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Well-posedness of the classical solution for the Kuramto–Sivashinsky equation with anisotropy effects

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Abstract. The Kuramto–Sivashinsky equation with anisotropy effects models the spinodal decomposition of phase separating systems in an external field, the spatiotemporal evolution of the morphology of steps on crystal surfaces and the growth of thermodynamically unstable crystal surfaces with strongly anisotropic surface tension. Written in terms of the step slope, it can be represented in a form similar to a convective Cahn–Hilliard equation. In this paper, we prove the well-posedness of the classical solutions for the Cauchy problem, associated with this equation.

Mathematics Subject Classification. 35G25, 35K55.

Keywords. Existence, Uniqueness, Stability, The Kuramto–Sivashinsky equation with anisotropy effects, Cauchy problem.

1. Introduction

In this paper, we investigate the well-posedness of the following Cauchy problem:

$$\begin{cases} \partial_t u+\alpha\partial_x^2 u+\beta^2\partial_x^4 u-\gamma^2(\partial_x u)^2\partial_x^2 u+\tau\partial_x u\partial_x^2 u & \\ \quad +\kappa(\partial_x u)^4+q(\partial_x u)^2+\delta\partial_x u\partial_x^3 u=0, & t>0, \quad x\in\mathbb{R}, \\ u(0,x)=u_0(x), & x\in\mathbb{R}, \end{cases} \tag{1.1}$$

with α, β, γ, τ, κ, q, $\delta\in\mathbb{R}$, β, $\gamma\neq 0$, such that

$$\delta^2<4\beta^2\gamma^2. \tag{1.2}$$

On the initial datum, we assume

$$u_0\in H^{\ell}(\mathbb{R}), \quad \ell\in\{2,3,4\}. \tag{1.3}$$

Observe that, using the variable (see [24,57])

$$v=\partial_x u, \tag{1.4}$$

Equation (1.1) is equivalent to the following one:

$$\partial_t v+\alpha\partial_x^2 v+\beta^2\partial_x^4 u-\frac{\gamma^2}{3}\partial_x^2\left(v^3\right)+\frac{\tau}{2}\partial_x^2\left(v^2\right)+\kappa\partial_x v^4+q\partial_x v^2+\delta\partial_x\left(v\partial_x^2 v\right)=0, \tag{1.5}$$

which is known as the convective Cahn–Hilliard equation (see [24,33]).

From a physical point of view, (1.1) and (1.5) model the spinodal decomposition of phase separating systems in an external field [19,42,64], the spatiotemporal evolution of the morphology of steps on

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crystal surfaces [24,33,52], and the growth of thermodynamically unstable crystal surfaces with strongly anisotropic surface tension [25,26,28,46].

In the case of a growing crystal surface with strongly anisotropic surface tension, the function u represents is the surface slope, while the constants κ and q are the growth driving forces proportional to the difference between the bulk chemical potentials of the solid and fluid phases. They were also obtained by Watson [61] as a small-slope approximation of the crystal growth model obtained in [17].

Observe that, in [52], the authors deduce (1.1) in the case $\tau = \kappa = \delta = 0$, while, in [24], (1.1) is done with $\delta = 0$. The general case is considered in [33]. In particular, in [24,33], the authors show the dependence of the coefficients on the anisotropy of the surface tension and on the velocity of the solidification front. It allows one to assess the effects of these parameters on the evolution of the instability.

Assuming $\kappa = q = \delta = 0$, (1.5) reads

$$\partial_t v + \alpha\partial_x^2 v + \beta^2\partial_x^4 u - \frac{\gamma^2}{3}\partial_x^2\left(v^3\right) + \frac{\tau}{2}\partial_x^2\left(v^2\right) = 0, \tag{1.6}$$

known as the Cahn–Hilliard equation [8,9,50,51]. It describes the process of spinodal decomposition. In this case, the function u is the concentration of one of the components of an alloy. [51] shows that (1.6) has an exact solution that describes the final stage of the spinodal decomposition, the formation of the interface between two stable state of an alloy with different concentrations.

It also describes the coarsening dynamics of the faceting of thermodynamically unstable surfaces [31,56]. Moreover, [34] shows that Eq. (1.6) can be an effective tool in technological applications to design nanostructured materials.

From a mathematical point of view, in [2], the existence of some extremely slowly evolving solutions for (1.5) is proven, considering a bounded domain, while, in [6,22], the problem of a global attractor is studied. Instead, in [27,65], numerical schemes for (1.5) are analyzed, while, in [60], an approximate analytical solution is studied.

Observe that Eq. (1.5) is has been studied in the multidimensional case in the papers [7,18,66] and their references.

Taking $\tau = \kappa = \delta = 0$ in (1.5), we have the following equation

$$\partial_t v + \alpha\partial_x^2 v + \beta^2\partial_x^4 u - \frac{\gamma^2}{3}\partial_x^2\left(v^3\right) + q\partial_x v^2 = 0. \tag{1.7}$$

(1.7) describes a spinodal decomposition in the presence of an external (e.g., gravitational or electric) field, when the dependence of the mobility factor on the order parameter is important [19,24,42,64].

From a mathematical point of view, the coarsening dynamics for (1.7) has been studied in the limit $0 < q \ll 1$ in [19,26] and analytically in [62].

In [1], a numerical scheme is studied for (1.7), while the existence of the periodic solution are analyzed in [20,36]. In [42,47], the existence of exact solutions for (1.7) and its viscous form have been investigated. Moreover, [26] shows that, when $q \to \infty$, (1.7) reduces to the Kuramoto–Sivashinsky equation (see Eq. (1.8)). Physically, it means that, with the growth of the driving force, there must be a transition from the coarsening dynamics to a chaotic spatiotemporal behavior.

Assuming $\gamma = \tau = \kappa = \delta = 0$ in (1.5), we have the following equation:

$$\partial_t v + \alpha\partial_x^2 v + \beta^2\partial_x^4 u + q\partial_x v^2 = 0, \tag{1.8}$$

(1.8) arises in interesting physical situations, for example as a model for long waves on a viscous fluid flowing down an inclined plane [59] and to derive drift waves in a plasma [16]. Equation (1.8) was derived also independently by Kuramoto [37–39] as a model for phase turbulence in reaction-diffusion systems and by Sivashinsky [55] as a model for plane flame propagation, describing the combined influence of diffusion and thermal conduction of the gas on the stability of a plane flame front.

Equation (1.8) also describes incipient instabilities in a variety of physical and chemical systems [11,29,40]. Moreover, (1.8), which is also known as the Benney–Lin equation [4,43], was derived by Kuramoto in the study of phase turbulence in the Belousov–Zhabotinsky reaction [44].

The dynamical properties and the existence of exact solutions for (1.8) have been investigated in [21,32,35,48,49,63]. In [3,10,23], the control problem for (1.8) with periodic boundary conditions, and on a bounded interval are studied, respectively. In [12], the problem of global exponential stabilization of (1.8) with periodic boundary conditions is analyzed. A generalization of optimal control theory for (1.8) was proposed in [30], while in [45] the problem of global boundary control of (1.8) is considered. In [53], the existence of solitonic solutions for (1.8) is proven. In [5,13,57], the well-posedness of the Cauchy problem for (1.8) is proven, using the energy space technique, a priori estimates together with an application of the Cauchy–Kovalevskaya and the fixed point methods, respectively. Finally, following [14,41,54], in [15], the convergence of the solution of (1.8) to the unique entropy one of the Burgers equation is proven when $\alpha,\ \beta \to 0$.

Before stating our main result it is important to comment our assumption (1.2) on the coefficients. That condition guarantees the conservation of the H^2 norm of the solution in time, in other words thanks to (1.2) the map $t \mapsto u(t,\cdot)$ never leaves the energy space, that is H^2.

We use the following definition of solution.

Definition 1.1. A function $u : [0,\infty) \to \mathbb{R}$ is a solution of (1.1) if

$$u \in L^\infty(0,T;H^2(\mathbb{R})), \quad T > 0,$$

and for every test function with compact support $\varphi \in C^\infty(\mathbb{R}^2)$

$$\begin{aligned}\int_0^\infty\int_{\mathbb{R}} \Big(& u\partial_t\varphi - \alpha u\partial_x^2\varphi - \beta^2\varphi\partial_x^4\varphi - \frac{\gamma^2}{3}(\partial_x u)^3\partial_x\varphi + \frac{\tau}{2}(\partial_x u)^2\partial_x\varphi \\ & - \kappa(\partial_x u)^4\varphi - q(\partial_x u)^2\varphi + \delta(\partial_x^2 u)^2 + \delta u\partial_x^2 u\partial_x\varphi\Big)\mathrm{d}t\mathrm{d}x + \int_{\mathbb{R}} u_0(x)\varphi(0,x)\mathrm{d}x = 0.\end{aligned}$$

The main result of this paper is the following theorem.

Theorem 1.1. *Fix $T > 0$. If (1.2) and*

$$u_0 \in H^4(\mathbb{R}), \tag{1.9}$$

hold there exists a unique solution u of (1.1), such that

$$u \in H^1((0,T)\times\mathbb{R}) \cap L^\infty(0,T;H^4(\mathbb{R})). \tag{1.10}$$

Moreover, if u_1 and u_2 are two solutions of (1.1), we have that

$$\|u_1(t,\cdot) - u_2(t,\cdot)\|_{H^1(\mathbb{R})} \le e^{C(T)t}\|u_{1,0} - u_{2,0}\|_{H^1(\mathbb{R})}, \tag{1.11}$$

for some suitable $C(T) > 0$, and every $0 \le t \le T$.

Assuming (1.2) and

$$u_0 \in H^3(\mathbb{R}), \quad \delta = 0, \tag{1.12}$$

there exists a unique solution u of (1.1), such that

$$u \in H^1((0,T)\times\mathbb{R}) \cap L^\infty(0,T;H^3(\mathbb{R})). \tag{1.13}$$

Moreover, if u_1 and u_2 are two solutions of (1.1), we have that

$$\|u_1(t,\cdot) - u_2(t,\cdot)\|_{L^2(\mathbb{R})} \le e^{C(T)t}\|u_{1,0} - u_{2,0}\|_{L^2(\mathbb{R})}, \tag{1.14}$$

Under Assumptions (1.2) and

$$u_0 \in H^2(\mathbb{R}), \tag{1.15}$$

there exists a solution u of (1.1), such that

$$u \in H^1((0,T)\times\mathbb{R}) \cap L^\infty(0,T;H^2(\mathbb{R})). \tag{1.16}$$

The argument of Theorem 1.1 relies on deriving suitable a priori estimates together with an application of the Cauchy–Kovalevskaya Theorem [58]. Moreover, observe that the models studied in [24,52] correspond to the case $\delta = 0$ and satisfy (1.2).

The paper is organized as follows. In Sect. 2, we prove some a priori estimates of (1.1). Those play a key role in the proof of our main result, that is given in Sect. 3.

2. A priori estimates

In this section, we prove some a priori estimates on u. We denote with C_0 the constants which depend only on the initial data, and with $C(T)$ the constants which depend also on T.

We begin by proving the following result

Lemma 2.1. *Fix $T > 0$. There exists a constant $C(T) > 0$, such that*

$$\|\partial_x u(t,\cdot)\|_{L^2(\mathbb{R})} \le C(T), \tag{2.1}$$

$$\int_0^t \left\|\partial_x^3 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} \mathrm{d}s \le C(T), \tag{2.2}$$

$$\int_0^t \left\|\partial_x u(s,\cdot)\partial_x^2 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} \mathrm{d}s \le C(T), \tag{2.3}$$

$$\int_0^t \left\|\partial_x^2 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} \mathrm{d}s \le C(T), \tag{2.4}$$

for every $0 \le t \le T$.

Proof. Let $0 \le t \le T$. Multiplying (1.1) by $-2\partial_x^2 u$, an integration on $\mathbb{R}$ gives

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})} &= -2\int_{\mathbb{R}} \partial_x^2 u \partial_t u \mathrm{d}x \\
&= 2\alpha\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\beta^2\int_{\mathbb{R}} \partial_x^2 u\partial_x^4 u\mathrm{d}x - 2\gamma^2\left\|\partial_x u(t,\cdot)\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\
&\quad + 2\tau\int_{\mathbb{R}} \partial_x u(\partial_x^2 u)^2 \mathrm{d}x + 2\kappa\int_{\mathbb{R}} (\partial_x u)^4\partial_x^2 u\mathrm{d}x + q\int_{\mathbb{R}} (\partial_x u)^2\partial_x^2 u\mathrm{d}x \\
&\quad + 2\delta\int_{\mathbb{R}} \partial_x u\partial_x^2 u\partial_x^3 u\mathrm{d}x \\
&= 2\alpha\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} - 2\beta^2\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} - 2\gamma^2\left\|\partial_x u(t,\cdot)\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\
&\quad + 2\tau\int_{\mathbb{R}} \partial_x u(\partial_x^2 u)^2 \mathrm{d}x + 2\delta\int_{\mathbb{R}} \partial_x u\partial_x^2 u\partial_x^3 u\mathrm{d}x.
\end{aligned}$$

Therefore, we have that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} &+ 2\beta^2\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\gamma^2\left\|\partial_x u(t,\cdot)\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ &= 2\alpha\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\tau\int_{\mathbb{R}}\partial_x u(\partial_x^2 u)^2\mathrm{d}x + 2\delta\int_{\mathbb{R}}\partial_x u\partial_x^2 u\partial_x^3 u\mathrm{d}x.\end{aligned}\tag{2.5}$$

Due to the Young inequality,

$$\begin{aligned}2|\delta|\int_{\mathbb{R}}|\partial_x u\partial_x^2 u||\partial_x^3 u|\mathrm{d}x &= 2\int_{\mathbb{R}}\left|\frac{\delta\partial_x u\partial_x^2 u}{\sqrt{D_1}}\right|\left|\sqrt{D_1}\partial_x^3 u\right|\mathrm{d}x\\ &\le \frac{\delta^2}{D_1}\left\|\partial_x u(t,\cdot)\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + D_1\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\end{aligned}$$

where D_1 is a positive constant, which will be specified later. It follows from (2.5) that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} &+ \left(2\beta^2 - D_1\right)\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ &+ \left(2\gamma^2 - \frac{\delta^2}{D_1}\right)\left\|\partial_x u(t,\cdot)\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ &\le 2|\alpha|\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2|\tau|\int_{\mathbb{R}}|\partial_x u|(\partial_x^2 u)^2\mathrm{d}x.\end{aligned}\tag{2.6}$$

We search D_1 such that,

$$2\beta^2 - D_1 > 0, \qquad 2\gamma^2 - \frac{\delta^2}{D_1} > 0,$$

that is

$$D_1 < 2\beta^2, \qquad D_1 > \frac{\delta^2}{2\gamma^2}.\tag{2.7}$$

By (2.7), we have that

$$\frac{\delta^2}{2\gamma^2} < D_1 < 2\beta^2.\tag{2.8}$$

Thanks to (1.2), D_1 does exist. Therefore, by (1.2), (2.6), (2.7) and (2.8), we have that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} &+ K_1^2\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ &+ K_2^2\left\|\partial_x u(t,\cdot)\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ &\le C_0\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2|\tau|\int_{\mathbb{R}}|\partial_x u|(\partial_x^2 u)^2\mathrm{d}x.\end{aligned}\tag{2.9}$$

where K_1^2, K_2^2 are two appropriate positive constants. Due to the Young inequality,

$$\begin{aligned}2|\tau|\int_{\mathbb{R}}|\partial_x u|(\partial_x^2 u)^2\mathrm{d}x &= \int_{\mathbb{R}}\left|K_2\partial_x u\partial_x^2 u\right|\left|\frac{2\tau\partial_x^2 u}{K_2}\right|\mathrm{d}x\\ &\le \frac{K_2^2}{2}\left\|\partial_x u(t,\cdot)\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{2\tau^2}{K_2^2}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.\end{aligned}$$

Consequently, by (2.9),

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} &+ K_1^2\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ &+\frac{K_2^2}{2}\left\|\partial_x u(t,\cdot)\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\le C_0\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.\end{aligned}\tag{2.10}$$

Observe that

$$C_0\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} = C_0\int_{\mathbb{R}}\partial_x^2 u\partial_x^2 u\mathrm{d}x = -C_0\int_{\mathbb{R}}\partial_x u\partial_x^3 u\mathrm{d}x.$$

Therefore, by the Young inequality,

$$\begin{aligned}C_0\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} &\le \int_{\mathbb{R}}\left|\frac{C_0\partial_x u}{K_1}\right|\left|K_1\partial_x^3 u\right|\mathrm{d}x\\ &\le C_0\left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\frac{K_1^2}{2}\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.\end{aligned}\tag{2.11}$$

Consequently, by (2.10),

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} &+ \frac{K_1^2}{2}\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ &+\frac{K_2^2}{2}\left\|\partial_x u(t,\cdot)\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\le C_0\left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.\end{aligned}\tag{2.12}$$

Integrating on $(0,t)$, by the Gronwall Lemma and (1.3), we have that

$$\begin{aligned}&\left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\frac{K_1^2e^{C_0t}}{2}\int_0^t e^{-C_s}\left\|\partial_x^3 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})}\mathrm{d}s\\ &\quad+\frac{K_2^2e^{C_0t}}{2}\int_0^t e^{-C_0s}\left\|\partial_x u(s,\cdot)\partial_x^2 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})}\mathrm{d}s\le C_0e^{C_0t}\le C(T),\end{aligned}$$

which gives (2.1), (2.2), (2.3).

