathbb{N}}\}. \tag{2.5}$$

Remark 2.11. In other terms, $S_p((u_n)_{n\in\mathbb{N}})$ can be defined as the value of the sum in (2.4) extended to all possible profiles counted as many times as their multiplicity.

The following lemma gives a bound on S_p and in particular allows to deduce that profiles of a bounded sequence in $L^p(\mathbb{R}^N)$ have a finite multiplicity.

Lemma 2.12. *Let $(u_n)_{n\in\mathbb{N}} \subset L^p(\mathbb{R}^N)$ be given. Then*

$$\mathcal{S}_p((u_n)_{n\in\mathbb{N}}) \leq \liminf_{n\to+\infty} \|u_n\|_p^p. \tag{2.6}$$

Proof. It is not restrictive to prove that $s_p((\varphi_i)_{i\in I}) \leq \liminf_{n\to+\infty} \|u_n\|_p^p$ for all finite profile system. Let $(\varphi_i)_{i\in I}$ be a finite profile system of the sequence $(u_n)_{n\in\mathbb{N}}$ and let $(\boldsymbol{\rho}_i)_{i\in I}$ be a related s.t.s. system (see Definition 2.7). Set for all $i \in I$,

$$\psi_i = |\varphi_i|^{p-2}\varphi_i \in L^{p'}(\mathbb{R}^N)$$

so that

$$\langle\varphi_i, \psi_i\rangle := \int_{\mathbb{R}^N} \varphi_i(x)\psi_i(x)\,dx = \|\varphi_i\|_p^p = \|\psi_i\|_{p'}^{p'}. \tag{2.7}$$

Then, by using (2.3), we get

$$\forall\varepsilon > 0\ \exists\nu \in \mathbb{N} \text{ such that } \forall n \geq \nu : \quad \Big|\Big\|\sum_{i\in I} (\rho_n^i)'(\psi_i)\Big\|_{p'}^{p'} - \sum_{i\in I} \|\psi_i\|_{p'}^{p'}\Big| < \varepsilon. \tag{2.8}$$

So, by Hölder inequality, (2.2), (2.1) and the last equality in (2.7), we get that

$$\begin{aligned}
\sum_{i\in I} \|\varphi_i\|_p^p &= \sum_{i\in I} \int_{\mathbb{R}^N} \varphi_i(x)\psi_i(x)\,dx = \sum_{i\in I} \int_{\mathbb{R}^N} (\rho_n^i)^{-1}(u_n)(x)\psi_i(x)\,dx \\
&= \sum_{i\in I} \int_{\mathbb{R}^N} u_n(x)(\rho_n^i)'(\psi_i)(x)\,dx = \int_{\mathbb{R}^N} u_n(x) \sum_{i\in I} (\rho_n^i)'(\psi_i)(x)\,dx \\
&\leq \|u_n\|_p \Big\|\sum_{i\in I} (\rho_n^i)'(\psi_i)\Big\|_{p'} \leq \|u_n\|_p \Big(\sum_{i\in I} \|\psi_i\|_{p'}^{p'}\Big)^{\frac{1}{p'}} = \|u_n\|_p \Big(\sum_{i\in I} \|\varphi_i\|_p^p\Big)^{\frac{1}{p'}},
\end{aligned} \tag{2.9}$$

modulo an infinitesimal term in n. □

The following definition matches Remark 2.6.

Definition 2.13 (Complete Profile System, Profile Converging Sequence). We say that a (possibly empty) profile system $(\varphi_i)_{i\in I}$ of a bounded sequence $(u_n)_{n\in\mathbb{N}}$ is *complete* if no subsequence $(u_{k_n})_{n\in\mathbb{N}}$ has a richer profile system. If a sequence admits a complete profile system we shall say that it is *profile converging*.

In other terms a given bounded sequence $(u_n)_{n\in\mathbb{N}}$ is *profile converging* if $(u_n)_{n\in\mathbb{N}}$ does not admit any subsequence with a bigger number of profiles, or with profiles with a higher multiplicity.

Remark 2.14. Note that if $(\|u_n\|_p)_{n\in\mathbb{N}}$ converges and equality holds in (2.6), then the sequence $(u_n)_{n\in\mathbb{N}}$ is profile converging.

We recall the following definition given in [10, Section 1] or in [9, p. 4].

Definition 2.15 (G-Convergence). If $(u_n)_{n\in\mathbb{N}}$ is a sequence in $L^p(\mathbb{R}^N)$, we shall say that $(u_n)_{n\in\mathbb{N}}$ G-converges to 0, and we shall write $u_n \overset{G}{\rightharpoonup} 0$ if for any sequence of scalings $(\rho_n)_{n\in\mathbb{N}} \subset G$ we have $\rho_n(u_n) \rightharpoonup 0$.

Note that if $(u_n - v_n)_{n\in\mathbb{N}}$ G-converges to 0, then $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ admit the same profiles with the same related s.t.s. The same thing happens with the (complete) profile systems. Conversely, the following result holds.

