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A Note on the Solutions for a Higher-Order Convective Cahn–Hilliard-Type Equation

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Abstract: The higher-order convective Cahn–Hilliard equation describes the evolution of crystal surfaces faceting through surface electromigration, the growing surface faceting, and the evolution of dynamics of phase transitions in ternary oil–water–surfactant systems. In this paper, we study the H^3 solutions of the Cauchy problem and prove, under different assumptions on the constants appearing in the equation and on the mean of the initial datum, that they are well-posed.

Keywords: existence; uniqueness; stability; higher-order convective cahn–hilliard type equation; cauchy problem

MSC: 35G25; 35K55

1. Introduction

In this paper, we study the well-posedness of the Cauchy problem:

$$\begin{cases} \partial_t u + \kappa \partial_x u^2 + \gamma \partial_x^2 u - \beta^2 \partial_x^6 u + \alpha \partial_x^4 u + \delta^2 \partial_x^4 (u^3) = 0, & t > 0, \quad x \in \mathbb{R}, \\ u(0, x) = u_0(x), & x \in \mathbb{R}, \end{cases} \quad (1)$$

with

$$\kappa, \gamma, \beta, \alpha, \delta \in \mathbb{R}, \quad \gamma < 0, \quad \alpha \geq 0, \quad \kappa, \beta, \delta \neq 0, \quad \text{or}, \quad (2)$$

$$\beta, \alpha, \delta \in \mathbb{R}, \quad \kappa = \gamma = 0, \quad \alpha \geq 0, \quad \beta, \delta \neq 0, \quad \text{or}, \quad (3)$$

$$\kappa, \gamma, \beta, \alpha, \delta \in \mathbb{R}, \quad \beta \neq 0, \quad \delta = 0, \quad \text{or}, \quad (4)$$

$$\kappa, \gamma, \beta, \alpha, \delta \in \mathbb{R}, \quad \beta, \delta \neq 0. \quad (5)$$

On the initial datum, we assume

$$u_0 \in H^3(\mathbb{R}), \quad \text{or}, \quad (6)$$

$$u_0 \in H^3(\mathbb{R}) \cap L^1(\mathbb{R}), \quad \int_{\mathbb{R}} u_0(x) dx = 0. \quad (7)$$

Inspired by [1–12], in light of (7), we define the following function:

$$P_0(x) = \int_{-\infty}^x u_0(y) dy, \quad (8)$$

on which we assume

$$\|P_0\|_{L^2(\mathbb{R})}^2 = \int_{\mathbb{R}} \left(\int_{-\infty}^x u_0(y) dy \right)^2 dx < \infty. \tag{9}$$

The equation in (1) is derived in [13] to model the evolution of a crystalline surface with small slopes that undergoes faceting. The unknown u gives the surface slope, the constant κ is proportional to the atomic flux deposition strength and the convective term $\kappa \partial_x u^2$ arises from the deposited atoms normal impingement. The sixth-order linear term $\partial_x^6 u$ regularizes the equation, taking into account the surface curvature and the anisotropy of the surface energy under the surface diffusion.

From a mathematical point of view, the existence and uniqueness of weak solutions of (1) with periodic boundary conditions is proven in [14], under the assumptions $\kappa > 0$ and $\gamma = 0$. In the same setting, a similar result is proven in two space dimensions in [15]. In [16], the authors derived the stationary solutions of (1), again assuming $\kappa > 0$ and $\gamma = 0$. In [17], the existence of a global-in-time attractor is studied, while the well-posedness of the classical solutions of (1) is proven in [18], requiring (7)–(9), and $\gamma = 0$. In this paper, we will prove that, if (2) or (3) hold, we have the well-posedness of the classical solutions of (1) assuming (6), while if (5) holds, we have the well-posedness of (1) assuming (7)–(9).

Taking $\kappa = 0$, (1) becomes

$$\partial_t u + \gamma \partial_x^2 u - \beta^2 \partial_x^6 u + \alpha \partial_x^4 u + \delta^2 \partial_x^4 (u^3) = 0, \tag{10}$$

which is a Cahn-Hilliard type equation [19–21]. It was deduced in [22] to describe the evolution of crystal surfaces faceting through surface electromigration. It also describes the phase transition development in ternary oil-water-surfactant systems. One part of the surfactant is hydrophilic, and the other one (termed amphiphile) is lipophilic. Oil, water, and microemulsion (i.e., a homogeneous, isotropic mixture of oil and water) can coexist in equilibrium. The unknown u , in (10), gives the local difference between oil and water concentrations.

From a mathematical point of view, in [23] the initial-boundary-value problem for (10) is analyzed, under appropriate assumptions on $\gamma, \beta, \alpha, \delta$. In [24], the authors analyze the existence of a global-in-time attractor. The existence of weak solutions for the initial-boundary-value problem for (10) is proven in the case of degenerate mobility in [25]. Finally, in [18], the well-posedness of the classical solution of the Cauchy problem of (10) is proven, assuming (7)–(9), with $\gamma = 0$. In this paper, we will show that the classical solutions of the Cauchy problem of (10) are well-posed, assuming (6), if $\gamma \leq 0$ and $\alpha > 0$, while in the general case, we will prove the same result assuming (7)–(9).

Observe that in [13], it is proven that, as $\kappa \rightarrow \infty$, (1) reduces

$$\partial_t u + \partial_x u^2 + a_1 \partial_x^2 u - b_1^2 \partial_x^6 u + c_1 \partial_x^4 u = 0, \tag{11}$$

which is known as the Kuramoto-Sivashinsky equation (see [26–28]). In Section 4, we will prove the well-posedness of the Cauchy problem for (11), assuming (6).

When $\beta = \delta = 0$ and $\alpha = f^2 \neq 0$, (1) reads

$$\partial_t u + \kappa \partial_x u^2 + \gamma \partial_x^2 u + f^2 \partial_x^4 u = 0. \tag{12}$$

(12) appears in several physical situations; for example, it models long waves on a viscous fluid flowing down an inclined plane [29] and drift waves in a plasma [30]. (12) was also independently deduced by Kuramoto [27,31,32] to describe the phase turbulence in reaction-diffusion systems, and by Sivashinsky [28] to describe plane flame propagation, taking into account the combined influence of diffusion and thermal conduction of the gas on the stability of a plane flame front.

Equation (12) can be used to study incipient instabilities in several physical and chemical systems [33–35]. Moreover, (12) is also termed the Benney-Lin equation [36,37], and was deduced by Kuramoto as a model for phase turbulence in the Belousov-Zhabotinsky reaction [38].

The existence and the dynamical properties of the exact solutions for (12) can be found in [39–44]. The control problem for (12) with periodic boundary conditions, and on a bounded interval, are studied in [45–47]. The problem of global-in-time exponential stabilization of (12) with periodic boundary conditions is analyzed in [48]. A generalization of the optimal control theory for (12) is proposed in [49], while the global boundary controllability of (12) is considered in [50]. The existence of solitonic solutions for (12) is proven in [51]. The well-posedness of the Cauchy problem for (12) is proven in [52–54], using the energy space technique, a priori estimates together with an application of the Cauchy-Kovalevskaya and the fixed point method, respectively. Instead, the initial-boundary value problem for (1) is studied, using a priori estimates together with an application of the Cauchy-Kovalevskaya, and the energy space technique in [55–57]. Inspired by [58–60], the convergence of the solution of (12) to the unique entropy one of the Burgers equation is proven in [61].

Finally, due to its general structure, we conjecture that (1) can have a possible application in machine learning (see [62,63]).

2. Results and Organization of the Paper

In this paper, we improve and complete the results of [14,16–18] working with H^3 initial data and having general assumptions on the constants appearing in (1). The main result of this paper is the following theorem. We prove the global-in-time existence, uniqueness, and stability of the solutions of the Cauchy problem (1).

Theorem 1. Fix $T > 0$. Assuming one of the following

- (i) (2) and (6),
- (ii) (3) and (6),
- (iii) (4) and (6),
- (iv) (5) and (7),

and (9), there exists a unique solution u of (1) such that

$$u \in H^1((0, T) \times \mathbb{R}) \cap L^\infty(0, T; H^3(\mathbb{R})). \tag{13}$$

In particular, under the Assumptions (7) and (9), we have that

$$\int_{\mathbb{R}} u(t, x) dx = 0. \tag{14}$$

Moreover, if u_1 and u_2 are two solutions of (1), we have that

$$\|u_1(t, \cdot) - u_2(t, \cdot)\|_{L^2(\mathbb{R})} \leq e^{C(T)t} \|u_{1,0} - u_{2,0}\|_{L^2(\mathbb{R})}, \tag{15}$$

for some suitable $C(T) > 0$, and every $0 \leq t \leq T$.

The well-posedness of (1) is guaranteed for a short time by the Cauchy-Kowaleskaya Theorem [64]. The solutions are indeed global, thanks to suitable a priori estimates.

The paper is organized as follows. In Section 3 we prove Theorem 1, assuming (i) or (ii). In Section 4 we prove Theorem 1, assuming (iii). In Section 5 we prove Theorem 1, assuming (iv).

3. Proof of Theorem 1 Assuming (i) or (ii)

In this section, we prove Theorem 1, assuming (i) or (ii). For the sake of notational simplicity, define

$$\gamma = -a^2, \quad \alpha = b^2, \tag{16}$$

and then (1) reads

$$\begin{cases} \partial_t u + \kappa \partial_x u^2 - a^2 \partial_x^2 u - \beta^2 \partial_x^6 u + b^2 \partial_x^4 u + \delta^2 \partial_x^4 (u^3) = 0, & t > 0, \quad x \in \mathbb{R}, \\ u(0, x) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{17}$$

Since the short time well-posedness of (17) is guaranteed by the Cauchy-Kowaleskaya Theorem [64], here we need to prove some suitable global a priori estimates.

From now on, we denote with C_0 the constants which depend only on the initial data, and with $C(T)$, the constants which depend also on T .

Following [65] (Lemma 2.2), we begin with the following energy estimate in the space $L^\infty(0, \infty; H^1(\mathbb{R})) \cap L^2(0, \infty; H^2(\mathbb{R}))$.

Lemma 1. *Assuming (2), for each $t > 0$, we have that*

$$\begin{aligned} & \frac{\beta^2}{2} \|\partial_x u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \frac{\delta^2}{4} \|u(t, \cdot)\|_{L^4(\mathbb{R})}^4 + \frac{b^2}{2} \|u(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\ & + K_1^2 \int_0^t \|\partial_x^2 u(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds + K_2^2 \int_0^t \|u(s, \cdot) \partial_x u(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds \\ & + a^2 b^2 \int_0^t \|\partial_x u(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds \\ & + \int_0^t \int_{\mathbb{R}} [b^2 \partial_x^2 u + \delta^2 \partial_x^2 (u^3) - \beta^2 \partial_x^4 u]^2 ds dx \leq C_0, \end{aligned} \tag{18}$$

where K_1^2, K_2^2 are two appropriate positive constants.

Assuming (3), for each $t > 0$, we have that

$$\begin{aligned} & \frac{\beta^2}{2} \|\partial_x u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \frac{\delta^2}{4} \|u(t, \cdot)\|_{L^4(\mathbb{R})}^4 + \frac{b^2}{2} \|u(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\ & + \int_0^t \int_{\mathbb{R}} [b^2 \partial_x^2 u + \delta^2 \partial_x^2 (u^3) - \beta^2 \partial_x^4 u]^2 ds dx \leq C_0. \end{aligned} \tag{19}$$

Moreover, there exists $C_0 > 0$, independent on κ, a, b , such that, for each $t \geq 0$,

$$\|u(t, \cdot)\|_{L^\infty(\mathbb{R})} \leq C_0. \tag{20}$$

Proof. We begin by proving (18). Assume (2). Multiplying (17) by $-\beta^2 \partial_x^2 u + \delta^2 u^3 + b^2 u$, we have that

$$\begin{aligned} & (-\beta^2 \partial_x^2 u + \delta^2 u^3 + b^2 u) \partial_t u + 2\kappa (-\beta^2 \partial_x^2 u + \delta^2 u^3 + b^2 u) u \partial_x u \\ & - a^2 (-\beta^2 \partial_x^2 u + \delta^2 u^3 + b^2 u) \partial_x^2 u - \beta^2 (-\beta^2 \partial_x^2 u + \delta^2 u^3 + b^2 u) \partial_x^6 u \\ & + b^2 (-\beta^2 \partial_x^2 u + \delta^2 u^3 + b^2 u) \partial_x^4 u + \delta^2 (-\beta^2 \partial_x^2 u + \delta^2 u^3 + b^2 u) \partial_x^4 (u^3) = 0. \end{aligned} \tag{21}$$

