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This is a pre-print of the following article

Original Citation:

Characterizing Hermitian varieties in three- and four-dimensional projective spaces / Aguglia, Angela. - In: JOURNAL OF THE AUSTRALIAN MATHEMATICAL SOCIETY. - ISSN 1446-7887. - STAMPA. - 107:1(2019), pp. 1-8.
[10.1017/S1446788718000253]

Availability:

This version is available at <http://hdl.handle.net/11589/185710> since: 2022-06-20

Published version

DOI:10.1017/S1446788718000253

Publisher:

Terms of use:

(Article begins on next page)

CHARACTERIZING HERMITIAN VARIETIES IN 3- AND 4-DIMENSIONAL PROJECTIVE SPACES

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(27 July 2017)

Abstract

We characterize Hermitian cones among the surfaces of degree $q + 1$ of $\text{PG}(3, q^2)$ by their intersection numbers with planes. We then use this result and provide a characterization of non-singular Hermitian varieties of $\text{PG}(4, q^2)$, among quasi-Hermitian ones.

Keywords and phrases: Hermitian variety; quasi-Hermitian variety; algebraic variety.

Mathematics Subject Classification (2010): 05B25, 51E20, 94B05.

1. Introduction

An m -character set in the projective space $\text{PG}(n, q)$, q any prime power, is a set of points of $\text{PG}(n, q)$ with the property that the intersection number with any hyperplane only takes m values, where m is a positive integer.

A non-singular Hermitian variety $\mathcal{H}(n, q^2)$ of $\text{PG}(n, q^2)$ is a remarkable example of a two-character set, precisely a set of $(q^{n+1} + (-1)^n)(q^n - (-1)^n)/(q^2 - 1)$ points of $\text{PG}(n, q)$ with the property that a hyperplane Π meets it in either

$$(q^n + (-1)^{n-1})(q^{(n-1)} - (-1)^{(n-1)})/(q^2 - 1)$$

points, in case Π is a non-tangent hyperplane to $\mathcal{H}(n, q^2)$ or,

$$1 + q^2(q^{n-1} + (-1)^n)(q^{(n-2)} - (-1)^n)/(q^2 - 1)$$

points, in case Π is a tangent hyperplane to $\mathcal{H}(n, q^2)$; see [21].

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The author was supported by the Italian National Group for Algebraic and Geometric Structures and their Applications (GNSAGA - INdAM).

Quasi-Hermitian varieties are generalizations of non-singular Hermitian varieties such that they have the same size and the same intersection numbers with respect to hyperplanes.

Actually, a point set \mathcal{S} of $\text{PG}(n, q^2)$, $n > 2$, having the same intersection numbers with respect to hyperplanes as a non-singular Hermitian variety $\mathcal{H}(n, q^2)$ has also the same number of points of $\mathcal{H}(n, q^2)$; for $n = 2$ the size of \mathcal{S} can be either $q^3 + 1$ that is, the size of a Hermitian curve also called a classical unital or, $q^2 + q + 1$ which is the number of points of a Baer subplane of $\text{PG}(2, q^2)$; see [7].

As far as we know, the only quasi-Hermitian varieties of $\text{PG}(n, q^2)$, which are not isomorphic to Hermitian varieties were constructed in the following series of papers [1, 2, 5, 6, 18, 19].

The definition of quasi-Hermitian variety can be extended to that of a singular quasi-Hermitian variety, that is point sets which have the same number of points and the same intersection numbers with respect to hyperplanes as singular Hermitian varieties. Each cone over a quasi-Hermitian variety is a singular quasi-Hermitian variety thus, a natural question is also whether such a cone is isomorphic to a singular Hermitian variety.

Various characterizations of a non-singular Hermitian variety among the quasi-Hermitian ones in $\text{PG}(n, q^2)$, with $n \in \{2, 3\}$ have been given, but very few in higher dimensional cases; see [3, 7, 8, 17]. In [3] singular Hermitian varieties were also characterized among singular quasi-Hermitian ones.

Here we first consider point sets of $\text{PG}(3, q^2)$ such that their intersection numbers with respect to planes takes three values as well as the Hermitian cone with one singular point that is, $q^2 + 1$, $q^3 + 1$ or $q^3 + q^2 + 1$.

Combining geometric and combinatorial arguments with algebraic geometry we prove the following result.

