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A New Class of Consensus Protocols for Agent Networks with Discrete Time Dynamics ⋆

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Abstract

The paper introduces a new class of consensus protocols to reach an agreement in networks of agents with discrete time dynamics. In order to guarantee the convergence of the proposed algorithms, some general results are proved in the framework of non-negative matrix theory. Moreover, we characterize the set of the consensus protocols and we specify the algorithm that each agent has to employ. Furthermore, we show that in the case of balanced graphs, the agents can apply the consensus protocols by a decentralized and scalable computation. The convergence properties are studied by a set of tests that show the good performance of the proposed algorithm for different network topologies, even in the cases in which the standard protocols do not exhibit satisfying performances. In particular, a rigorous theoretical analysis of the proposed protocol convergence for networks with ring topology is provided and compared with the standard algorithm.

Key words: Sensor networks, Multi-agent systems, Consensus algorithms, Convergence analysis.

1 Introduction

In recent years the study of the consensus problem has received a great effort by the scientific community involving several fields and many applications. More precisely, in networks of autonomous agents, consensus means to reach an agreement regarding a certain quantity of interest that depends on the state of all the agents. A consensus algorithm (or protocol) is an interaction rule that specifies the information exchange between an agent and all of its neighbors on the network.
In a pioneering contribution, Jadbabaie et al.[7] provided the theoretical framework for the problem of reaching an agreement on network systems with topology described by undirected graphs. Olfati-Saber and Murray in [10] and [11] show that the discrete time model of the consensus network is described by a directed or undirected graph and the associated graph Laplacian matrix L plays an important role in the convergence and alignment analysis. Indeed, the nominal state evolution of the agents is governed by a discrete time consensus equation defined as $x(k+1) = (I-\epsilon L)x(k)$, where I is the identity matrix and $\epsilon > 0$ is a stepsize parameter. However, such standard protocols exhibit two drawbacks: the convergence is affected by the choice of the step-size parameter and has a low speed of reaching a consensus for particular graph topologies (i.e., graphs constituted by periodic strong components [3]). An alternative form of the standard Laplacian matrix is presented in [5], but the proposed algorithm does not converge for periodic graphs. The convergence speed of consensus protocols is an important topic that has received significant attention in recent years [4]. In [16] the authors find the general conditions to determine the weight to be associated to each node for the linear iteration to converge to the average and to make the convergence as fast as possible. However, an optimization problem has to be solved in a centralized approach and the solution can be applied to a particular graph topology. In addition, some authors demonstrated that predictive consensus algorithms can

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converge much faster [1], [9], [12]. In particular, Oreshkin et al. [12] provide a theoretical demonstration that adding a local prediction component to the update rule can significantly improve the convergence rate of the distributed average algorithm. However, the computation of the prediction component needs an overhead in order to evaluate some parameters requiring the spectrum knowledge of the original iteration matrix.
For fast consensus seeking, Jin and Murray [8] propose protocols that enlarge the algebraic connectivity without physically changing the network topology. Moreover, network communication delays that may occur while exchanging data among multiple agents can degrade the system performances. In this context, Fang et al. [4] introduce the weighted average prediction into existing consensus protocol to simultaneously impose the robustness to communication delay and the convergence speed achieving the consensus. In addition, the technical note [14] addresses the consensus problem of discrete-time networked multi-agent systems with network transmission delays, based on a networked predictive control scheme.
In order to investigate consensus protocols with fast asymptotic convergence, we proposed new consensus algorithms in [2]. In particular, we consider the linear system $x = (I - \epsilon L)x$ and, according to the approach of the Point Jacobi and Gauss-Seidel iterative methods to solve large systems of linear equations [15], we presented some consensus algorithms that are based on a positive splitting of matrix $P_\epsilon = (I - \epsilon L)$. However, such protocols did not converge for any network topology.
In this paper, we relax the condition of the positive splitting of matrix P_ϵ and we propose a new class of protocols that are based on a triangular splitting of P_ϵ. The nonnegative matrices theory [15] provides the framework for analyzing the convergence properties of the proposed consensus algorithms. Furthermore, we determine in closed form the protocol that exhibits the following main properties: i) for each agent network with topology described by a strongly connected graph there exists a triangular splitting that guarantees the convergence at the group decision value; ii) the consensus algorithm is independent from the value of ϵ ; iii) in each step the agents update the state in a fixed sequence in order to employ the updated state values of the upstream nodes. The convergence properties are studied and compared with the algorithms proposed in the related literature by means of a number of tests. The results show the good performances of the presented algorithm, even in the cases in which the standard consensus protocol exhibits low convergence speed, i.e., for network topologies described by periodic graphs. In particular, a rigorous theoretical analysis of the proposed protocol convergence is provided for networks with ring topology (a common type of periodic graphs), and compared with the standard algorithm.
The paper is organized as follows. Section 2 describes the problem and Section 3 introduces the new class of consensus algorithms and proves its convergence. Then Section 4 characterizes the triangular splitting that guarantees the convergence. Moreover, Section 5 provides a rigorous comparison for ring topologies between the convergence of the proposed protocol and the standard protocols. Finally, Section 6 summarizes the conclusions.