Finally, we prove (2.4). Due to (2.2) and (2.11),

$$C_0\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\le C(T)+\frac{K_1^2}{2}\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\tag{2.13}$$

Integrating on $(0,t)$, by (2.2), we have (2.4). □

Lemma 2.2. *Fix $T>0$. There exist a constant $C(T)>0$, such that*

$$\|u\|_{L^\infty((0,T)\times\mathbb{R})}\le C(T),\tag{2.14}$$

$$\|u(t,\cdot)\|_{L^2(\mathbb{R})}\le C(T),\tag{2.15}$$

$$\int_0^t\|\partial_x u(s,\cdot)\|^4_{L^4(\mathbb{R})}\mathrm{d}s\le C(T),\tag{2.16}$$

for every $0\le t\le T$.

The proof of this lemma is based on the following result.

Lemma 2.3. *We have that*

$$\|\partial_x u(t,\cdot)\|^4_{L^4(\mathbb{R})}\le 9\int_{\mathbb{R}}u^2(\partial_x^2 u)^2\mathrm{d}x.\tag{2.17}$$

Proof. We begin by observing that

$$\int_{\mathbb{R}} (\partial_x u)^4 \mathrm{d}x = \int_{\mathbb{R}} \partial_x u (\partial_x u)^3 \mathrm{d}x = -3 \int_{\mathbb{R}} u (\partial_x u)^2 \partial_x^2 u \mathrm{d}x. \tag{2.18}$$

By the Young inequality,

$$3 \int_{\mathbb{R}} |u| (\partial_x u)^2 |\partial_x^2 u| \mathrm{d}x \leq \frac{1}{2} \int_{\mathbb{R}} (\partial_x u)^4 \mathrm{d}x + \frac{9}{2} \int_{\mathbb{R}} u^2 (\partial_x^2 u)^2 \mathrm{d}x.$$

It follows from (2.18) that

$$\frac{1}{2} \int_{\mathbb{R}} (\partial_x u)^4 \mathrm{d}x \leq \frac{9}{2} \int_{\mathbb{R}} u^2 (\partial_x^2 u)^2 \mathrm{d}x,$$

which gives (2.17). □

Proof of Lemma 2.2. Let $0 \leq t \leq T$. Multiplying (1.1) by $2u$, an integration on $\mathbb{R}$ gives

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \|u(t,\cdot)\|^2_{L^2(\mathbb{R})} &= 2 \int_{\mathbb{R}} u \partial_t u \mathrm{d}x \\
&= -2 \int_{\mathbb{R}} u \partial_x^2 u \mathrm{d}x - 2\beta^2 \int_{\mathbb{R}} u \partial_x^4 u \mathrm{d}x + 2\gamma^2 \int_{\mathbb{R}} u (\partial_x u) \partial_x^2 u \mathrm{d}x \\
&\quad - 2\tau \int_{\mathbb{R}} u \partial_x u \partial_x^2 u \mathrm{d}x - 2\kappa \int_{\mathbb{R}} u (\partial_x u)^4 \mathrm{d}x - 2q \int_{\mathbb{R}} u (\partial_x u)^2 \mathrm{d}x \\
&\quad - 2\delta \int_{\mathbb{R}} u \partial_x u \partial_x^3 u \mathrm{d}x \\
&= 2\alpha \|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\beta \int_{\mathbb{R}} \partial_x u \partial_x^3 u \mathrm{d}x - \frac{2\gamma^2}{3} \|\partial_x u(t,\cdot)\|^4_{L^4(\mathbb{R})} \\
&\quad - 2\tau \int_{\mathbb{R}} u \partial_x u \partial_x^2 u \mathrm{d}x - 2\kappa \int_{\mathbb{R}} u (\partial_x u)^4 \mathrm{d}x - 2q \int_{\mathbb{R}} u (\partial_x u)^2 \mathrm{d}x \\
&\quad - 2\delta \int_{\mathbb{R}} u \partial_x u \partial_x^3 u \mathrm{d}x \\
&= 2\alpha \|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})} - 2\beta^2 \left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} - \frac{2\gamma^2}{3} \|\partial_x u(t,\cdot)\|^4_{L^4(\mathbb{R})} \\
&\quad - 2\tau \int_{\mathbb{R}} u \partial_x u \partial_x^2 u \mathrm{d}x - 2\kappa \int_{\mathbb{R}} u (\partial_x u)^4 \mathrm{d}x - 2q \int_{\mathbb{R}} u (\partial_x u)^2 \mathrm{d}x \\
&\quad - 2\delta \int_{\mathbb{R}} u \partial_x u \partial_x^3 u \mathrm{d}x.
\end{aligned}$$

Therefore, we have that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|u(t,\cdot)\|^2_{L^2(\mathbb{R})}+2\beta^2\left\|\partial_x^2u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\frac{2\gamma^2}{3}\|\partial_xu(t,\cdot)\|^4_{L^4(\mathbb{R})}\\
&\quad=2\alpha\|\partial_xu(t,\cdot)\|^2_{L^2(\mathbb{R})}-2\tau\int_{\mathbb{R}}u\partial_xu\partial_x^2u\mathrm{d}x-2\kappa\int_{\mathbb{R}}u(\partial_xu)^4\mathrm{d}x\\
&\qquad-2q\int_{\mathbb{R}}u(\partial_xu)^2\mathrm{d}x-2\delta\int_{\mathbb{R}}u\partial_xu\partial_x^3u\mathrm{d}x.
\end{aligned}\tag{2.19}$$

Due to (2.2) and the Young inequality,

$$\begin{aligned}
2\tau\int_{\mathbb{R}}|u\partial_xu||\partial_x^2u|\mathrm{d}x&\le\tau^2\int_{\mathbb{R}}u^2(\partial_xu)^2\mathrm{d}x+\left\|\partial_x^2u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\le\tau^2\|u\|^2_{L^\infty((0,T)\times\mathbb{R})}\|\partial_xu(t,\cdot)\|^2_{L^2(\mathbb{R})}+\left\|\partial_x^2u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\le C(T)\|u\|^2_{L^\infty((0,T)\times\mathbb{R})}+\left\|\partial_x^2u(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\
2|q|\int_{\mathbb{R}}|u|(\partial_xu)^2\mathrm{d}x&\le2|q|\|u\|_{L^\infty((0,T)\times\mathbb{R})}\|\partial_xu(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&\le C(T)\|u\|_{L^\infty((0,T)\times\mathbb{R})}\le C(T)\|u\|^2_{L^\infty((0,T)\times\mathbb{R})}+C(T),\\
2|\delta|\int_{\mathbb{R}}|u\partial_xu||\partial_x^3u|\mathrm{d}x&\\
&\le\delta^2\int_{\mathbb{R}}u^2(\partial_xu)^2\mathrm{d}x+\left\|\partial_x^3u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\le\delta^2\|u\|^2_{L^\infty((0,T)\times\mathbb{R})}\|\partial_xu(t,\cdot)\|^2_{L^2(\mathbb{R})}+\left\|\partial_x^3u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\le C(T)\|u\|^2_{L^\infty((0,T)\times\mathbb{R})}+\left\|\partial_x^3u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.
\end{aligned}$$

It follows from (2.2) and (2.19) that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|u(t,\cdot)\|^2_{L^2(\mathbb{R})}+2\beta^2\left\|\partial_x^2u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\frac{2\gamma^2}{3}\|\partial_xu(t,\cdot)\|^4_{L^4(\mathbb{R})}\\
&\quad\le2|\alpha|\|\partial_xu(t,\cdot)\|^2_{L^2(\mathbb{R})}+\left\|\partial_x^3u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+C(T)\|u\|^2_{L^\infty((0,T)\times\mathbb{R})}\\
&\qquad+2|\kappa|\int_{\mathbb{R}}|u|(\partial_xu)^4\mathrm{d}x+C(T)\\
&\quad\le\left\|\partial_x^3u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+C(T)\left(1+\|u\|^2_{L^\infty((0,T)\times\mathbb{R})}\right)+2|\kappa|\int_{\mathbb{R}}|u|(\partial_xu)^4\mathrm{d}x\\
&\qquad+C(T)\|u\|^2_{L^\infty((0,T)\times\mathbb{R})}+\left\|\partial_x^2u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.
\end{aligned}\tag{2.20}$$

Thanks to (2.17), we have that

$$\begin{aligned}
2|\kappa|\int_{\mathbb{R}}|u|(\partial_xu)^4\mathrm{d}x&\le2|\kappa|\|u\|_{L^\infty((0,T)\times\mathbb{R})}\|\partial_xu(t,\cdot)\|^4_{L^4(\mathbb{R})}\\
&\le18|\kappa|\|u\|_{L^\infty((0,T)\times\mathbb{R})}\int_{\mathbb{R}}u^2(\partial_x^2u)^2\mathrm{d}x
\end{aligned}$$

$$\leq 18|\kappa| \|u\|^3_{L^\infty((0,T)\times\mathbb{R})} \left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.$$

Consequently, by (2.20),

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|u(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\beta^2\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{2\gamma^2}{3}\|\partial_x u(t,\cdot)\|^4_{L^4(\mathbb{R})}\\
&\quad\leq \left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + C(T)\left(1+\|u\|^2_{L^\infty((0,T)\times\mathbb{R})}\right)\\
&\qquad + 18|\kappa|\|u\|^3_{L^\infty((0,T)\times\mathbb{R})}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\qquad + C(T)\|u\|^2_{L^\infty((0,T)\times\mathbb{R})} + \left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.
\end{aligned} \tag{2.21}$$

Integrating on $(0,t)$, by (1.3), (2.2) and (2.4), we have that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|u(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\beta^2\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{2\gamma^2}{3}\|\partial_x u(t,\cdot)\|^4_{L^4(\mathbb{R})}\\
&\quad\leq C_0 + \int_0^t \left\|\partial_x^3 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})}\mathrm{d}s + C(T)\left(1+\|u\|^2_{L^\infty((0,T)\times\mathbb{R})}\right)t\\
&\qquad + 18|\kappa|\|u\|^3_{L^\infty((0,T)\times\mathbb{R})}\int_0^t\left\|\partial_x^2 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})}\mathrm{d}s\\
&\qquad + C(T)\|u\|^2_{L^\infty((0,T)\times\mathbb{R})}t + \int_0^t\left\|\partial_x^2 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})}\mathrm{d}s\\
&\quad\leq C(T)\left(1+\|u\|^2_{L^\infty((0,T)\times\mathbb{R})}+\|u\|^3_{L^\infty((0,T)\times\mathbb{R})}\right).
\end{aligned} \tag{2.22}$$

Due to the Young inequality,

$$\begin{aligned}
\|u\|^3_{L^\infty((0,T)\times\mathbb{R})} &= \sqrt{D_2}\|u\|_{L^\infty((0,T)\times\mathbb{R})}\frac{1}{\sqrt{D_2}}\|u\|^2_{L^\infty((0,T)\times\mathbb{R})}\\
&\leq \frac{D_2}{2}\|u\|^2_{L^\infty((0,T)\times\mathbb{R})} + \frac{1}{2D_2}\|u\|^4_{L^\infty((0,T)\times\mathbb{R})},
\end{aligned}$$

where D_2 is a positive constant, which will be specified later. Therefore, by (2.22),

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|u(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\beta^2\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{2\gamma^2}{3}\|\partial_x u(t,\cdot)\|^4_{L^4(\mathbb{R})}\\
&\quad\leq C(T)\left(1+\left(1+\frac{D_2}{2}\right)\|u\|^2_{L^\infty((0,T)\times\mathbb{R})} + \frac{2}{D_2}\|u\|^4_{L^\infty((0,T)\times\mathbb{R})}\right).
\end{aligned} \tag{2.23}$$

We prove (2.14). Thanks to (2.2), (2.23) and the Hölder inequality,

$$\begin{aligned}
u(t,x)^2 = 2\int_{-\infty}^x u\partial_x u\mathrm{d}y \leq 2\int_{\mathbb{R}}|u||\partial_x u|\mathrm{d}x \leq 2\|u(t,\cdot)\|_{L^2(\mathbb{R})}\|\partial_x u(t,\cdot)\|_{L^2(\mathbb{R})}\\
\leq C(T)\sqrt{\left(1+\left(1+\frac{D_2}{2}\right)\|u\|^2_{L^\infty((0,T)\times\mathbb{R})}+\frac{2}{D_2}\|u\|^4_{L^\infty((0,T)\times\mathbb{R})}\right)}.
\end{aligned}$$

Therefore,

$$\left(1-\frac{C(T)}{2D_2}\right)\|u\|^4_{L^\infty((0,T)\times\mathbb{R})} - C(T)\left(1+\frac{D_2}{2}\right)\|u\|^2_{L^\infty((0,T)\times\mathbb{R})} - C(T) \leq 0.$$

Taking

$$D_2 = C(T), \tag{2.24}$$

we have that

$$\frac{1}{2}\|u\|^4_{L^\infty((0,T)\times\mathbb{R})} - C(T)\|u\|^2_{L^\infty((0,T)\times\mathbb{R})} - C(T) \le 0,$$

which gives (2.14).

Finally, (2.15) follows from (2.14), (2.23) and (2.24). □

Lemma 2.4. *Fix $T > 0$. There exist a constant $C(T) > 0$, such that*

$$\|\partial_x u\|_{L^\infty((0,T)\times\mathbb{R})} \le C(T), \tag{2.25}$$

$$\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \beta^2\int_0^t \left\|\partial_x^4 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} \mathrm{d}s \le C(T), \tag{2.26}$$

for every $0 \le t \le T$.