Performing some integration by parts, we gain

$$\begin{aligned} & \int_{\mathbb{R}} (-\beta^2 \partial_x^2 u + \delta^2 u^3 + b^2 u) \partial_t u dx \\ & = \frac{d}{dt} \left(\frac{\beta^2}{2} \|\partial_x u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \frac{\delta^2}{4} \|u(t, \cdot)\|_{L^4(\mathbb{R})}^4 + \frac{b^2}{2} \|u(t, \cdot)\|_{L^2(\mathbb{R})}^2 \right), \\ & 2\kappa \int_{\mathbb{R}} (-\beta^2 \partial_x^2 u + \delta^2 u^3 + b^2 u) u \partial_x u dx = -2\kappa \beta^2 \int_{\mathbb{R}} u \partial_x u \partial_x^2 u dx, \\ & - a^2 \int_{\mathbb{R}} (-\beta^2 \partial_x^2 u + \delta^2 u^3 + b^2 u) \partial_x^2 u dx \end{aligned} \tag{22}$$

$$\begin{aligned}
 &= a^2 \beta^2 \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 3a^2 \delta^2 \|u(t, \cdot) \partial_x u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + a^2 b^2 \|\partial_x u(t, \cdot)\|_{L^2(\mathbb{R})}^2, \\
 &- \beta^2 \int_{\mathbb{R}} \left(-\beta^2 \partial_x^2 u + \delta^2 u^3 + b^2 u \right) \partial_x^6 u dx \\
 &= -\beta^4 \int_{\mathbb{R}} \partial_x^3 u \partial_x^5 u dx + \beta^2 \delta^2 \int_{\mathbb{R}} \partial_x (u^3) \partial_x^5 u dx + \beta^2 b^2 \int_{\mathbb{R}} \partial_x u \partial_x^5 u dx \\
 &= \beta^4 \int_{\mathbb{R}} (\partial_x^4 u)^4 dx - \beta^2 \delta^2 \int_{\mathbb{R}} \partial_x^2 (u^3) \partial_x^4 u dx - \beta^2 a^2 \int_{\mathbb{R}} \partial_x^2 u \partial_x^4 u dx, \\
 &b^2 \int_{\mathbb{R}} \left(-\beta^2 \partial_x^2 u + \delta^2 u^3 + b^2 u \right) \partial_x^4 u dx \\
 &= -\beta^2 b^2 \int_{\mathbb{R}} \partial_x^2 u \partial_x^4 u dx - \delta^2 b^2 \int_{\mathbb{R}} \partial_x (u^3) \partial_x^3 u dx - b^4 \int_{\mathbb{R}} \partial_x u \partial_x^3 u dx \\
 &= -\beta^2 b^2 \int_{\mathbb{R}} \partial_x^2 u \partial_x^4 u dx + \delta^2 b^2 \int_{\mathbb{R}} \partial_x^2 (u^3) \partial_x^2 u dx + b^4 \int_{\mathbb{R}} (\partial_x^2 u)^2 dx, \\
 &\delta^2 \int_{\mathbb{R}} \left(-\beta^2 \partial_x^2 u + \delta^2 u^3 + b^2 u \right) \partial_x^4 (u^3) dx \\
 &= \beta^2 \delta^2 \int_{\mathbb{R}} \partial_x^3 u \partial_x^3 (u^3) dx - \delta^4 \int_{\mathbb{R}} \partial_x (u^3) \partial_x^3 (u^3) dx - b^2 \delta^2 \int_{\mathbb{R}} \partial_x u \partial_x^3 (u^3) dx \\
 &= -\beta^2 \delta^2 \int_{\mathbb{R}} \partial_x^4 u \partial_x^2 (u^3) dx + \delta^4 \int_{\mathbb{R}} \left[\partial_x^2 (u^3) \right]^2 dx + b^2 \delta^2 \int_{\mathbb{R}} \partial_x^2 u \partial_x^2 (u^3) dx.
 \end{aligned}$$

Therefore, thanks to (22), an integration of (21) on \mathbb{R} gives

$$\begin{aligned}
 &\frac{d}{dt} \left(\frac{\beta^2}{2} \|\partial_x u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \frac{\delta^2}{4} \|u(t, \cdot)\|_{L^4(\mathbb{R})}^4 + \frac{b^2}{2} \|u(t, \cdot)\|_{L^2(\mathbb{R})}^2 \right) \\
 &+ a^2 \beta^2 \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 3a^2 \delta^2 \|u(t, \cdot) \partial_x u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + a^2 b^2 \|\partial_x u(t, \cdot)\|_{L^2(\mathbb{R})}^2 \tag{23} \\
 &+ \beta^4 \int_{\mathbb{R}} (\partial_x^4 u)^4 dx - 2\beta^2 \delta^2 \int_{\mathbb{R}} \partial_x^2 (u^3) \partial_x^4 u dx - 2\beta^2 a^2 \int_{\mathbb{R}} \partial_x^2 u \partial_x^4 u dx \\
 &+ 2\delta^2 b^2 \int_{\mathbb{R}} \partial_x^2 (u^3) \partial_x^2 u dx + b^4 \int_{\mathbb{R}} (\partial_x^2 u)^2 dx + \delta^4 \int_{\mathbb{R}} \left[\partial_x^2 (u^3) \right]^2 dx \\
 &= 2\kappa \beta^2 \int_{\mathbb{R}} u \partial_x u \partial_x^2 u dx.
 \end{aligned}$$

Since

$$\begin{aligned}
 &b^4 \int_{\mathbb{R}} (\partial_x^2 u)^2 dx + \delta^4 \int_{\mathbb{R}} \left[\partial_x^2 (u^3) \right]^2 dx + \beta^4 \int_{\mathbb{R}} (\partial_x^4 u)^4 dx \\
 &+ 2\delta^2 b^2 \int_{\mathbb{R}} \partial_x^2 (u^3) \partial_x^2 u dx - 2\beta^2 a^2 \int_{\mathbb{R}} \partial_x^2 u \partial_x^4 u dx - 2\beta^2 \delta^2 \int_{\mathbb{R}} \partial_x^2 (u^3) \partial_x^4 u dx \\
 &= \int_{\mathbb{R}} \left[b^2 \partial_x^2 u + \delta^2 \partial_x^2 (u^3) - \beta^2 \partial_x^4 u \right]^2 dx,
 \end{aligned}$$

(23) becomes

$$\begin{aligned}
 &\frac{d}{dt} \left(\frac{\beta^2}{2} \|\partial_x u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \frac{\delta^2}{4} \|u(t, \cdot)\|_{L^4(\mathbb{R})}^4 + \frac{b^2}{2} \|u(t, \cdot)\|_{L^2(\mathbb{R})}^2 \right) \\
 &+ a^2 \beta^2 \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 3a^2 \delta^2 \|u(t, \cdot) \partial_x u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + a^2 b^2 \|\partial_x u(t, \cdot)\|_{L^2(\mathbb{R})}^2 \tag{24} \\
 &+ \int_{\mathbb{R}} \left[b^2 \partial_x^2 u + \delta^2 \partial_x^2 (u^3) - \beta^2 \partial_x^4 u \right]^2 dx = 2\kappa \beta^2 \int_{\mathbb{R}} u \partial_x u \partial_x^2 u dx.
 \end{aligned}$$

Due to the Young inequality,

$$2|\kappa| \beta^2 \int_{\mathbb{R}} |u \partial_x u| |\partial_x^2 u| dx = 2\beta^2 \int_{\mathbb{R}} \left| \frac{\kappa u \partial_x u}{\sqrt{D_1}} \right| \left| \sqrt{D_1} \partial_x^2 u \right| dx$$

$$\leq \frac{\beta^2 \kappa^2}{D_1} \|u(t, \cdot) \partial_x u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \beta^2 D_1 \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2,$$

where D_1 is a positive constant, which will be specified later. Consequently, by (24),

$$\begin{aligned} & \frac{d}{dt} \left(\frac{\beta^2}{2} \|\partial_x u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \frac{\delta^2}{4} \|u(t, \cdot)\|_{L^4(\mathbb{R})}^4 + \frac{b^2}{2} \|u(t, \cdot)\|_{L^2(\mathbb{R})}^2 \right) \\ & + \beta^2 (a^2 - D_1) \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \left(3a^2 \delta^2 - \frac{\beta^2 \kappa^2}{D_1} \right) \|u(t, \cdot) \partial_x u(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\ & + a^2 b^2 \|\partial_x u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \int_{\mathbb{R}} \left[b^2 \partial_x^2 u + \delta^2 \partial_x^2 (u^3) - \beta^2 \partial_x^4 u \right]^2 dx \leq 0. \end{aligned} \tag{25}$$

We search D_1 , such that

$$a^2 - D_1 > 0, \quad 3a^2 \delta^2 - \frac{\beta^2 \kappa^2}{D_1} > 0. \tag{26}$$

Since, after a rescaling, $|a|$ can be taken very big, D_1 does exist and (26) holds.

Therefore, by (25) and (26),

$$\begin{aligned} & \frac{d}{dt} \left(\frac{\beta^2}{2} \|\partial_x u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \frac{\delta^2}{4} \|u(t, \cdot)\|_{L^4(\mathbb{R})}^4 + \frac{b^2}{2} \|u(t, \cdot)\|_{L^2(\mathbb{R})}^2 \right) \\ & + K_1^2 \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + K_2^2 \|u(t, \cdot) \partial_x u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + a^2 b^2 \|\partial_x u(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\ & + \int_{\mathbb{R}} \left[b^2 \partial_x^2 u + \delta^2 \partial_x^2 (u^3) - \beta^2 \partial_x^4 u \right]^2 dx \leq 0, \end{aligned}$$

where K_1^2, K_2^2 are two appropriate positive constants. Integrating on $(0, t)$, by (6), we have that

$$\begin{aligned} & \frac{\beta^2}{2} \|\partial_x u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \frac{\delta^2}{4} \|u(t, \cdot)\|_{L^4(\mathbb{R})}^4 + \frac{b^2}{2} \|u(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\ & + K_1^2 \int_0^t \left\| \partial_x^2 u(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 ds + K_2^2 \int_0^t \|u(s, \cdot) \partial_x u(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds \\ & + a^2 b^2 \int_0^t \|\partial_x u(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds \\ & + \int_0^t \int_{\mathbb{R}} \left[b^2 \partial_x^2 u + \delta^2 \partial_x^2 (u^3) - \beta^2 \partial_x^4 u \right]^2 ds dx \leq C_0, \end{aligned}$$

that is, (18).

We continue by proving (19). Assume (3). Since $\kappa = \gamma = 0$, (17) reads

$$\partial_t u - \beta^2 \partial_x^6 u + b^2 \partial_x^4 u + \delta^2 \partial_x^4 (u^3) = 0 \tag{27}$$

. Multiplying (27), by $-\beta^2 \partial_x^2 u + \delta^2 u^3 + b^2 u$, arguing as in the previous case, an integration on \mathbb{R} gives

$$\begin{aligned} & \frac{d}{dt} \left(\frac{\beta^2}{2} \|\partial_x u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \frac{\delta^2}{4} \|u(t, \cdot)\|_{L^4(\mathbb{R})}^4 + \frac{b^2}{2} \|u(t, \cdot)\|_{L^2(\mathbb{R})}^2 \right) \\ & + \int_{\mathbb{R}} \left[b^2 \partial_x^2 u + \delta^2 \partial_x^2 (u^3) - \beta^2 \partial_x^4 u \right]^2 dx = 0. \end{aligned}$$

(6) and an integration on $(0, t)$ give (19).

Finally, we prove (20). Thanks to (18) or (19) and the Hölder inequality,

$$|u(t, x)|^3 = 3 \left| \int_{-\infty}^x u^2 \partial_x u dy \right| \leq 3 \int_{\mathbb{R}} u^2 |\partial_x u| dx$$

$$\leq 3 \|u(t, \cdot)\|_{L^4(\mathbb{R})}^2 \|\partial_x u(t, \cdot)\|_{L^2(\mathbb{R})}^2 \leq C_0.$$

Hence,

$$\|u(t, \cdot)\|_{L^\infty(\mathbb{R})}^3 \leq C_0,$$

which gives (20). \square

We continue by proving an L^2 - estimate, which is independent on κ, a, b .

Lemma 2. Fix $T > 0$ and assume (2) or (3). There exists a constant $C(T) > 0$, independent on κ, a, b , such that

$$\|u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2a^2 \int_{\mathbb{R}} \|\partial_x u(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds + \beta^2 \int_{\mathbb{R}} \|\partial_x^3 u(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds \tag{28}$$

$$+ 2b^2 \int_0^t \|\partial_x^2 u(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds \leq C(T),$$

$$\int_0^t \|\partial_x^2 u(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds \leq C(T), \tag{29}$$

$$\int_0^t \|u(s, \cdot)(\partial_x u(s, \cdot))^2\|_{L^2(\mathbb{R})}^2 ds \leq C(T), \tag{30}$$

$$\int_0^t \|\partial_x u(s, \cdot)\|_{L^4(\mathbb{R})}^4 ds \leq C(T), \tag{31}$$

for every $0 \leq t \leq T$.

Proof. Let $0 \leq t \leq T$. We begin by observing that

$$\partial_x u^3 = 3u^2 \partial_x u, \quad \partial_x^2 (u^3) = 6u(\partial_x u)^2 + 3u^2 \partial_x^2 u. \tag{32}$$

Multiplying (17) by $2u$, an integration on \mathbb{R} and several integrations by part give

$$\begin{aligned} \frac{d}{dt} \|u(t, \cdot)\|_{L^2(\mathbb{R})}^2 &= 2 \int_{\mathbb{R}} u \partial_t u dx \\ &= -4\kappa \int_{\mathbb{R}} u^2 \partial_x u dx + 2a^2 \int_{\mathbb{R}} u \partial_x^2 u dx + 2\beta^2 \int_{\mathbb{R}} u \partial_x^6 u dx \\ &\quad - 2b^2 \int_{\mathbb{R}} u \partial_x^4 u dx - 2\delta^2 \int_{\mathbb{R}} u \partial_x^4 (u^3) dx \\ &= -2a^2 \|\partial_x u(t, \cdot)\|_{L^2(\mathbb{R})}^2 - 2\beta^2 \int_{\mathbb{R}} \partial_x u \partial_x^5 u dx + 2b^2 \int_{\mathbb{R}} \partial_x u \partial_x^3 u dx \\ &\quad + 2\delta^2 \int_{\mathbb{R}} \partial_x u \partial_x^3 (u^3) dx \\ &= -2a^2 \|\partial_x u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2\beta^2 \int_{\mathbb{R}} \partial_x^2 u \partial_x^4 u dx - 2b^2 \|\partial_x^2 u(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\ &\quad - 2\delta^2 \int_{\mathbb{R}} \partial_x^2 u \partial_x^2 (u^3) dx \\ &= -2a^2 \|\partial_x u(t, \cdot)\|_{L^2(\mathbb{R})}^2 - 2\beta^2 \|\partial_x^3 u(t, \cdot)\|_{L^2(\mathbb{R})}^2 - 2b^2 \|\partial_x^2 u(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\ &\quad - 2\delta^2 \int_{\mathbb{R}} \partial_x^2 u \partial_x^2 (u^3) dx, \end{aligned}$$

that is,

$$\begin{aligned} \frac{d}{dt} \|u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2a^2 \|\partial_x u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2\beta^2 \|\partial_x^3 u(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\ + 2b^2 \|\partial_x^2 u(t, \cdot)\|_{L^2(\mathbb{R})}^2 = -2\delta^2 \int_{\mathbb{R}} \partial_x^2 u \partial_x^2 (u^3) dx. \end{aligned} \tag{33}$$