2 Problem Statement

Consider a network of n autonomous agents labelled by an index $i \in V$ with $V = \{1, 2, .., n\}$. Let $x_i \in \Re$ denote the state of the agent i that can represent a physical quantity, such as altitude, position, temperature, voltage, and so on. The interaction topology of the network of agents is represented by a directed graph $G = (V, E)$ where V is the set of nodes and $E \subseteq V \times V$ is the set of edges. Moreover, matrix $A = [a_{ij}]$, with $a_{ij} \in \{0, 1\}$, denotes the adjacency matrix of G, $N_i = \{j \in V : a_{ij} = 1\}$ is the set of neighbours of agent i and $|N_i|$ is its cardinality. More precisely, in the accepted assumption of the related literature, setting $a_{ij} = 1$ denotes the fact that node i can receive information from node j [13], [11], [10]. We say that the nodes of a network have reached a consensus if and only if (iff) $x_i = x_j$ for all $i, j \in V$. Furthermore, we define the degree matrix D as the diagonal matrix whose diagonal entries are $D_{ii} = |N_i|$, i.e., the valence of vertex i within the graph. Whenever the agents of a network are all in agreement, the common value of all nodes is called the agreement state and can be expressed as $x^* = \alpha\mathbf{1}$, where $\mathbf{1} = [1, \ldots, 1]^T$ and α is a collective decision of the group of the agents. A standard consensus algorithm that solves the agreement problem in a network of agents with discrete-time model is [11]:

$$x(k + 1) = P_\epsilon \, x(k) \tag{1}$$

where matrix $P_\epsilon = (I - \epsilon L) = [p_{\epsilon ij}]$ is the iteration matrix, ϵ is the step-size parameter, I is the identity matrix and $L = (D - A) = [l_{ij}]$ is the graph Laplacian induced by the graph G. The convergence analysis of the discrete-time consensus algorithm heavily relies on the nonnegative matrix theory [15]. Denoting by $\Delta = max_i \, l_{ii}$ the maximum node out-degree of G, if G is strongly connected, then P_ϵ is a stochastic and irreducible matrix for all $\epsilon \in (0, 1/\Delta)$. Moreover, the decision value is $x^* = lim_{k\to\infty} x(k) = vw^T x(0)$, where $v = \mathbf{1}$ and $w > 0$ are respectively the right and left eigenvectors of P_ϵ associated with the eigenvalue $\lambda = 1$.

3 The New Class of Consensus Algorithms

This section introduces new consensus algorithms that solve an agreement in networks with fixed or switching topology and zero-communication time delay.
Consider the consensus algorithm (1) and define the following splitting of P_ϵ.

Definition 1 *We denote as the triangular splitting of matrix P_ϵ a pair of matrices belonging to the following set:*

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$Q(\epsilon) = \{R \in \Re^{n\times n}, S \in \Re^{n\times n} | R$ is a lower triangular matrix with $r_{ii} \neq 1$ and $r_{ii} \neq 0$ for $i = 1, \dots n$, S is an upper non-negative triangular matrix and $R + S = P_\epsilon\}$.

The following lemma allows us to prove a property of the triangular splitting of P_ϵ.

Lemma 2 *Let P_ϵ be a stochastic and irreducible matrix and $(R, S) \in Q(\epsilon)$. Then the matrix $(I - R)^{-1}$ exists and is non-negative.*

PROOF. By definition it holds $r_{ii} \neq 1$ and $r_{ii} \neq 0$ for $i = 1, \dots, n$. Moreover, since P_ϵ is irreducible and stochastic, then $0 \leq s_{ii} + r_{ii} < 1$. Since by Definition 1 $s_{ii} \geq 0$ and R is a lower triangular matrix, then $r_{ii} < 1$ and $(I - R)$ is non-singular.
Now, in order to prove that $(I - R)^{-1}$ is non-negative, we have to show that $\forall b \geq 0 \; \exists x \geq 0$ such that $(I - R)^{-1}x = b$ [6].
Consider $b = [b_1 \dots b_n]^T = b_1 e_1 + \dots + b_n e_n$, where e_i for $i = 1, \dots, n$ is the canonical basis of $\Re^n$. Denoting by $x_i = [x_i^1 \dots x_i^n]^T$ the solution of the iteration $(I - R)x_i = b_i e_i$, we obtain:

$$x_i^j = 0 \quad \text{for } j = 1, \dots i-1 \quad \text{and} \quad i = 2, \dots n \tag{2}$$

$$x_i^j = (1 - r_{ii})^{-1} b_i \quad \text{for } j = i \quad \text{and} \quad i = 1, \dots n \tag{3}$$

$$x_i^j = (1 - r_{jj})^{-1} \sum_{k=1}^{j-1} r_{jk} x_i^k \text{ for } j > i \text{ and } i = 1, \dots n \tag{4}$$

Since $r_{ii} < 1$ for $i = 1, \dots, n$, it is easy to infer by (2)-(4) that $x_i \geq 0$ for $i = 1, \dots, n$ and $x = x_1 e_1 + \dots + x_n e_n \geq 0$, then matrix $(I - R)^{-1}$ is non-negative.