THEOREM 1.1. *Let \mathcal{S} be a surface of $\text{PG}(3, q^2)$, of degree $q + 1$. If every plane meets \mathcal{S} in either $q^2 + 1$, $q^3 + 1$, or $q^3 + q^2 + 1$ points of $\text{PG}(3, q^2)$ then, \mathcal{S} is a cone projecting a Hermitian curve in a plane π from a point V not in π .*

Next, we also provide the following characterization of non-singular Hermitian varieties of $\text{PG}(4, q^2)$.

THEOREM 1.2. *Let \mathcal{S} be a quasi-Hermitian variety of $\text{PG}(4, q^2)$. If \mathcal{S} is a hypersurface of degree $q + 1$ then \mathcal{S} is a non-singular Hermitian variety.*

2. Preliminaries

Let $\Sigma = \text{PG}(n, q^2)$ be the Desarguesian projective space of dimension n over $\text{GF}(q^2)$ and denote by $X = (x_1, x_2, \dots, x_{n+1})$ homogeneous coordinates for its points.

We use σ to write the involutory automorphism of $\text{GF}(q^2)$ which leaves all the elements of the subfield $\text{GF}(q)$ invariant. A Hermitian variety $\mathcal{H}(n, q^2)$ is the set of all points X of Σ which are self conjugate under a Hermitian polarity h . If H is the Hermitian $(n+1) \times (n+1)$ -matrix associated with h , then the Hermitian variety $\mathcal{H}(n, q^2)$ has equation

$$XH(X^\sigma)^T = 0.$$

When H is non-singular, the corresponding Hermitian variety is non-singular, whereas if H has rank $r+1$, with $r < n$, the related Hermitian variety is singular and it is a cone $\Pi_{n-r-1}\mathcal{H}(r, q^2)$ with vertex an $n-r-1$ -space Π_{n-r-1} and basis a non-singular Hermitian variety $\mathcal{H}(r, q^2)$ of an r -space disjoint from Π_{n-r-1} .

A *d-singular quasi-Hermitian variety* is a subset of points of $\text{PG}(n, q^2)$ having the same number of points and the same intersection sizes with hyperplanes as a singular Hermitian variety with a singular space of dimension d .

Non-singular Hermitian varieties of $\text{PG}(n, q^2)$ are in particular hypersurfaces. We recall that a projective *hypersurface* \mathcal{S} of degree d is a set of points of $\text{PG}(n, q^2)$ whose homogenous coordinates satisfy

$$F(X_0, X_1, \dots, X_n) = 0, \tag{1}$$

where F is a form of degree d over $\text{GF}(q^2)$.

However, to understand the geometry of the hypersurface \mathcal{S} , the zeros of F over $\text{GF}(q^2)$ and over any extension of $\text{GF}(q^2)$ are required. Thus, \mathcal{S} is viewed as a hypersurface over the algebraic closure of $\text{GF}(q^2)$ and a point of $\text{PG}(n, q^2)$ in \mathcal{S} is called a $\text{GF}(q^2)$ -point or a *rational point* of \mathcal{S} ; in general a $\text{GF}(q^{2i})$ -point of \mathcal{S} is a point $P(a_0, \dots, a_n)$ in $\text{PG}(n, q^{2i})$ such that $F(a_0, \dots, a_n) = 0$. The number of $\text{GF}(q^{2i})$ -point of \mathcal{S} is denoted by $N_{q^{2i}}(\mathcal{S})$. When $n = 2$, a projective hypersurface \mathcal{S} is called a *projective plane curve*, whereas when $n = 3$, \mathcal{S} is called a *projective surface*.

The following results will be crucial to our proof.

LEMMA 2.1 ([20]). *Let d be an integer with $1 \leq d \leq q+1$ and \mathcal{C} be a curve of degree d in $\text{PG}(2, q)$ defined over $\text{GF}(q)$, which may have $\text{GF}(q)$ -linear components. Then the number $N_{q^2}(\mathcal{C})$ of rational points of \mathcal{C} is at most $dq+1$ and $N_q(\mathcal{C}) = dq+1$ if and only if \mathcal{C} is a pencil of d lines of $\text{PG}(2, q)$.*

LEMMA 2.2 ([13, 14, 15]). *Let d be an integer with $2 \leq d \leq q+2$, and \mathcal{C} be a curve of degree d in $\text{PG}(2, q)$ without $\text{GF}(q)$ -line components. Then the number of rational points of \mathcal{C} is at most $(d-1)q+1$ except for a class of plane curves of degree 4 over $\text{GF}(4)$ having 14 points.*

LEMMA 2.3 ([11]). Suppose $q \neq 2$. Let \mathcal{C} be a plane curve over $\text{GF}(q^2)$ of degree $q + 1$ without $\text{GF}(q^2)$ -line components. If \mathcal{C} has $q^3 + 1$ points over $\text{GF}(q^2)$, then \mathcal{C} is a Hermitian curve.