Due to the Young inequality, we can estimate the right-hand side of (33), as follows:

$$2\delta^2 \int_{\mathbb{R}} |\partial_x^2 u| |\partial_x^2 (u^3)| dx \leq \|\partial_x^2 u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \delta^4 \int_{\mathbb{R}} [\partial_x^2 (u^3)]^2 dx.$$

Consequently, (33) becomes

$$\begin{aligned} \frac{d}{dt} \|u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2a^2 \|\partial_x u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2\beta^2 \|\partial_x^3 u(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\ + 2b^2 \|\partial_x^2 u(t, \cdot)\|_{L^2(\mathbb{R})}^2 \leq \|\partial_x^2 u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \delta^4 \int_{\mathbb{R}} [\partial_x^2 (u^3)]^2 dx. \end{aligned} \tag{34}$$

Observe that, by (32),

$$\begin{aligned} \int_{\mathbb{R}} [\partial_x^2 (u^3)]^2 dx &= 36 \int_{\mathbb{R}} u^2 (\partial_x u)^4 dx + 9 \|u^2(t, \cdot) \partial_x^2 u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 36 \int_{\mathbb{R}} u^3 (\partial_x u)^2 \partial_x^2 u dx \\ &= 36 \int_{\mathbb{R}} u^2 (\partial_x u)^4 dx + 9 \|u^2(t, \cdot) \partial_x^2 u(t, \cdot)\|_{L^2(\mathbb{R})}^2 - 36 \int_{\mathbb{R}} u^2 (\partial_x u)^4 dx \\ &= 9 \|u^2(t, \cdot) \partial_x^2 u(t, \cdot)\|_{L^2(\mathbb{R})}^2 = 9 \int_{\mathbb{R}} u^4 (\partial_x^2 u)^2 dx. \end{aligned} \tag{35}$$

Using (20),

$$\int_{\mathbb{R}} [\partial_x^2 (u^3)]^2 dx \leq 9 \|u(t, \cdot)\|_{L^\infty(\mathbb{R})}^4 \|\partial_x^2 u(t, \cdot)\|_{L^2(\mathbb{R})}^2 \leq C_0 \|\partial_x^2 u(t, \cdot)\|_{L^2(\mathbb{R})}^2. \tag{36}$$

It follows from (34) and (36) that

$$\begin{aligned} \frac{d}{dt} \|u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2a^2 \|\partial_x u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2\beta^2 \|\partial_x^3 u(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\ + 2b^2 \|\partial_x^2 u(t, \cdot)\|_{L^2(\mathbb{R})}^2 \leq C_0 \|\partial_x^2 u(t, \cdot)\|_{L^2(\mathbb{R})}^2. \end{aligned} \tag{37}$$

Since

$$C_0 \|\partial_x^2 u(t, \cdot)\|_{L^2(\mathbb{R})}^2 = C_0 \int_{\mathbb{R}} \partial_x^2 u \partial_x^2 u dx = -C_0 \int_{\mathbb{R}} \partial_x u \partial_x^3 u dx,$$

Lemma 1 and the Young inequality give

$$\begin{aligned} C_0 \|\partial_x^2 u(t, \cdot)\|_{L^2(\mathbb{R})}^2 &\leq 2 \int_{\mathbb{R}} \left| \frac{C_0 \partial_x u}{2\beta} \right| |\beta \partial_x^3 u| dx \\ &\leq C_0 \|\partial_x u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \beta^2 \|\partial_x^3 u(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\ &\leq C_0 + \beta^2 \|\partial_x^3 u(t, \cdot)\|_{L^2(\mathbb{R})}^2. \end{aligned} \tag{38}$$

Consequently, by (37),

$$\frac{d}{dt} \|u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2a^2 \|\partial_x u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \beta^2 \|\partial_x^3 u(t, \cdot)\|_{L^2(\mathbb{R})}^2$$

$$+ 2b^2 \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \leq C(T).$$

Integrating on $(0, t)$, by (6), we have that

$$\begin{aligned} & \|u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2a^2 \int_0^t \|\partial_x u(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds + \beta^2 \int_0^t \|\partial_x^3 u(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds \\ & + 2b^2 \int_0^t \|\partial_x^2 u(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds \leq C_0 + C(T)t \leq C(T), \end{aligned}$$

which gives (28).

(29) follows from (28), (38) and an integration on $(0, t)$.

We prove (30). We begin by observing that, by (35) and (36), we have that

$$\begin{aligned} 36 \int_{\mathbb{R}} u^2 (\partial_x u)^4 dx & \leq C_0 \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 - 9 \left\| u^2(t, \cdot) \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \\ & - 36 \int_{\mathbb{R}} u^3 (\partial_x u)^2 \partial_x^2 u dx. \end{aligned} \tag{39}$$

Thanks to the Young inequality,

$$36 \int_{\mathbb{R}} |u|^3 (\partial_x u)^2 |\partial_x^2 u| dx \leq 18 \int_{\mathbb{R}} u^2 (\partial_x u)^4 dx + 18 \left\| u^2(t, \cdot) \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2.$$

Consequently, by (39),

$$18 \int_{\mathbb{R}} u^2 (\partial_x u)^4 dx \leq C_0 \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 9 \left\| u^2(t, \cdot) \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2. \tag{40}$$

Observe that, by (20),

$$9 \left\| u^2(t, \cdot) \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 = 9 \int_{\mathbb{R}} u^4 (\partial_x^2 u)^2 dx \leq C_0 \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2.$$

Therefore, by (40),

$$18 \left\| u(t, \cdot) (\partial_x u(t, \cdot))^2 \right\|_{L^2(\mathbb{R})}^2 \leq C_0 \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2. \tag{41}$$

Integrating (41) on $(0, t)$, by (29), we have (30).

Finally, we prove (31). We begin by observing that [66] (Lemma 2.3) says that

$$\|\partial_x u(t, \cdot)\|_{L^4(\mathbb{R})}^4 \leq 6 \left(\|u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \|\partial_x u(t, \cdot)\|_{L^2(\mathbb{R})}^2 \right) \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2.$$

Therefore, by Lemma 1 and (28), we have that

$$\|\partial_x u(t, \cdot)\|_{L^4(\mathbb{R})}^4 \leq C(T) \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2.$$

Integrating on $(0, t)$, by (29), we have (31). \square

We continue with an a priori estimate in the space $L^2(0, \infty; H^4(\mathbb{R}))$.

Lemma 3. Fix $T > 0$ and assume (2) or (3). There exists a constant $C(T) > 0$, independent on κ, a, b , such that

$$\begin{aligned} \|\partial_x u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2a^2 \int_0^t \|\partial_x^2 u(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds + \beta^2 \int_0^t \|\partial_x^4 u(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds \\ + 2b^2 \int_0^t \|\partial_x^3 u(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds \leq C(T) \end{aligned} \tag{42}$$

for every $0 \leq t \leq T$.

Proof. Let $0 \leq t \leq T$. Multiplying (17) by $-2\partial_x^2 u$, an integration on \mathbb{R} gives

$$\begin{aligned} \frac{d}{dt} \|\partial_x u(t, \cdot)\|_{L^2(\mathbb{R})}^2 &= -2 \int_{\mathbb{R}} \partial_x^2 u \partial_t u dx \\ &= 4\kappa \int_{\mathbb{R}} u \partial_x u \partial_x^2 u dx - 2a^2 \|\partial_x^2 u(t, \cdot)\|_{L^2(\mathbb{R})}^2 - 2\beta^2 \int_{\mathbb{R}} \partial_x^2 u \partial_x^6 u dx \\ &\quad + 2b^2 \int_{\mathbb{R}} \partial_x^2 u \partial_x^4 u dx + 2\delta^2 \int_{\mathbb{R}} \partial_x^2 u \partial_x^4 (u^3) dx \\ &= -2\kappa \int_{\mathbb{R}} (\partial_x u)^3 dx - 2a^2 \|\partial_x^2 u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2\beta^2 \int_{\mathbb{R}} \partial_x^3 u \partial_x^5 u dx \\ &\quad - 2b^2 \|\partial_x^3 u(t, \cdot)\|_{L^2(\mathbb{R})}^2 - 2\delta^2 \int_{\mathbb{R}} \partial_x^3 u \partial_x^3 (u^3) dx \\ &= -2\kappa \int_{\mathbb{R}} (\partial_x u)^3 dx - 2a^2 \|\partial_x^2 u(t, \cdot)\|_{L^2(\mathbb{R})}^2 - 2\beta^2 \|\partial_x^4 u(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\ &\quad - 2b^2 \|\partial_x^3 u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2\delta^2 \int_{\mathbb{R}} \partial_x^4 u \partial_x^2 (u^3) dx, \end{aligned}$$

that is,

$$\begin{aligned} \frac{d}{dt} \|\partial_x u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2a^2 \|\partial_x^2 u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2\beta^2 \|\partial_x^4 u(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\ + 2b^2 \|\partial_x^3 u(t, \cdot)\|_{L^2(\mathbb{R})}^2 = 2\kappa \int_{\mathbb{R}} (\partial_x u)^3 dx + 2\delta^2 \int_{\mathbb{R}} \partial_x^4 u \partial_x^2 (u^3) dx. \end{aligned} \tag{43}$$

Due to Lemma 1, (36) and the Young inequality,

$$\begin{aligned} 2|\kappa| \int_{\mathbb{R}} |\partial_x u|^3 dx &\leq \kappa^2 \|\partial_x u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \|\partial_x u(t, \cdot)\|_{L^4(\mathbb{R})}^4 \\ &\leq C_0 + \|\partial_x u(t, \cdot)\|_{L^4(\mathbb{R})}^4, \\ 2\delta^2 \int_{\mathbb{R}} |\partial_x^4 u| |\partial_x^2 (u^3)| dx &= 2 \int_{\mathbb{R}} \left| \beta \partial_x^4 u \right| \left| \frac{\delta^2 \partial_x^2 (u^3)}{\beta} \right| dx \\ &\leq \beta^2 \|\partial_x^4 u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \frac{\delta^4}{\beta^2} \int_{\mathbb{R}} \left[\partial_x^2 (u^3) \right]^2 dx \\ &\leq \beta^2 \|\partial_x^4 u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + C_0 \|\partial_x^2 u(t, \cdot)\|_{L^2(\mathbb{R})}^2. \end{aligned}$$

Therefore, by (43),

$$\begin{aligned} \frac{d}{dt} \|\partial_x u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2a^2 \|\partial_x^2 u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \beta^2 \|\partial_x^4 u(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\ + 2b^2 \|\partial_x^3 u(t, \cdot)\|_{L^2(\mathbb{R})}^2 \leq \|\partial_x u(t, \cdot)\|_{L^4(\mathbb{R})}^4 + C_0 \|\partial_x^2 u(t, \cdot)\|_{L^2(\mathbb{R})}^2. \end{aligned}$$

Integrating on $(0, t)$, by (6), (29) and (31),

$$\begin{aligned} & \|\partial_x u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2a^2 \int_0^t \|\partial_x^2 u(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds + \beta^2 \int_0^t \|\partial_x^4 u(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds \\ & \quad + 2b^2 \int_0^t \|\partial_x^3 u(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds \\ & \leq C_0 + \int_0^t \|\partial_x u(s, \cdot)\|_{L^4(\mathbb{R})}^4 ds + C_0 \int_0^t \|\partial_x^2 u(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds \leq C(T), \end{aligned}$$

which gives (42). \square

We continue with an a priori estimate in the space $L^2(0, \infty; H^5(\mathbb{R}))$.

Lemma 4. Fix $T > 0$ and assume (2) or (3). There exists a constant $C(T) > 0$, independent on κ, a, b ,

$$\|\partial_x u\|_{L^\infty((0,T) \times \mathbb{R})} \leq C(T), \tag{44}$$

$$\begin{aligned} & \|\partial_x^2 u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2a^2 \int_0^t \|\partial_x^4 u(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds \\ & + \beta^2 \int_0^t \|\partial_x^5 u(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds + 2b^2 \int_0^t \|\partial_x^4 u(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds \leq C(T), \end{aligned} \tag{45}$$

for every $0 \leq t \leq T$, where $C(T)$ is independent on κ, a, b .