Let us consider the linear system $x = P_\epsilon x$. According to the approach of the Point Jacobi and Gauss-Seidel iterative methods to solve large systems of linear equations [15], we can carry out the following iterative method derived by each splitting $(R, S) \in Q(\epsilon)$:

$$(I - R)x(k+1) = Sx(k) \; , \; k \geq 0. \tag{5}$$

By Lemma 2 the matrix $(I - R)$ is non-singular and the discrete-time collective dynamics of the network can be written as follows:

$$x(k+1) = (I - R)^{-1} Sx(k) \; , \; k \geq 0. \tag{6}$$

Since each triangular splitting $(R, S) \in Q(\epsilon)$ induces an iterative method that is characterized by the iteration matrix $\Gamma = (I - R)^{-1}$, we say that a class of new consensus protocols is introduced: if Γ is irreducible and acyclic (i.e., primitive) then the iteration scheme is convergent [15].

3.1 Convergence of the consensus algorithms

The following results characterize the convergence properties of the obtained consensus algorithms. In particular, we show that under some conditions on the triangular splitting, algorithm (6) converges to the same group decision value x^* of protocol (1). To this aim we prove the conditions to obtain a primitive iteration matrix Γ that assures the convergence of the consensus algorithm. The proofs show the following properties of matrix Γ:

P1) Γ is a stochastic matrix;
P2) $\lambda = 1$ is a simple eigenvalue of Γ;
P3) there exists a set of triangular splitting $(R, S) \in Q(\epsilon)$, such that Γ is irreducible and acyclic.

The following lemma proves property P1.

Lemma 3 *Consider $(R, S) \in Q(\epsilon)$; if P_ϵ is stochastic, then matrix $\Gamma = (I - R)^{-1}S$ is stochastic too.*

PROOF. By Lemma 2 it holds $(I - R)^{-1} \geq 0$. Observing that S is a non-negative matrix, it immediately follows that Γ is non-negative too. Since $P_\epsilon \mathbf{1} = (R + S)\mathbf{1} = \mathbf{1}$, it holds $(I - R)\mathbf{1} = S\mathbf{1}$ and $(I - R)^{-1}S\mathbf{1} = \mathbf{1}$. Then $v = \mathbf{1}$ is the right eigenvector associated with the eigenvalue $\lambda = 1$ and Γ is a stochastic matrix.

Now, the following theorem proves P2.

Theorem 4 *Let P_ϵ be a stochastic irreducible matrix and w be the left eigenvector of P_ϵ associated with the eigenvalue $\lambda = 1$. Consider the triangular splitting $(R, S) \in Q(\epsilon)$; then for matrix $\Gamma = (I - R)^{-1}S$ the following statements hold true:*

i) *the spectral radius of Γ is $\rho(\Gamma) = 1$;*
ii) *$\lambda = 1$ is a simple eigenvalue of Γ;*
iii) *the left eigenvector of matrix $\Gamma = (I - R)^{-1}S$ associated with the eigenvalue $\lambda = 1$ is $w'^T = w^T S$.*

PROOF. Statement *i)* is a direct consequence of Lemma 3. Statement *ii)* follows from the fact that there is a unique right eigenvector $v = \mathbf{1}$ corresponding to the simple eigenvalue $\lambda = 1$ of the irreducible matrix P_ϵ. On the other hand, $P_\epsilon v = v$ implies and is implied by $(I - R)^{-1}Sv = v$. Hence, P_ϵ and $(I - R)^{-1}S$ have the same number of independent right eigenvectors associated with the eigenvalue $\lambda = 1$. Therefore, the geometric multiplicity of $\lambda = 1$ is the same for both matrices and it equals one: statement *ii)* is proved.
To prove *iii)* we assume that vector w is the left eigenvector of P_ϵ associated with the eigenvalue λ=1, i.e., $w^T(R + S)$=w^T. After some easy passages, we infer:

$$w^T S(I - R)^{-1} S = w^T S. \tag{7}$$

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Hence $w^T S$ is the left eigenvector of $(I-R)^{-1}S$ associated with the eigenvalue $\lambda = 1$.

Given a state set and a stochastic matrix, there exists a Markov chain associated with them. Hence, let MC be the Markov chain associated with the stochastic matrix $(I-R)^{-1}S$. By Theorem 4, Γ has only one eigenvalue equal to 1, consequently MC has only one recurrent class. The following proposition proves a sufficient condition assuring Γ irreducible with $|\lambda| < 1$ for each eigenvalue $|\lambda| \neq 1$ of Γ, i.e., Γ is primitive.