LEMMA 2.4 ([16]). Let \mathcal{S} be a surface in $\text{PG}(3, q^2)$ without $\text{GF}(q^2)$ -plane components. If the degree of \mathcal{S} is $q + 1$ and the number of its rational points is $(q^3 + 1)(q^2 + 1)$ then \mathcal{S} is a non-singular Hermitian surface.

Finally, a hyperplane of $\text{PG}(n, q^2)$ intersecting a point set \mathcal{S} of the projective space in i points will be called an i -hyperplane whereas, a line meeting \mathcal{S} in s points will be called an s -secant line if $s \geq 1$ or an *external line* to \mathcal{S} if $s = 0$.

LEMMA 2.5 ([8]). If each intersection number with planes and hyperplanes of a point set \mathcal{H} in $\text{PG}(4, q^2)$ is also an intersection number with planes and hyperplanes of $\mathcal{H}(4, q^2)$, then \mathcal{H} is a non-singular Hermitian variety $\mathcal{H}(4, q^2)$.

3. Hermitian cones of $\text{PG}(3, q^2)$

THEOREM 3.1. Let \mathcal{S} be a surface of $\text{PG}(3, q^2)$, of degree $q + 1$, q any prime power. If every plane meets \mathcal{S} in either $q^2 + 1$, $q^3 + 1$, or $q^3 + q^2 + 1$ points of $\text{PG}(3, q^2)$ then \mathcal{S} is a cone projecting a Hermitian curve in a plane π from a point V not in π .

PROOF. Let π be a $q^3 + q^2 + 1$ -plane. As \mathcal{S} is a surface of degree $q + 1$ then $C = \mathcal{S} \cap \pi$ is a plane curve of degree $q + 1$. By Lemma 2.2, C must have some $\text{GF}(q^2)$ -line component and thus by Lemma 2.1, C turns out to be a pencil of $q + 1$ lines of π . Furthermore, each line of π has to meet \mathcal{S} in 1, $q + 1$ or $q^2 + 1$ rational points and in particular, the surface \mathcal{S} contains lines of $\text{PG}(3, q^2)$.

Now, assume that the plane π is a $q^3 + 1$ -plane which does not have any $\text{GF}(q^2)$ -line of \mathcal{S} . In this case $C = \pi \cap \mathcal{S}$ is a plane curve of degree $q + 1$ without $\text{GF}(q^2)$ -line components and it has $q^3 + 1$ $\text{GF}(q^2)$ -points; thus, by Lemma 2.3, C is a non-singular Hermitian curve for $q \neq 2$.

We are going to prove that \mathcal{S} meets every line of $\text{PG}(3, q^2)$ that is, \mathcal{S} is a blocking set with respect to lines of the projective space. First, we assume $q \neq 2$ and consider a line r of $\text{PG}(3, q^2)$. If r is on a $q^3 + q^2 + 1$ -plane then r is at least a 1-secant line of \mathcal{S} . In the case in which r lies on a $q^3 + 1$ -plane, say π , either π contains some line of \mathcal{S} or $\pi \cap \mathcal{S}$ is a Hermitian unital of π ; in both cases r turns out to be at least 1-secant line of \mathcal{S} .

Thus, if r is an external line to \mathcal{S} , all planes through r have to be $q^2 + 1$ -planes and the number $N_{q^2}(\mathcal{S})$ of rational points of \mathcal{S} is $(q^2 + 1)^2$. Let t be a 1-secant line of \mathcal{S} lying in some $q^3 + q^2 + 1$ -plane and let t_i denote the

numbers of i -planes through t . Counting the number of $\text{GF}(q^2)$ -points of \mathcal{S} by using all planes through t we obtain

$$(q^2 + 1)^2 = t_{q^2+1}q^2 + t_{q^3+1}q^3 + t_{q^3+q^2+1}(q^3 + q^2) + 1$$

that gives

$$1 = (q - 1)t_{q^3+1} + qt_{q^3+q^2+1},$$

namely $t_{q^3+1} = 0$ and $t_{q^3+q^2+1} = 1/q$, a contradiction.