Proof. Let $0 \leq t \leq T$. We begin by observing that, by (32), we have that

$$\partial_x^3(u^3) = 6(\partial_x u)^3 + 18u\partial_x u\partial_x^2 u + 3u^2\partial_x^3 u. \tag{46}$$

Multiplying (17) by $2\partial_x^4 u$, thanks to (46), an integration on \mathbb{R} gives

$$\begin{aligned} \frac{d}{dt} \|\partial_x^2 u(t, \cdot)\|_{L^2(\mathbb{R})}^2 & = 2 \int_{\mathbb{R}} \partial_x^4 u \partial_t u dx \\ & = -2\kappa \int_{\mathbb{R}} u \partial_x u \partial_x^4 u dx + 2a^2 \int_{\mathbb{R}} \partial_x^2 u \partial_x^4 u dx + 2\beta^2 \int_{\mathbb{R}} \partial_x^4 u \partial_x^6 u dx \\ & \quad - 2b^2 \|\partial_x^4 u(t, \cdot)\|_{L^2(\mathbb{R})}^2 - 2\delta^2 \int_{\mathbb{R}} \partial_x^4 u \partial_x^4(u^3) dx \\ & = -2\kappa \int_{\mathbb{R}} u \partial_x u \partial_x^4 u dx - 2a^2 \|\partial_x^3 u(t, \cdot)\|_{L^2(\mathbb{R})}^2 - 2\beta^2 \|\partial_x^5 u(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\ & \quad - 2b^2 \|\partial_x^4 u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2\delta^2 \int_{\mathbb{R}} \partial_x^5 u \partial_x^3(u^3) dx \\ & = -2\kappa \int_{\mathbb{R}} u \partial_x u \partial_x^4 u dx - 2a^2 \|\partial_x^3 u(t, \cdot)\|_{L^2(\mathbb{R})}^2 - 2\beta^2 \|\partial_x^5 u(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\ & \quad - 2b^2 \|\partial_x^4 u(t, \cdot)\|_{L^2(\mathbb{R})}^2 - 12\delta^2 \int_{\mathbb{R}} (\partial_x u)^3 \partial_x^5 u dx - 36 \int_{\mathbb{R}} u \partial_x u \partial_x^2 u \partial_x^5 u dx \\ & \quad - 6 \int_{\mathbb{R}} u^2 \partial_x^3 u \partial_x^5 u dx, \end{aligned}$$

that is,

$$\begin{aligned} & \frac{d}{dt} \|\partial_x^2 u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2a^2 \|\partial_x^4 u(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\ & \quad + 2\beta^2 \|\partial_x^5 u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2b^2 \|\partial_x^4 u(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\ & = -2\kappa \int_{\mathbb{R}} u \partial_x u \partial_x^4 u dx - 12\delta^2 \int_{\mathbb{R}} (\partial_x u)^3 \partial_x^5 u dx \end{aligned} \tag{47}$$

$$- 36\delta^2 \int_{\mathbb{R}} u \partial_x u \partial_x^2 u \partial_x^5 u dx - 6\delta^2 \int_{\mathbb{R}} u^2 \partial_x^3 u \partial_x^5 u dx.$$

Due to Lemma 1 and the Young inequality,

$$\begin{aligned} 2|\kappa| \int_{\mathbb{R}} |u| |\partial_x u| |\partial_x^4 u| dx &\leq \kappa^2 \int_{\mathbb{R}} u^2 (\partial_x u)^2 dx + \left\| \partial_x^4 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \\ &\leq \kappa^2 \|u(t, \cdot)\|_{L^\infty(\mathbb{R})}^2 \|\partial_x u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \left\| \partial_x^4 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \\ &\leq C_0 \|\partial_x u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \left\| \partial_x^4 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \leq C_0 + \left\| \partial_x^4 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2, \\ 12\delta^2 \int_{\mathbb{R}} |\partial_x u|^3 |\partial_x^5 u| dx &= 2 \int_{\mathbb{R}} \left| \frac{6\delta^2 (\partial_x u)^3}{\beta \sqrt{D_2}} \right| \left| \beta \sqrt{D_2} \partial_x^5 u \right| dx \\ &\leq \frac{36\delta^4}{\beta^2 D_2} \int_{\mathbb{R}} (\partial_x u)^6 dx + \beta^2 D_2 \left\| \partial_x^5 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \\ &\leq \frac{36\delta^4}{\beta^2 D_2} \|\partial_x u\|_{L^\infty((0,T) \times \mathbb{R})}^2 \|\partial_x u(t, \cdot)\|_{L^4(\mathbb{R})}^4 + \beta^2 D_2 \left\| \partial_x^5 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2, \\ 36\delta^2 \int_{\mathbb{R}} |u| |\partial_x u \partial_x^2 u| |\partial_x^5 u| dx &\leq 36\delta^2 \|u(t, \cdot)\|_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} |\partial_x u \partial_x^2 u| |\partial_x^5 u| dx \\ &\leq 2C_0 \int_{\mathbb{R}} |\partial_x u \partial_x^2 u| |\partial_x^5 u| dx = 2 \int_{\mathbb{R}} \left| \frac{C_0 \partial_x u \partial_x^2 u}{\beta \sqrt{D_2}} \right| \left| \beta \sqrt{D_2} \partial_x^5 u \right| dx \\ &\leq \frac{C_0}{D_2} \int_{\mathbb{R}} (\partial_x u)^2 (\partial_x^2 u)^2 dx + \beta^2 D_2 \left\| \partial_x^5 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \\ &\leq \frac{C_0}{D_2} \|\partial_x u\|_{L^\infty((0,T) \times \mathbb{R})}^2 \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \beta^2 D_2 \left\| \partial_x^5 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2, \\ 6\delta^2 \int_{\mathbb{R}} u^2 |\partial_x^3 u| |\partial_x^5 u| dx &\leq 6\delta^2 \|u(t, \cdot)\|_{L^\infty(\mathbb{R})}^2 \int_{\mathbb{R}} |\partial_x^3 u| |\partial_x^5 u| dx \\ &\leq 2C_0 \int_{\mathbb{R}} |\partial_x^3 u| |\partial_x^5 u| dx \leq 2 \int_{\mathbb{R}} \left| \frac{C_0 \partial_x^3 u}{\beta \sqrt{D_2}} \right| \left| \beta \sqrt{D_2} \partial_x^5 u \right| dx \\ &\leq \frac{C_0}{D_2} \left\| \partial_x^3 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \beta^2 D_2 \left\| \partial_x^5 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2, \end{aligned}$$

where D_2 is a positive constant, which will be specified later. Therefore, by (47),

$$\begin{aligned} &\frac{d}{dt} \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 2a^2 \left\| \partial_x^4 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \\ &\quad + \beta^2 (2 - 3D_2) \left\| \partial_x^5 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 2b^2 \left\| \partial_x^4 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \\ &\leq C_0 + \left\| \partial_x^4 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \frac{36\delta^4}{\beta^2 D_2} \|\partial_x u\|_{L^\infty((0,T) \times \mathbb{R})}^2 \|\partial_x u(t, \cdot)\|_{L^4(\mathbb{R})}^4 \\ &\quad + \frac{C_0}{D_2} \|\partial_x u\|_{L^\infty((0,T) \times \mathbb{R})}^2 \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \frac{C_0}{D_2} \left\| \partial_x^3 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2, \end{aligned}$$

and taking $D_2 = \frac{1}{3}$

$$\begin{aligned} &\frac{d}{dt} \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 2a^2 \left\| \partial_x^4 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \\ &\quad + \beta^2 \left\| \partial_x^5 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 2b^2 \left\| \partial_x^4 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \\ &\leq C_0 + \left\| \partial_x^4 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \frac{108\delta^4}{\beta^2} \|\partial_x u\|_{L^\infty((0,T) \times \mathbb{R})}^2 \|\partial_x u(t, \cdot)\|_{L^4(\mathbb{R})}^4 \end{aligned}$$

$$+ C_0 \|\partial_x u\|_{L^\infty((0,T)\times\mathbb{R})}^2 \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + C_0 \left\| \partial_x^3 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2.$$

Integrating on $(0, t)$, by (6), (29), (31) and (42), we obtain that

$$\begin{aligned} & \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 2a^2 \int_0^t \left\| \partial_x^4 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \\ & + \beta^2 \int_0^t \left\| \partial_x^5 u(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 ds + 2b^2 \int_0^t \left\| \partial_x^4 u(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 ds \\ & \leq C_0 + C_0 t + \int_0^t \left\| \partial_x^4 u(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 ds \\ & + \frac{108\delta^4}{\beta^2} \|\partial_x u\|_{L^\infty((0,T)\times\mathbb{R})}^2 \int_0^t \|\partial_x u(s, \cdot)\|_{L^4(\mathbb{R})}^4 ds \\ & + C_0 \|\partial_x u\|_{L^\infty((0,T)\times\mathbb{R})}^2 \int_0^t \left\| \partial_x^2 u(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 ds + C_0 \int_0^t \left\| \partial_x^3 u(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 ds \\ & \leq C(T) \left(1 + \|\partial_x u\|_{L^\infty((0,T)\times\mathbb{R})}^2 \right). \end{aligned} \tag{48}$$

We prove (44). Thanks to Lemma 1, (48) and the Hölder inequality,

$$\begin{aligned} (\partial_x u(t, x))^2 &= 2 \int_{-\infty}^x \partial_x u \partial_x^2 u dx \leq 2 \int_{\mathbb{R}} |\partial_x u| |\partial_x^2 u| dx \leq 2 \|\partial_x u(t, \cdot)\|_{L^2(\mathbb{R})} \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})} \\ &\leq C(T) \sqrt{\left(1 + \|\partial_x u\|_{L^\infty((0,T)\times\mathbb{R})}^2 \right)}. \end{aligned}$$

Hence,

$$\|\partial_x u\|_{L^4((0,T)\times\mathbb{R})}^4 - C(T) \|\partial_x u\|_{L^\infty((0,T)\times\mathbb{R})}^2 - C(T) \leq 0,$$

which gives (44).

Finally, (45) follows from (44) and (48). □

We continue with an a priori estimate in the space $L^4(0, \infty; W^{2,4}(\mathbb{R}))$.

Lemma 5. Fix $T > 0$ and assume (2) or (3). There exists a constant $C(T) > 0$, independent on κ, a, b , such that

$$\int_0^t \left\| \partial_x^2 u(s, \cdot) \right\|_{L^4(\mathbb{R})}^4 ds \leq C(T), \tag{49}$$

for every $0 \leq t \leq T$.

Proof. Let $0 \leq t \leq T$. We begin by observing that

$$\int_{\mathbb{R}} (\partial_x^2 u)^4 dx = \int_{\mathbb{R}} (\partial_x^2 u)^3 \partial_x^2 u dx = -3 \int_{\mathbb{R}} \partial_x u (\partial_x^2 u)^2 \partial_x^3 u dx. \tag{50}$$

Due to the Young inequality,

$$3 \int_{\mathbb{R}} |u| |\partial_x u (\partial_x^2 u)^2| |\partial_x^3 u| dx \leq \frac{9}{2} \int_{\mathbb{R}} (\partial_x u)^2 (\partial_x^3 u)^2 dx + \frac{1}{2} \int_{\mathbb{R}} (\partial_x^2 u)^4 dx.$$

It follows from (50) that

$$\int_{\mathbb{R}} (\partial_x^2 u)^4 dx \leq 9 \int_{\mathbb{R}} (\partial_x u)^2 (\partial_x^3 u)^2 dx.$$

By (44), we have that

$$\left\| \partial_x^2 u(t, \cdot) \right\|_{L^4(\mathbb{R})}^4 \leq 9 \|\partial_x u\|_{L^\infty((0,T)\times\mathbb{R})}^2 \left\| \partial_x^3 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \leq C(T) \left\| \partial_x^3 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2.$$

Integrating on $(0, t)$, by (28), we have that

$$\int_0^t \left\| \partial_x^2 u(s, \cdot) \right\|_{L^4(\mathbb{R})}^4 ds \leq C(T) \int_0^t \left\| \partial_x^3 u(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 ds \leq C(T),$$

which gives (49). \square

We continue with an a priori estimate in the space $L^2(0, \infty; H^6(\mathbb{R}))$.

Lemma 6. Fix $T > 0$ and assume (2) or (3). There exists a constant $C(T) > 0$, independent on κ, a, b , such that

$$\begin{aligned} \left\| \partial_x^3 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 2a^2 \int_0^t \left\| \partial_x^4 u(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 ds \\ + \beta^2 \int_0^t \left\| \partial_x^6 u(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 ds + 2b^2 \int_0^t \left\| \partial_x^5 u(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 ds \leq C(T) \end{aligned} \tag{51}$$

In particular, we have that

$$\left\| \partial_x^2 u \right\|_{L^\infty((0,T) \times \mathbb{R})} \leq C(T), \tag{52}$$

where $C(T)$ is independent on κ, a, b .

Proof. Let $0 \leq t \leq T$. We begin by observing that, by (46),

$$\partial_x^4 \left(u^3 \right) = 36(\partial_x u)^2 \partial_x^2 u + 18u(\partial_x^2 u)^2 + 24u \partial_x u \partial_x^3 u + 3u^2 \partial_x^4 u \tag{53}$$

Multiplying (17) by $-2\partial_x^6 u$, thanks to (46), an integration on \mathbb{R} gives

$$\begin{aligned} \frac{d}{dt} \left\| \partial_x^3 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 &= -2 \int_{\mathbb{R}} \partial_x^6 u \partial_t u dx \\ &= 4\kappa \int_{\mathbb{R}} u \partial_x u \partial_x^6 u dx - 2a^2 \int_{\mathbb{R}} \partial_x^2 u \partial_x^6 u dx - 2\beta^2 \left\| \partial_x^6 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \\ &\quad + 2b^2 \int_{\mathbb{R}} \partial_x^4 u \partial_x^6 u dx + 2\delta^2 \int_{\mathbb{R}} \partial_x^5 u \partial_x^4 \left(u^3 \right) dx \\ &= 4\kappa \int_{\mathbb{R}} u \partial_x u \partial_x^6 u dx + 2a^2 \int_{\mathbb{R}} \partial_x^3 u \partial_x^5 u dx - 2\beta^2 \left\| \partial_x^6 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \\ &\quad - 2b^2 \left\| \partial_x^5 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 72\delta^2 \int_{\mathbb{R}} (\partial_x u)^2 \partial_x^2 u \partial_x^6 u dx \\ &\quad + 36\delta^2 \int_{\mathbb{R}} u (\partial_x^2 u)^2 \partial_x^6 u dx + 48\delta^2 \int_{\mathbb{R}} u \partial_x u \partial_x^3 u \partial_x^6 u dx \\ &\quad + 6\delta^2 \int_{\mathbb{R}} u^2 \partial_x^4 u \partial_x^6 u dx \\ &= 4\kappa \int_{\mathbb{R}} u \partial_x u \partial_x^6 u dx - 2a^2 \left\| \partial_x^4 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 - 2\beta^2 \left\| \partial_x^6 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \\ &\quad - 2b^2 \left\| \partial_x^5 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 72\delta^2 \int_{\mathbb{R}} (\partial_x u)^2 \partial_x^2 u \partial_x^6 u dx \\ &\quad + 36\delta^2 \int_{\mathbb{R}} u (\partial_x^2 u)^2 \partial_x^6 u dx + 48\delta^2 \int_{\mathbb{R}} u \partial_x u \partial_x^3 u \partial_x^6 u dx \\ &\quad + 6\delta^2 \int_{\mathbb{R}} u^2 \partial_x^4 u \partial_x^6 u dx. \end{aligned}$$