Proof. Let $0 \le t \le T$. Multiplying (1.1) by $2\partial_x^4 u$, an integration on $\mathbb{R}$ gives

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} &= 2\int_{\mathbb{R}} \partial_x^4 u\partial_t u \mathrm{d}x \\
&= -2\alpha\int_{\mathbb{R}} \partial_x^2 u\partial_x^4 u\mathrm{d}x - 2\beta^2\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\gamma^2\int_{\mathbb{R}}(\partial_x u)^2\partial_x^2 u\partial_x^4 u\mathrm{d}x \\
&\quad - 2\tau\int_{\mathbb{R}} \partial_x u\partial_x^2 u\partial_x^4 u\mathrm{d}x - 2\kappa\int_{\mathbb{R}}(\partial_x u)^4\partial_x^4 u\mathrm{d}x - 2q\int_{\mathbb{R}}(\partial_x u)^2\partial_x^4 u\mathrm{d}x \\
&\quad - 2\delta\int_{\mathbb{R}} \partial_x u\partial_x^3 u\partial_x^4 u\mathrm{d}x \\
&= 2\alpha\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} - 2\beta^2\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\gamma^2\int_{\mathbb{R}}(\partial_x u)^2\partial_x^2 u\partial_x^4 u\mathrm{d}x \\
&\quad - 2\tau\int_{\mathbb{R}} \partial_x u\partial_x^2 u\partial_x^4 u\mathrm{d}x + 8\kappa\int_{\mathbb{R}}(\partial_x u)^3\partial_x^2 u\partial_x^3 u\mathrm{d}x - 2q\int_{\mathbb{R}}(\partial_x u)^2\partial_x^4 u\mathrm{d}x \\
&\quad - 2\delta\int_{\mathbb{R}} \partial_x u\partial_x^3 u\partial_x^4 u\mathrm{d}x.
\end{aligned}$$

Therefore, we have that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\beta^2\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\
&\quad = 2\alpha\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\gamma^2\int_{\mathbb{R}}(\partial_x u)^2\partial_x^2 u\partial_x^4 u\mathrm{d}x - 2\tau\int_{\mathbb{R}} \partial_x u\partial_x^2 u\partial_x^4 u\mathrm{d}x \\
&\qquad + 8\kappa\int_{\mathbb{R}}(\partial_x u)^3\partial_x^2 u\partial_x^3 u\mathrm{d}x - 2q\int_{\mathbb{R}}(\partial_x u)^2\partial_x^4 u\mathrm{d}x - 2\delta\int_{\mathbb{R}} \partial_x u\partial_x^3 u\partial_x^4 u\mathrm{d}x.
\end{aligned} \tag{2.27}$$

Due to the Young inequality,

$$2\gamma^2\int_{\mathbb{R}}(\partial_x u)^2|\partial_x^2 u|\partial_x^4 u\mathrm{d}x = 2\int_{\mathbb{R}}\left|\frac{\gamma^2(\partial_x u)^2\partial_x^2 u}{\beta\sqrt{D_3}}\right|\left|\beta\sqrt{D_3}\partial_x^4 u\right|\mathrm{d}x$$

$$
\begin{aligned}
&\leq \frac{\gamma^4}{\beta^2 D_3}\int_{\mathbb{R}}(\partial_x u)^4(\partial_x^2 u)^2\mathrm{d}x+\beta^2 D_3\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\leq \frac{\gamma^4}{\beta^2 D_3}\left\|\partial_x u\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\left\|\partial_x u(t,\cdot)\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\beta^2 D_3\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})},
\end{aligned}
$$

$$
\begin{aligned}
2|\tau|\int_{\mathbb{R}}|\partial_x u\partial_x^2 u||\partial_x^4 u|\mathrm{d}x&=2\int_{\mathbb{R}}\left|\frac{\tau\partial_x u\partial_x^2 u}{\beta\sqrt{D_3}}\right|\left|\beta\partial_x^4 u\right|\mathrm{d}x\\
&\leq \frac{\tau^2}{\beta^2 D_3}\left\|\partial_x u(t,\cdot)\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+D_3\beta^2\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})},
\end{aligned}
$$

$$
\begin{aligned}
8|\kappa|\int_{\mathbb{R}}|\partial_x u|^3|\partial_x^2 u||\partial_x^3 u|\mathrm{d}x&\leq 8|\kappa|\left\|\partial_x u\right\|^3_{L^\infty((0,T)\times\mathbb{R})}\int_{\mathbb{R}}|\partial_x^2 u||\partial_x^3 u|\mathrm{d}x\\
&\leq 4\kappa^2\left\|\partial_x u\right\|^3_{L^\infty((0,T)\times\mathbb{R})}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+4\left\|\partial_x u\right\|^3_{L^\infty((0,T)\times\mathbb{R})}\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})},
\end{aligned}
$$

$$
\begin{aligned}
2|q|\int_{\mathbb{R}}(\partial_x u)^2|\partial_x^4 u|\mathrm{d}x&=2\int_{\mathbb{R}}\left|\frac{q(\partial_x u)^2}{\beta\sqrt{D_3}}\right|\left|\beta\sqrt{D_3}\partial_x^4 u\right|\mathrm{d}x\\
&\leq \frac{q^2}{\beta^2 D_3}\left\|\partial_x u(t,\cdot)\right\|^4_{L^4(\mathbb{R})}+\beta^2 D_3\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})},
\end{aligned}
$$

$$
\begin{aligned}
2|\delta|\int_{\mathbb{R}}|\partial_x u\partial_x^3 u||\partial_x^4 u|\mathrm{d}x&=2\int_{\mathbb{R}}\left|\frac{\delta\partial_x u\partial_x^3 u}{\beta\sqrt{D_3}}\right|\left|\beta\sqrt{D_3}\partial_x^4 u\right|\mathrm{d}x\\
&\leq \frac{\delta^2}{\beta^2 D_3}\int_{\mathbb{R}}(\partial_x u)^2(\partial_x^3 u)^2\mathrm{d}x+\beta^2 D_3\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\leq \frac{\delta^2}{\beta^2 D_3}\left\|\partial_x u\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\beta^2 D_3\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})},
\end{aligned}
$$

where D_3 is a positive constant, which will be specified later. It follows from (2.27) that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+2\left(1-2D_3\right)\beta^2\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad\leq C_0\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\left(\frac{\tau^2}{\beta^2 D_3}+\frac{\gamma^4}{\beta^2 D_3}\left\|\partial_x u\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\right)\left\|\partial_x u(t,\cdot)\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\qquad+4\kappa^2\left\|\partial_x u\right\|^3_{L^\infty((0,T)\times\mathbb{R})}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\frac{q^2}{\beta^2 D_3}\left\|\partial_x u(t,\cdot)\right\|^4_{L^4(\mathbb{R})}\\
&\qquad+\frac{\delta^2}{\beta^2 D_3}\left\|\partial_x u\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+4\left\|\partial_x u\right\|^3_{L^\infty((0,T)\times\mathbb{R})}\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.
\end{aligned}
$$

Taking $D_3=\frac{1}{4}$, we obtain that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\beta^2\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad\leq C_0\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\left(\frac{4\tau^2}{\beta^2}+\frac{4\gamma^4}{\beta^2}\right)\left\|\partial_x u\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\left\|\partial_x u(t,\cdot)\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\qquad+4\kappa^2\left\|\partial_x u\right\|^3_{L^\infty((0,T)\times\mathbb{R})}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\frac{4q^2}{\beta^2}\left\|\partial_x u(t,\cdot)\right\|^4_{L^4(\mathbb{R})}\\
&\qquad+\frac{4\delta^2}{\beta^2}\left\|\partial_x u\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+4\left\|\partial_x u\right\|^3_{L^\infty((0,T)\times\mathbb{R})}\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.
\end{aligned}
$$

Integrating on $(0,t)$, by (1.3), (2.2), (2.4) and (2.15), we have that

$$\begin{aligned}
&\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \beta^2 \int_0^t \left\|\partial_x^4 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} \mathrm{d}s \\
&\quad \le C_0 + C_0 \int_0^t \left\|\partial_x^3 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} \mathrm{d}s + \frac{4q^2}{\beta^2} \int_0^t \left\|\partial_x u(s,\cdot)\right\|^4_{L^4(\mathbb{R})} \mathrm{d}s \\
&\qquad + \left(\frac{4\tau^2}{\beta^2} + \frac{4\gamma^4}{\beta^2} \left\|\partial_x u\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\right) \int_0^t \left\|\partial_x u(s,\cdot)\partial_x^2 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} \mathrm{d}s \\
&\qquad + 4\kappa^2 \left\|\partial_x u\right\|^3_{L^\infty((0,T)\times\mathbb{R})} \int_0^t \left\|\partial_x^2 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} \mathrm{d}s \\
&\qquad + \frac{4\delta^2}{\beta^2} \left\|\partial_x u\right\|^2_{L^\infty((0,T)\times\mathbb{R})} \int_0^t \left\|\partial_x^3 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} \mathrm{d}s \\
&\qquad + 4 \left\|\partial_x u\right\|^3_{L^\infty((0,T)\times\mathbb{R})} \int_0^t \left\|\partial_x^3 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} \mathrm{d}s \\
&\quad \le C(T)\left(1 + \left\|\partial_x u\right\|^2_{L^\infty((0,T)\times\mathbb{R})} + \left\|\partial_x u\right\|^3_{L^\infty((0,T)\times\mathbb{R})}\right). \qquad (2.28)
\end{aligned}$$

Due to the Young inequality,

$$\begin{aligned}
\left\|\partial_x u\right\|^3_{L^\infty((0,T)\times\mathbb{R})} &= \sqrt{D_4}\left\|\partial_x u\right\|_{L^\infty((0,T)\times\mathbb{R})} \frac{1}{\sqrt{D_3}} \left\|\partial_x u\right\|^2_{L^\infty((0,T)\times\mathbb{R})} \\
&\le \frac{D_4}{2} \left\|\partial_x u\right\|^2_{L^\infty((0,T)\times\mathbb{R})} + \frac{1}{2D_2} \left\|\partial_x u\right\|^4_{L^\infty((0,T)\times\mathbb{R})},
\end{aligned}$$

where D_4 is a positive constant, which will be specified later. Therefore, by (2.28),

$$\begin{aligned}
&\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \beta^2 \int_0^t \left\|\partial_x^4 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} \mathrm{d}s \\
&\quad \le C(T)\left(1 + \left(1 + \frac{D_4}{2}\right) \left\|\partial_x u\right\|^2_{L^\infty((0,T)\times\mathbb{R})} + \frac{1}{2D_4} \left\|\partial_x u\right\|^4_{L^\infty((0,T)\times\mathbb{R})}\right).
\end{aligned} \qquad (2.29)$$

We prove (2.25). Thanks to (2.2), (2.29) and the Hölder inequality,

$$\begin{aligned}
(\partial_x u(t,x))^2 &= 2\int_{-\infty}^x \partial_x u \partial_x^2 u \mathrm{d}x = 2\int_{\mathbb{R}} |\partial_x u||\partial_x^2 u| \mathrm{d}x \le 2\left\|\partial_x u(t,\cdot)\right\|_{L^2(\mathbb{R})} \left\|\partial_x^2 u(t,\cdot)\right\|_{L^2(\mathbb{R})} \\
&\le C(T)\sqrt{\left(1 + \left(1 + \frac{D_4}{2}\right) \left\|\partial_x u\right\|^2_{L^\infty((0,T)\times\mathbb{R})} + \frac{1}{2D_4} \left\|\partial_x u\right\|^4_{L^\infty((0,T)\times\mathbb{R})}\right)}.
\end{aligned}$$

Therefore,

$$\left(1 - \frac{C(T)}{2D_4}\right) \left\|\partial_x u\right\|^4_{L^\infty((0,T)\times\mathbb{R})} - C(T)\left(1 + \frac{D_4}{2}\right) \left\|\partial_x u\right\|^2_{L^\infty((0,T)\times\mathbb{R})} - C(T) \le 0.$$

Taking

$$D_4 = C(T), \qquad (2.30)$$

we have that

$$\frac{1}{2}\|\partial_x u\|^4_{L^\infty((0,T)\times\mathbb{R})} - C(T)\|\partial_x u\|^4_{L^\infty((0,T)\times\mathbb{R})} - C(T) \le 0,$$

which gives (2.25).

Finally, (2.26) follows from (2.25), (2.29) and (2.30). ☐

Lemma 2.5. *Fix $T > 0$. There exist a constant $C(T) > 0$, such that*

$$\beta^2\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \int_0^t \|\partial_t u(s,\cdot)\|^2_{L^2(\mathbb{R})}\,\mathrm{d}s \le C(T), \tag{2.31}$$

for every $0 \le t \le T$.

Proof. Let $0 \le t \le T$. Multiplying (1.1) by $2\partial_t u$, an integration on $\mathbb{R}$ gives

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\beta^2\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} - \alpha\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})}\right)\\
&\quad= -2\beta^2\int_{\mathbb{R}}\partial_x^4 u\partial_t u\mathrm{d}x + 2\alpha\int_{\mathbb{R}}\partial_x^2 u\partial_t u\mathrm{d}x\\
&\quad= -2\|\partial_t u(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\gamma^2\int_{\mathbb{R}}(\partial_x u)^2\partial_x^2 u\partial_t u\mathrm{d}x - 2\tau\int_{\mathbb{R}}\partial_x u\partial_x^2 u\partial_t u\mathrm{d}x\\
&\qquad- 2\kappa\int_{\mathbb{R}}(\partial_x u)^4\partial_t u\mathrm{d}x - 2q\int_{\mathbb{R}}(\partial_x u)^2\partial_t u\mathrm{d}x - 2\delta\int_{\mathbb{R}}\partial_x u\partial_x^3 u\partial_t u\mathrm{d}x.
\end{aligned}$$

Therefore, we have that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\beta^2\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} - \alpha\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})}\right) + 2\|\partial_t u(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&\quad= 2\gamma^2\int_{\mathbb{R}}(\partial_x u)^2\partial_x^2 u\partial_t u\mathrm{d}x - 2\tau\int_{\mathbb{R}}\partial_x u\partial_x^2 u\partial_t u\mathrm{d}x - 2\kappa\int_{\mathbb{R}}(\partial_x u)^4\partial_t u\mathrm{d}x\\
&\qquad- 2q\int_{\mathbb{R}}(\partial_x u)^2\partial_t u\mathrm{d}x - 2\delta\int_{\mathbb{R}}\partial_x u\partial_x^3 u\partial_t u\mathrm{d}x.
\end{aligned} \tag{2.32}$$