Proposition 2.16. *If two sequences $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ have a common complete profile system and a common related s.t.s. system, then $u_n - v_n \overset{G}{\rightharpoonup} 0$.*

Proof. Let $\boldsymbol{\rho} = (\rho_n)_{n\in\mathbb{N}} \in \mathcal{G}$ be given, note that it is sufficient to prove that $(\rho_n^{-1}(u_n - v_n))_{n\in\mathbb{N}}$ has a subsequence that weakly converges to 0. Modulo subsequences, we have to face two possible cases:
(1) $\boldsymbol{\rho}$ is almost orthogonal to all $\boldsymbol{\rho}_i$,
(2) there exist $\bar{\imath} \in I$ and (a unique) $g \in G$ such that $\boldsymbol{\rho}$ is scale equivalent to $(g \circ \rho_n^{\bar{\imath}})_{n\in\mathbb{N}}$.
If case (1) applies, since $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ do not admit any subsequence which is better profiled, we have $\rho_n^{-1}(u_n), \rho_n^{-1}(v_n) \rightharpoonup 0$. Assume now case (2). By Remark 2.4, $\boldsymbol{\rho}$ is, for both sequences, an s.t.s. of the profile $g(\varphi_{\bar{\imath}})$, i.e. $\rho_n^{-1}(u_n), \rho_n^{-1}(v_n) \rightharpoonup g(\varphi_{\bar{\imath}})$. □

3 Multiscale Weak Compactness

The aim of this section is to prove the following result.

Theorem 3.1 (Multiscale Weak Compactness). *Any bounded sequence in $L^p(\mathbb{R}^N)$ admits a profile converging subsequence.*

The proof is rather easy and technically, it can be reached by taking at each step a richer profile system obtained by an argument similar to that used in the proof of [8, Theorem 2], or by a maximality argument. Since we shall choose here the maximality argument, we need to introduce an ordering.

Definition 3.2. Let E denote the space of bounded sequences in $L^p(\mathbb{R}^N)$. Given $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}} \in E$, we say that $(v_n)_{n\in\mathbb{N}}$ is *better profiled* than $(u_n)_{n\in\mathbb{N}}$, and we shall write $(u_n)_{n\in\mathbb{N}} \preceq (v_n)_{n\in\mathbb{N}}$, if $(v_n)_{n\in\mathbb{N}} = (u_n)_{n\in\mathbb{N}}$ or if $(v_n)_{n\in\mathbb{N}}$ is a subsequence of $(u_n)_{n\in\mathbb{N}}$ with the possible exception of finitely many terms (i.e. there exist $\nu \in \mathbb{N}$ and an extraction law $(k_n)_{n\in\mathbb{N}} \subset \mathbb{N}$ such that, for all $n \geq \nu$, $v_n = u_{k_n}$) and $\mathcal{S}_p((u_n)_{n\in\mathbb{N}}) < \mathcal{S}_p((v_n)_{n\in\mathbb{N}})$.

Remark 3.3. The binary relation $\preceq$ is an ordering and a sequence $(u_n)_{n\in\mathbb{N}} \subset L^p(\mathbb{R}^N)$ is profile converging if and only if it is maximal with respect to $\preceq$.

Proof of Theorem 3.1. By Remark 3.3, we shall prove the existence of a maximal element, by using [7, Theorem A.1], thanks to the increasing (with respect to $\preceq$) real-valued function S_p defined by (2.5). To this aim we just need to prove that the ordered set $(E, \preceq)$ is countably inductive (in the sense of [7, Appendix A]). So we fix an increasing sequence with respect to $\preceq$. Note that, if it is constant for large n, then it clearly has an upper bound. Otherwise, after removing a finite number of terms from each element, we have a sequence of sequences which are all extracted from the previous one. Then we take the diagonal selection and use the monotonicity of S_p in order to conclude that it is an upper bound of the whole sequence. □

4 Profile Reconstruction

In this section we deal with the following question which can be seen as an "inverse problem" which will be solved at the end of Section 5 below: Given a family of nonnull functions $(\varphi_i)_{i\in I}$ and a family $(\boldsymbol{\rho}_i)_{i\in I}$ of mutually diverging scalings, we look for a bounded sequence $(v_n)_{n\in\mathbb{N}} \subset L^p(\mathbb{R}^N)$ of which $(\varphi_i)_{i\in I}$ is a complete profile system and $(\boldsymbol{\rho}_i)_{i\in I}$ is a related s.t.s. system. According to Definition 2.7, we shall assume that the elements of the family $(\varphi_i)_{i\in I}$ which are copies of a given function φ are equal and that the value of the function s_p defined in (2.4) is finite. Finally, we shall make use of the following assumption.

Definition 4.1 (Routed Sequences of Scalings). Let $(\varphi_i)_{i\in I}$ be a given family of functions and let $(\boldsymbol{\rho}_i)_{i\in I}$ be a family of mutually diverging sequences of scalings. We shall say that the family $(\boldsymbol{\rho}_i)_{i\in I}$ is *routed* (in $L^p(\mathbb{R}^N)$ with respect to $(\varphi_i)_{i\in I}$) if the sum $\sum_{i\in I}\rho_n^i(\varphi_i)$ is unconditionally convergent (in L^p) with respect to i, uniformly with respect to n.