Proposition 5 *Let P_ϵ be a stochastic irreducible matrix and $(R,S) \in Q(\epsilon)$. If S has no null column, then Γ=$(I-R)^{-1}S$ is irreducible and acyclic.*

PROOF. Consider the vector w'^T=$w^T S$ and let MC be the Markov chain associated with Γ. Since w' is the left eigenvector of Γ=$(I-R)^{-1}S$ associated with the eigenvalue λ=1, w' is proportional to the steady state vector of MC. Now let us observe that $w > 0$ is the steady-state probability vector of the recurrent states of the Markov chain associated with the stochastic irreducible matrix P_ϵ: then the i-th entry of w' is zero if the i-th column of S has all zero entries. Remarking that only states in recurrent classes can occur with positive steady state probability, since S has no null column, $w' > 0$ and Γ is irreducible.
Now, we prove by contradiction that Γ is acyclic. Let us assume that Γ is cyclic, and therefore it has all zero entries along the main diagonal. Recalling that R is a lower triangular matrix, by Lemma 2 matrix $(I-R)^{-1}$ is non-negative lower triangular. Hence, all the entries along the main diagonal of $(I-R)^{-1}$ are positive. Consequently, the first element of matrix Γ is zero iff the first column of S (that is upper triangular) is zero: this contradicts the assumption and the proposition is proved.

The following theorem guarantees the convergence to the group decision value $x^* = vw^T x(0)$ of the algorithm (6) that is induced by a triangular splitting.

Theorem 6 *Let P_ϵ be a stochastic irreducible matrix and w the left eigenvector of P_ϵ associated with the eigenvalue $\lambda = 1$. If there exist $(R,S) \in Q(\epsilon)$ and $\mu > 0$ such that S has no null column and $w^T S$=μw^T, then algorithm (6) converges for all the initial states and the group decision value is $x^* = vw^T x(0)$.*

PROOF. Assume that $(R,S) \in Q(\epsilon)$ and P_ϵ be a stochastic irreducible matrix. If we select S with no null column, then by Proposition 5, $(I-R)^{-1}S$ is primitive with right and left eigenvectors $v = \mathbf{1}$ and $w'^T = w^T S$ respectively, associated with the eigenvalue $\lambda = 1$. Consequently, the algorithm (6) converges and gives the decision value $lim_{k\to\infty} x(k) = vw^T Sx(0)$. Moreover, if there exists $\mu > 0$ such that $w^T S$=μw^T (i.e., w^T is the left eigenvector of S associated with the eigenvalue $\mu > 0$), then, choosing w such that $\mathbf{1}^T \frac{w^T}{\mu} = 1$, it holds $lim_{k\to\infty} x(k) = vw^T x(0)$. This proves the theorem.

4 Consensus Protocols

In this section we find the set of the triangular splitting of P_ϵ that guarantees the convergence of the consensus algorithm (6) to the group decision value. To this aim, the following set of linear algebraic constraints $\varphi(P_\epsilon, w)$ allows obtaining S and μ that satisfy the hypothesis of Theorem 6:

$$\varphi(P_\epsilon, w) =$$

$$= \begin{cases} \mu > 0 & (8.a) \\ s_{ii} \geq 0 \text{ for } i = 1,\dots,n & (8.b) \\ s_{ij} = 0 \text{ for } i > j,\ i,j = 1,\dots,n & (8.c) \\ s_{ij} = -\epsilon l_{ij} \text{ for } i < j,\ i,j = 1,\dots,n & (8.d) \\ \sum_{j=1}^{i} w_j s_{ji} - w_i \mu = 0 \text{ for } i = 1,\dots,n & (8.e) \\ \mathbf{1}^T S > 0 & (8.f) \end{cases} \quad (8)$$

Hence, algorithm (6), with any triangular splitting $(R,S) \in Q(\epsilon)$ that satisfies the linear algebraic constraints $\varphi(P_\epsilon, w)$, gives a consensus protocol associated with the network topology G. The following result provides matrix S that satisfies the linear algebraic constraints $\varphi(P_\epsilon, w)$.

Proposition 7 *Let P_ϵ be a stochastic irreducible matrix and w the left eigenvector of P_ϵ associated with the eigenvalue $\lambda = 1$. Let S be defined as follows:*

$$s_{ij} = 0 \text{ for } i > j, \quad s_{ij} = -\epsilon l_{ij} \text{ for } i < j \text{ with } i,j = 1,\dots,n \quad (9)$$

$$s_{11} = \eta \max_i \sum_{j=1}^{i-1} \frac{w_j}{w_i} s_{ji} = \eta \sum_{j=1}^{i*-1} \frac{w_j}{w_{i*}} s_{ji*} \quad (10)$$

where $\eta \in \Re^+$, $\eta \geq 1$ and i^* is the index of the i^*-th column corresponding to the maximum value of $\sum_{j=1}^{i-1} \frac{w_j}{w_i} s_{ji}$,

$$s_{ii} = \eta \sum_{j=1}^{i*-1} \frac{w_j}{w_{i*}} s_{ji*} - \sum_{j=1}^{i-1} \frac{w_j}{w_i} s_{ji} \text{ for } i = 2,\dots,n. \quad (11)$$

Then S satisfies the linear algebraic constraints $\varphi(P_\epsilon, w)$.