Due to (2.2), (2.25), (2.26) and the Young inequality,

$$\begin{aligned}
2\gamma^2\int_{\mathbb{R}}(\partial_x u)^2|\partial_x^2 u||\partial_t u|\mathrm{d}x &\le 2\gamma^2\|\partial_x u\|^2_{L^\infty((0,T)\times\mathbb{R})}\int_{\mathbb{R}}|\partial_x^2 u||\partial_t u|\mathrm{d}x\\
&\le 2C(T)\int_{\mathbb{R}}|\partial_x^2 u||\partial_t u|\mathrm{d}x = 2\int_{\mathbb{R}}\left|\frac{C(T)\partial_x^2 u}{\sqrt{D_5}}\right|\left|\sqrt{D_5}\partial_t u\right|\mathrm{d}x\\
&\le \frac{C(T)}{D_5}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + D_5\|\partial_t u(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&\le \frac{C(T)}{D_5} + D_5\|\partial_t u(t,\cdot)\|^2_{L^2(\mathbb{R})},\\
2|\tau|\int_{\mathbb{R}}|\partial_x u||\partial_x^2 u||\partial_t u|\mathrm{d}x &\le 2|\tau|\|\partial_x u\|_{L^\infty((0,T)\times\mathbb{R})}\int_{\mathbb{R}}|\partial_x^2 u||\partial_t u|\mathrm{d}x\\
&\le 2C(T)\int_{\mathbb{R}}|\partial_x^2 u||\partial_t u|\mathrm{d}x = 2\int_{\mathbb{R}}\left|\frac{C(T)\partial_x^2 u}{\sqrt{D_5}}\right|\left|\sqrt{D_5}\partial_t u\right|\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{C(T)}{D_5}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + C(T)\left\|\partial_t u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\leq \frac{C(T)}{D_5} + C(T)\left\|\partial_t u(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\
2|\kappa|\int_{\mathbb{R}}(\partial_x u)^4\partial_t u \mathrm{d}x &\leq 2|\kappa|\left\|\partial_x u\right\|^3_{L^\infty((0,T)\times\mathbb{R})}\int_{\mathbb{R}}|\partial_x u||\partial_t u|\mathrm{d}x\\
&\leq 2C(T)\int_{\mathbb{R}}|\partial_x u||\partial_t u|\mathrm{d}x = 2\int_{\mathbb{R}}\left|\frac{C(T)\partial_x u}{\sqrt{D_5}}\right|\left|\sqrt{D_5}\partial_t u\right|\mathrm{d}x\\
&\leq \frac{C(T)}{D_5}\left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + D_5\left\|\partial_t u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\leq \frac{C(T)}{D_5} + D_5\left\|\partial_t u(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\
2|q|\int_{\mathbb{R}}(\partial_x u)^2|\partial_t u|\mathrm{d}x &= 2|q|\left\|\partial_x u\right\|_{L^\infty((0,T)\times\mathbb{R})}\int_{\mathbb{R}}|\partial_x u||\partial_t u|\mathrm{d}x\\
&\leq 2C(T)\int_{\mathbb{R}}|\partial_x u||\partial_t u|\mathrm{d}x = 2\int_{\mathbb{R}}\left|\frac{C(T)\partial_x u}{\sqrt{D_5}}\right|\left|\sqrt{D_5}\partial_t u\right|\mathrm{d}x\\
&\leq \frac{C(T)}{D_5}\left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + D_5\left\|\partial_t u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\leq \frac{C(T)}{D_5} + D_5\left\|\partial_t u(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\
2|\delta|\int_{\mathbb{R}}|\partial_x u||\partial_x^3 u||\partial_t u|\mathrm{d}x &= 2|\delta|\left\|\partial_x u\right\|_{L^\infty((0,T)\times\mathbb{R})}\int_{\mathbb{R}}|\partial_x^3 u||\partial_t u|\mathrm{d}x\\
&\leq 2C(T)\int_{\mathbb{R}}|\partial_x^3 u||\partial_t u|\mathrm{d}x = 2\int_{\mathbb{R}}\left|\frac{C(T)\partial_x^3 u}{\sqrt{D_5}}\right|\left|\sqrt{D_5}\partial_t u\right|\mathrm{d}x\\
&\leq \frac{C(T)}{D_5}\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + D_5\left\|\partial_t u(t,\cdot)\right\|^2_{L^2(\mathbb{R})},
\end{aligned}
$$

where D_5 is a positive constant, which will be specified later. It follows from (2.32) that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\beta^2\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} - \alpha\left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\right) + (2-5D_5)\left\|\partial_t u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad\leq \frac{C(T)}{D_5} + \frac{C(T)}{D_5}\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.
\end{aligned}
$$

Taking $D_5 = \frac{1}{5}$, we have that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\beta^2\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} - \alpha\left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\right) + \left\|\partial_t u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad\leq C(T) + C(T)\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.
\end{aligned}
$$

(1.3), (2.2) and an integration on $(0,t)$ give

$$
\beta^2\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} - \alpha\left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \int_0^t\left\|\partial_t u(s,\cdot)\right\|^2_{L^2(\mathbb{R})}\mathrm{d}s
$$

$$\leq C_0 + C(T)t + C(T)\int_0^t \left\|\partial_x^3 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} \mathrm{d}s \leq C(T).$$

Therefore, by (2.2),

$$\begin{aligned}
&\beta^2 \left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \int_0^t \left\|\partial_t u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} \mathrm{d}s \\
&\quad \leq C(T) + \alpha \left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \leq C(T) + |\alpha| \left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \leq C(T),
\end{aligned}$$

which gives (2.31). □

Lemma 2.6. *Fix $T > 0$ and assume (1.3), with $\ell \in \{3, 4\}$. There exist a constant $C(T) > 0$, such that*

$$\left\|\partial_x^2 u\right\|_{L^\infty((0,T)\times\mathbb{R})} \leq C(T). \tag{2.33}$$

In particular,

$$\beta^2 \left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \int_0^t \left\|\partial_t \partial_x u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} \mathrm{d}s \leq C(T), \tag{2.34}$$

for every $0 \leq t \leq T$.

Proof. Let $0 \leq t \leq T$. Multiplying (1.1) by $-2\partial_t\partial_x^2 u$, an integration on $\mathbb{R}$ gives

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\beta^2 \left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} - \alpha \left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\right) \\
&\quad = -2\beta^2 \int_{\mathbb{R}} \partial_x^4 u \partial_t \partial_x^2 u \mathrm{d}x + \alpha \int_{\mathbb{R}} \partial_x^2 u \partial_t \partial_x^2 u \mathrm{d}x \\
&\quad = 2\int_{\mathbb{R}} \partial_t \partial_x^2 u \partial_t u \mathrm{d}x - 2\gamma^2 \int_{\mathbb{R}} (\partial_x u)^2 \partial_x^2 u \partial_t \partial_x^2 u \mathrm{d}x + 2\tau \int_{\mathbb{R}} \partial_x u \partial_x^2 u \partial_t \partial_x^2 u \mathrm{d}x \\
&\qquad + 2\kappa \int_{\mathbb{R}} (\partial_x u)^4 \partial_t \partial_x^2 u \mathrm{d}x + 2q \int_{\mathbb{R}} (\partial_x u)^2 \partial_t \partial_x^2 u \mathrm{d}x + 2\delta \int_{\mathbb{R}} \partial_x u \partial_x^3 u \partial_t \partial_x^2 u \mathrm{d}x \\
&\quad = -2 \left\|\partial_t \partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 4\gamma^2 \int_{\mathbb{R}} \partial_x u (\partial_x^2 u)^2 \partial_t \partial_x u \mathrm{d}x + 2\gamma^2 \int_{\mathbb{R}} (\partial_x u)^2 \partial_x^3 u \partial_t \partial_x u \mathrm{d}x \\
&\qquad - 2\tau \int_{\mathbb{R}} (\partial_x^2 u)^2 \partial_t \partial_x u \mathrm{d}x - 2\tau \int_{\mathbb{R}} \partial_x \partial_x^3 u \partial_t \partial_x u \mathrm{d}x - 8\kappa \int_{\mathbb{R}} (\partial_x u)^3 \partial_x^2 u \partial_t \partial_x u \mathrm{d}x \\
&\qquad - 4q \int_{\mathbb{R}} \partial_x u \partial_x^2 u \partial_t \partial_x u \mathrm{d}x - 2\delta \int_{\mathbb{R}} \partial_x^2 u \partial_x^3 u \partial_t \partial_x u \mathrm{d}x - 2\delta \int_{\mathbb{R}} \partial_x u \partial_x^4 u \partial_t \partial_x u \mathrm{d}x.
\end{aligned}$$

Therefore, we have that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\beta^2 \left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} - \alpha \left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\right) + 2 \left\|\partial_t \partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\
&\quad = 4\gamma^2 \int_{\mathbb{R}} \partial_x u (\partial_x^2 u)^2 \partial_t \partial_x u \mathrm{d}x + 2\gamma^2 \int_{\mathbb{R}} (\partial_x u)^2 \partial_x^3 u \partial_t \partial_x u \mathrm{d}x \qquad (2.35) \\
&\qquad - 2\tau \int_{\mathbb{R}} (\partial_x^2 u)^2 \partial_t \partial_x u \mathrm{d}x - 2\tau \int_{\mathbb{R}} \partial_x u \partial_x^3 u \partial_t \partial_x u \mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&- 8\kappa \int_{\mathbb{R}} (\partial_x u)^3 \partial_x^2 u \partial_t \partial_x u \mathrm{d}x - 4q \int_{\mathbb{R}} \partial_x u \partial_x^2 u \partial_t \partial_x u \mathrm{d}x \\
&- 2\delta \int_{\mathbb{R}} \partial_x^2 u \partial_x^3 u \partial_t \partial_x u \mathrm{d}x - 2\delta \int_{\mathbb{R}} \partial_x u \partial_x^4 u \partial_t \partial_x u \mathrm{d}x.
\end{aligned}
$$

Due to (2.2), (2.25), (2.26) and the Young inequality,

$$
\begin{aligned}
4\gamma^2 &\int_{\mathbb{R}} |\partial_x u| (\partial_x^2 u)^2 |\partial_t \partial_x u| \mathrm{d}x \leq 4\gamma^2 \left\| \partial_x u \right\|_{L^\infty((0,T)\times\mathbb{R})} \int_{\mathbb{R}} (\partial_x^2 u)^2 |\partial_t \partial_x u| \mathrm{d}x \\
&\leq 2C(T) \int_{\mathbb{R}} (\partial_x^2 u)^2 |\partial_t \partial_x u| \mathrm{d}x = 2 \int_{\mathbb{R}} \left| \frac{C(T)(\partial_x^2 u)^2}{\sqrt{D_6}} \right| \left| \sqrt{D_6} \partial_t \partial_x u \mathrm{d}x \right| \mathrm{d}x \\
&\leq \frac{C(T)}{D_6} \int_{\mathbb{R}} (\partial_x^2 u)^4 \mathrm{d}x + D_6 \left\| \partial_t \partial_x u(t,\cdot) \right\|^2_{L^2(\mathbb{R})} \\
&\leq \frac{C(T)}{D_6} \left\| \partial_x^2 u \right\|^2_{L^\infty((0,T)\times\mathbb{R})} \left\| \partial_x^2 u(t,\cdot) \right\|^2_{L^2(\mathbb{R})} + D_6 \left\| \partial_t \partial_x u(t,\cdot) \right\|^2_{L^2(\mathbb{R})} \\
&\leq \frac{C(T)}{D_6} \left\| \partial_x^2 u \right\|^2_{L^\infty((0,T)\times\mathbb{R})} + D_6 \left\| \partial_t \partial_x u(t,\cdot) \right\|^2_{L^2(\mathbb{R})}, \\
2\gamma^2 &\int_{\mathbb{R}} (\partial_x u)^2 |\partial_x^3 u| |\partial_t \partial_x u| \mathrm{d}x \leq 2\gamma^2 \left\| \partial_x u \right\|^2_{L^\infty((0,T)\times\mathbb{R})} \int_{\mathbb{R}} |\partial_x^3 u| \partial_t \partial_x u| \mathrm{d}x \\
&\leq 2C(T) \int_{\mathbb{R}} |\partial_x^3 u| |\partial_t \partial_x u| \mathrm{d}x = 2 \int_{\mathbb{R}} \left| \frac{C(T)\partial_x^3 u}{\sqrt{D_6}} \right| \left| \sqrt{D_6} \partial_t \partial_x u \mathrm{d}x \right| \mathrm{d}x \\
&\leq \frac{C(T)}{D_6} \left\| \partial_x^3 u(t,\cdot) \right\|^2_{L^2(\mathbb{R})} + D_6 \left\| \partial_t \partial_x u(t,\cdot) \right\|^2_{L^2(\mathbb{R})}, \\
2|\tau| &\int_{\mathbb{R}} (\partial_x^2 u)^2 |\partial_t \partial_x u| \mathrm{d}x = 2 \int_{\mathbb{R}} \left| \frac{\tau (\partial_x^2 u)^2 \mathrm{d}x}{\sqrt{D_6}} \right| \left| \sqrt{D_6} \partial_t \partial_x u \right| \mathrm{d}x \\
&\leq \frac{\tau^2}{D_6} \int_{\mathbb{R}} (\partial_x^2 u)^4 \mathrm{d}x + D_6 \left\| \partial_t \partial_x u(t,\cdot) \right\|^2_{L^2(\mathbb{R})} \\
&\leq \frac{\tau^2}{D_6} \left\| \partial_x^2 u \right\|^2_{L^\infty((0,T)\times\mathbb{R})} \left\| \partial_x^2 u(t,\cdot) \right\|^2_{L^2(\mathbb{R})} + D_6 \left\| \partial_t \partial_x u(t,\cdot) \right\|^2_{L^2(\mathbb{R})} \\
&\leq \frac{C(T)}{D_6} \left\| \partial_x^2 u \right\|^2_{L^\infty((0,T)\times\mathbb{R})} + D_6 \left\| \partial_t \partial_x u(t,\cdot) \right\|^2_{L^2(\mathbb{R})}, \\
2|\tau| &\int_{\mathbb{R}} |\partial_x u| |\partial_x^3 u| |\partial_t \partial_x u| \mathrm{d}x = 2|\tau| \left\| \partial_x u \right\|_{L^\infty((0,T)\times\mathbb{R})} \int_{\mathbb{R}} |\partial_x^3 u| |\partial_t \partial_x u| \mathrm{d}x \\
&\leq 2C(T) \int_{\mathbb{R}} |\partial_x^3 u| |\partial_t \partial_x u| \mathrm{d}x = \int_{\mathbb{R}} \left| \frac{C(T)\partial_x^3 u}{\sqrt{D_6}} \right| \left| \sqrt{D_6} \partial_t \partial_x u \right| \mathrm{d}x \\
&\leq \frac{C(T)}{D_6} \left\| \partial_x^3 u(t,\cdot) \right\|^2_{L^2(\mathbb{R})} + D_6 \left\| \partial_t \partial_x u(t,\cdot) \right\|^2_{L^2(\mathbb{R})}, \\
8|\kappa| &\int_{\mathbb{R}} |\partial_x u|^3 |\partial_x^2 u| |\partial_t \partial_x u| \mathrm{d}x \leq 8|\kappa| \left\| \partial_x u \right\|^3_{L^\infty((0,T)\times\mathbb{R})} \int_{\mathbb{R}} |\partial_x^2 u| |\partial_t \partial_x u| \mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&\leq 2C(T)\int_{\mathbb{R}} |\partial_x^2 u||\partial_t\partial_x u|\mathrm{d}x \leq 2\int_{\mathbb{R}} \left|\frac{C(T)\partial_x^2 u}{\sqrt{D_6}}\right|\left|\sqrt{D_6}\partial_t\partial_x u\right|\mathrm{d}x\\
&\leq \frac{C(T)}{D_6}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + D_6\left\|\partial_t\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\leq \frac{C(T)}{D_6} + D_6\left\|\partial_t\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})},
\end{aligned}
$$