Remark 4.2. When the family $(\boldsymbol{\rho}_i)_{i\in I}$ is routed with respect to $(\varphi_i)_{i\in I}$, for any $\varepsilon > 0$ there exists $F \subset I$, F finite, such that

$$\Big\|\sum_{i\in I\setminus F}\rho_n^i(\varphi_i)\Big\|_p < \varepsilon, \quad \text{for all } n \in \mathbb{N}, \tag{4.1}$$

and so, roughly speaking, we will be able to treat the sum $\|\sum_{i\in I}\rho_n^i(\varphi_i)\|_p$ as if the set of indexes I were finite (and use, for instance, (2.3)).

Definition 4.3 (Profile Reconstruction). Let $(\varphi_i)_{i\in I}$ be a family of functions such that $\sum_{i\in I}\|\varphi_i\|_p^p < +\infty$ and let $(\boldsymbol{\rho}_i)_{i\in I}$ be a routed family of sequences of scalings. The sequence $(v_n)_{n\in\mathbb{N}}$, defined by setting for all $n \in \mathbb{N}$,

$$v_n = \sum_{i\in I}\rho_n^i(\varphi_i), \tag{4.2}$$

will be called *profile reconstruction* determined by $(\varphi_i)_{i\in I}$ and $(\boldsymbol{\rho}_i)_{i\in I}$.

Taking into account (4.1) and (2.3), we get

$$\lim_{n\to+\infty}\|v_n\|_p^p = \sum_{i\in I}\|\varphi_i\|_p^p. \tag{4.3}$$

Lemma 4.4. *For all $i \in I$, we have $(\rho_n^i)^{-1}(v_n) \rightharpoonup \varphi_i$, i.e. φ_i is a profile of $(v_n)_{n\in\mathbb{N}}$ and $\boldsymbol{\rho}_i = (\rho_n^i)_{n\in\mathbb{N}}$ is a related s.t.s. sequence.*

Proof. The assertion easily follows from (4.1) and the divergence of $((\rho_n^i)^{-1}\circ\rho_n^j)_{n\in\mathbb{N}}$ for $i \neq j$. □

Corollary 4.5. *$(\varphi_i)_{i\in I}$ is a complete profile system of $(v_n)_{n\in\mathbb{N}}$ and $(\boldsymbol{\rho}_i)_{i\in I}$ is a related s.t.s. system.*

Proof. The completeness of $(\varphi_i)_{i\in I}$ follows from (4.3) and Remark 2.14. □

If we assume that the family $(\varphi_i)_{i\in I}$ has been already found as a profile system of a given sequence $(u_n)_{n\in\mathbb{N}}$, then (2.6) can be restated as

$$\lim_{n\to+\infty}\|v_n\|_p \le \liminf_{n\to+\infty}\|u_n\|_p. \tag{4.4}$$

The following lemma gives a multiscale version of the Kadec–Klee property of L^p spaces.

Lemma 4.6. *The reminder $(u_n - v_n)_{n\in\mathbb{N}}$ is infinitesimal in $L^p(\mathbb{R}^N)$ if and only if*

$$\limsup_{n\to+\infty}\|u_n\|_p \le \lim_{n\to+\infty}\|v_n\|_p. \tag{4.5}$$

Proof. The first implication is trivial. To prove the converse implication we shall assume (4.5), so we can set

$$s := \lim_{n\to+\infty}\|u_n\|_p = \lim_{n\to+\infty}\|v_n\|_p.$$

For any fixed $\varepsilon > 0$ there exists a finite $F \subset I$ such that

$$\sum_{i\in F}\|\varphi_i\|_p^p > \sum_{i\in I}\|\varphi_i\|_p^p - \varepsilon.$$

Making use of the same notation introduced in the proof of Lemma 2.12, we set

$$v_{n,\varepsilon} = \sum_{i\in F}(\rho_n^i)'(\psi_i),$$

and remark that, modulo an infinitesimal term in n, by (2.8) and (2.7),

$$\|v_{n,\varepsilon}\|_{p'}^{p'} = \sum_{i\in F} \|\varphi_i\|_p^p > (s-\varepsilon)^p.$$

Since $(\varphi_i)_{i\in I}$ is also a profile system of $(v_n)_{n\in\mathbb{N}}$, we deduce from the first two lines of (2.9) that

$$\langle v_{n,\varepsilon}, u_n\rangle \geq \sum_{i\in F} \|\varphi_i\|_p^p \quad \text{and} \quad \langle v_{n,\varepsilon}, v_n\rangle \geq \sum_{i\in F} \|\varphi_i\|_p^p,$$

and then, by Hölder Inequality,

$$\sum_{i\in F} \|\varphi_i\|_p^p \leq \Big\langle v_{n,\varepsilon}, \frac{u_n+v_n}{2}\Big\rangle \leq \Big(\sum_{i\in F} \|\varphi_i\|_p^p\Big)^{\frac{1}{p'}} \Big\|\frac{u_n+v_n}{2}\Big\|_p,$$

so

$$\frac{1}{2}\|u_n+v_n\|_p \geq \Big(\sum_{i\in F} \|\varphi_i\|_p^p\Big)^{\frac{1}{p}} = s-\varepsilon.$$

Then, by using the uniform convexity of $L^p(\mathbb{R}^N)$ (and in particular the Clarkson inequalities) we deduce $\|u_n - v_n\|_p^p \to 0$ and so the thesis follows. □

Corollary 4.7. *A (bounded) sequence $(v_n)_{n\in\mathbb{N}}$ which admits $(\varphi_i)_{i\in I}$ as a (complete) profile system, $(\boldsymbol{\rho}_i)_{i\in I}$ as a related s.t.s. system and satisfies* (4.3) *is uniquely determined modulo an infinitesimal term in L^p.*

Remark 4.8. If the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in a suitable Sobolev space, we can easily prove, by an iterated application of the Brezis–Lieb lemma (see [1]), that

$$\|u_n\|_p^p = \|v_n\|_p^p + \|u_n - v_n\|_p^p + o(1),$$

getting in particular (4.4) and Lemma 4.6.