Now we assume $q = 2$. An algebraic plane curve of degree 3 in $\text{PG}(2, 4)$, with 9 rational points, without $\text{GF}(4)$ -line components is a unital or is projectively equivalent to the curve $\mathcal{C}' : X_0^3 + wX_1^2 + w^2X_2^3 = 0$, which meets each line in either 0, 2 or 3 rational points; see [10, §11]. Therefore, if r is an external line to \mathcal{S} then, r could be contained either in 5-planes or in 9-planes. Suppose that there is at least a planar section of \mathcal{S} which consists of 5 rational points on a line. In this case, a 9-plane never can intersect \mathcal{S} in an algebraic plane curve which is projectively equivalent to \mathcal{C}' therefore, only 5-planes can pass through an external line r of \mathcal{S} . Arguing as in the case $q \neq 2$, we get a contradiction.

Hence, each planar section of \mathcal{S} with 5 points has to be an absolutely irreducible cubic curve with a cusp or a non-singular cubic with one rational inflexion; see [10, §11]. Thus, a line of \mathcal{S} lies either on a 9-plane or on a 13-plane, whereas a 2-secant line lies either on a 5-plane or on a 9-plane. Let m be a 2-secant line of a 5-plane, that we know to exist and denote by x_m the number of 5-plane through m . Next, take a line ℓ of \mathcal{S} and denote by x_ℓ the number of 9-planes through ℓ . Counting the number of $\text{GF}(4)$ -points of \mathcal{S} by using all planes through ℓ and all planes through m we get

$$x_\ell(9 - 5) + (5 - x_\ell)(13 - 5) + 5 = x_m(5 - 2) + (5 - x_m)(9 - 2) + 2,$$

that gives $x_\ell = x_m + 2$. As $x_m \geq 1$, we obtain $x_\ell \in \{3, 4, 5\}$. Consequently, the number of rational points $N_4(\mathcal{S}) \in \{33, 29, 25\}$. In order to prove that none of the previous possibilities can occur for $N_4(\mathcal{S})$, we count in double way the number of planes, the number of pairs (P, π) , where $P \in \text{PG}(3, 4)$ and π is a plane through P , and the number of pairs $((P, Q), \pi)$, where $P, Q \in \text{PG}(3, 4)$ and π is a plane through P and Q . Let x, y, z denote the numbers of 5- 9- 13- planes respectively, we get the following equations

$$\begin{cases} x + y + z = 85 \\ 5x + 9y + 13z = 21N_4(\mathcal{S}) \\ 20x + 72y + 156z = 5N_4(\mathcal{S})(N_4(\mathcal{S}) - 1). \end{cases} \quad (2)$$

For $N_4(\mathcal{S}) = 25$ or $N_4(\mathcal{S}) = 29$, (2) provide $z = 0$ or $z = -1$ respectively, in both cases a contradiction. When $N_4(\mathcal{S}) = 33$, (2) give $z = 3$ that is,

there are 3 13-planes, each of which meets \mathcal{S} in 3 concurrent lines. On the other hand, exactly 2 13-planes have to pass through each line of \mathcal{S} and hence we get a contradiction. Thus, \mathcal{S} is a blocking set with respect to lines of $\text{PG}(3, q^2)$ for all prime power q .

We recall that a blocking set with respect to lines of $\text{PG}(2, q^2)$ which consists of $q^2 + 1$ points is a line; see [4]. Thus, if π is a $q^2 + 1$ -plane then $\pi \cap \mathcal{S}$ consists of $q^2 + 1$ points on a line.

Furthermore, each line which is not contained in \mathcal{S} meets \mathcal{S} in i points with $1 \leq i \leq q + 1$ as \mathcal{S} is a surface of degree $q + 1$ over $\text{GF}(q^2)$.