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PROOF. The constraints (8.c) and (8.d) are trivially satisfied by (9). Now, we consider the constraints (8.e) that can be written as follows:

$$s_{11} = \mu \tag{12}$$

$$s_{ii} = \mu - \sum_{j=1}^{i-1} \frac{w_j}{w_i} s_{ji} \quad \text{for } i = 2, \ldots, n \tag{13}$$

In order to satisfy (8.a) and (8.b) by taking into account (12) and (13), we choose $\mu = \eta \max_i \sum_{j=1}^{i-1} \frac{w_j}{w_i} s_{ji} = \eta \sum_{j=1}^{i*-1} \frac{w_j}{w_{i*}} s_{ji*}$ with $\eta \geq 1$.
If $\eta > 1$, then $s_{ii} > 0$ for $i = 1, \ldots, n$ and the constraints (8.f) are satisfied too.
On the other hand, if $\eta = 1$, then it holds $s_{i^*i^*} = 0$. Assume by contradiction that the i^*-th column of S is equal to zero and (8.f) is not true. This implies that $\sum_{j=1}^{i*-1} \frac{w_j}{w_{i*}} s_{ji*} = 0$ and by (10) it holds $\sum_{j=1}^{i-1} \frac{w_j}{w_i} s_{ji} = 0$ for $i = 2, \ldots, n$. Now, since S is a lower triangular matrix with $s_{ij} = -\epsilon l_{ij}$ for $i < j$ with $i, j = 1, \ldots, n$, if $s_{ij} = 0$ for $i > j$, then P_ϵ is upper triangular: this contradicts the assumption that P_ϵ is irreducible.

The convergence of protocol (1) is affected by the choice of ϵ and is guaranteed if $\epsilon \in (0, 1/\Delta)$ [10], [11]. We remark that if the triangular splitting $(R, S) \in Q(\epsilon)$ is determined by (9)-(11), then the iteration scheme is not affected by the choice of the step-size parameter ϵ. Indeed, let us denote:

$$L_i = -\sum_{j=1}^{i-1} \frac{w_j}{w_i} l_{ji} = \sum_{j=1}^{i-1} \frac{w_j}{w_i} a_{ji} \text{ for } i = 2, \ldots, n \tag{14}$$

$$L_{i^*} = \max_i L_i. \tag{15}$$

By definition it holds $s_{ij} = -\epsilon l_{ij} = \epsilon a_{ij}$ for $i < j$, $r_{ij} = -\epsilon l_{ij} = \epsilon a_{ij}$ for $i > j$ and with $i, j = 1, \ldots, n$. Now by (9)-(11) and taking into account (14)-(15), we write:

$$s_{11} = \mu = \eta \epsilon L_{i^*} \tag{16}$$

$$s_{ii} = \epsilon(\eta L_{i^*} - L_i) \text{ for } i = 2, \ldots, n \tag{17}$$

$$1 - r_{11} = \epsilon(l_{11} + \eta L_{i^*}) \tag{18}$$

$$1 - r_{ii} = \epsilon(l_{ii} + \eta L_{i^*} - L_i) \text{ for } i = 2, \ldots, n \tag{19}$$

Therefore, it is easy to infer that $\Gamma = (I - R)^{-1} S$ is independent from ϵ.

4.1 Consensus Discrete Time Dynamics

In this sub-section we specify how the consensus algorithm can be applied by a set of autonomous agents. Now we denote:

$$\alpha_1 = L_{i^*} = \eta \sum_{j=1}^{i*-1} \frac{w_j}{w_{i*}} a_{ji*} \tag{20}$$

$$\alpha_i = \eta L_{i^*} - L_i = \eta \sum_{j=1}^{i*-1} \frac{w_j}{w_{i*}} a_{ji*} - \sum_{j=1}^{i-1} \frac{w_j}{w_i} a_{ji} \quad \text{for } i = 2, \ldots, n \tag{21}$$

$$\beta_1 = l_{11} + \eta L_{i^*} = \sum_{h=2}^{n} a_{1h} + \eta \sum_{j=1}^{i*-1} \frac{w_j}{w_{i*}} a_{ji*} \tag{22}$$

$$\begin{aligned} \beta_i = l_{ii} + \eta L_{i^*} - L_i = \textstyle\sum_{h=1,h\neq i}^{n} a_{ih} + \\ +\eta \textstyle\sum_{j=1}^{i*-1} \frac{w_j}{w_{i*}} a_{ji*} - \sum_{j=1}^{i-1} \frac{w_j}{w_i} a_{ji} \text{ for } i = 2, \ldots, n \end{aligned} \tag{23}$$