Therefore, we have that

$$\frac{d}{dt} \left\| \partial_x^3 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 2a^2 \left\| \partial_x^4 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 2\beta^2 \left\| \partial_x^6 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 2b^2 \left\| \partial_x^5 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \tag{54}$$

$$\begin{aligned}
 &= 4\kappa \int_{\mathbb{R}} u \partial_x u \partial_x^6 u dx + 72\delta^2 \int_{\mathbb{R}} (\partial_x u)^2 \partial_x^2 u \partial_x^6 u dx \\
 &\quad + 36\delta^2 \int_{\mathbb{R}} u (\partial_x^2 u)^2 \partial_x^6 u dx + 48\delta^2 \int_{\mathbb{R}} u \partial_x u \partial_x^3 u \partial_x^6 u dx + 6\delta^2 \int_{\mathbb{R}} u^2 \partial_x^4 u \partial_x^6 u dx.
 \end{aligned}$$

Due to Lemma 1, (44), (45) and the Young inequality,

$$\begin{aligned}
 &4|\kappa| \int_{\mathbb{R}} |u \partial_x u| |\partial_x^6 u| dx \leq 4|\kappa| \|u(t, \cdot)\|_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} |\partial_x u| |\partial_x^6 u| dx \\
 &\leq 2C_0 \int_{\mathbb{R}} |\partial_x u| |\partial_x^6 u| dx = 2 \int_{\mathbb{R}} \left| \frac{C_0 \partial_x u}{\beta \sqrt{D_3}} \right| \left| \beta \sqrt{D_3} \partial_x^6 u \right| dx \\
 &\leq \frac{C_0}{D_3} \|\partial_x u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \beta^2 D_3 \|\partial_x^6 u(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\
 &\leq \frac{C_0}{D_3} + \beta^2 D_3 \|\partial_x^6 u(t, \cdot)\|_{L^2(\mathbb{R})}^2, \\
 &72\delta^2 \int_{\mathbb{R}} (\partial_x u)^2 |\partial_x^2 u| |\partial_x^6 u| dx \leq 72\delta^2 \|\partial_x u\|_{L^\infty((0,T) \times \mathbb{R})}^2 \int_{\mathbb{R}} |\partial_x^2 u| |\partial_x^6 u| dx \\
 &\leq 2C(T) \int_{\mathbb{R}} |\partial_x^2 u| |\partial_x^6 u| dx = 2 \int_{\mathbb{R}} \left| \frac{C(T) \partial_x^2 u}{\beta \sqrt{D_3}} \right| \left| \beta \sqrt{D_3} \partial_x^6 u \right| dx \\
 &\leq \frac{C(T)}{D_3} \|\partial_x^2 u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \beta^2 D_3 \|\partial_x^6 u(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\
 &\leq \frac{C(T)}{D_3} + \beta^2 D_3 \|\partial_x^6 u(t, \cdot)\|_{L^2(\mathbb{R})}^2, \\
 &36\delta^2 \int_{\mathbb{R}} |u| (\partial_x^2 u)^2 |\partial_x^6 u| dx = 36\delta^2 \|u\|_{L^\infty((0,T) \times \mathbb{R})} \int_{\mathbb{R}} (\partial_x^2 u)^2 |\partial_x^6 u| dx \\
 &\leq 2C(T) \int_{\mathbb{R}} (\partial_x^2 u)^2 |\partial_x^6 u| dx = 2 \int_{\mathbb{R}} \left| \frac{C(T) (\partial_x^2 u)^2}{\beta \sqrt{D_3}} \right| \left| \beta \sqrt{D_3} \partial_x^6 u \right| dx \\
 &\leq \frac{C(T)}{D_3} \|\partial_x^2 u(t, \cdot)\|_{L^4(\mathbb{R})}^4 + \beta^2 D_3 \|\partial_x^6 u(t, \cdot)\|_{L^2(\mathbb{R})}^2, \\
 &48\delta^2 \int_{\mathbb{R}} |u| |\partial_x u| |\partial_x^3 u| |\partial_x^6 u| dx \leq 48\delta^2 \|u(t, \cdot)\|_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} |\partial_x u| |\partial_x^3 u| |\partial_x^6 u| dx \\
 &\leq 2C_0 \int_{\mathbb{R}} |\partial_x u| |\partial_x^3 u| |\partial_x^6 u| dx \leq 2C_0 \|\partial_x u\|_{L^\infty((0,T) \times \mathbb{R})} \int_{\mathbb{R}} |\partial_x^3 u| |\partial_x^6 u| dx \\
 &\leq 2C(T) \int_{\mathbb{R}} |\partial_x^3 u| |\partial_x^6 u| dx \leq 2 \int_{\mathbb{R}} \left| \frac{C(T) \partial_x^3 u}{\beta \sqrt{D_3}} \right| \left| \beta \sqrt{D_3} \partial_x^6 u \right| dx \\
 &\leq \frac{C(T)}{D_3} \|\partial_x^3 u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \beta^2 D_3 \|\partial_x^6 u(t, \cdot)\|_{L^2(\mathbb{R})}^2, \\
 &6\delta^2 \int_{\mathbb{R}} u^2 |\partial_x^4 u| |\partial_x^6 u| dx \leq 6\delta^2 \|u(t, \cdot)\|_{L^\infty(\mathbb{R})}^2 \int_{\mathbb{R}} |\partial_x^4 u| |\partial_x^6 u| dx \\
 &\leq 2C_0 \int_{\mathbb{R}} |\partial_x^4 u| |\partial_x^6 u| dx = 2 \int_{\mathbb{R}} \left| \frac{C_0 \partial_x^4 u}{\beta \sqrt{D_3}} \right| \left| \beta \sqrt{D_3} \partial_x^6 u \right| dx \\
 &\leq \frac{C_0}{D_3} \|\partial_x^4 u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \beta^2 D_3 \|\partial_x^6 u(t, \cdot)\|_{L^2(\mathbb{R})}^2,
 \end{aligned}$$

where D_3 is a positive constant, which will be specified later. It follows from (54) that

$$\begin{aligned}
 &\frac{d}{dt} \|\partial_x^3 u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2a^2 \|\partial_x^4 u(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\
 &\quad + \beta^2 (2 - 5D_3) \|\partial_x^6 u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2b^2 \|\partial_x^5 u(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\
 &\leq \frac{C(T)}{D_3} + \frac{C(T)}{D_3} \|\partial_x^2 u(t, \cdot)\|_{L^4(\mathbb{R})}^4 + \frac{C(T)}{D_3} \|\partial_x^3 u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \frac{C_0}{D_3} \|\partial_x^4 u(t, \cdot)\|_{L^2(\mathbb{R})}^2.
 \end{aligned}$$

Taking $D_3 = \frac{1}{5}$, we have that

$$\begin{aligned} & \frac{d}{dt} \left\| \partial_x^3 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 2a^2 \left\| \partial_x^4 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \\ & \quad + \beta^2 \left\| \partial_x^6 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 2b^2 \left\| \partial_x^5 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \\ & \leq C(T) + C(T) \left\| \partial_x^2 u(t, \cdot) \right\|_{L^4(\mathbb{R})}^4 + C(T) \left\| \partial_x^3 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + C_0 \left\| \partial_x^4 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2. \end{aligned}$$

Integrating on $(0, t)$, by (6), (28), (42), (49), we have that

$$\begin{aligned} & \left\| \partial_x^3 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 2a^2 \int_0^t \left\| \partial_x^4 u(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 ds \\ & \quad + \beta^2 \int_0^t \left\| \partial_x^6 u(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 ds + 2b^2 \int_0^t \left\| \partial_x^5 u(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 ds \\ & \leq C_0 + C(T) \int_0^t \left\| \partial_x^2 u(s, \cdot) \right\|_{L^4(\mathbb{R})}^4 ds + C(T) \int_0^t \left\| \partial_x^3 u(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 ds + C_0 \int_0^t \left\| \partial_x^4 u(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 ds \\ & \leq C(T), \end{aligned}$$

which gives (51).

Finally, we prove (52). Thanks to (45), (51) and the Hölder inequality,

$$\begin{aligned} (\partial_x^2 u(t, x))^2 &= 2 \int_{-\infty}^x \partial_x^2 u \partial_x^3 u dy \leq 2 \int_{\mathbb{R}} |\partial_x^2 u| |\partial_x^3 u| dx \\ &\leq 2 \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})} \left\| \partial_x^3 u(t, \cdot) \right\|_{L^2(\mathbb{R})} \leq C(T). \end{aligned}$$

Hence,

$$\left\| \partial_x^2 u \right\|_{L^\infty((0,T) \times \mathbb{R})}^2 \leq C(T),$$

which gives (52). □

We continue with an a priori estimate in the space $H^1((0, \infty) \times \mathbb{R})$.

Lemma 7. Fix $T > 0$ and assume (2) or (3). There exists a constant $C(T) > 0$, independent on κ, a, b , such that

$$a^2 \left\| \partial_x u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \beta^2 \left\| \partial_x^3 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + b^2 \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \int_0^t \left\| \partial_t u(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 ds \leq C(T), \tag{55}$$

for every $0 \leq t \leq T$.

Proof. Let $0 \leq t \leq T$. Multiplying (17) by $2\partial_t u$, we have that

$$2(\partial_t u)^2 + 4\kappa u \partial_x u \partial_t u - 2a^2 \partial_x^2 u \partial_t u - 2\beta^2 \partial_x^6 u \partial_t u + 2b^2 \partial_x^4 u \partial_t u + 2\delta^2 \partial_t u \partial_x^4 (u^3) = 0. \tag{56}$$

Since,

$$\begin{aligned} -2a^2 \int_{\mathbb{R}} \partial_x^2 u \partial_t u &= a^2 \frac{d}{dt} \left\| \partial_x u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2, \\ -2\beta^2 \int_{\mathbb{R}} \partial_x^6 u \partial_t u dx &= \beta^2 \frac{d}{dt} \left\| \partial_x^3 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2, \\ 2b^2 \int_{\mathbb{R}} \partial_x^4 u \partial_t u dx &= b^2 \frac{d}{dt} \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2, \end{aligned}$$

thanks to (53), an integration of (56) on \mathbb{R} gives

$$\begin{aligned} & \frac{d}{dt} \left(a^2 \|\partial_x u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \beta^2 \|\partial_x^3 u(t, \cdot)\|_{L^2(\mathbb{R})}^2 \right) + b^2 \frac{d}{dt} \|\partial_x^2 u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2 \|\partial_t u(t, \cdot)\|_{L^2(\mathbb{R})}^2 \quad (57) \\ &= -4\kappa \int_{\mathbb{R}} u \partial_x u \partial_t u dx - 72\delta^2 \int_{\mathbb{R}} (\partial_x u)^2 \partial_x^2 u \partial_t u dx \\ & \quad + 36\delta^2 \int_{\mathbb{R}} u (\partial_x^2 u)^2 \partial_t u dx - 48\delta^2 \int_{\mathbb{R}} u \partial_x u \partial_x^2 u \partial_t u dx - 6\delta^2 \int_{\mathbb{R}} u^2 \partial_x^4 u \partial_t u dx. \end{aligned}$$

Due to Lemma 1, (44), (45), (52) and the Young inequality,

$$\begin{aligned} 4|\kappa| \int_{\mathbb{R}} |u| |\partial_x u| |\partial_t u| dx &\leq 4|\kappa| \|u(t, \cdot)\|_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} |\partial_x u| |\partial_t u| dx \\ &\leq 2C_0 \int_{\mathbb{R}} |\partial_x u| |\partial_t u| dx = 2 \int_{\mathbb{R}} \left| \frac{C_0 \partial_x u}{\sqrt{D_3}} \right| \left| \sqrt{D_3} \partial_t u \right| dx \\ &\leq \frac{C_0}{D_3} \|\partial_x u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + D_3 \|\partial_t u(t, \cdot)\|_{L^2(\mathbb{R})}^2 \leq \frac{C_0}{D_3} + D_3 \|\partial_t u(t, \cdot)\|_{L^2(\mathbb{R})}^2, \\ 72\delta^2 \int_{\mathbb{R}} (\partial_x u)^2 |\partial_x^2 u| |\partial_t u| dx &\leq 72\delta^2 \|\partial_x u\|_{L^\infty((0,T) \times \mathbb{R})}^2 \int_{\mathbb{R}} |\partial_x^2 u| |\partial_t u| dx \\ &\leq 2C(T) \int_{\mathbb{R}} |\partial_x^2 u| |\partial_t u| dx = 2 \int_{\mathbb{R}} \left| \frac{C(T) \partial_x^2 u}{\sqrt{D_3}} \right| \left| \sqrt{D_3} \partial_t u \right| dx \\ &\leq \frac{C(T)}{D_3} \|\partial_x^2 u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + D_3 \|\partial_t u(t, \cdot)\|_{L^2(\mathbb{R})}^2 \leq \frac{C(T)}{D_3} + D_3 \|\partial_t u(t, \cdot)\|_{L^2(\mathbb{R})}^2, \\ 36\delta^2 \int_{\mathbb{R}} |u| (\partial_x^2 u)^2 |\partial_t u| dx &\leq 36\delta^2 \|u(t, \cdot)\|_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} (\partial_x^2 u)^2 |\partial_t u| dx \\ &\leq 2C_0 \int_{\mathbb{R}} (\partial_x^2 u)^2 |\partial_t u| dx = 2C_0 \|\partial_x^2 u\|_{L^\infty((0,T) \times \mathbb{R})} \int_{\mathbb{R}} |\partial_x^2 u| |\partial_t u| dx \\ &\leq 2C(T) \int_{\mathbb{R}} |\partial_x^2 u| |\partial_t u| dx = 2 \int_{\mathbb{R}} \left| \frac{C(T) \partial_x^2 u}{\sqrt{D_3}} \right| \left| \sqrt{D_3} \partial_t u \right| dx \\ &\leq \frac{C(T)}{D_3} \|\partial_x^2 u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + D_3 \|\partial_t u(t, \cdot)\|_{L^2(\mathbb{R})}^2 \leq \frac{C(T)}{D_3} + D_3 \|\partial_t u(t, \cdot)\|_{L^2(\mathbb{R})}^2, \\ 48\delta^2 \int_{\mathbb{R}} |u| |\partial_x u| |\partial_x^2 u| |\partial_t u| dx &\leq 48\delta^2 \|u(t, \cdot)\|_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} |\partial_x u| |\partial_x^2 u| |\partial_t u| dx \\ &\leq 2C_0 \int_{\mathbb{R}} |\partial_x u| |\partial_x^2 u| |\partial_t u| dx \leq 2C_0 \|\partial_x u\|_{L^\infty((0,T) \times \mathbb{R})} \int_{\mathbb{R}} |\partial_x^2 u| |\partial_t u| dx \\ &\leq 2C(T) \int_{\mathbb{R}} |\partial_x^2 u| |\partial_t u| dx = 2 \int_{\mathbb{R}} \left| \frac{C(T) \partial_x^2 u}{\sqrt{D_3}} \right| \left| \sqrt{D_3} \partial_t u \right| dx \\ &\leq \frac{C(T)}{D_3} \|\partial_x^2 u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + D_3 \|\partial_t u(t, \cdot)\|_{L^2(\mathbb{R})}^2 \leq \frac{C(T)}{D_3} + D_3 \|\partial_t u(t, \cdot)\|_{L^2(\mathbb{R})}^2, \\ 6\delta^2 \int_{\mathbb{R}} u^2 |\partial_x^4 u| |\partial_t u| dx &= 6\delta^2 \|u(t, \cdot)\|_{L^\infty(\mathbb{R})}^2 \int_{\mathbb{R}} |\partial_x^4 u| |\partial_t u| dx \\ &\leq 2C_0 \int_{\mathbb{R}} |\partial_x^4 u| |\partial_t u| dx = 2 \int_{\mathbb{R}} \left| \frac{C_0 \partial_x^4 u}{\sqrt{D_3}} \right| \left| \sqrt{D_3} \partial_t u \right| dx \\ &\leq \frac{C_0}{D_3} \|\partial_x^4 u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + D_3 \|\partial_t u(t, \cdot)\|_{L^2(\mathbb{R})}^2, \end{aligned}$$