Next step is to prove that each line meets \mathcal{S} either in one, or $q + 1$ or $q^2 + 1$ $\text{GF}(q^2)$ -points. By way of contradiction, assume that there is an i -secant line to \mathcal{S} , say m , with $2 \leq i \leq q$. Then, each plane through m has to be a $q^3 + 1$ -plane containing some line of \mathcal{S} . Counting the number of $\text{GF}(q^2)$ -points of \mathcal{S} by using all planes through m we obtain $N_{q^2}(\mathcal{S}) = (q^2 + 1)(q^3 + 1 - i) + i = q^5 + q^3 + q^2 - iq^2 + 1$. Let us consider a $q^2 + 1$ -plane, say α and let ℓ be the line $\alpha \cap \mathcal{S}$. By considering that each plane through ℓ has at most $q^3 + q^2 + 1$ $\text{GF}(q^2)$ -points of \mathcal{S} we get that $N_{q^2}(\mathcal{S}) \leq q^2 q^3 + q^2 + 1 = q^5 + q^2 + 1$. Then, $i \geq q$ and hence $i = q$ which gives $N_{q^2}(\mathcal{S}) = q^5 + q^2 + 1$. In particular each line of \mathcal{S} is contained in at most one $q^2 + 1$ -plane. Now, let x_i denote the numbers of i -planes with respect to \mathcal{S} . In this case double counting arguments give

$$\begin{cases} \sum_i x_i = (q^4 + 1)(q^2 + 1) \\ \sum_i i x_i = (q^5 + q^2 + 1)(q^4 + q^2 + 1) \\ \sum_{i=1} i(i-1)x_i = (q^5 + q^2 + 1)(q^5 + q^2)(q^2 + 1). \end{cases} \quad (3)$$

By solving (3) we obtain in particular that the number x_{q^2+1} of $q^2 + 1$ -planes is $q^3 + 1$.

Denote by $\Sigma = \{\alpha_1, \dots, \alpha_{q^3+1}\}$ the set of all $q^2 + 1$ -planes to \mathcal{S} and set $\ell_i = \alpha_i \cap \mathcal{S}$ for all $\alpha_i \in \Sigma$. We observe that any two lines ℓ_i and ℓ_j , with $i \neq j$ intersect in a point and never three of these lines form a triangle. In fact, a triangle PQR of such lines would be contained in a $q^3 + 1$ -plane π ; since every line of π would meet \mathcal{S} in at least two points we would obtain in particular that every line of π through P would be at least a q -secant to \mathcal{S} and hence we would get $|\pi \cap \mathcal{S}| \geq (q^2 - 1)(q - 1) + 2q^2 + 1 > q^3 + 1$, a contradiction.

This means that the $q^3 + 1$ lines contained in \mathcal{S} are concurrent at a point V . Since \mathcal{S} has exactly $q^2(q^3 + 1) + 1$ rational points, each other line contained in \mathcal{S} cannot pass through V and has to meet $q^2 + 1$ lines among the lines ℓ_i , with $1 \leq i \leq q^3 + 1$. Thus, we find a $\text{GF}(q^2)$ -planar component of \mathcal{S} which is excluded. Hence, \mathcal{S} contains exactly $q^3 + 1$ lines and for each line ℓ contained in \mathcal{S} exactly one $q^2 + 1$ -plane through it exists whereas no plane

through ℓ are $q^3 + 1$ -plane. But then, there are no $q^3 + 1$ -planes containing some line of \mathcal{S} , a contradiction.

Thus, each line which is not contained in \mathcal{S} meets \mathcal{S} in either 1, or $q + 1$ rational points. For $q \neq 2$, from [12, Th. 23.5.1] \mathcal{S} has to be a cone $\Pi_0 \mathcal{S}'$ with \mathcal{S}' of type

- I. a unital;
- II. a subplane $\text{PG}(2, q)$;
- III. a set of type $(0, q)$ plus an external line;
- IV. the complement of a set of type $(0, q^2 - q)$.

As the possible intersection sizes with planes of $\text{PG}(3, q^2)$ are $q^2 + 1, q^3 + 1, q^3 + q^2 + 1$, possibilities II, III, and IV must be excluded, since their sizes cannot be possible. This implies that $\mathcal{S} = \Pi_0 \mathcal{S}'$, where \mathcal{S}' is a unital. On the other hand \mathcal{S}' turns out to be an algebraic curve of degree $q + 1$ without linear components and with $q^3 + 1$ points over $\text{GF}(q^2)$. Thus, for $q \neq 2$ Lemma 2.3 applies and \mathcal{S}' has to be a Hermitian curve.