We shall be concerned with the case of a bounded sequence in a Sobolev space in Section 6 below, but we shall need to apply the results in this section also to the sequence $(\nabla u_n)_{n\in\mathbb{N}}$ and therefore even in that section we shall need the results established in the above setting. We can collect Lemma 2.12, Proposition 2.16 and Lemma 4.6 in the following statement.

Theorem 4.9. *Let $(u_n)_{n\in\mathbb{N}} \subset L^p(\mathbb{R}^N)$ be a bounded sequence. Let $(\varphi_i)_{i\in I}$ be a complete profile system in $L^p(\mathbb{R}^N)$ and let $(\boldsymbol{\rho}_i)_{i\in I} = ((\rho_n^i)_{n\in\mathbb{N}})_{i\in I}$ be a related routed s.t.s. system. Then*

$$\sum_{i\in I} \|\varphi_i\|_p^p \leq \liminf_{n\to+\infty} \|u_n\|_p^p \tag{4.6}$$

and

$$u_n - \sum_{i\in I} \rho_n^i(\varphi_i) \quad G\text{-converges to } 0. \tag{4.7}$$

Moreover,

$$u_n - \sum_{i\in I} \rho_n^i(\varphi_i) \to 0 \text{ in } L^p \quad \text{if and only if} \quad \lim_{n\to+\infty} \|u_n\|_p \text{ exists and equality holds in (4.6).} \tag{4.8}$$

5 Routing an s.t.s. System

In this section we shall prove, in particular, that any profile system admits a routed s.t.s. system.

Lemma 5.1. *Let $(\varphi_i)_{i\in I}$ be as in Section 4. Then there exists a family of scalings $\overline{\boldsymbol{\rho}} = (\overline{\rho}_i)_{i\in I} \in \mathcal{G}$ such that for all $F \subset I$, F finite,*

$$\Big|\Big\|\sum_{j\in F} \overline{\rho}_j(\varphi_j)\Big\|_p^p - \sum_{j\in F} \|\varphi_j\|_p^p\Big| \leq 2^{-\min(F)} - 2^{-\max(F)} < 2^{-\min(F)}. \tag{5.1}$$

Proof. We shall construct the sequence of scalings $(\overline{\rho}_i)_{i\in I}$ recursively by arguing on the first inequality of (5.1). In particular, we shall prove that if, for $i \in \mathbb{N}$, $\overline{\rho}_1, \dots, \overline{\rho}_i$ have been already defined and if (5.1) holds true for any set $F \subset \{1, \dots, i\}$, by choosing $\overline{\rho}_{i+1}$ so that for all $H \subset \{1, \dots, i\}$,

$$\Big| \Big\| \sum_{j\in H\cup\{i+1\}} \overline{\rho}_j(\varphi_j) \Big\|_p^p - \Big\| \sum_{j\in H} \overline{\rho}_j(\varphi_j) \Big\|_p^p - \|\varphi_{i+1}\|_p^p \Big| < 2^{-i-1}, \tag{5.2}$$

then (5.1) holds true for all $F \subset \{1, \dots, i+1\}$. (There is no problem in choosing $\overline{\rho}_{i+1}$ as in (5.2), it is indeed enough to select a term of sufficiently large index from any diverging sequence of scalings.) Let $F \subset \{1, \dots, i+1\}$. If $\max(F) \le i$, then the assertion follows by induction assumptions. Therefore, we assume $\max(F) = i+1$, then by using (5.2) with $H = F \setminus \{i+1\}$ and by using induction assumptions, we get

$$\begin{aligned} \Big| \Big\| \sum_{j\in F} \overline{\rho}_j(\varphi_j) \Big\|_p^p - \sum_{j\in F} \|\varphi_j\|_p^p \Big| &\le 2^{-i-1} + \Big| \Big\| \sum_{j\in F\setminus\{i+1\}} \overline{\rho}_j(\varphi_j) \Big\|_p^p - \sum_{j\in F\setminus\{i+1\}} \|\varphi_j\|_p^p \Big| \\ &\le 2^{-i-1} + 2^{-\min(F\setminus\{i+1\})} - 2^{-\max(F\setminus\{i+1\})}. \end{aligned}$$