where D_3 is a positive constant, which will be specified later. As a consequence, (57) becomes

$$\begin{aligned} & \frac{d}{dt} \left(a^2 \|\partial_x u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \beta^2 \|\partial_x^3 u(t, \cdot)\|_{L^2(\mathbb{R})}^2 \right) \\ & \quad + b^2 \frac{d}{dt} \|\partial_x^2 u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2(1 - 2D_3) \|\partial_t u(t, \cdot)\|_{L^2(\mathbb{R})}^2 \end{aligned}$$

$$\leq \frac{C(T)}{D_3} + \frac{C_0}{D_3} \left\| \partial_x^4 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2,$$

and taking $D_3 = \frac{1}{2}$,

$$\begin{aligned} \frac{d}{dt} \left(a^2 \left\| \partial_x u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \beta^2 \left\| \partial_x^3 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \right) + b^2 \frac{d}{dt} \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \left\| \partial_t u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \\ \leq C(T) + C_0 \left\| \partial_x^4 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2. \end{aligned}$$

Integrating on $(0, t)$, by (6) and (42), we have that

$$\begin{aligned} a^2 \left\| \partial_x u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \beta^2 \left\| \partial_x^3 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + b^2 \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \int_0^t \left\| \partial_t u(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 ds \\ \leq C_0 + C(T)t + C_0 \int_0^t \left\| \partial_x^4 u(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 ds \leq C(T), \end{aligned}$$

which gives (55). \square

We are finally ready to prove Theorem 1, assuming (i) or (ii).

Proof of Theorem 1 assuming (i) or (ii). The well-posedness of (1) is guaranteed for a short time by the Cauchy-Kowaleskaya Theorem [64]. Thanks to the a priori estimates proved in Lemmas 1–7, we have that the global-in-time existence of a is the solution of (1) that satisfies (13).

The stability estimates (15) can be proved using the same arguments of [18] (Theorem 1). \square

4. Proof of Theorem 1 Assuming (iii)

In this section, we prove Theorem 1 assuming (iii). Due to (4), here, (1) becomes

$$\begin{cases} \partial_t u + \kappa \partial_x u^2 + \gamma \partial_x^2 u - \beta^2 \partial_x^6 u + \alpha \partial_x^4 u = 0, & t > 0, \quad x \in \mathbb{R}, \\ u(0, x) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{58}$$

The argument of this section is analogous to that of the previous one. We deduce the local-in-time well-posedness from the Cauchy-Kowaleskaya Theorem [64], and we improve the local-in-time existence to the global-in-time one, proving some suitable a priori estimates on u .

We begin with an energy estimate in the space $L_{loc}^\infty(0, \infty; L^2(\mathbb{R})) \cap L_{loc}^2(0, \infty; H^3(\mathbb{R}))$.

Lemma 8. Fix $T > 0$. There exists a constant $C(T) > 0$, such that

$$\left\| u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \beta^2 e^{C_0 t} \int_0^t e^{-C_s} \left\| \partial_x^3 u(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 ds \leq C(T), \tag{59}$$

for every $0 \leq t \leq T$. In particular, (29) holds. Moreover, we have that

$$\int_0^t \left\| \partial_x u(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 ds \leq C(T), \tag{60}$$

for every $0 \leq t \leq T$.

Proof. Let $0 \leq t \leq T$. Multiplying (58) by $2u$, an integration on \mathbb{R} gives

$$\begin{aligned} \frac{d}{dt} \left\| u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 &= 2 \int_{\mathbb{R}} u \partial_t u dx \\ &= 4\kappa \int_{\mathbb{R}} u^2 \partial_x u dx - 2\gamma \int_{\mathbb{R}} u \partial_x^2 u dx + 2\beta^2 \int_{\mathbb{R}} u \partial_x^6 u dx - 2\alpha \int_{\mathbb{R}} u \partial_x^4 u dx \end{aligned}$$

$$\begin{aligned}
 &= -2\gamma \int_{\mathbb{R}} u \partial_x^2 u dx - 2\beta^2 \int_{\mathbb{R}} \partial_x u \partial_x^5 u dx + 2\alpha \int_{\mathbb{R}} \partial_x u \partial_x^3 u dx \\
 &= -2\gamma \int_{\mathbb{R}} u \partial_x^2 u dx + 2\beta^2 \int_{\mathbb{R}} \partial_x^2 u \partial_x^4 u dx - 2\alpha \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \\
 &= -2\gamma \int_{\mathbb{R}} u \partial_x^2 u dx - 2\beta^2 \left\| \partial_x^3 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 - 2\alpha \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2,
 \end{aligned}$$

that is,

$$\frac{d}{dt} \|u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2\beta^2 \left\| \partial_x^3 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 = -2\gamma \int_{\mathbb{R}} u \partial_x^2 u dx - 2\alpha \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2. \tag{61}$$

Since, using the Young inequality,

$$2|\gamma| \int_{\mathbb{R}} |u| |\partial_x^2 u| dx \leq \gamma^2 \|u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2,$$

we can pass from (61) to

$$\frac{d}{dt} \|u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2\beta^2 \left\| \partial_x^3 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \leq C_0 \|u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + C_0 \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2. \tag{62}$$

Observe that

$$C_0 \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 = C_0 \int_{\mathbb{R}} \partial_x^2 u \partial_x^2 u dx = -C_0 \int_{\mathbb{R}} \partial_x u \partial_x^3 u dx.$$

Therefore, by the Young inequality,

$$\begin{aligned}
 C_0 \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 &\leq 2 \int_{\mathbb{R}} \left| \frac{C_0 \partial_x u}{2\sqrt{D_4}} \right| \left| \sqrt{D_4} \partial_x^3 u \right| dx \\
 &\leq \frac{C_0}{D_4} \|\partial_x u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + D_4 \left\| \partial_x^3 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2,
 \end{aligned} \tag{63}$$

where D_4 is a positive constant, which will be specified later. Observe again that

$$\frac{C_0}{D_4} \|\partial_x u(t, \cdot)\|_{L^2(\mathbb{R})}^2 = \frac{C_0}{D_4} \int_{\mathbb{R}} \partial_x u \partial_x u dx = -\frac{C_0}{D_4} \int_{\mathbb{R}} u \partial_x^2 u dx.$$

Consequently, by the Young inequality,

$$\begin{aligned}
 \frac{C_0}{D_4} \|\partial_x u(t, \cdot)\|_{L^2(\mathbb{R})}^2 &\leq 2 \int_{\mathbb{R}} \left| \frac{C_0 D_4 u}{2\sqrt{D_5}} \right| \left| \sqrt{D_5} \partial_x^2 u \right| dx \\
 &\leq \frac{C_0 D_4^2}{D_5} \|u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + D_5 \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2,
 \end{aligned} \tag{64}$$

where D_5 is a positive constant, which will be specified later. It follows from (63) and (64) that

$$(C_0 - D_5) \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \leq \frac{C_0 D_4^2}{D_5} \|u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + D_4 \left\| \partial_x^3 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2.$$

Choosing

$$D_5 = \frac{C_0}{2}, \tag{65}$$

we have that

$$\frac{C_0}{2} \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \leq 2D_4^2 \|u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + D_4 \left\| \partial_x^3 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2,$$

that is

$$C_0 \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \leq 4D_4^2 \|u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2D_4 \left\| \partial_x^3 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \tag{66}$$

It follows from (62) and (66) that

$$\frac{d}{dt} \|u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2(\beta^2 - D_4) \left\| \partial_x^3 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \leq (C_0 + 4D_4^2) \|u(t, \cdot)\|_{L^2(\mathbb{R})}^2.$$

Choosing

$$D_4 = \frac{\beta^2}{2}, \tag{67}$$

we have that

$$\frac{d}{dt} \|u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \beta^2 \left\| \partial_x^3 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \leq C_0 \|u(t, \cdot)\|_{L^2(\mathbb{R})}^2.$$

By the the Gronwall Lemma and (6), we get

$$\|u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \beta^2 e^{C_0 t} \int_0^t e^{-C_0 s} \left\| \partial_x^3 u(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 ds \leq C_0 e^{C_0 t} \leq C(T),$$

which gives (59).

We prove (29). Thanks to (59), (66) and (67),

$$C_0 \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \leq C(T) + \frac{\beta^2}{2} \left\| \partial_x^3 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2.$$

Integrating on $(0, t)$, by (59), we have that

$$C_0 \int_0^t \left\| \partial_x^2 u(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 ds \leq C(T)t + \frac{\beta^2}{2} \int_0^t \left\| \partial_x^3 u(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 ds \leq C(T),$$

which gives (29).

Finally, we prove (60). Thanks to (59), (64) and (67),

$$C_0 \left\| \partial_x u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \leq C(T) + C_0 \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2.$$

Integrating on $(0, t)$, by (29), we have that

$$C_0 \int_0^t \left\| \partial_x u(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 ds \leq C(T)t + C_0 \int_0^t \left\| \partial_x^2 u(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 ds \leq C(T),$$

which gives (60). \square

We continue with an energy estimate in the space $L^\infty_{loc}(0, \infty; H^1(\mathbb{R})) \cap L^2_{loc}(0, \infty; H^4(\mathbb{R}))$.

Lemma 9. Fix $T > 0$. There exists a constant $C(T) > 0$, such that

$$\|u\|_{L^\infty((0,T) \times \mathbb{R})} \leq C(T), \tag{68}$$

$$\left\| \partial_x u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 2\beta^2 \int_0^t \left\| \partial_x^4 u(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 ds \leq C(T), \tag{69}$$

for every $0 \leq t \leq T$.

Proof. Let $0 \leq t \leq T$. Multiplying (58) by $-2\partial_x^2 u$, an integration on \mathbb{R} gives

$$\begin{aligned} \frac{d}{dt} \left\| \partial_x u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 &= -2 \int_{\mathbb{R}} \partial_x^2 u \partial_t u dx \\ &= 4\kappa \int_{\mathbb{R}} u \partial_x u \partial_x^2 u dx + 2\gamma \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 - 2\beta^2 \int_{\mathbb{R}} \partial_x^6 u \partial_x^2 u dx + 2\alpha \int_{\mathbb{R}} \partial_x^4 u \partial_x^2 u dx \end{aligned}$$

$$\begin{aligned}
 &= 4\kappa \int_{\mathbb{R}} u \partial_x u \partial_x^2 u dx + 2\gamma \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 2\beta^2 \int_{\mathbb{R}} \partial_x^5 u \partial_x^3 u dx - 2\alpha \left\| \partial_x^3 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \\
 &= 4\kappa \int_{\mathbb{R}} u \partial_x u \partial_x^2 u dx + 2\gamma \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 - 2\beta^2 \left\| \partial_x^4 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 - 2\alpha \left\| \partial_x^3 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2.
 \end{aligned}$$

Consequently, we have that

$$\begin{aligned}
 &\frac{d}{dt} \left\| \partial_x u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 2\beta^2 \left\| \partial_x^4 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \\
 &= 4\kappa \int_{\mathbb{R}} u \partial_x u \partial_x^2 u dx + 2\gamma \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 - 2\alpha \left\| \partial_x^3 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2.
 \end{aligned} \tag{70}$$

Due to the Young inequality,

$$\begin{aligned}
 4\kappa \int_{\mathbb{R}} |u \partial_x u| |\partial_x^2 u| dx &\leq 2\kappa^2 \int_{\mathbb{R}} u^2 (\partial_x u)^2 dx + 2 \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \\
 &\leq 2\kappa^2 \|u\|_{L^\infty((0,T) \times \mathbb{R})}^2 \left\| \partial_x u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 2 \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2.
 \end{aligned}$$

Therefore, by (70),

$$\begin{aligned}
 &\frac{d}{dt} \left\| \partial_x u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 2\beta^2 \left\| \partial_x^4 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \\
 &\leq 2\kappa^2 \|u\|_{L^\infty((0,T) \times \mathbb{R})}^2 \left\| \partial_x u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 2(1 + |\gamma|) \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 2|\alpha| \left\| \partial_x^3 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \\
 &\leq C_0 \|u\|_{L^\infty((0,T) \times \mathbb{R})}^2 \left\| \partial_x u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + C_0 \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + C_0 \left\| \partial_x^3 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2.
 \end{aligned}$$

(6), (29), (59), (60) and an integration on (0, t) give

$$\begin{aligned}
 &\left\| \partial_x u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 2\beta^2 \int_0^t \left\| \partial_x^4 u(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 ds \\
 &\leq C_0 + C_0 \|u\|_{L^\infty((0,T) \times \mathbb{R})}^2 \int_0^t \left\| \partial_x u(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 ds \\
 &\quad + C_0 \int_0^t \left\| \partial_x^2 u(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 ds + C_0 \int_0^t \left\| \partial_x^3 u(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 ds \\
 &\leq C(T) \left(1 + \|u\|_{L^\infty((0,T) \times \mathbb{R})}^2 \right).
 \end{aligned} \tag{71}$$

We prove (68). Thanks to (59), (69), and the Hölder inequality,

$$\begin{aligned}
 u^2(t, x) &= 2 \int_{-\infty}^x u \partial_x u dy \leq 2 \int_{\mathbb{R}} |u| |\partial_x u| dx \\
 &\leq \|u(t, \cdot)\|_{L^2(\mathbb{R})} \left\| \partial_x u(t, \cdot) \right\|_{L^2(\mathbb{R})} \leq C(T) \sqrt{\left(1 + \|u\|_{L^\infty((0,T) \times \mathbb{R})}^2 \right)}.
 \end{aligned}$$

Hence,

$$\|u\|_{L^\infty((0,T) \times \mathbb{R})}^4 - C(T) \|u\|_{L^\infty((0,T) \times \mathbb{R})}^2 - C(T) \leq 0,$$

which gives (68).