$$
\begin{aligned}
4|q|\int_{\mathbb{R}} |\partial_x u||\partial_x^2 u||\partial_t\partial_x u|\mathrm{d}x &\leq 4|q|\left\|\partial_x u\right\|_{L^\infty((0,T)\times\mathbb{R})}\int_{\mathbb{R}} |\partial_x^2 u||\partial_t\partial_x u|\mathrm{d}x\\
&\leq 2C(T)\int_{\mathbb{R}} |\partial_x^2 u||\partial_t\partial_x u|\mathrm{d}x = 2\int_{\mathbb{R}} \left|\frac{C(T)\partial_x^2 u}{\sqrt{D_6}}\right|\left|\sqrt{D_6}\partial_t\partial_x u\right|\mathrm{d}x\\
&\leq \frac{C(T)}{D_6}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + D_6\left\|\partial_t\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\leq \frac{C(T)}{D_6} + D_6\left\|\partial_t\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})},
\end{aligned}
$$

$$
\begin{aligned}
2|\delta|\int_{\mathbb{R}} |\partial_x^2 u\partial_x^3 u||\partial_t\partial_x u|\mathrm{d}x &= 2\int_{\mathbb{R}} \left|\frac{\delta\partial_x^2 u\partial_x^3 u}{\sqrt{D_6}}\right|\left|\sqrt{D_6}\partial_t\partial_x u\right|\mathrm{d}x\\
&\leq \frac{\delta^2}{D_6}\int_{\mathbb{R}} (\partial_x^2 u)^2(\partial_x^3 u)^2\mathrm{d}x + D_6\left\|\partial_t\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\leq \frac{\delta^2}{D_6}\left\|\partial_x^2 u\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + D_6\left\|\partial_t\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})},
\end{aligned}
$$

$$
\begin{aligned}
2|\delta|\int_{\mathbb{R}} |\partial_x u||\partial_x^4 u||\partial_t\partial_x u|\mathrm{d}x &= 2|\delta|\left\|\partial_x u\right\|_{L^\infty((0,T)\times\mathbb{R})}\int_{\mathbb{R}} |\partial_x^4 u||\partial_t\partial_x u|\mathrm{d}x\\
&\leq 2C(T)\int_{\mathbb{R}} |\partial_x^4 u||\partial_t\partial_x u|\mathrm{d}x = 2\int_{\mathbb{R}} \left|\frac{C(T)\partial_x^4 u}{\sqrt{D_6}}\right|\left|\sqrt{D_6}\partial_t\partial_x u\right|\mathrm{d}x\\
&\leq \frac{C(T)}{D_6}\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + D_6\left\|\partial_t\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})},
\end{aligned}
$$

where D_6 is a positive constant, which will be specified later. it follows from (2.35) that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\beta^2\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} - \alpha\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\right) + 2\left(1-4D_6\right)\left\|\partial_t\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad\leq \frac{C(T)}{D_6}\left(1+\left\|\partial_x^2 u\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\right) + \frac{C(T)}{D_6}\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad + \frac{C(T)}{D_6}\left(1+\left\|\partial_x^2 u\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\right)\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.
\end{aligned}
$$

Taking $D_6 = \frac{1}{8}$, we have that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\beta^2\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} - \alpha\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\right) + \left\|\partial_t\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad\leq C(T)\left(1+\left\|\partial_x^2 u\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\right) + C(T)\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad + C(T)\left(1+\left\|\partial_x^2 u\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\right)\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.
\end{aligned}
$$

Integrating on $(0,t)$, by (1.3), (2.2) and (2.26), we obtain that

$$\begin{aligned}
&\beta^2 \left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} - \alpha \left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \int_0^t \|\partial_t\partial_x u(s,\cdot)\|^2_{L^2(\mathbb{R})}\,\mathrm{d}s \\
&\quad\le C_0 + C(T)\left(1+\left\|\partial_x^2 u\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\right)t + C(T)\int_0^t \left\|\partial_x^4 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})}\,\mathrm{d}s \\
&\qquad + C(T)\left(1+\left\|\partial_x^2 u\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\right)\int_0^t \left\|\partial_x^3 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})}\,\mathrm{d}s \\
&\quad\le C(T)\left(1+\left\|\partial_x^2 u\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\right).
\end{aligned}$$

Therefore, by (2.26),

$$\begin{aligned}
&\beta^2 \left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \int_0^t \|\partial_t\partial_x u(s,\cdot)\|^2_{L^2(\mathbb{R})}\,\mathrm{d}s \\
&\quad\le C(T)\left(1+\left\|\partial_x^2 u\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\right) + |\alpha|\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\
&\quad\le C(T)\left(1+\left\|\partial_x^2 u\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\right).
\end{aligned} \tag{2.36}$$

We prove (2.33). Thanks to (2.26), (2.36) and the Hölder inequality,

$$\begin{aligned}
(\partial_x^2 u(t,x))^2 &= 2\int_{-\infty}^x \partial_x^2 u\partial_x^3 u\mathrm{d}y \le 2\int_{\mathbb{R}} |\partial_x^2 u||\partial_x^3 u|\mathrm{d}x \le 2\left\|\partial_x^2 u(t,\cdot)\right\|_{L^2(\mathbb{R})}\left\|\partial_x^3 u(t,\cdot)\right\|_{L^2(\mathbb{R})} \\
&\le C(T)\sqrt{\left(1+\|\partial_x^2 u\|^2_{L^\infty((0,T)\times\mathbb{R})}\right)}.
\end{aligned}$$

Hence,

$$\left\|\partial_x^2 u\right\|^4_{L^\infty((0,T)\times\mathbb{R})} - C(T)\left\|\partial_x^2 u\right\|^2_{L^\infty((0,T)\times\mathbb{R})} - C(T) \le 0,$$

which gives (2.33).

Finally, (2.34) follows from (2.33) and (2.36). □

Lemma 2.7. *Fix $T>0$ and assume (1.3), with $\ell = 4$. There exist a constant $C(T)>0$, such that*

$$\left\|\partial_x^3 u\right\|_{L^\infty((0,T)\times\mathbb{R})} \le C(T), \tag{2.37}$$

$$\begin{aligned}
&\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{\beta^2}{2}\int_0^t \left\|\partial_x^6 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})}\,\mathrm{d}s \\
&\quad + 2\gamma^2\int_0^t \left\|\partial_x u(s,\cdot)\partial_x^5 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})}\,\mathrm{d}s \le C(T),
\end{aligned} \tag{2.38}$$

$$\int_0^t \left\|\partial_x^5 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \le C(T), \tag{2.39}$$

for every $0\le t\le T$.

Proof. Let $0 \leq t \leq T$. Multiplying (1.1) by $2\partial_x^8 u$, we have that

$$\begin{aligned} &2\partial_x^8 u\partial_t u + 2\alpha\partial_x^2 u\partial_x^8 u + 2\beta^2\partial_x^4 u\partial_x^8 u - 2\gamma^2(\partial_x u)^2\partial_x^2 u\partial_x^8 u \\ &\quad + 2\tau\int_{\mathbb{R}} \partial_x u\partial_x^2 u\partial_x^8 u \mathrm{d}x + 2\kappa(\partial_x u)^4\partial_x^8 u + 2q(\partial_x u)^2\partial_x^8 u \\ &\quad + 2\delta\partial_x u\partial_x^3 u\partial_x^8 u = 0. \end{aligned} \tag{2.40}$$

Observe that

$$\begin{aligned} 2\int_{\mathbb{R}} \partial_x^8 u\partial_t u \mathrm{d}x &= -2\int_{\mathbb{R}} \partial_x^7 u\partial_t\partial_x u = 2\int_{\mathbb{R}} \partial_x^6 u\partial_t\partial_x^2 u \mathrm{d}x \\ &= -2\int_{\mathbb{R}} \partial_x^5 u\partial_t\partial_x^3 u \mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}, \\ 2\alpha\int_{\mathbb{R}} \partial_x^2 u\partial_x^8 u \mathrm{d}x &= -2\alpha\int_{\mathbb{R}} \partial_x^3 u\partial_x^7 u \mathrm{d}x = 2\alpha\int_{\mathbb{R}} \partial_x^4 u\partial_x^6 u \mathrm{d}x \\ &= -2\alpha\left\|\partial_x^5 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}, \\ 2\beta^2\int_{\mathbb{R}} \partial_x^4 u\partial_x^8 u \mathrm{d}x &= -2\beta^2\int_{\mathbb{R}} \partial_x^5 u\partial_x^7 u dx = 2\beta^2\left\|\partial_x^6 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}, \\ -2\gamma^2\int_{\mathbb{R}} (\partial_x u)^2\partial_x^2 u\partial_x^8 u &= 4\gamma^2\int_{\mathbb{R}} \partial_x u(\partial_x^2 u)^2\partial_x^7 u \mathrm{d}x + 2\gamma^2\int_{\mathbb{R}} (\partial_x u)^2\partial_x^3 u\partial_x^7 u \mathrm{d}x \\ &= -4\gamma^2\int_{\mathbb{R}} (\partial_x^3 u)^3\partial_x^6 u \mathrm{d}x - 12\gamma^2\int_{\mathbb{R}} \partial_x u\partial_x^2 u\partial_x^3 u\partial_x^6 u \mathrm{d}x \\ &\quad - 2\gamma^2\int_{\mathbb{R}} (\partial_x u)^2\partial_x^4 u\partial_x^6 u \mathrm{d}x \\ &= -4\gamma^2\int_{\mathbb{R}} (\partial_x^3 u)^3\partial_x^6 u \mathrm{d}x - 12\gamma^2\int_{\mathbb{R}} \partial_x u\partial_x^2 u\partial_x^3 u\partial_x^6 u \mathrm{d}x \\ &\quad + 4\gamma^2\int_{\mathbb{R}} \partial_x u\partial_x^2 u\partial_x^4 u\partial_x^5 u \mathrm{d}x + 2\gamma^2\left\|\partial_x u(t,\cdot)\partial_x^5 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}, \\ 2\tau\int_{\mathbb{R}} \partial_x u\partial_x^2 u\partial_x^8 u \mathrm{d}x &= -2\tau\int_{\mathbb{R}} (\partial_x^2 u)^2\partial_x^7 u \mathrm{d}x - 2\tau\int_{\mathbb{R}} \partial_x u\partial_x^3 u\partial_x^7 u \mathrm{d}x \\ &= 6\tau\int_{\mathbb{R}} \partial_x^2 u\partial_x^3 u\partial_x^6 u \mathrm{d}x + 2\tau\int_{\mathbb{R}} \partial_x u\partial_x^4 u\partial_x^6 u \mathrm{d}x, \\ 2\kappa\int_{\mathbb{R}} (\partial_x u)^4\partial_x^8 u \mathrm{d}x &= -8\kappa\int_{\mathbb{R}} (\partial_x u)^3\partial_x^2 u\partial_x^7 u \mathrm{d}x \\ &= 24\kappa\int_{\mathbb{R}} (\partial_x u)^2(\partial_x^2 u)^2\partial_x^6 u \mathrm{d}x + 8\kappa\int_{\mathbb{R}} (\partial_x u)^3\partial_x^3 u\partial_x^6 u \mathrm{d}x \\ 2q\int_{\mathbb{R}} (\partial_x u)^2\partial_x^8 u \mathrm{d}x &= -4q\int_{\mathbb{R}} \partial_x u\partial_x^2 u\partial_x^7 u \mathrm{d}x \end{aligned}$$

$$
\begin{aligned}
&= 4q\int_{\mathbb{R}}(\partial_x^2u)^2\partial_x^6u\mathrm{d}x + 4q\int_{\mathbb{R}}\partial_xu\partial_x^3u\partial_x^6u\mathrm{d}x,\\
2\delta\int_{\mathbb{R}}\partial_xu\partial_x^3u\partial_x^8u\mathrm{d}x &= -2\delta\int_{\mathbb{R}}\partial_x^2u\partial_x^3u\partial_x^7u\mathrm{d}x - 2\delta\int_{\mathbb{R}}\partial_xu\partial_x^4u\partial_x^7u\mathrm{d}x\\
&= 2\delta\int_{\mathbb{R}}(\partial_x^3u)^2\partial_x^6u\mathrm{d}x + 4\delta\int_{\mathbb{R}}\partial_x^2u\partial_x^4u\partial_x^6u\mathrm{d}x\\
&\quad + 2\delta\int_{\mathbb{R}}\partial_xu\partial_x^5u\partial_x^6u\mathrm{d}x\\
&= -4\delta\int_{\mathbb{R}}\partial_x^3u\partial_x^4u\partial_x^5u\mathrm{d}x + 4\delta\int_{\mathbb{R}}\partial_x^2u\partial_x^4u\partial_x^6u\mathrm{d}x\\
&\quad - \delta\int_{\mathbb{R}}\partial_x^2u(\partial_x^5u)^2\mathrm{d}x. \qquad (2.41)
\end{aligned}
$$

Therefore, thanks to (2.41), an integration of (2.40) on $\mathbb{R}$ gives

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left\|\partial_x^4u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\beta^2\left\|\partial_x^6u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\gamma^2\left\|\partial_xu(t,\cdot)\partial_x^5u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad = 2\alpha\left\|\partial_x^5u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 4\gamma^2\int_{\mathbb{R}}(\partial_x^3u)^3\partial_x^6u\mathrm{d}x + 12\gamma^2\int_{\mathbb{R}}\partial_xu\partial_x^2u\partial_x^3u\partial_x^6u\mathrm{d}x\\
&\qquad - 4\gamma^2\int_{\mathbb{R}}\partial_xu\partial_x^2u\partial_x^4u\partial_x^5u\mathrm{d}x - 6\tau\int_{\mathbb{R}}\partial_x^2u\partial_x^3u\partial_x^6u\mathrm{d}x - 2\tau\int_{\mathbb{R}}\partial_xu\partial_x^4u\partial_x^6u\mathrm{d}x\\
&\qquad - 24\kappa\int_{\mathbb{R}}(\partial_xu)^2(\partial_x^2u)^2\partial_x^6u\mathrm{d}x - 8\kappa\int_{\mathbb{R}}(\partial_xu)^3\partial_x^3u\partial_x^6u\mathrm{d}x\\
&\qquad - 4q\int_{\mathbb{R}}(\partial_x^2u)^2\partial_x^6u\mathrm{d}x - 4q\int_{\mathbb{R}}\partial_xu\partial_x^3u\partial_x^6u\mathrm{d}x\\
&\qquad + 4\delta\int_{\mathbb{R}}\partial_x^3u\partial_x^4u\partial_x^5u\mathrm{d}x - 4\delta\int_{\mathbb{R}}\partial_x^2u\partial_x^4u\partial_x^6u + \delta\int_{\mathbb{R}}\partial_x^2u(\partial_x^5u)^2\mathrm{d}x. \qquad (2.42)
\end{aligned}
$$