Choosing S according to (9)-(11), the consensus discrete-time dynamics of the network described by (6) is expressed by the following consensus algorithm:

$$\begin{aligned} &x_1(k+1) = \tfrac{1}{\beta_1} \left(\alpha_1 x_1(k) + \textstyle\sum_{j=2}^{n} a_{1j} x_j(k) \right) \\ &x_i(k+1) = \tfrac{1}{\beta_i} \left(\textstyle\sum_{j=1}^{i-1} a_{ij} x_j(k+1) + \alpha_i x_i(k) + \right. \\ &\left. + \textstyle\sum_{j=i+1}^{n} a_{ij} x_j(k) \right) \text{ for } i = 2, \ldots, n \text{ and } k \geq 0 \end{aligned} \tag{24}$$

The algorithm (24) establishes an order to update the values of each agent state. More precisely, to update the state at the time $k+1$, the i-th agent uses the already determined values of the states $x_j(k+1)$ for $j = 1, \ldots, i-1$ for $i > 1$.
Hence, it is clear that each agent i needs to know two elements to update his state by (24): i) the associated order in the sequence; ii) the values of the parameters α_i and β_i. Now, considering the agent order $i = 1, 2, \ldots, n$, we denote by $US_i = \{j | j \in N_i \text{ and } j < i\}$ the set of the upstream agents with which agent i communicates. Moreover, in order to determine the parameters α_i and β_i, the i-th agent has to know the values L_{i^*}, w_i and w_j for each $j \in US_i$. Hence, before applying the consensus protocol, each agent has to perform a *Start-up algorithm* that is composed of two phases. In the first phase (*Assignment phase*) each node receives by a coordinator agent an identification number i and the entries w_i and w_j for each $j \in US_i$. In the second phase (*Communication phase*) the agent finds out the values of L_{i^*}, α_i and β_i by a communication protocol. In the following we list the steps of the *Start-up algorithm*.

Start-up algorithm

Assignment phase

A1) Assign an order among the agents: each agent is associated with an identification number $id = i$ with $i \in \{1, \ldots, n\}$.

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A2) Assign to each agent $i \in \{1, \dots, n\}$ the values w_i and w_j $\forall j \in US_i$.

Communication phase
Determining L_{i^*}, α_i and β_i

C1) If $i > 1$ **then set** $L_i = \sum_{j=1}^{i-1} \frac{w_j}{w_i} a_{ji}$ **else set** $L_1 = 0$.
C2) Set $L_i^{max}(0) = L_i$
C3) For $k = 1, n$
C4) **Receive** $L_j^{max}(k-1)$ from each $j \in N_i$
C5) **Set** $L_i^{max}(k) = \max_{j \in N_i \cup \{i\}} L_j^{max}(k-1)$
C6) End for
C7) Set $L_{i^*} = L_i^{max}(n)$
C8) Determine α_i and β_i according to (20)-(23)
C9) End

Note that the communication phase is simultaneously performed by all the agents in the same time instant. More precisely, at time $k > 0$, the i-th agent updates the value $L_i^{max}(k)$ by comparing $L_i^{max}(k-1)$ with $L_j^{max}(k-1)$ for each $j \in N_i$. Since the graph is strongly connected, each agent reaches the same value L_{i^*} by at most n iterations. This procedure can exploit a meaningful and convenient property, as specified by the following statement.

Proposition 8 *If graph G is balanced, then the value of L_i is equal to the cardinality of US_i for $i = 1, \dots, n$.*

PROOF. The proof is straightforward taking into account (14), and recalling that $w_i = 1$ for $i = 1, \dots, n$ for balanced graphs.

Consequently, if graph G is balanced then the *Start-up* algorithm skips step A2 and the i-th agent can autonomously determine the parameters L_{i^*}, α_i and β_i by the communication phase, without the intervention of the coordinator agent.

5 Convergence Rate Analysis

In this section we first study and compare the convergence properties of the proposed algorithm class considering network topologies described by generic and periodic graphs. Then, a rigorous theoretical analysis of the proposed protocol convergence is provided for networks with ring topology (a common type of periodic graphs), and compared with the standard algorithm.

5.1 Convergence Properties for Generic and Periodic Graphs

We point out that the convergence rate of the proposed algorithm can be affected by the selection of η in (10) (and the consequent choice of μ and S). In order to evaluate the best choice of η, we consider 10^4 random generated topologies of networks with 10 and 100 agents. More precisely, the adjacency matrices of the strongly connected aperiodic graphs are randomly generated. For each topology, we determine the iteration matrix $\Gamma = (I - R)^{-1} S$ taking into account (16) - (19) with $\eta = 1, 1.1, 1.3, 1.5, 1.7$ and 2. The convergence properties of algorithm (24) as function of η is eveluated by computing the asymptotic convergence factor as the second largest (subdominant) eigenvalue $\lambda_{2\Gamma}$ of Γ: the smaller $\lambda_{2\Gamma}$ is, the faster the algorithm is [10], [11]. Hence, for each η we calculate the value $\overline{\lambda}_{2\Gamma}(\eta)$ averaged on the 10^4 generated topologies. The results of the convergence study are reported in Table 1 confirming that in the 100% of the considered cases $\eta = 1$ guarantees the minimum values of $\overline{\lambda}_{2\Gamma}$. Such results authorize to conclude that the minimum possible value of μ provides a good convergence in any case.