Finally, (69) follows from (68) and (71). □

We continue with an energy estimate in the space $L_{loc}^\infty(0, \infty; H^2(\mathbb{R})) \cap L_{loc}^2(0, \infty; H^5(\mathbb{R}))$.

Lemma 10. Fix $T > 0$. There exists a constant $C(T) > 0$, such that

$$\left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \beta^2 \int_{\mathbb{R}} \left\| \partial_x^5 u(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 ds \leq C(T), \tag{72}$$

for every $0 \leq t \leq T$. In particular, (44) holds.

Proof. Let $0 \leq t \leq T$. Multiplying (58) by $2\partial_x^4 u$, an integration on \mathbb{R} gives

$$\begin{aligned} \frac{d}{dt} \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 &= 2 \int_{\mathbb{R}} \partial_x^4 u \partial_t u dx \\ &= -4\kappa \int_{\mathbb{R}} u \partial_x u \partial_x^4 u dx - 2\gamma \int_{\mathbb{R}} \partial_x^2 u \partial_x^4 u dx + 2\beta^2 \int_{\mathbb{R}} \partial_x^6 u \partial_x^4 u dx - 2\alpha \left\| \partial_x^4 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \\ &= -4\kappa \int_{\mathbb{R}} u \partial_x u \partial_x^4 u dx + 2\gamma \left\| \partial_x^3 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 - 2\beta^2 \left\| \partial_x^5 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 - 2\alpha \left\| \partial_x^4 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2. \end{aligned}$$

Therefore, we have that

$$\begin{aligned} \frac{d}{dt} \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 2\beta^2 \left\| \partial_x^5 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \\ = -4\kappa \int_{\mathbb{R}} u \partial_x u \partial_x^4 u dx + 2\gamma \left\| \partial_x^3 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 - 2\alpha \left\| \partial_x^4 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2. \end{aligned} \tag{73}$$

Due to (68), (69) and the Young inequality,

$$\begin{aligned} 4|\kappa| \int_{\mathbb{R}} |u \partial_x u| |\partial_x^4 u| dx &\leq 4|\kappa| \|u\|_{L^\infty((0,T) \times \mathbb{R})} \int_{\mathbb{R}} |\partial_x u| |\partial_x^4 u| dx \\ &\leq C(T) \int_{\mathbb{R}} |\partial_x u| |\partial_x^4 u| dx \leq C(T) \|\partial_x u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + C(T) \left\| \partial_x^4 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \\ &\leq C(T) + C(T) \left\| \partial_x^4 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2. \end{aligned}$$

It follows from (73) that

$$\begin{aligned} \frac{d}{dt} \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 2\beta^2 \left\| \partial_x^5 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \\ \leq C(T) + C(T) \left\| \partial_x^4 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 2|\gamma| \left\| \partial_x^3 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 2|\alpha| \left\| \partial_x^4 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \\ \leq C(T) + C(T) \left\| \partial_x^4 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + C(T) \left\| \partial_x^3 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2. \end{aligned}$$

Integrating on $(0, t)$, by (6), (59) and (69), we have

$$\begin{aligned} \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 2\beta^2 \int_0^t \left\| \partial_x^5 u(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 ds \\ \leq C_0 + C(T)t + C(T) \int_0^t \left\| \partial_x^4 u(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 ds + C(T) \int_0^t \left\| \partial_x^3 u(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 ds \\ \leq C(T), \end{aligned}$$

which gives (72).

Finally, by (69), (72) and the Hölder inequality, we have (44). \square

We continue with an energy estimate in the space $L_{loc}^\infty(0, \infty; H^3(\mathbb{R})) \cap L_{loc}^2(0, \infty; H^6(\mathbb{R}))$.

Lemma 11. Fix $T > 0$. There exists a constant $C(T) > 0$, such that

$$\left\| \partial_x^3 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \beta^2 \int_0^t \left\| \partial_x^6 u(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 ds \leq C(T), \tag{74}$$

for every $0 \leq t \leq T$. In particular, (52) holds.

Proof. Let $0 \leq t \leq T$. Multiplying (58) by $-2\partial_x^6 u$, an integration on \mathbb{R} gives

$$\begin{aligned} \frac{d}{dt} \left\| \partial_x^3 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 &= -2 \int_{\mathbb{R}} \partial_x^6 u \partial_t u dx \\ &= 4\kappa \int_{\mathbb{R}} u \partial_x u \partial_x^6 u dx + 2\gamma \int_{\mathbb{R}} \partial_x^6 u \partial_x^2 u dx - 2\beta^2 \left\| \partial_x^6 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 2\alpha \int_{\mathbb{R}} \partial_x^6 u \partial_x^4 u dx \\ &= 4\kappa \int_{\mathbb{R}} u \partial_x u \partial_x^6 u dx - 2\gamma \int_{\mathbb{R}} \partial_x^5 u \partial_x^3 u dx - 2\beta^2 \left\| \partial_x^6 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 - 2\alpha \left\| \partial_x^5 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \\ &= 4\kappa \int_{\mathbb{R}} u \partial_x u \partial_x^6 u dx + 2\gamma \left\| \partial_x^4 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 - 2\beta^2 \left\| \partial_x^6 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 - 2\alpha \left\| \partial_x^5 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2. \end{aligned}$$

Therefore, we have that

$$\begin{aligned} \frac{d}{dt} \left\| \partial_x^3 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 2\beta^2 \left\| \partial_x^6 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \\ = 4\kappa \int_{\mathbb{R}} u \partial_x u \partial_x^6 u dx + 2\gamma \left\| \partial_x^4 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 - 2\alpha \left\| \partial_x^5 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2. \end{aligned} \tag{75}$$

Due to (68), (69) and the Young inequality,

$$\begin{aligned} 4|\kappa| \int_{\mathbb{R}} |u| |\partial_x u| |\partial_x^6 u| dx &\leq 2|\kappa| \|u\|_{L^\infty((0,T) \times \mathbb{R})} \int_{\mathbb{R}} |\partial_x u| |\partial_x^6 u| dx \\ &\leq 2C(T) \int_{\mathbb{R}} |\partial_x u| |\partial_x^6 u| dx = 2 \int_{\mathbb{R}} \left| \frac{C(T) \partial_x u}{\beta} \right| |\beta \partial_x^6 u| dx \\ &\leq C(T) \left\| \partial_x u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \beta^2 \left\| \partial_x^6 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \\ &\leq C(T) + \beta^2 \left\| \partial_x^6 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2. \end{aligned}$$

It follows from (75) that

$$\begin{aligned} \frac{d}{dt} \left\| \partial_x^3 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \beta^2 \left\| \partial_x^6 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \\ \leq C(T) + 2|\gamma| \left\| \partial_x^4 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 2|\alpha| \left\| \partial_x^5 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \\ \leq C(T) + C(T) \left\| \partial_x^4 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + C(T) \left\| \partial_x^5 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2. \end{aligned}$$

By (6), (69), (72), and an integration on $(0, t)$,

$$\begin{aligned} \left\| \partial_x^3 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \beta^2 \int_0^t \left\| \partial_x^6 u(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 ds \\ \leq C_0 + C(T)t + C(T) \int_0^t \left\| \partial_x^4 u(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 ds + C(T) \int_0^t \left\| \partial_x^5 u(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 ds \\ \leq C(T), \end{aligned}$$

which gives (74).

Finally, arguing as in Lemma 6, we have (52). \square

We continue with an energy estimate in the space $L_{loc}^\infty(0, \infty; H^3(\mathbb{R})) \cap H_{loc}^1((0, \infty) \times \mathbb{R})$.

Lemma 12. Fix $T > 0$. There exists a constant $C(T) > 0$, such that

$$\beta^2 \left\| \partial_x^3 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \int_0^t \left\| \partial_t u(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 ds \leq C(T). \tag{76}$$

for every $0 \leq t \leq T$.

Proof. Let $0 \leq t \leq T$. Multiplying (58) by $2\partial_t u$, an integration on \mathbb{R} gives

$$\begin{aligned} & \frac{d}{dt} \left(\beta^2 \left\| \partial_x^3 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \alpha \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 - \gamma \left\| \partial_x u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \right) \\ &= -2\beta^2 \int_{\mathbb{R}} \partial_t u \partial_x^6 u dx + 2\alpha \int_{\mathbb{R}} \partial_t u \partial_x^4 u dx + 2\gamma \int_{\mathbb{R}} \partial_t \partial_x u \partial_x^2 u dx \\ &= -2 \left\| \partial_t u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 - 4\kappa \int_{\mathbb{R}} u \partial_x u \partial_t u dx. \end{aligned}$$

Therefore, we have that

$$\begin{aligned} & \frac{d}{dt} \left(\beta^2 \left\| \partial_x^3 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \alpha \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 - \gamma \left\| \partial_x u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \right) \\ &+ 2 \left\| \partial_t u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 = -4\kappa \int_{\mathbb{R}} u \partial_x u \partial_t u dx. \end{aligned} \tag{77}$$

Due to (68), (69), and the Young inequality,

$$\begin{aligned} 4|\kappa| \int_{\mathbb{R}} |u| |\partial_x u| |\partial_t u| dx &\leq 4|\kappa| \|u\|_{L^\infty((0,T) \times \mathbb{R})} \int_{\mathbb{R}} |\partial_x u| |\partial_t u| dx \\ &\leq 2C(T) \int_{\mathbb{R}} |\partial_x u| |\partial_t u| dx \leq C(T) \left\| \partial_x u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \left\| \partial_t u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \\ &\leq C(T) + \left\| \partial_t u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2. \end{aligned}$$

Consequently, by (77),

$$\begin{aligned} & \frac{d}{dt} \left(\beta^2 \left\| \partial_x^3 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \alpha \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 - \gamma \left\| \partial_x u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \right) \\ &+ \left\| \partial_t u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \leq C(T). \end{aligned}$$

(6) and an integration on $(0, t)$ give

$$\begin{aligned} & \beta^2 \left\| \partial_x^3 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \alpha \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 - \gamma \left\| \partial_x u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \\ &+ \int_0^t \left\| \partial_t u(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 ds \leq C_0 + C(T)t \leq C(T). \end{aligned}$$

Therefore, by (69), (72), we have that

$$\begin{aligned} & \beta^2 \left\| \partial_x^3 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \int_0^t \left\| \partial_t u(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 ds \\ &\leq C(T) + |\alpha| \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + |\gamma| \left\| \partial_x u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \leq C(T), \end{aligned}$$

which gives (76). \square

We are finally ready to prove Theorem 1 assuming (iii).

Proof of Theorem 1 assuming (iii). The well-posedness of (1) is guaranteed for a short time by the Cauchy-Kowaleskaya Theorem [64]. Thanks to the a priori estimates proved in Lemmas 8–12, we have that the global-in-time existence of a is solution of (1) that satisfies (13).

The stability estimates (15) can be proved using the same arguments of [18] (Theorem 1). \square

5. Proof of Theorem 1 Assuming (iv)

In this final section, we prove Theorem 1 assuming (iv).

The argument is again analogous to the one of the previous sections. We deduce the local-in-time well-posedness from the Cauchy-Kowaleskaya Theorem [64], and we improve the local-in-time existence to the global-in-time one proving some suitable a priori estimates on u .

We begin with the zero mean estimate.

Lemma 13. For each $t > 0$, we have (14).

Proof. Integrating (1) on \mathbb{R} , we have that

$$\int_{\mathbb{R}} \partial_t u(t, x) dx = \frac{d}{dt} \int_{\mathbb{R}} u(t, x) dx = 0 \tag{78}$$

(14) follows from (6) and (78). \square

Remark 1. In light of (14), we can consider the following equation:

$$P(t, x) = \int_{-\infty}^x u(t, y) dy. \tag{79}$$

Moreover, again by (14), we have that

$$P(t, -\infty) = P(t, \infty) = 0. \tag{80}$$

We continue by proving some energy estimates on the function P .