□

Of course every s.t.s. system related to a finite profile system is routed (actually any s.t.s. related to a profile system $(\varphi_i)_{i\in I}$ such that the sum of $\|\varphi_i\|_p$ is finite is routed). So, if $(\varphi_i)_{i\in I}$ is a profile system and if $(\boldsymbol{\rho}_i)_{i\in I} = ((\rho_n^i)_{n\in\mathbb{N}})_{i\in I}$ is a related s.t.s. system, we can focus the case $I = \mathbb{N}$. Since the s.t.s. are mutually diverging, by taking into account (2.3), we deduce that for any $i \in I$ there exists $\nu \in \mathbb{N}$ such that for all $n \ge \nu$, and for all $F \subset \{1, 2, \dots, i\}$, we have

$$\Big| \Big\| \sum_{j\in F} \rho_n^j(\varphi_j) \Big\|_p^p - \sum_{j\in F} \|\varphi_j\|_p^p \Big| < 2^{-i}. \tag{5.3}$$

Then, set for any $i \in I$,

$$n(i) = \max(i, \min\{\nu \mid (5.3) \text{ holds for all } n \ge \nu \text{ and for all } F \subset \{1, \dots, i\}\}),$$

so that (5.3) holds if $n(i) \le n$ for every upper bound i of F. Since the sequence $(n(i))_{i\in I}$ is a diverging nondecreasing sequence of natural numbers, we can consider the "left inverse" sequence $(i(n))_{n\in\mathbb{N}}$ of $(n(i))_{i\in I}$, where for any $n \in \mathbb{N}$,

$$i(n) = \max\{j \in I \mid n(j) \le n\} \le n, \tag{5.4}$$

so that $n \ge n(i)$ if and only if $i \le i(n)$ and (5.3) holds with $i = i(n)$ when $i(n)$ is an upper bound of F.

Proposition 5.2. *Let $(\varphi_i)_{i\in I}$ be as in Section 4 and let $(\boldsymbol{\rho}_i)_{i\in I} = ((\rho_n^i)_{n\in\mathbb{N}})_{i\in I}$ be a family of mutually diverging sequences of scalings. Then there exists a family $(\boldsymbol{\sigma}_i)_{i\in I} = ((\sigma_n^i)_{n\in\mathbb{N}})_{i\in I}$ which is routed with respect to $(\varphi_i)_{i\in I}$ and such that for all $i \in I$, $\sigma_n^i = \rho_n^i$, with the exception of a finite number of indexes n (and therefore such that $\boldsymbol{\sigma}_i \in [\boldsymbol{\rho}_i]_S$ for all $i \in I$).*

Proof. For any fixed $i \in I$ and $n \in \mathbb{N}$ set

$$\sigma_n^i = \begin{cases} \rho_n^i & \text{if } n \ge n(i), \text{ (i.e. if } i \le i(n)), \\ \overline{\rho}_i & \text{if } n < n(i), \text{ (i.e. if } i(n) < i), \end{cases} \tag{5.5}$$

where $(\overline{\rho}_i)_{i\in I}$ is the sequence provided by Lemma 5.1. Note that the last part of the assertion then follows by construction since, for any $i \in I$, $n(i) \in \mathbb{N}$.

In order to prove that $(\boldsymbol{\sigma}_i)_{i\in I}$ is routed, in correspondence to $\varepsilon > 0$ we fix $i_\varepsilon \in I$ large enough to have,

$$\sum_{i=i_\varepsilon+1}^{+\infty} \|\varphi_i\|_p^p + 2^{-i_\varepsilon} < \Big(\frac{\varepsilon}{2}\Big)^p. \tag{5.6}$$

Let $F \subset I$ be such that $\min(F) > i_\varepsilon$, then we shall deduce the first part of the assertion by proving that, for all $n \in \mathbb{N}$, $\| \sum_{i\in F} \sigma_n^i(\varphi_i)\|_p < \varepsilon$. Given $n \in \mathbb{N}$, we set

$$F_1 = \{j \in F \mid j \le i(n)\} \quad \text{and} \quad F_2 = F \setminus F_1.$$

By (5.5), we have

$$\Big\|\sum_{j\in F}\sigma_n^j(\varphi_j)\Big\|_p \le \Big\|\sum_{j\in F_1}\rho_n^j(\varphi_j)\Big\|_p + \Big\|\sum_{j\in F_2}\overline{\rho}_j(\varphi_j)\Big\|_p. \tag{5.7}$$

Since (5.3) holds for $F = F_1$ (with $i = i(n)$), and since by definition $i(n)$ is an upper bound of F_1 and $i(n) > i_\varepsilon$ if $F_1 \neq \emptyset$, we immediately see from (5.6) that the first term on the right-hand side of (5.7) is bounded by $\frac{\varepsilon}{2}$. The same conclusion holds for the second one by (5.1). □

Applying Proposition 5.2, one gets the following results. The first one is an easy corollary of Theorem 3.1 and Theorem 4.9.