Table 1
Subdominant Eigenvalue Analysis.

$\overline{\lambda}_{2\Gamma}(\eta)$	$\eta = 1$	$\eta = 1.1$	$\eta = 1.3$	$\eta = 1.5$	$\eta = 1.7$	$\eta = 2$
$n = 10$	0.83	0.85	0.87	0.88	0.89	0.91
$n = 100$	0.63	0.65	0.69	0.72	0.75	0.78

Now we turn to compare protocol (24) with algorithm (1). The case of non-periodic graphs is considered in [2]: the results confirm that the proposed protocol improves the convergence obtained by algorithm (1) with $\epsilon = \frac{0.2}{\Delta}$, $\epsilon = \frac{0.5}{\Delta}$ and $\epsilon = \frac{0.9}{\Delta}$. Here, we analyse a set of cases where the network topologies are described by periodic graphs of 24 nodes. More precisely, we consider 5 cases of periodic graphs with period d=2, 4, 6, 12 and 24. Table 2 reports for each case the subdominant eigenvalue: the results show that the standard protocol converges slowly. On the other hand, the proposed protocol results in the fastest convergence in any examined case.

Table 2
Subdominant Eigenvalues of Periodic Graphs.

Algorithm	$d = 2$	$d = 4$	$d = 6$	$d = 12$	$d = 24$
P_ϵ, $\epsilon = \frac{0.2}{\Delta}$	0.993	0.970	0.925	0.978	0.994
P_ϵ, $\epsilon = \frac{0.5}{\Delta}$	0.983	0.926	0.866	0.966	0.991
P_ϵ, $\epsilon = \frac{0.9}{\Delta}$	0.969	0.865	0.967	0.991	0.998
Γ	0.967	0.821	0.716	0.834	0.940

5.2 Convergence Analysis for Ring Topology

In this subsection we consider networks described by ring topology of n nodes (i.e., periodic graphs of period n). Then, we provide a theoretical analysis of the convergence by determining the analytical expression of the subdominant eigenvalues of P_ϵ and $\Gamma = (I - R)^{-1} S$, as a function of the parameters ϵ and η, respectively.

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Theorem 9 *The eigenvalues of matrix P_ϵ associated with a ring topology of n nodes are*

$$\lambda_k = 1 - \epsilon + \epsilon e^{j\frac{2\pi k}{n}} \quad \text{for } k = 0, \ldots, n-1. \tag{25}$$

PROOF. The characteristic polynomial of matrix P_ϵ associated with a ring of n nodes is $p_{P_\epsilon}(\lambda)=(1-\epsilon-\lambda)^n-(-\epsilon)^n$. Hence, its roots λ_k satisfy the following equation:

$$\left(\frac{1-\epsilon-\lambda_k}{-\epsilon}\right)^n = 1.$$

The result (25) easily follows by considering the roots of units.

Proposition 10 *Let us consider a ring topology of n nodes. The minimum value of the subdominant eigenvalue of P_ϵ as a function of parameter ϵ is:*

$$\lambda_{2\epsilon min} = \frac{1}{2}\left(1 + e^{j\frac{2\pi}{n}}\right). \tag{26}$$

PROOF. By (25) the subdominant eigenvalue of P_ϵ is

$$\lambda_{2\epsilon} = 1 - \epsilon + \epsilon e^{j\frac{2\pi}{n}}. \tag{27}$$

Considering that for a ring it holds $0 < \epsilon < 1$, it is easy to infer that the minimum of (27) is obtained for $\epsilon = 0.5$ and (26) follows.

Proposition 11 *Let us consider a ring topology of n nodes. The roots of the characteristic polynomial of $\Gamma = (I-R)^{-1}S$ as a function of the parameter η satisfy the following relation:*

$$\left(\frac{-\lambda}{\eta-(1+\eta)\lambda}\right)^{n-1} = -\frac{1-2\lambda}{\eta-(1+\eta)\lambda}(\eta-1-\lambda\eta). \tag{28}$$

PROOF. The characteristic polynomial of Γ is the determinant of the matrix $S + \lambda R - \lambda I$. By taking in account (16)- (19), we obtain:

$$\begin{aligned} p_\Gamma(\lambda) = (1-2\lambda)(\eta-1-\eta\lambda)(\eta-\lambda(1+\eta))^{n-2}+ \\ +(-\lambda)^{n-1}, \end{aligned} \tag{29}$$

from which the result follows.

By the analysis of the roots of equation (28), we infer that the subdominant eigenvalue of Γ increases for $\eta > 2$. Then, we determine an approximation of the subdominant eigenvalue of Γ for $1 \leq \eta \leq 2$.