Lemma 14. Let $T > 0$. There exists a constant $C(T) > 0$, such that

$$\begin{aligned} & \|P(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2\beta^2 e^{2t} \int_0^t e^{-2s} \left\| \partial_x^2 u(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 ds \\ & + 2\delta^2 e^t \int_0^{2t} e^{-2s} \|u(s, \cdot) \partial_x u(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds \\ & \leq C(T) + C(T) \int_0^t \|u(s, \cdot)\|_{L^4(\mathbb{R})}^4 ds + C(T) \int_0^t \|\partial_x u(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds, \end{aligned} \tag{81}$$

for every $0 \leq t \leq T$.

Proof. Let $0 \leq t \leq T$. Integrating (1) on $(-\infty, x)$, we have that

$$\int_{-\infty}^x \partial_t u dx + \kappa u^2 + \gamma \partial_x u - \beta^2 \partial_x^5 u + \alpha \partial_x^3 u + \delta^2 \partial_x^3 (u^3) = 0. \tag{82}$$

Differentiating (79) with respect to t , we obtain that

$$\partial_t P(t, x) = \frac{d}{dt} \int_{-\infty}^x u(t, y) dy = \int_{-\infty}^x \partial_t u(t, y) dy. \tag{83}$$

It follows from (82) and (83) that

$$\partial_t P + \kappa u^2 + \gamma \partial_x u - \beta^2 \partial_x^5 u + \alpha \partial_x^3 u + \delta^2 \partial_x^3 (u^3) = 0. \tag{84}$$

Arguing as in [18] (Lemma 2), we have that

$$\begin{aligned}
 -2\beta^2 \int_{\mathbb{R}} P\partial_x^5 u dx &= 2\beta^2 \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2, \\
 2\alpha \int_{\mathbb{R}} P\partial_x^3 u dx &= 2\alpha \left\| \partial_x u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \\
 2\delta^2 \int_{\mathbb{R}} P\partial_x^3 (u^3) dx &= 6\delta^2 \left\| u(t, \cdot) \partial_x u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2.
 \end{aligned}
 \tag{85}$$

Therefore, multiplying (84) by $2P$, thanks to (85), an integration on \mathbb{R} gives

$$\begin{aligned}
 \frac{d}{dt} \left\| P(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 2\beta^2 \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 6\delta^2 \left\| u(t, \cdot) \partial_x u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \\
 = -2\kappa \int_{\mathbb{R}} Pu^2 dx - 2\gamma \int_{\mathbb{R}} P\partial_x u dx - 2\alpha \left\| \partial_x u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2.
 \end{aligned}
 \tag{86}$$

Due to the Young inequality,

$$\begin{aligned}
 2|\kappa| \int_{\mathbb{R}} |P|u^2 dx &\leq \left\| P(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \kappa^2 \left\| u(t, \cdot) \right\|_{L^4(\mathbb{R})}^4, \\
 2|\gamma| \int_{\mathbb{R}} |P|\partial_x u dx &\leq \left\| P(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \gamma^2 \left\| \partial_x u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2.
 \end{aligned}$$

It follows from (86) that

$$\begin{aligned}
 \frac{d}{dt} \left\| P(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 2\beta^2 \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 6\delta^2 \left\| u(t, \cdot) \partial_x u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \\
 \leq 2 \left\| P(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + C_0 \left\| u(t, \cdot) \right\|_{L^4(\mathbb{R})}^4 + C_0 \left\| \partial_x u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2.
 \end{aligned}$$

Therefore, by the Gronwall Lemma and (9), we have that

$$\begin{aligned}
 \left\| P(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 2\beta^2 e^{2t} \int_0^t e^{-2s} \left\| \partial_x^2 u(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 ds \\
 + 6\delta^2 e^{2t} \int_0^t e^{-2s} \left\| u(s, \cdot) \partial_x u(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 ds \\
 \leq C_0 + C_0 e^{2t} \int_0^t e^{-2s} \left\| u(s, \cdot) \right\|_{L^4(\mathbb{R})}^4 ds + C_0 e^{2t} \int_0^t e^{-2s} \left\| \partial_x u(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 ds \\
 \leq C(T) + C(T) \int_0^t \left\| u(s, \cdot) \right\|_{L^4(\mathbb{R})}^4 ds + C(T) \int_0^t \left\| \partial_x u(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 ds,
 \end{aligned}$$

which gives (81). \square

Lemma 15. *Let $T > 0$. There exists a constant $C(T) > 0$, such that*

$$\frac{\beta^2}{6} \left\| \partial_x u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \frac{\delta^2}{4} \left\| u(t, \cdot) \right\|_{L^4(\mathbb{R})}^2 \leq C(T),
 \tag{87}$$

for every $0 \leq t \leq T$. In particular, we have (29), (31),

$$\begin{aligned}
 \left\| P(t, \cdot) \right\|_{L^2(\mathbb{R})} &\leq C(T), \\
 \left\| u(t, \cdot) \right\|_{L^2(\mathbb{R})} &\leq C(T), \\
 \int_0^t \left\| u(s, \cdot) \partial_x u(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 ds &\leq C(T), \\
 \int_0^t \int_{\mathbb{R}} \left[\beta^2 (\partial_x^4 u)^2 - \delta^2 \partial_x^3 (u^3) \right]^2 ds dx &\leq C(T),
 \end{aligned}$$

for every $0 \leq t \leq T$. Moreover,

$$\begin{aligned} \|P\|_{L^\infty((0,T) \times \mathbb{R})} &\leq C(T) \\ \|u\|_{L^\infty((0,T) \times \mathbb{R})} &\leq C(T). \end{aligned}$$

Proof. Let $0 \leq t \leq T$. Consider an real constant A , which will be specified later. Observe that

$$\begin{aligned} \delta^2 \gamma \int_{\mathbb{R}} u^3 \partial_x^2 u dx &= -3\delta^2 \gamma \|u(t, \cdot) \partial_x u(t, \cdot)\|_{L^2(\mathbb{R})}^2, \\ 2A\gamma \int_{\mathbb{R}} u \partial_x^2 u dx &= -2A \|\partial_x u(t, \cdot)\|_{L^2(\mathbb{R})}^2. \end{aligned} \tag{88}$$

Multiplying (1) by

$$-\beta^2 \partial_x^2 u + \delta^2 u^3 + Au,$$

thanks to (88) and arguing as in [18] (Lemma 3), an integration on \mathbb{R} gives

$$\begin{aligned} \frac{d}{dt} &\left(\frac{\beta^2}{2} \|\partial_x u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \frac{\delta^2}{4} \|u(t, \cdot)\|_{L^4(\mathbb{R})}^4 + \frac{A}{2} \|u(t, \cdot)\|_{L^2(\mathbb{R})}^2 \right) \\ &+ \int_{\mathbb{R}} \left[\beta^2 (\partial_x^4 u)^2 - \delta^2 \partial_x^2 (u^3) \right]^2 dx \\ &= -\beta^2 (A + \alpha) \|\partial_x^3 u(t, \cdot)\|_{L^2(\mathbb{R})}^2 - \delta^2 (A + \alpha) \int_{\mathbb{R}} \partial_x^2 u \partial_x^2 (u^3) dx \\ &- (A\alpha - \gamma\beta^2) \|\partial_x^2 u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 3\delta^2 \gamma \|u(t, \cdot) \partial_x u(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\ &+ 2\beta^2 \kappa \int_{\mathbb{R}} u \partial_x u \partial_x^2 u dx. \end{aligned}$$

Taking

$$A = -\alpha,$$

we have that

$$\begin{aligned} \frac{d}{dt} &\left(\frac{\beta^2}{2} \|\partial_x u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \frac{\delta^2}{4} \|u(t, \cdot)\|_{L^4(\mathbb{R})}^4 - \frac{\alpha}{2} \|u(t, \cdot)\|_{L^2(\mathbb{R})}^2 \right) \\ &+ \int_{\mathbb{R}} \left[\beta^2 (\partial_x^4 u)^2 - \delta^2 \partial_x^2 (u^3) \right]^2 dx \\ &= (\alpha^2 + \gamma\beta^2) \|\partial_x^2 u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 3\delta^2 \gamma \|u(t, \cdot) \partial_x u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2\beta^2 \kappa \int_{\mathbb{R}} u \partial_x u \partial_x^2 u dx. \end{aligned} \tag{89}$$

Due to the Young inequality,

$$2\beta^2 |\kappa| \int_{\mathbb{R}} |u \partial_x u| |\partial_x^2 u| dx \leq \beta^2 \kappa^2 \|u(t, \cdot) \partial_x u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \beta^2 \|\partial_x^2 u(t, \cdot)\|_{L^2(\mathbb{R})}^2.$$

It follows from (89) that

$$\begin{aligned} \frac{d}{dt} &\left(\frac{\beta^2}{2} \|\partial_x u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \frac{\delta^2}{4} \|u(t, \cdot)\|_{L^4(\mathbb{R})}^4 - \frac{\alpha}{2} \|u(t, \cdot)\|_{L^2(\mathbb{R})}^2 \right) + \int_{\mathbb{R}} \left[\beta^2 (\partial_x^4 u)^2 - \delta^2 \partial_x^2 (u^3) \right]^2 dx \\ &\leq C_0 \|\partial_x^2 u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + C_0 \|u(t, \cdot) \partial_x u(t, \cdot)\|_{L^2(\mathbb{R})}^2. \end{aligned}$$

Integrating on $(0, t)$, by (7), we have that

$$\frac{\beta^2}{2} \|\partial_x u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \frac{\delta^2}{4} \|u(t, \cdot)\|_{L^4(\mathbb{R})}^4 - \frac{\alpha}{2} \|u(t, \cdot)\|_{L^2(\mathbb{R})}^2$$

$$\begin{aligned}
 & + \int_0^t \int_{\mathbb{R}} \left[\beta^2 (\partial_x^4 u)^2 - \delta^2 \partial_x^2 (u^3) \right]^2 ds dx \\
 & \leq C_0 + C_0 \int_0^t \left\| \partial_x^2 u(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 ds + C_0 \int_0^t \|u(s, \cdot) \partial_x u(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds,
 \end{aligned}$$

that is

$$\begin{aligned}
 & \frac{\beta^2}{2} \|\partial_x u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \frac{\delta^2}{4} \|u(t, \cdot)\|_{L^4(\mathbb{R})}^4 + \int_0^t \int_{\mathbb{R}} \left[\beta^2 (\partial_x^4 u)^2 - \delta^2 \partial_x^2 (u^3) \right]^2 ds dx \tag{90} \\
 & \leq C_0 + C_0 \int_0^t \left\| \partial_x^2 u(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 ds + C_0 \int_0^t \|u(s, \cdot) \partial_x u(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds + \frac{\alpha}{2} \|u(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\
 & \leq C_0 + C_0 \int_0^t \left\| \partial_x^2 u(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 ds + C_0 \int_0^t \|u(s, \cdot) \partial_x u(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds + C_0 \|u(t, \cdot)\|_{L^2(\mathbb{R})}^2.
 \end{aligned}$$

Observe that, by (79) and (80),

$$C_0 \|u(t, \cdot)\|_{L^2(\mathbb{R})}^2 = C_0 \int_{\mathbb{R}} u u dx = -C_0 \int_{\mathbb{R}} P \partial_x u dx.$$

Therefore, by the Young inequality,

$$\begin{aligned}
 C_0 \|u(t, \cdot)\|_{L^2(\mathbb{R})}^2 & \leq 2 \int_{\mathbb{R}} \left| \frac{\sqrt{3} C_0 P}{2\beta} \right| \left| \frac{\beta \partial_x u}{\sqrt{3}} \right| dx \\
 & \leq C_0 \|P(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \frac{\beta^2}{3} \|\partial_x u(t, \cdot)\|_{L^2(\mathbb{R})}^2.
 \end{aligned} \tag{91}$$

It follows from (81), (90) and (91) that

$$\begin{aligned}
 & \frac{\beta^2}{6} \|\partial_x u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \frac{\delta^2}{4} \|u(t, \cdot)\|_{L^4(\mathbb{R})}^4 + \int_0^t \int_{\mathbb{R}} \left[\beta^2 (\partial_x^4 u)^2 - \delta^2 \partial_x^2 (u^3) \right]^2 ds dx \\
 & \leq C_0 + C_0 \int_0^t \left\| \partial_x^2 u(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 ds + C_0 \int_0^t \|u(s, \cdot) \partial_x u(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds + C_0 \|P(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\
 & \leq C(T) + C(T) \int_0^t \|u(s, \cdot)\|_{L^4(\mathbb{R})}^4 ds + C(T) \int_0^t \|\partial_x u(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds \\
 & \leq C(T) + C(T) \left(\frac{\beta^2}{6} \int_0^t \|\partial_x u(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds + \frac{\delta^2}{4} \int_0^t \|u(s, \cdot)\|_{L^4(\mathbb{R})}^4 ds \right).
 \end{aligned}$$

Therefore, we have that

$$\begin{aligned}
 & \frac{\beta^2}{6} \|\partial_x u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \frac{\delta^2}{4} \|u(t, \cdot)\|_{L^4(\mathbb{R})}^4 \\
 & \leq C(T) + C(T) \left(\frac{\beta^2}{6} \int_0^t \|\partial_x u(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds + \frac{\delta^2}{4} \int_0^t \|u(s, \cdot)\|_{L^4(\mathbb{R})}^4 ds \right).
 \end{aligned}$$

The Gronwall Lemma and (7) give (87).

Finally, arguing as in [18] (Lemma 3), the proof is concluded. \square

Arguing as in [18] (Theorem 1), we have Theorem 1.

6. Conclusions

This paper is dedicated to the well-posedness of a solution to the Cauchy problem for a higher-order convective Cahn-Hilliard equation. Such an equation models the evolution of crystal surfaces faceting through surface electromigration, the growing surface faceting, and the evolution of dynamics of phase transitions in ternary oil-water-surfactant systems. The well-posedness of (1) is

proved for a short time by the Cauchy-Kowaleskaya Theorem [64]. The global-in-time well-posedness is thus proved, proving several a priori estimates.

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