Corollary 5.3. *Let $(u_n)_{n\in\mathbb{N}} \subset L^p(\mathbb{R}^N)$ be a bounded sequence. Replacing $(u_n)_{n\in\mathbb{N}}$ by a suitable subsequence, we can find a complete profile system $(\varphi_i)_{i\in I}$ in $L^p(\mathbb{R}^N)$ and a related routed s.t.s. system $(\boldsymbol{\rho}_i)_{i\in I} = ((\rho_n^i)_{n\in\mathbb{N}})_{i\in I}$ such that* (4.6), (4.7) *and* (4.8) *hold.*

Proposition 5.4. *Given any $(\varphi_i)_{i\in I}$ as in Section 4, for any family $(\boldsymbol{\rho}_i)_{i\in I}$ of mutually diverging sequences of scalings, there exists a "profile reconstruction" $(v_n)_{n\in\mathbb{N}}$ which satisfies the assumptions of Corollary 4.7.*

Proof. It is enough to replace $(\boldsymbol{\rho}_i)_{i\in I}$ by the s.t.s. system $(\boldsymbol{\sigma}_i)_{i\in I}$ provided by Proposition 5.2 and apply Corollary 4.5 and Remark 2.9. □

As already pointed out in Corollary 4.7, this profile reconstruction is uniquely determined modulo an infinitesimal term and therefore it does not need to be defined exclusively by (4.2) (with ρ_n^i replaced by σ_n^i of course). For instance, finite sums with diverging number of terms work in the same way. We shall discuss these variants in details in the case of Sobolev spaces (see Corollaries 6.3 and 6.4 below).

Corollary 5.5. *The results in Section 4 are in general false (the sequence $(v_n)_{n\in\mathbb{N}}$ does not exist) if the assumption that $(\boldsymbol{\rho}_i)_{i\in I}$ is routed is removed.*

Proof. Let $(a_i)_{i\in\mathbb{N}}$ be any sequence of positive real numbers such that $\sum_{i\in\mathbb{N}} a_i = +\infty$ and $\sum_{i\in\mathbb{N}} a_i^p < +\infty$. Let $\varphi \in L^p(\mathbb{R}^N)$, $\varphi \neq 0$. Let $\varphi_i = a_i\varphi$ and let $(\boldsymbol{\rho}_i)_{i\in\mathbb{N}}$ be any sequence of mutually diverging sequences of scalings. (Note that, since $\sum_{i\in I}\|\varphi_i\|_p^p = (\sum_{i\in I} a_i^p)\|\varphi\|_p^p < +\infty$, by Corollary 4.7, $(\varphi_i)_{i\in I}$ is a complete profile system of a suitable bounded sequence $(u_n)_{n\in\mathbb{N}}$ and $(\boldsymbol{\rho}_i)_{i\in I}$ is a corresponding s.t.s. system.) We can easily get $(\boldsymbol{\rho}_i)_{i\in I}$ "derouted" by applying the same procedure of Proposition 5.2, defining $\boldsymbol{\sigma}_i$ as in (5.5) but taking $\overline{\rho}_i$ equal to the identity function instead of the value given by Lemma 5.1. Then, $(\boldsymbol{\sigma}_i)_{i\in I}$ is another s.t.s. system of $(\varphi_i)_{i\in I}$, but

$$v_n = \sum_{i\in I}\sigma_n^i(\varphi_i) = \sum_{i=0}^{i(n)}\rho_n^i(\varphi_i) + \Big(\sum_{i=i(n)+1}^{+\infty} a_i\Big)\varphi$$

does not exist. □

6 Profile Decomposition Theorems in Sobolev Spaces $H^{1,p}$

In this section we shall apply the results obtained so far in L^p spaces to the Sobolev space $H^{1,p}(\mathbb{R}^N)$ (with $1 < p < N$), equipped with the homogeneous norm

$$\|u\|_{1,p} = \|\nabla u\|_p,$$

with respect to which $H^{1,p}(\mathbb{R}^N)$ is embedded into $L^{p^*}(\mathbb{R}^N)$, at the light of [8, Theorem 1]. Such "cocompactness result", thanks to the Sobolev embedding in Lorentz spaces, admits a simple proof which is carried out on the Marcinkiewicz space of index p^*. This is not just a technical device because this result is false in the case of the optimal embedding in the category of Lorentz spaces, as analogously happens with Rellich theorem in the category of Lebesque spaces, see [8]. The result can be reformulated as follows with the terminology introduced in this paper.

Proposition 6.1. *Let $(w_n)_{n\in\mathbb{N}}$ be a bounded sequence in $H^{1,p}(\mathbb{R}^N)$ then $(w_n)_{n\in\mathbb{N}}$ is infinitesimal in $L^{p^*}(\mathbb{R}^N)$ if and only if $(w_n)_{n\in\mathbb{N}}$ G-converges to zero.*

Taking into account that the gradient operator ∇ is linear and weakly continuous, we deduce that a function φ is a profile in $L^{p^*}(\mathbb{R}^N)$ of a sequence $(u_n)_{n\in\mathbb{N}} \subset H^{1,p}(\mathbb{R}^N)$ if and only if $\nabla\varphi$ is a profile of $(\nabla u_n)_{n\in\mathbb{N}}$ in $L^p(\mathbb{R}^N)$. Moreover, for any $u \in H^{1,p}$, $\tilde\rho(\nabla u) = \nabla(\rho(u))$ where ρ is any L^{p^*}-invariant scaling and $\tilde\rho$ is the corresponding L^p-invariant scaling having the same center and modulus of ρ. In the remaining part of this section we shall denote by G and respectively $\tilde G$ the group of L^{p^*} and L^p-invariant scalings. From the above equality we deduce that $\boldsymbol{\rho} = (\rho_n)_{n\in\mathbb{N}} \subset G$ is an s.t.s. of φ in L^{p^*}, if and only if $\tilde{\boldsymbol{\rho}} = (\tilde\rho_n)_{n\in\mathbb{N}} \subset \tilde G$ is an s.t.s. of $\nabla\varphi$ in L^p. Therefore, if $(\varphi_i)_{i\in I}$ is a (complete) profile system of $(u_n)_{n\in\mathbb{N}}$ in $L^{p^*}(\mathbb{R}^N)$ and $(\boldsymbol{\rho}_i)_{i\in I}$ is a related s.t.s. system, then $(\nabla\varphi_i)_{i\in I}$ is a (complete) profile system of $(\nabla u_n)_{n\in\mathbb{N}}$ in $L^p(\mathbb{R}^N)$ and $(\tilde{\boldsymbol{\rho}}_i)_{i\in I}$ is a related s.t.s. system.