Due to (2.2), (2.25), (2.26), (2.33), (2.34) and the Young inequality,

$$
\begin{aligned}
4\gamma^2\int_{\mathbb{R}}|\partial_x^3u|^3|\partial_x^6u|\mathrm{d}x &= 4\gamma^2\left\|\partial_xu\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\int_{\mathbb{R}}|\partial_xu|\partial_x^6u|\mathrm{d}x\\
&\le 2C(T)\int_{\mathbb{R}}|\partial_xu||\partial_x^6u|\mathrm{d}x = 2\int_{\mathbb{R}}\left|\frac{C(T)\partial_xu}{\beta\sqrt{D_7}}\right|\left|\beta\sqrt{D_7}\partial_x^6u\right|\mathrm{d}x\\
&\le \frac{C(T)}{D_7}\left\|\partial_xu(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \beta^2D_7\left\|\partial_x^6u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\le \frac{C(T)}{D_7} + \beta^2D_7\left\|\partial_x^6u(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\
12\gamma^2\int_{\mathbb{R}}|\partial_xu||\partial_x^2u||\partial_x^3u||\partial_x^6u|\mathrm{d}x &= 12\gamma^2\left\|\partial_xu\right\|_{L^\infty((0,T)\times\mathbb{R})}\int_{\mathbb{R}}|\partial_x^2u||\partial_x^3u||\partial_x^6u|\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&\leq 2C(T)\int_{\mathbb{R}}|\partial_x^2 u||\partial_x^3 u||\partial_x^6 u|\mathrm{d}x \leq 2C(T)\left\|\partial_x^2 u\right\|_{L^\infty((0,T)\times\mathbb{R})}\int_{\mathbb{R}}|\partial_x^3 u||\partial_x^6 u|\mathrm{d}x\\
&\leq 2C(T)\int_{\mathbb{R}}|\partial_x^3 u||\partial_x^6 u|\mathrm{d}x = 2\int_{\mathbb{R}}\left|\frac{C(T)\partial_x^3 u}{\beta\sqrt{D_7}}\right|\left|\beta\sqrt{D_7}\partial_x^6 u\right|\mathrm{d}x\\
&\leq \frac{C(T)}{D_7}\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \beta^2 D_7\left\|\partial_x^6 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\leq \frac{C(T)}{D_7} + \beta^2 D_7\left\|\partial_x^6 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})},
\end{aligned}
$$

$$
\begin{aligned}
4\gamma^2\int_{\mathbb{R}}&|\partial_x u||\partial_x^2 u||\partial_x^4 u||\partial_x^5 u|\mathrm{d}x \leq 4\gamma^2\left\|\partial_x u\right\|_{L^\infty((0,T)\times\mathbb{R})}\int_{\mathbb{R}}|\partial_x^2 u||\partial_x^4 u||\partial_x^5 u|\mathrm{d}x\\
&\leq 2C(T)\int_{\mathbb{R}}|\partial_x^2 u||\partial_x^4 u||\partial_x^5 u|\mathrm{d}x \leq 2C(T)\left\|\partial_x^2 u\right\|_{L^\infty((0,T)\times\mathbb{R})}\int_{\mathbb{R}}|\partial_x^4 u||\partial_x^5 u|\mathrm{d}x\\
&\leq 2C(T)\int_{\mathbb{R}}|\partial_x^4 u||\partial_x^5 u|\mathrm{d}x \leq C(T)\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + C(T)\left\|\partial_x^5 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})},
\end{aligned}
$$

$$
\begin{aligned}
24|\kappa|\int_{\mathbb{R}}&(\partial_x u)^2(\partial_x^2 u)^2|\partial_x^6 u|\mathrm{d}x \leq 24|\kappa|\left\|\partial_x u\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\int_{\mathbb{R}}(\partial_x^2 u)^2|\partial_x^6 u|\mathrm{d}x\\
&\leq 2C(T)\int_{\mathbb{R}}(\partial_x^2 u)^2|\partial_x^6 u|\mathrm{d}x \leq 2C(T)\left\|\partial_x^2 u\right\|_{L^\infty((0,T)\times\mathbb{R})}\int_{\mathbb{R}}|\partial_x^2 u||\partial_x^6 u|\mathrm{d}x\\
&\leq 2C(T)\int_{\mathbb{R}}|\partial_x^2 u||\partial_x^6 u|\mathrm{d}x = 2\int_{\mathbb{R}}\left|\frac{C(T)\partial_x^2 u}{\beta\sqrt{D_7}}\right|\left|\beta\sqrt{D_7}\partial_x^6 u\right|\mathrm{d}x\\
&\leq \frac{C(T)}{D_7}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \beta^2 D_7\left\|\partial_x^6 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\leq \frac{C(T)}{D_7} + \beta^2 D_7\left\|\partial_x^6 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})},
\end{aligned}
$$

$$
\begin{aligned}
6|\tau|\int_{\mathbb{R}}&|\partial_x^2 u||\partial_x^3 u|||\partial_x^6 u|\mathrm{d}x \leq 6|\tau|\left\|\partial_x^2 u\right\|_{L^\infty((0,T)\times\mathbb{R})}\int_{\mathbb{R}}|\partial_x^3 u||\partial_x^6 u|\mathrm{d}x\\
&\leq 2C(T)\int_{\mathbb{R}}|\partial_x^3 u||\partial_x^6 u|\mathrm{d}x = 2\int_{\mathbb{R}}\left|\frac{C(T)\partial_x^3 u}{\beta\sqrt{D_7}}\right|\left|\beta\sqrt{D_7}\partial_x^6 u\right|\mathrm{d}x\\
&\leq \frac{C(T)}{D_7}\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \beta^2 D_7\left\|\partial_x^6 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\leq \frac{C(T)}{D_7} + \beta^2 D_7\left\|\partial_x^6 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})},
\end{aligned}
$$

$$
\begin{aligned}
2|\tau|\int_{\mathbb{R}}&|\partial_x u||\partial_x^4 u||\partial_x^6 u|\mathrm{d}x \leq 2|\tau|\left\|\partial_x u\right\|_{L^\infty((0,T)\times\mathbb{R})}\int_{\mathbb{R}}|\partial_x^4 u||\partial_x^6 u|\mathrm{d}x\\
&\leq 2C(T)\int_{\mathbb{R}}|\partial_x^4 u||\partial_x^6 u|\mathrm{d}x = 2\int_{\mathbb{R}}\left|\frac{C(T)\partial_x^4 u}{\beta\sqrt{D_7}}\right|\left|\beta\sqrt{D_7}\partial_x^6 u\right|\mathrm{d}x\\
&\leq \frac{C(T)}{D_7}\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \beta^2 D_7\left\|\partial_x^6 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})},
\end{aligned}
$$

$$
\begin{aligned}
8|\kappa| \int_{\mathbb{R}} |\partial_x u|^3 |\partial_x^3 u||\partial_x^6 u| \mathrm{d}x &\leq 8\kappa \left\|\partial_x u\right\|^3_{L^\infty((0,T)\times\mathbb{R})} \int_{\mathbb{R}} |\partial_x^3 u||\partial_x^6 u| \mathrm{d}x \\
&\leq 2C(T) \int_{\mathbb{R}} |\partial_x^3 u||\partial_x^6 u| \mathrm{d}x = 2\int_{\mathbb{R}} \left|\frac{C(T)\partial_x^3 u}{\beta\sqrt{D_7}}\right| \left|\beta\sqrt{D_7}\partial_x^6 u\right| \mathrm{d}x \\
&\leq \frac{C(T)}{D_7} \left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \beta^2 D_7 \left\|\partial_x^6 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\
&\leq \frac{C(T)}{D_7} + \beta^2 D_7 \left\|\partial_x^6 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}, \\
4|q| \int_{\mathbb{R}} (\partial_x^2 u)^2 |\partial_x^6 u| \mathrm{d}x &\leq 4|q| \left\|\partial_x^2 u\right\|_{L^\infty((0,T)\times\mathbb{R})} \int_{\mathbb{R}} |\partial_x^2 u||\partial_x^6 u| \mathrm{d}x \\
&\leq 2C(T) \int_{\mathbb{R}} |\partial_x^2 u||\partial_x^6 u| \mathrm{d}x \leq 2\int_{\mathbb{R}} \left|\frac{C(T)\partial_x^2 u}{\beta\sqrt{D_7}}\right| \left|\beta\sqrt{D_7}\partial_x^6 u\right| \mathrm{d}x \\
&\leq \frac{C(T)}{D_7} \left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \beta^2 D_7 \left\|\partial_x^6 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\
&\leq \frac{C(T)}{D_7} + \beta^2 D_7 \left\|\partial_x^6 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}, \\
4|q| \int_{\mathbb{R}} |\partial_x u||\partial_x^3 u||\partial_x^6 u| \mathrm{d}x &\leq 4|q| \left\|\partial_x u\right\|_{L^\infty((0,T)\times\mathbb{R})} \int_{\mathbb{R}} |\partial_x^3 u||\partial_x^6 u| \mathrm{d}x \\
&\leq 2C(T) \int_{\mathbb{R}} |\partial_x^3 u||\partial_x^6 u| \mathrm{d}x = 2\int_{\mathbb{R}} \left|\frac{C(T)\partial_x^3 u}{\beta\sqrt{D_7}}\right| \left|\beta\sqrt{D_7}\partial_x^6 u\right| \mathrm{d}x \\
&\leq \frac{C(T)}{D_7} \left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \beta^2 D_7 \left\|\partial_x^6 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\
&\leq \frac{C(T)}{D_7} + \beta^2 D_7 \left\|\partial_x^6 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}, \\
4|\delta| \int_{\mathbb{R}} |\partial_x^3 u \partial_x^4 u||\partial_x^5 u| \mathrm{d}x &\leq 2\delta^2 \int_{\mathbb{R}} (\partial_x^3 u)^2 (\partial_x^4 u)^2 \mathrm{d}x + 2\left\|\partial_x^5 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\
&\leq 2\delta^2 \left\|\partial_x^3 u\right\|^2_{L^\infty((0,T)\times\mathbb{R})} \left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\left\|\partial_x^5 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}, \\
4|\delta| \int_{\mathbb{R}} |\partial_x^2 u||\partial_x^4 u||\partial_x^6 u| \mathrm{d}x &\leq 4|\delta| \left\|\partial_x^2 u\right\|_{L^\infty((0,T)\times\mathbb{R})} \int_{\mathbb{R}} |\partial_x^4 u||\partial_x^6 u| \mathrm{d}x \\
&\leq 2C(T) \int_{\mathbb{R}} |\partial_x^4 u||\partial_x^6 u| \mathrm{d}x = 2\int_{\mathbb{R}} \left|\frac{C(T)\partial_x^4 u}{\beta\sqrt{D_7}}\right| \left|\beta\sqrt{D_7}\partial_x^6 u\right| \mathrm{d}x \\
&\leq \frac{C(T)}{D_7} \left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \beta^2 D_7 \left\|\partial_x^6 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}, \\
|\delta| \int_{\mathbb{R}} |\partial_x^2 u|(\partial_x^5 u)^2 \mathrm{d}x &|\delta| \left\|\partial_x^2 u\right\|_{L^\infty((0,T)\times\mathbb{R})} \left\|\partial_x^5 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\
&\leq C(T) \left\|\partial_x^5 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})},
\end{aligned}
$$

where D_7 is a positive constant, which will be specified later. It follows from (2.42) that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\beta^2\left(2-7D_7\right)\left\|\partial_x^6 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+2\gamma^2\left\|\partial_x u(t,\cdot)\partial_x^5 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\&\quad\le\frac{C(T)}{D_7}+C(T)\left\|\partial_x^5 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\&\qquad+C(T)\left(1+\frac{1}{D_7}+\left\|\partial_x^3 u\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\right)\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.\end{aligned}$$

Taking $D_7=\frac{1}{7}$, we have that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\beta^2\left\|\partial_x^6 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+2\gamma^2\left\|\partial_x u(t,\cdot)\partial_x^5 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\&\quad\le C(T)+C(T)\left(1+\left\|\partial_x^3 u\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\right)\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\&\qquad+C(T)\left\|\partial_x^5 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.\end{aligned}\tag{2.43}$$

Observe that

$$C(T)\left\|\partial_x^5 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}=C(T)\int_{\mathbb{R}}\partial_x^5 u\partial_x^5 u\mathrm{d}x=-C(T)\int_{\mathbb{R}}\partial_x^4 u\partial_x^6 u\mathrm{d}x.$$

Therefore, by the Young inequality,

$$\begin{aligned}C(T)\left\|\partial_x^5 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}&\le\int_{\mathbb{R}}\left|\frac{C(T)\partial_x^4 u}{\beta}\right|\left|\beta\partial_x^6 u\right|\mathrm{d}x\\&\le C(T)\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\frac{\beta^2}{2}\left\|\partial_x^6 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.\end{aligned}\tag{2.44}$$

Consequently, by (2.43),

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\frac{\beta^2}{2}\left\|\partial_x^6 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+2\gamma^2\left\|\partial_x u(t,\cdot)\partial_x^5 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\&\quad\le C(T)+C(T)\left(1+\left\|\partial_x^3 u\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\right)\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.\end{aligned}$$

Integrating on $(0,t)$, by (2.26), we have that

$$\begin{aligned}&\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\frac{\beta^2}{2}\int_0^t\left\|\partial_x^6 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})}\mathrm{d}s+2\gamma^2\int_0^t\left\|\partial_x u(s,\cdot)\partial_x^5 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})}\mathrm{d}s\\&\quad\le C_0+C(T)t+C(T)\left(1+\left\|\partial_x^3 u\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\right)\int_0^t\left\|\partial_x^4 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})}\mathrm{d}s\\&\quad\le C(T)\left(1+\left\|\partial_x^3 u\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\right).\end{aligned}\tag{2.45}$$

We prove (2.37). Thanks to (2.34), (2.45) and the Hölder inequality,

$$\begin{aligned}(\partial_x^3 u(t,x))^2&=2\int_{-\infty}^x\partial_x^3 u\partial_x^4 u\mathrm{d}y\le 2\int_{\mathbb{R}}|\partial_x^3 u||\partial_x^4 u|\mathrm{d}x\\&\le 2\left\|\partial_x^3 u(t,\cdot)\right\|_{L^2(\mathbb{R})}\left\|\partial_x^4 u(t,\cdot)\right\|_{L^2(\mathbb{R})}\le C(T)\sqrt{\left(1+\|\partial_x^3 u\|^2_{L^\infty((0,T)\times\mathbb{R})}\right)}.\end{aligned}$$

Hence,

$$\left\|\partial_x^3 u\right\|^4_{L^\infty((0,T)\times\mathbb{R})}-C(T)\left\|\partial_x^3 u\right\|^2_{L^\infty((0,T)\times\mathbb{R})}-C(T)\le 0,$$

which gives (2.37).

Finally, (2.38) follows from (2.37) and (2.45), while (2.26), (2.38) and an integration on $(0,t)$ gives (2.39). $\square$

Lemma 2.8. *Fix $T>0$ and assume (1.3), with $\ell = 4$. There exist a constant $C(T)>0$, such that*

$$\beta^2 \left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{1}{42}\int_0^t \left\|\partial_t\partial_x^2 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} \mathrm{d}s \le C(T), \tag{2.46}$$

for every $0 \le t \le T$.