For $q = 2$, there is just one point set in $\text{PG}(3, 4)$ up to equivalence, meeting each line in 1, 3 or 5 points and each plane in 5, 9 or 13 points, that is the Hermitian cone, see [9, Theorem 19.6.8]. Thus also for $q = 2$ our theorem follows. \square

As an easy consequence of Theorem 3.1 we get the following.

COROLLARY 3.2. *Let \mathcal{S} be a surface of $\text{PG}(3, q^2)$ of degree d . If every plane meets \mathcal{S} in either $q^2 + 1, q^3 + 1$ or $q^3 + q^2 + 1$ points over $\text{GF}(q^2)$ then, $d \geq q + 1$. If $d = q + 1$ then \mathcal{S} is a cone over a Hermitian curve.*

4. A characterization of $\mathcal{H}(4, q^2)$

THEOREM 4.1. *Let \mathcal{S} be a quasi-Hermitian variety of $\text{PG}(4, q^2)$. If \mathcal{S} is a hypersurface of degree $q + 1$ then \mathcal{S} is a non-singular Hermitian variety.*

PROOF. We recall that \mathcal{S} has $q^7 + q^5 + q^2 + 1$ rational points and its intersection numbers with respect to hyperplanes over $\text{GF}(q^2)$ are $q^5 + q^2 + 1$ or $q^5 + q^3 + q^2 + 1$.

First we prove that \mathcal{S} does not contain any plane of $\text{PG}(4, q^2)$. Suppose on the contrary that there is a plane α which is contained in \mathcal{S} . Let us denote by x the number of hyperplanes through α meeting \mathcal{S} in $q^5 + q^2 + 1$ $\text{GF}(q^2)$ -points.

Counting the number $N_{q^2}(\mathcal{S})$ of $\text{GF}(q^2)$ -points of \mathcal{S} by using all hyperplanes through α we obtain

$$q^7 + q^5 + q^2 + 1 = N_{q^2}(\mathcal{S}) = (q^2 + 1 - x)(q^5 + q^3 - q^4) + x(q^5 - q^4) + q^4 + q^2 + 1$$

that is,

$$xq^3 = -q^6 + q^5 + q^2,$$

a contradiction. This implies that $\Sigma = \mathcal{S} \cap \Pi$ is an algebraic surface of degree $q + 1$ without $\text{GF}(q^2)$ -plane components. In the case in which $N_{q^2}(\Sigma) = q^5 + q^3 + q^2 + 1$, by Lemma 2.4, Σ is a non-singular Hermitian surface.

Now let Π' be a hyperplane of $\text{PG}(4, q^2)$ meeting \mathcal{S} in $q^5 + q^2 + 1$ rational points and set $\Sigma' = \Pi' \cap \mathcal{S}$. We are going to study the planar sections of Σ' . Thus, let us denote by α a plane contained in Π' . If at least one $q^5 + q^3 + q^2 + 1$ -hyperplane passes through α then, $\alpha \cap \mathcal{S}$ is either a Hermitian curve or a pencil of $q + 1$ concurrent lines and hence, $N_{q^2}(\alpha \cap \Sigma') = q^3 + 1$ or $N_{q^2}(\alpha \cap \Sigma') = q^3 + q^2 + 1$.

Suppose that all hyperplanes containing α meet \mathcal{S} in $q^5 + q^2 + 1$ rational points and set $y = N_{q^2}(\alpha \cap \mathcal{S})$. Then we obtain,

$$q^7 + q^5 + q^2 + 1 = N_{q^2}(\mathcal{S}) = (q^2 + 1)(q^5 + q^2 + 1 - y) + y$$

namely $y = q^2 + 1$ and thus, $N_{q^2}(\alpha \cap \Sigma') = q^2 + 1$.

Theorem 3.1 applies and Σ' turns out to be a cone over a Hermitian curve. Then, each intersection number over $\text{GF}(q^2)$ with planes and hyperplanes of \mathcal{S} is also an intersection number with planes and hyperplanes of $\mathcal{H}(4, q^2)$. By Lemma 2.5, \mathcal{S} has to be a non-singular Hermitian variety of $\text{PG}(4, q^2)$. \square

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