Proposition 12 *Let us consider a ring topology of n nodes. The modulus of the subdominant eigenvalue of $\Gamma = (I-R)^{-1}S$ as a function of the parameter η with $1 \leq \eta \leq 2$ and $n > 2$ can be approximated by:*

$$|\lambda_{2\Gamma}| \approx \left|\frac{\eta e^{\frac{2\pi j}{n-1}}}{(1-\eta)e^{\frac{2\pi j}{n-1}} - 1}\right|. \tag{30}$$

PROOF. The roots of the polynomial $p_\Gamma(\lambda)$ satisfy equation (28). For $1 \leq \eta \leq 2$ both $\left|\frac{1-2\lambda}{\eta-(1+\eta)\lambda}\right|$ and $|(\eta-1-\lambda\eta)|$ are equal or less than 1. Hence we compute as an approximation of $|\lambda_{2\Gamma}|$ the modulus of the subdominant solution of the following equation for $n > 2$:

$$\left(\frac{-\lambda}{\eta-(1+\eta)\lambda}\right)^{n-1} = 1.$$

The result (30) follows by again considering the roots of units.

We observe that for $\eta = 1$ and $n = 2$ the subdominant eigenvalue of the matrix Γ is $\lambda_{2\Gamma} = 0$. Moreover, deriving (30) with respect to η, the minimum value of the approximation of $|\lambda_{2\Gamma}|$ for $1 \leq \eta \leq 2$ is obtained for $\eta = 1$:

$$|\hat{\lambda}_{2\Gamma min}| = \left|\frac{e^{\frac{2\pi j}{n-1}}}{2e^{\frac{2\pi j}{n-1}} - 1}\right|. \tag{31}$$

Now, Figure 1 depicts, as a function of $n > 2$, the difference between $|\lambda_{2\epsilon min}|$ and the modulus of the subdominant eigenvalue of Γ. Moreover, the figure depicts the difference between $|\lambda_{2\epsilon min}|$ and the approximation $|\hat{\lambda}_{2\Gamma min}|$. Two basic results are enlightened: i) for each value of $n > 2$, the modulus of the subdominant eigenvalue of Γ is minor than the corresponding value $|\lambda_{2\epsilon min}|$; ii) the approximation $|\hat{\lambda}_{2\Gamma min}|$ is very close to the modulus of the subdominant eigenvalue of Γ.

The following proposition analytically shows the relation between $\lambda_{2\epsilon min}$ and $\hat{\lambda}_{2\Gamma min}$ as a function of $n > 2$.

Proposition 13 *Let us consider a ring topology of n nodes, then it holds $|\hat{\lambda}_{2\Gamma min}| < |\lambda_{2\epsilon min}|$ $\forall n > 2$.*

PROOF. By (26) and (31) we have to check the following inequality:

$$\left|\frac{e^{\frac{2\pi j}{n-1}}}{2e^{\frac{2\pi j}{n-1}} - 1}\right| < \left|\frac{1}{2}\left(1 + e^{\frac{2\pi j}{n}}\right)\right|,$$

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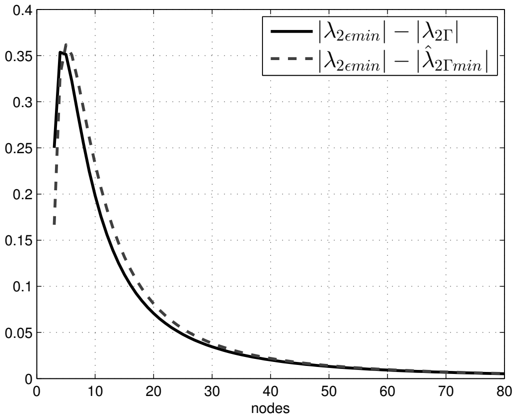

Fig. 1. Comparison between the modulus of the subdominant eigenvalues.

which is equivalent to:

$$\frac{1}{5 - 4\cos\frac{2\pi}{n-1}} < \frac{1}{2}\left(1 + \cos\frac{2\pi}{n}\right). \tag{32}$$

It is easy to show that (32) is satisfied for $n > 2$.

Hence, the theoretical results confirm that in the case of network with ring topology the proposed protocol exhibits a faster convergence than the standard protocol when $\eta = 1$.

6 Conclusions

This paper investigates new and fast alignment protocols that can be applied to the discrete time model of consensus networks. In particular, a class of consensus algorithms that are based on the triangular splitting of the standard iteration matrix is presented. The convergence of the proposed discrete-time consensus algorithms is proved in the framework of the non-negative matrix theory.
Moreover, we determine a particular triangular splitting that guarantees the following main properties.

- Any choice of $\eta \geq 1$ (independently from the network topology) guarantees the convergence of the presented algorithm and, in particular, $\eta = 1$ gives good (and optimal in some cases) convergence factors. On the contrary, in the standard consensus algorithm if $\epsilon \geq 1/\Delta$ then the convergence is not guaranteed.
- The proposed protocol has good convergence rate for particular graph topologies, i.e. periodic graphs. We observe that the standard consensus algorithm (1) has low convergence speed for these graphs.
- A rigorous theoretical analysis of the convergence speed is provided in the case of ring topology, the results prove that the proposed algorithm exhibits faster convergence than the standard protocol.

Future research will focus on the analytical study of the performance analysis of the proposed consensus protocols in relation with other network topologies.

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