Proof. Let $0\le t\le T$. Multiplying (1.1) by $2\partial_t\partial_x^4 u$, an integration on $\mathbb{R}$ gives

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\beta^2 \left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} - \alpha\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\right)\\
&\quad= 2\beta^2\int_{\mathbb{R}} \partial_x^4 u\partial_t\partial_x^4 u \mathrm{d}x + 2\alpha\int_{\mathbb{R}} \partial_x^2 u\partial_t\partial_x^4 u\mathrm{d}x\\
&\quad= -2\int_{\mathbb{R}}\partial_t u\partial_t\partial_x^4 u\mathrm{d}x + 2\gamma^2\int_{\mathbb{R}}(\partial_x u)^2\partial_x^2 u\partial_t\partial_x^4 u\mathrm{d}x - 2\tau\int_{\mathbb{R}}\partial_x u\partial_x^2 u\partial_t\partial_x^4 u\mathrm{d}x\\
&\qquad - 2\kappa\int_{\mathbb{R}}(\partial_x u)^4\partial_t\partial_x^4 u\mathrm{d}x - 2q\int_{\mathbb{R}}(\partial_x u)^2\partial_t\partial_x^4 u\mathrm{d}x - 2\delta\int_{\mathbb{R}}\partial_x u\partial_x^3 u\partial_t\partial_x^4 u\mathrm{d}x\\
&\quad= 2\int_{\mathbb{R}}\partial_t\partial_x u\partial_t\partial_x^3 u\mathrm{d}x - 4\gamma^2\int_{\mathbb{R}}\partial_x u(\partial_x^2 u)^2\partial_t\partial_x^3 u\mathrm{d}x - 2\gamma^2\int_{\mathbb{R}}(\partial_x u)^2\partial_x^3 u\partial_t\partial_x^3 u\mathrm{d}x\\
&\qquad + 2\tau\int_{\mathbb{R}}(\partial_x^2 u)^2\partial_t\partial_x^3 u\mathrm{d}x + 2\tau\int_{\mathbb{R}}\partial_x u\partial_x^3 u\partial_t\partial_x^3 u\mathrm{d}x + 8\kappa\int_{\mathbb{R}}(\partial_x u)^3\partial_x^2 u\partial_t\partial_x^3 u\mathrm{d}x\\
&\qquad + 4q\int_{\mathbb{R}}\partial_x u\partial_x^2 u\partial_t\partial_x^3 u\mathrm{d}x + 2\delta\int_{\mathbb{R}}\partial_x^2 u\partial_x^3 u\partial_t\partial_x^3 u\mathrm{d}x + 2\delta\int_{\mathbb{R}}\partial_x u\partial_x^4 u\partial_t\partial_x^3 u\mathrm{d}x\\
&\quad= -2\left\|\partial_t\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 4\gamma^2\int_{\mathbb{R}}(\partial_x^2 u)^3\partial_t\partial_x^2 u\mathrm{d}x + 12\gamma^2\int_{\mathbb{R}}\partial_x u\partial_x^2 u\partial_x^3 u\partial_t\partial_x^2 u\mathrm{d}x\\
&\qquad + 2\gamma^2\int_{\mathbb{R}}(\partial_x u)^2\partial_x^4 u\partial_t\partial_x^2 u\mathrm{d}x - 4\tau\int_{\mathbb{R}}\partial_x^2 u\partial_x^3 u\partial_t\partial_x^2 u\mathrm{d}x - 2\tau\int_{\mathbb{R}}\partial_x u\partial_x^4 u\partial_t\partial_x^2 u\mathrm{d}x\\
&\qquad - 24\kappa\int_{\mathbb{R}}(\partial_x u)^2(\partial_x^2 u)^2\partial_t\partial_x^2 u\mathrm{d}x - 8\kappa\int_{\mathbb{R}}(\partial_x u)^3\partial_x^3 u\partial_t\partial_x^2 u\mathrm{d}x - 4q\int_{\mathbb{R}}(\partial_x^2 u)^2\partial_t\partial_x^2 u\mathrm{d}x\\
&\qquad - 4q\int_{\mathbb{R}}\partial_x u\partial_x^3 u\partial_t\partial_x^2 u\mathrm{d}x - 2\delta\int_{\mathbb{R}}(\partial_x^3 u)^2\partial_t\partial_x^2 u\mathrm{d}x - 4\delta\int_{\mathbb{R}}\partial_x^2 u\partial_x^4 u\partial_t\partial_x^2 u\mathrm{d}x\\
&\qquad - 2\delta\int_{\mathbb{R}}\partial_x u\partial_x^5 u\partial_t\partial_x^2 u\mathrm{d}x\\
&\quad= -2\left\|\partial_t\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \gamma^2\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}(\partial_x^2 u)^4\mathrm{d}x - 12\gamma^2\int_{\mathbb{R}}(\partial_x^2 u)^2\partial_x^3 u\partial_t\partial_x u\mathrm{d}x\\
&\qquad - 12\gamma^2\int_{\mathbb{R}}\partial_x u(\partial_x^3 u)^2\partial_t\partial_x u\mathrm{d}x - 16\gamma^2\int_{\mathbb{R}}\partial_x u\partial_x^3 u\partial_x^4 u\partial_t\partial_x u\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&-2\gamma^2\int_{\mathbb{R}}(\partial_x u)^2\partial_x^5 u\partial_t\partial_x u\mathrm{d}x-4\tau\int_{\mathbb{R}}\partial_x^2 u\partial_x^3 u\partial_t\partial_x^2 u\mathrm{d}x\\
&-2\tau\int_{\mathbb{R}}\partial_x u\partial_x^4 u\partial_t\partial_x^2 u\mathrm{d}x+24\kappa\int_{\mathbb{R}}(\partial_x u)^2\partial_x^2 u\partial_x^3 u\partial_t\partial_x u\mathrm{d}x\\
&+8\kappa\int_{\mathbb{R}}(\partial_x u)^2\partial_x^4 u\partial_t\partial_x u\mathrm{d}x-\frac{4q}{3}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}(\partial_x^2 u)^3\mathrm{d}x+4q\int_{\mathbb{R}}\partial_x^2 u\partial_x^3 u\partial_t\partial_x u\mathrm{d}x\\
&+4q\int_{\mathbb{R}}\partial_x u\partial_x^4 u\partial_t\partial_x u\mathrm{d}x-2\delta\int_{\mathbb{R}}(\partial_x^3 u)^2\partial_t\partial_x^2 u\mathrm{d}x-4\delta\int_{\mathbb{R}}\partial_x^2 u\partial_x^4 u\partial_t\partial_x^2 u\mathrm{d}x\\
&-2\delta\int_{\mathbb{R}}\partial_x u\partial_x^5 u\partial_t\partial_x^2 u\mathrm{d}x.
\end{aligned}
$$

Therefore, we have that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\beta^2\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}-\alpha\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\right)\\
&\quad-\frac{\mathrm{d}}{\mathrm{d}t}\left(\gamma^2\int_{\mathbb{R}}(\partial_x^2 u)^4\mathrm{d}x-\frac{4q}{3}\int_{\mathbb{R}}(\partial_x^2 u)^3\mathrm{d}x\right)+2\left\|\partial_t\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&=-12\gamma^2\int_{\mathbb{R}}(\partial_x^2 u)^2\partial_x^3 u\partial_t\partial_x u\mathrm{d}x-12\gamma^2\int_{\mathbb{R}}\partial_x u(\partial_x^3 u)^2\partial_t\partial_x u\mathrm{d}x\\
&\quad-16\gamma^2\int_{\mathbb{R}}\partial_x u\partial_x^3 u\partial_x^4 u\partial_t\partial_x u\mathrm{d}x-2\gamma^2\int_{\mathbb{R}}(\partial_x u)^2\partial_x^5 u\partial_t\partial_x u\mathrm{d}x\\
&\quad-4\tau\int_{\mathbb{R}}\partial_x^2 u\partial_x^3 u\partial_t\partial_x^2 u\mathrm{d}x-2\tau\int_{\mathbb{R}}\partial_x u\partial_x^4 u\partial_t\partial_x^2 u\mathrm{d}x\\
&\quad+24\kappa\int_{\mathbb{R}}(\partial_x u)^2\partial_x^2 u\partial_x^3 u\partial_t\partial_x u\mathrm{d}x+8\kappa\int_{\mathbb{R}}(\partial_x u)^2\partial_x^4 u\partial_t\partial_x u\mathrm{d}x\\
&\quad+4q\int_{\mathbb{R}}\partial_x^2 u\partial_x^3 u\partial_t\partial_x u\mathrm{d}x+4q\int_{\mathbb{R}}\partial_x u\partial_x^4 u\partial_t\partial_x u\mathrm{d}x\\
&\quad-2\delta\int_{\mathbb{R}}(\partial_x^3 u)^2\partial_t\partial_x^2 u\mathrm{d}x-4\delta\int_{\mathbb{R}}\partial_x^2 u\partial_x^4 u\partial_t\partial_x^2 u\mathrm{d}x\\
&\quad-2\delta\int_{\mathbb{R}}\partial_x u\partial_x^5 u\partial_t\partial_x^2 u\mathrm{d}x.
\end{aligned}
\tag{2.47}
$$

Due to (2.25), (2.33), (2.34), (2.37), (2.38) and the Young inequality,

$$
\begin{aligned}
12\gamma^2&\int_{\mathbb{R}}(\partial_x^2 u)^2|\partial_x^3 u||\partial_t\partial_x u|\mathrm{d}x\le 12\gamma^2\left\|\partial_x^2 u\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\int_{\mathbb{R}}|\partial_x^3 u\partial_t\partial_x u|\mathrm{d}x\\
&\le C(T)\int_{\mathbb{R}}|\partial_x^3 u\partial_t\partial_x u|\mathrm{d}x\le C(T)\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+C(T)\left\|\partial_t\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\le C(T)+C(T)\left\|\partial_t\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})},
\end{aligned}
$$

$$
\begin{aligned}
&12\gamma^2 \int_{\mathbb{R}} |\partial_x u|(\partial_x^3 u)^2 |\partial_t\partial_x u| \mathrm{d}x \le 12\gamma^2 \left\|\partial_x u\right\|_{L^\infty((0,T)\times\mathbb{R})} \int_{\mathbb{R}} (\partial_x^3 u)^2 |\partial_t\partial_x u| \mathrm{d}x\\
&\quad \le C(T)\int_{\mathbb{R}} (\partial_x^3 u)^2 |\partial_t\partial_x u| \mathrm{d}x \le C(T)\left\|\partial_x^3 u\right\|_{L^\infty((0,T)\times\mathbb{R})}\int_{\mathbb{R}} |\partial_x^3 u||\partial_t\partial_x u| \mathrm{d}x\\
&\quad \le C(T)\int_{\mathbb{R}} |\partial_x^3 u||\partial_t\partial_x u| \mathrm{d}x \le C(T)\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + C(T)\left\|\partial_t\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad \le C(T) + C(T)\left\|\partial_t\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\
&16\gamma^2 \int_{\mathbb{R}} |\partial_x u||\partial_x^3 u||\partial_x^4 u||\partial_t\partial_x u| \mathrm{d}x \le 16\gamma^2 \left\|\partial_x u\right\|_{L^\infty((0,T)\times\mathbb{R})} \int_{\mathbb{R}} |\partial_x^3 u||\partial_x^4 u||\partial_t\partial_x u| \mathrm{d}x\\
&\quad \le C(T)\int_{\mathbb{R}} |\partial_x^3 u||\partial_x^4 u||\partial_t\partial_x u| \mathrm{d}x \le C(T)\left\|\partial_x^3 u\right\|_{L^\infty((0,T)\times\mathbb{R})}\int_{\mathbb{R}} |\partial_x^4 u||\partial_t\partial_x u| \mathrm{d}x\\
&\quad \le C(T)\int_{\mathbb{R}} |\partial_x^4 u||\partial_t\partial_x u| \mathrm{d}x \le C(T)\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + C(T)\left\|\partial_t\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad \le C(T) + C(T)\left\|\partial_t\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\
&2\gamma^2 \int_{\mathbb{R}} (\partial_x u)^2|\partial_x^5 u||\partial_t\partial_x u| \mathrm{d}x \le 2\gamma^2 \left\|\partial_x u\right\|^2_{L^\infty((0,T)\times\mathbb{R})} \int_{\mathbb{R}} |\partial_x^5 u||\partial_t\partial_x u| \mathrm{d}x\\
&\quad \le C(T)\int_{\mathbb{R}} |\partial_x^5 u||\partial_t\partial_x u| \mathrm{d}x \le C(T)\left\|\partial_x^5 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + C(T)\left\|\partial_t\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\
&4|\tau| \int_{\mathbb{R}} |\partial_x^2 u||\partial_x^3 u||\partial_t\partial_x^2 u| \mathrm{d}x \le 4|\tau| \left\|\partial_x^2 u\right\|_{L^\infty((0,T)\times\mathbb{R})} \int_{\mathbb{R}} |\partial_x^3 u||\partial_t\partial_x^2 u| \mathrm{d}x\\
&\quad \le 2C(T)\int_{\mathbb{R}} |\partial_x^3 u||\partial_t\partial_x^2 u| \mathrm{d}x = 2\int_{\mathbb{R}} \left|\sqrt{3}C(T)\partial_x^3 u\right|\left|\frac{\partial_t\partial_x^2 u}{\sqrt{3}}\right| \mathrm{d}x\\
&\quad \le C(T)\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{1}{3}\left\|\partial_t\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad \le C(T) + \frac{1}{3}\left\|\partial_t\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\
&2|\tau| \int_{\mathbb{R}} |\partial_x u||\partial_x^4 u||\partial_t\partial_x^2 u| \mathrm{d}x \le 2|\tau| \left\|\partial_x u\right\|_{L^\infty((0,T)\times\mathbb{R})} \int_{\mathbb{R}} |\partial_x^4 u||\partial_t\partial_x^2 u| \mathrm{d}x\\
&\quad \le 2C(T)\int_{\mathbb{R}} |\partial_x^4 u||\partial_t\partial_x^2 u| \mathrm{d}x = 2\int_{\mathbb{R}} \left|\sqrt{7}C(T)\partial_x^4 u\right|\left|\frac{\partial_t\partial_x^2 u}{\sqrt{7}}\right| \mathrm{d}x\\
&\quad \le C(T)\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{1}{7}\left\|\partial_t\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad \le C(T) + \frac{1}{7}\left\|\partial_t\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\
&24|\kappa| \int_{\mathbb{R}} (\partial_x u)^2|\partial_x^2 u||\partial_x^3 u||\partial_t\partial_x u| \mathrm{d}x \le 24|\kappa| \left\|\partial_x u\right\|^2_{L^\infty((0,T)\times\mathbb{R})} \int_{\mathbb{R}} |\partial_x^2 u||\partial_x^3 u||\partial_t\partial_x u| \mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&\leq C(T)\int_{\mathbb{R}}|\partial_x^2 u||\partial_x^3 u||\partial_t\partial_x u|\mathrm{d}x \leq C(T)\left\|\partial_x^2 u\right\|_{L^\infty((0,T)\times\mathbb{R})}\int_{\mathbb{R}}|\partial_x^3 u||\partial_t\partial_x u|\mathrm{d}x\\
&\leq C(T)\int_{\mathbb{R}}|\partial_x^3 u||\partial_t\partial_x u|\mathrm{d}x \leq C(T)\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + C(T)\left\|\partial_t\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\leq C(T) + C(T)\left\|\partial_t\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\
8|\kappa|&\int_{\mathbb{R}}(\partial_x u)^2|\partial_x^4 u||\partial_t\partial_x u|\mathrm{d}x \leq 8|\kappa|\left\|\partial_x u\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\int_{\mathbb{R}}|\partial_x^4 u||\partial_t\partial_x u|\mathrm{d}x\\
&\leq C(T)\int_{\mathbb{R}}|\partial_x^4 u||\partial_t\partial_x u|\mathrm{d}x \leq C(T)\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + C(T)\left\|\partial_t\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\leq C(T) + C(T)\left\|\partial_t\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\
4|q|&\int_{\mathbb{R}}|\partial_x^2 u||\partial_x^3 u||\partial_t\partial_x u|\mathrm{d}x \leq 4|q|\left\|\partial_x^2 u\right\|_{L^\infty((0,T)\times\mathbb{R})}\int_{\mathbb{R}}|\partial_x^3 u||\partial_t\partial_x u|\mathrm{d}x\\
&\leq C(T)\int_{\mathbb{R}}|\partial_x^3 u||\partial_t\partial_x u|\mathrm{d}x \leq C(T)\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + C(T)\left\|\partial_t\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\leq C(T) + C(T)\left\|\partial_t\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\
4|q|&\int_{\mathbb{R}}|\partial_x u||\partial_x^4 u||\partial_t\partial_x u|\mathrm{d}x \leq 4|q|\left\|\partial_x u\right\|_{L^\infty((0,T)\times\mathbb{R})}\int_{\mathbb{R}}|\partial_x^4 u||\partial_t\partial_x u|\mathrm{d}x\\
&\leq C(T)\int_{\mathbb{R}}|\partial_x^4 u||\partial_t\partial_x u|\mathrm{d}x \leq C(T)\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + C(T)\left\|\partial_t\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\leq C(T) + C(T)\left\|\partial_t\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\
2|\delta|&\int_{\mathbb{R}}(\partial_x^3 u)^2\partial_t\partial_x^2 u\mathrm{d}x \leq 2|\delta|\left\|\partial_x^3 u\right\|_{L^\infty((0,T)\times\mathbb{R})}\int_{\mathbb{R}}|\partial_x^3 u||\partial_t\partial_x^2 u|\mathrm{d}x\\
&\leq C(T)\int_{\mathbb{R}}|\partial_x^3 u||\partial_t\partial_x^2 u|\mathrm{d}x \leq C(T)\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{1}{2}\left\|\partial_t\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\leq C(T) + \frac{1}{2}\left\|\partial_t\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\
4|\delta|&\int_{\mathbb{R}}|\partial_x^2 u||\partial_x^4 u||\partial_t\partial_x^2 u|\mathrm{d}x \leq 4|\delta|\left\|\partial_x^2 u\right\|_{L^\infty((0,T)\times\mathbb{R})}\int_{\mathbb{R}}|\partial_x^4 u||\partial_t\partial_x^2 u|\mathrm{d}x\\
&\leq C(T)\int_{\mathbb{R}}|\partial_x^4 u||\partial_t\partial_x^2 u|\mathrm{d}x \leq C(T)\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{1}{2}\left\|\partial_t\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\leq C(T) + \frac{1}{2}\left\|\partial_t\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\
2|\delta|&\int_{\mathbb{R}}|\partial_x u||\partial_x^5 u||\partial_t\partial_x^2 u|\mathrm{d}x \leq 2|\delta|\left\|\partial_x u\right\|_{L^\infty((0,T)\times\mathbb{R})}\int_{\mathbb{R}}|\partial_x^5 u||\partial_t\partial_x^2 u|\mathrm{d}x\\
&\leq C(T)\int_{\mathbb{R}}|\partial_x^5 u||\partial_t\partial_x^2 u|\mathrm{d}x \leq C(T)\left\|\partial_x^5 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{1}{2}\left\|\partial_t\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.
\end{aligned}
$$