We shall say that a family of s.t.s. $(\boldsymbol{\rho}_i)_{i\in I}$ is routed in $H^{1,p}(\mathbb{R}^N)$ if the family $(\tilde{\boldsymbol{\rho}}_i)_{i\in I}$ is routed in $L^p(\mathbb{R}^N)$. Then, by Sobolev embedding, $(\boldsymbol{\rho}_i)_{i\in I}$ is also routed in $L^{p^*}(\mathbb{R}^N)$. So, if $(\varphi_i)_{i\in I}$ is a complete profile system in $L^{p^*}(\mathbb{R}^N)$ of $(u_n)_{n\in\mathbb{N}} \subset H^{1,p}(\mathbb{R}^N)$ and if $(\boldsymbol{\rho}_i)_{i\in I} = ((\rho_n^i)_{n\in\mathbb{N}})_{i\in I}$ is a related s.t.s. system which is routed (in $H^{1,p}(\mathbb{R}^N)$), then, set v_n as in (4.2), we have that $(\nabla v_n)_{n\in\mathbb{N}}$ is the profile reconstruction of $(\nabla u_n)_{n\in\mathbb{N}}$ in $L^p(\mathbb{R}^N)$. So from Theorem 4.9 we get the following statement, which implies [8, Theorem 2] thanks to the results in Section 5 (in the second part of (6.2) below we prefer the sentence "G-converges to 0" instead of the more appropriate "$\tilde G$-converges to 0").

Theorem 6.2. *Let $(u_n)_{n\in\mathbb{N}} \subset H^{1,p}(\mathbb{R}^N)$ be a bounded sequence. Let $(\varphi_i)_{i\in I}$ be a complete profile system in $L^{p^*}(\mathbb{R}^N)$ and let $(\boldsymbol{\rho}_i)_{i\in I} = ((\rho_n^i)_{n\in\mathbb{N}})_{i\in I}$ be a related routed s.t.s. system. Then*

$$\sum_{i\in I} \|\varphi_i\|_{1,p}^p \le \varliminf_{n\to+\infty} \|u_n\|_{1,p}^p, \tag{6.1}$$

and

$$u_n - \sum_{i\in I} \rho_n^i(\varphi_i) \to 0 \text{ in } L^{p^*} \quad \text{and} \quad \nabla u_n - \sum_{i\in I} \nabla(\rho_n^i(\varphi_i)) \text{ } G\text{-converges to } 0 \text{ in } L^p. \tag{6.2}$$

Moreover,

$$u_n - \sum_{i\in I} \rho_n^i(\varphi_i) \to 0 \quad \text{in } H^{1,p}(\mathbb{R}^N)$$

if and only if equality holds in (6.1).

Proof. As remarked above we can apply Theorem 4.9 in $L^{p^*}(\mathbb{R}^N)$ to the sequences $(u_n)_{n\in\mathbb{N}}$, $(\varphi_i)_{i\in I}$, $(\boldsymbol{\rho}_i)_{i\in I}$ and in $L^p(\mathbb{R}^N)$ to the sequences $(\nabla u_n)_{n\in\mathbb{N}}$, $(\nabla\varphi_i)_{i\in I}$, $(\tilde{\boldsymbol{\rho}}_i)_{i\in I}$. The first convergence in (6.2) is strong by Proposition 6.1. □

Taking $I = \mathbb{N}$, we can also replace the infinite sum with a finite sum as stated in the following corollary which easily follows from the previous theorem thanks to the uniformity of the summability in the definition of v_n.

Corollary 6.3. *Under the assumptions of Theorem 6.2,* for any *sequence $(\ell_n)_{n\in\mathbb{N}} \subset \mathbb{N}$ such that $\ell_n \to +\infty$, by* (6.2)*, we have*

$$u_n - \sum_{i=0}^{\ell_n} \rho_n^i(\varphi_i) \to 0 \quad \text{in } L^{p^*}(\mathbb{R}^N). \tag{6.3}$$

Corollary 5.5 clearly shows that the above statements are false if the assumption that $(\boldsymbol{\rho}_i)_{i\in I}$ is routed is removed. In the case of a (nonrouted) s.t.s. system all we can say is a result of the type of [4, Theorem 1.1].

Corollary 6.4. *Let $(u_n)_{n\in\mathbb{N}} \subset H^{1,p}(\mathbb{R}^N)$ be a bounded sequence. Let $(\varphi_i)_{i\in I}$ be a complete profile system in $L^{p^*}(\mathbb{R}^N)$ and let $(\boldsymbol{\rho}_i)_{i\in I} = ((\rho_n^i)_{n\in\mathbb{N}})_{i\in I}$ be a related (eventually* nonrouted*) s.t.s. system. Then* there exists *a diverging sequence $(\ell_n)_{n\in\mathbb{N}}$ such that* (6.3) *holds true (see also* [4, Remark 1.2]*).*

Proof. It is sufficient to take, for any $n \in \mathbb{N}$, $\ell_n < i(n)$, where $i(n)$ is as in (5.4), and apply Corollary 6.3 to the routed s.t.s. $(\boldsymbol{\sigma}_i)_{i\in I}$ given by (5.5). □

7 Polar Profile Decomposition

In the recent work [9], see also [10], the profile decomposition theorem has been stated in the general context of Banach spaces and has been obtained by using the notion of polar convergence (actually Δ-convergence in the last version of the paper; see [2, 6]) instead of weak convergence. In such a case we shall speak of the polar profile decomposition theorem.

Applying the results obtained so far in L^p spaces, we get that starting from a bounded sequence $(u_n)_{n\in\mathbb{N}} \subset H^{1,p}(\mathbb{R}^N)$ it is possible to find a complete polar profile system $(\varphi_i)_{i\in I}$ of $(u_n)_{n\in\mathbb{N}}$ in L^{p^*} and another polar profile system $(\psi_j)_{j\in J}$ of $(\nabla u_n)_{n\in\mathbb{N}}$ in L^p. However, we need to put an important warning: in general, we cannot say that $\psi_j = \nabla\varphi_i$ since the gradient operator is not continuous with respect to polar convergence. Since every bounded sequence in Sobolev spaces admits a subsequence which is converging a.e. and, since for bounded a.e. converging sequences in L^p polar limit and weak limit agree (see [2, Remark 5.6]), we deduce that $(\varphi_i)_{i\in I}$ *is a complete profile system* also in the sense given in Section 2 and so we can apply all results proved in Section 6. In particular, $(\nabla\varphi_i)_{i\in I}$ is a complete system of $(\nabla u_n)_{n\in\mathbb{N}}$ which G-converges to 0. The circumstance that $\nabla\varphi_i$ can be distinct from any ψ_j is an immediate consequence of [2, Theorem 5.5] which allows to construct bounded sequences in L^p whose polar and weak limit do not coincide. However, this situation changes if one substitutes the L^p-norm of the gradient with an equivalent norm. Indeed, the polar convergence, differently from the weak one, is not invariant under the change of equivalent norms. In the case of Sobolev spaces we can pass to an equivalent norm, based on the Littlewood–Paley decomposition which induces a polar convergence equal to the weak one (in other terms it is a “van Dust norm”, see [14]). In such a case we can go back to the previous framework, which therefore enters in the more general theory developed in [9] and [10], to which we refer for more details. The change of norm makes the previous warning disappear.

Acknowledgment: The authors are grateful to K. Tintarev for several discussions on the subject.

References

[1] H. Brezis and E. Lieb, A relation between pointwise convergence of functions and convergence of functionals, *Proc. Amer. Math. Soc.* **88** (1983), 486–490.

[2] G. Devillanova, S. Solimini and K. Tintarev, On weak convergence in metric spaces, in: *Nonlinear Analysis and Optimization*, Contemp. Math. 659, American Mathematical Society, Providence (2016), 43–63.

[3] G. Devillanova, S. Solimini and K.Tintarev, Profile decomposition in metric spaces, in preparation.

[4] P. Gérard, Description du défaut de compacité de l'injection de Sobolev, *ESAIM Control Optim. Calc. Var.* **3** (1998), 213–233.

[5] S. Jaffard, Analysis of the lack of compactness in the critical Sobolev embeddings, *J. Funct. Anal.* **161** (1999), 384–396.

[6] T. C. Lim, Remarks on some fixed point theorems, *Proc. Amer. Math. Soc.* **60** (1976), 179–182.

[7] F. Maddalena and S. Solimini, Synchronic and asynchronic descriptions of irrigation problems, *Adv. Nonlinear Stud.* **13** (2013), no. 3, 583–623.

[8] S. Solimini, A note on compactness-type properties with respect to Lorentz norms of bounded subsets of a Sobolev space, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **12** (1995), no. 3, 319–337.

[9] S. Solimini and K. Tintarev, Concentration analysis in Banach spaces, *Commun. Contemp. Math.* **18** (2015), no. 3, Article ID 1550038.

[10] S. Solimini and K. Tintarev, On the defect of compactness in Banach Spaces, *C. R. Math. Acad. Sci. Paris* **353** (2015), no. 10, 899–903.

[11] M. Struwe, A global compactness result for elliptic boundary value problems involving limiting nonlinearities, *Math. Z.* **187** (1984), 511–517.

[12] T. Tao, Concentration compactness via nonstandard analysis, http://terrytao.wordpress.com/2010/11/29/concentration-compactness-via-nonstandard-analysis/.

[13] K. Tintarev and K.-H. Fieseler, *Concentration Compactness: Functional-Analitic Grounds and Applications*, Imperial College Press, London, 2007.

[14] D. van Dulst, Equivalent norms and the fixed point property for nonexpansive mappings, *J. Lond. Math. Soc. (2)* **25** (1982), 139–144.