It follows from (2.47) that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\beta^2\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}-\alpha\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\right)\\
&\quad-\frac{\mathrm{d}}{\mathrm{d}t}\left(\gamma^2\int_{\mathbb{R}}(\partial_x^2 u)^4\mathrm{d}x-\frac{4q}{3}\int_{\mathbb{R}}(\partial_x^2 u)^3\mathrm{d}x\right)+\frac{1}{42}\left\|\partial_t\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad\le C(T)+C(T)\left\|\partial_x^5 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+C(T)\left\|\partial_t\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.
\end{aligned}$$

(1.3), (2.34), (2.39) and an integration on $(0,t)$ give

$$\begin{aligned}
&\beta^2\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}-\alpha\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad-\gamma^2\int_{\mathbb{R}}(\partial_x^2 u)^4\mathrm{d}x+\frac{4q}{3}\int_{\mathbb{R}}(\partial_x^2 u)^3\mathrm{d}x+\frac{1}{42}\int_0^t\left\|\partial_t\partial_x^2 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})}\mathrm{d}s\\
&\quad\le C_0+C(T)\int_0^t\left\|\partial_x^5 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})}\mathrm{d}s+C(T)\int_0^t\left\|\partial_t\partial_x u(s,\cdot)\right\|^2_{L^2(\mathbb{R})}\mathrm{d}s\le C(T).
\end{aligned}$$

Therefore, by (2.26), (2.33) and (2.34),

$$\begin{aligned}
&\beta^2\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\frac{1}{42}\int_0^t\left\|\partial_t\partial_x^2 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})}\mathrm{d}s\\
&\quad\le C(T)+\alpha\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\gamma^2\int_{\mathbb{R}}(\partial_x^2 u)^4\mathrm{d}x-\frac{4q}{3}\int_{\mathbb{R}}(\partial_x^2 u)^3\mathrm{d}x\\
&\quad\le C(T)+|\alpha|\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\gamma^2\left\|\partial_x^2 u\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\qquad+\left|\frac{4q}{3}\right|\left\|\partial_x^2 u\right\|_{L^\infty((0,T)\times\mathbb{R})}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\le C(T),
\end{aligned}$$

which gives (2.46). □

3. Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1.

We begin by proving the following lemma.

Lemma 3.1. *Fix $T>0$. Under Assumptions (1.2) and (1.9), there exists a unique solution u of (1.1), such that (1.10) and (1.11) hold.*

Proof. Fix $T>0$. Thanks to Lemmas 2.1, 2.3, 2.4, 2.5, 2.6, 2.7, 2.8 and the Cauchy–Kovalevskaya Theorem [58], we have that u is solution of (1.1) and (1.10) holds.

We prove (1.11). Let u_1 and u_2 be two solutions of (1.1), which verify (1.10), that is

$$\begin{cases}\partial_t u_1+\alpha\partial_x^2 u_1+\beta^2\partial_x^4 u_1-\gamma^2(\partial_x u_1)^2\partial_x^2 u_1+\tau\partial_x u_1\partial_x^2 u_1\\
\quad+\kappa(\partial_x u_1)^4+q(\partial_x u_1)^2+\delta\partial_x u_1\partial_x^3 u_1=0, & t>0,\quad x\in\mathbb{R},\\
u_1(0,x)=u_{1,0}(x), & x\in\mathbb{R},\end{cases}$$

$$\begin{cases} \partial_t u_2 + \alpha\partial_x^2 u_2 + \beta^2\partial_x^4 u_2 - \gamma^2(\partial_x u_2)^2\partial_x^2 u_2 + \tau\partial_x u_2\partial_x^2 u_2 \\ \quad +\kappa(\partial_x u_2)^4 + q(\partial_x u_2)^2 + \delta\partial_x u_2\partial_x^3 u_2 = 0, & t>0, \quad x\in\mathbb{R}, \\ u_1(0,x) = u_{1,0}(x), & x\in\mathbb{R}, \end{cases}$$

Then, the function

$$\omega = u_1 - u_2 \tag{3.1}$$

is the solution of the following Cauchy problem:

$$\begin{cases} \partial_t\omega + \alpha\partial_x^2\omega + \beta^2\partial_x^4\omega - \gamma^2\left[(\partial_x u_1)^2\partial_x^2 u_1 - (\partial_x u_2)^2\partial_x^2 u_2\right] \\ \quad +\tau\left[\partial_x u_1\partial_x^2 u_1 - \partial_x u_2\partial_x^2 u_2\right] \\ \quad +\kappa\left[(\partial_x u_1)^4 - (\partial_x u_2)^4\right] + q\left[(\partial_x u_1)^2 - (\partial_x u_2)^2\right] \\ \quad +\delta\left(\partial_x u_1\partial_x^3 u_1 - \partial_x u_2\partial_x^3 u_2\right) = 0, & t>0, \quad x\in\mathbb{R}, \\ \omega(0,x) = u_{1,0}(x) - u_{2,0}(x), & x\in\mathbb{R}. \end{cases} \tag{3.2}$$

Observe that

$$\begin{aligned} &(\partial_x u_1)^2\partial_x^2 u_1 - (\partial_x u_2)^2\partial_x^2 u_2 \\ &\quad = (\partial_x u_1)^2\partial_x^2 u_1 - (\partial_x u_1)^2\partial_x^2 u_2 + (\partial_x u_1)^2\partial_x^2 u_2 - (\partial_x u_2)^2\partial_x^2 u_2 \\ &\quad = (\partial_x u_1)^2\partial_x^2\omega + \partial_x^2 u_2\left[(\partial_x u_1)^2 - (\partial_x u_2)^2\right] \\ &\quad = (\partial_x u)^2\partial_x^2\omega + \partial_x^2 u_2(\partial_x u_1 + \partial_x u_2)\partial_x\omega, \\ &\partial_x u_1\partial_x^2 u_1 - \partial_x u_2\partial_x^2 u_2 = \partial_x u_1\partial_x^2 u_1 - \partial_x u_1\partial_x^2 u_2 + \partial_x u_1\partial_x^2 u_2 - \partial_x u_2\partial_x^2 u_2 \\ &\quad = \partial_x u_1\partial_x^2\omega + \partial_x^2 u_2\partial_x\omega, \\ &(\partial_x u_1)^4 - (\partial_x u_2)^4 = \left[(\partial_x u_1)^2 + (\partial_x u_2)^2\right](\partial_x u_1 + \partial_x u_2)\omega, \\ &(\partial_x u_1)^2 - (\partial_x u_2)^2 = (\partial_x u_1 + \partial_x u_2)\partial_x\omega, \\ &\partial_x u_1\partial_x^3 u_1 - \partial_x u_2\partial_x^3 u_2 \\ &\quad = \partial_x u_1\partial_x^3 u_1 - \partial_x u_1\partial_x^3 u_2 + \partial_x u_1\partial_x^3 u_2 - \partial_x u_2\partial_x^3 u_2 \\ &\quad = \partial_x u_1\partial_x^3\omega + \partial_x^3 u_2\partial_x\omega. \end{aligned}$$

Therefore, (3.2) is equivalent the following one:

$$\begin{aligned} &\partial_t\omega + \alpha\partial_x^2\omega + \beta^2\partial_x^4\omega - \gamma^2(\partial_x u_1)^2\partial_x^2\omega - \gamma^2\partial_x^2 u_2(\partial_x u_1 + \partial_x u_2)\partial_x\omega \\ &\qquad + \tau\partial_x u_1\partial_x^2\omega + \tau\partial_x^2 u_2\partial_x\omega + \kappa\left[(\partial_x u_1)^2 + (\partial_x u_2)^2\right](\partial_x u_1 + \partial_x u_2)\partial_x\omega \\ &\qquad + q(\partial_x u_1 + \partial_x u_2)\partial_x\omega + \delta\partial_x u_1\partial_x^3\omega + \delta\partial_x^3 u_2\partial_x\omega = 0 \end{aligned} \tag{3.3}$$

Observe that, since $u_1,\, u_2 \in H^4(\mathbb{R})$, for every $0\le t\le T$, we have that

$$\begin{aligned} &\|\partial_x u_1\|_{L^\infty((0,T)\times\mathbb{R})},\, \|\partial_x u_2\|_{L^\infty((0,T)\times\mathbb{R})} \le C(T), \\ &\left\|\partial_x^2 u_2\right\|_{L^\infty((0,T)\times\mathbb{R})},\, \left\|\partial_x^3 u_2\right\|_{L^\infty((0,T)\times\mathbb{R})} \le C(T). \end{aligned} \tag{3.4}$$

Thanks to (3.4), we obtain

$$\begin{aligned} (\partial_x u_1)^2 &\le C(T), \\ |\partial_x^2 u_2||\partial_x u_1 + \partial_x u_2| &\le C(T), \\ \left[(\partial_x u_1)^2 + (\partial_x u_2)^2\right]|\partial_x u_1 + \partial_x u_2| &\le C(T), \\ |\partial_x u_1 + \partial_x u_2| &\le C(T). \end{aligned} \tag{3.5}$$

Since

$$2\int_{\mathbb{R}}(\omega-\partial_x^2\omega)\partial_t\omega\mathrm{d}x=\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})}+\|\partial_x\omega(t,\cdot)\|^2_{L^2(\mathbb{R})}\right)=\frac{\mathrm{d}}{\mathrm{d}t}\|\omega(t,\cdot)\|^2_{H^1(\mathbb{R})},$$

$$2\alpha\int_{\mathbb{R}}(\omega-\partial_x^2\omega)\partial_x^2\omega=-2\alpha\|\partial_x\omega(t,\cdot)\|^2_{L^2(\mathbb{R})}-2\alpha\left\|\partial_x^2\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})},$$

$$2\beta^2\int_{\mathbb{R}}(\omega-\partial_x^2\omega)\partial_x^4\omega\mathrm{d}x=2\beta^2\left\|\partial_x^2\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+2\beta^2\left\|\partial_x^3\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})},$$

multiplying (3.3) by $2\omega-2\partial_x^2\omega$, an integration on $\mathbb{R}$ gives,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|\omega(t,\cdot)\|^2_{H^1(\mathbb{R})}+2\beta^2\left\|\partial_x^2\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+2\beta^2\left\|\partial_x^3\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad=2\alpha\|\partial_x\omega(t,\cdot)\|^2_{L^2(\mathbb{R})}+2\alpha\left\|\partial_x^2\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+2\gamma^2\int_{\mathbb{R}}(\partial_xu_1)^2\omega\partial_x^2\omega\mathrm{d}x\\
&\qquad-2\gamma^2\int_{\mathbb{R}}(\partial_xu_1)^2(\partial_x^2\omega)^2\mathrm{d}x+2\gamma^2\int_{\mathbb{R}}\partial_x^2u_2(\partial_xu_1+\partial_xu_2)\omega\partial_x\omega\mathrm{d}x\\
&\qquad-2\tau\int_{\mathbb{R}}\partial_xu_1\omega\partial_x^2\omega\mathrm{d}x+2\tau\int_{\mathbb{R}}\partial_xu_1(\partial_x^2\omega)^2\mathrm{d}x\\
&\qquad-2\tau\int_{\mathbb{R}}\partial_x^2u_2\omega\partial_x\omega\mathrm{d}x+2\tau\int_{\mathbb{R}}\partial_x^2u_2\partial_x\omega\partial_x^2\omega\mathrm{d}x\\
&\qquad-2\kappa\int_{\mathbb{R}}\left[(\partial_xu_1)^2+(\partial_xu_2)^2\right](\partial_xu_1+\partial_xu_2)\omega\partial_x\omega\mathrm{d}x\\
&\qquad+2\kappa\int_{\mathbb{R}}\left[(\partial_xu_1)^2+(\partial_xu_2)^2\right](\partial_xu_1+\partial_xu_2)\partial_x\omega\partial_x^2u\mathrm{d}x\\
&\qquad-2q\int_{\mathbb{R}}(\partial_xu_1+\partial_xu_2)\omega\partial_x\omega\mathrm{d}x+2q\int_{\mathbb{R}}(\partial_xu_1+\partial_xu_2)\partial_x\omega\partial_x^2\omega\mathrm{d}x\\
&\qquad-2\delta\int_{\mathbb{R}}\partial_xu_1\omega\partial_x^3\omega\mathrm{d}x+2\delta\int_{\mathbb{R}}\partial_xu_1\partial_x^2\omega\partial_x^3\omega\mathrm{d}x\\
&\qquad-2\delta\int_{\mathbb{R}}\partial_x^3u_2\omega\partial_x\omega\mathrm{d}x+2\delta\int_{\mathbb{R}}\partial_x^3u_2\partial_x\omega\partial_x^2\omega\mathrm{d}x.
\end{aligned}\tag{3.6}$$

Due to (3.4), (3.5) and the Young inequality,

$$\begin{aligned}
&2\gamma^2\int_{\mathbb{R}}(\partial_xu_2)^2|\omega||\partial_x^2\omega|\mathrm{d}x\le C(T)\int_{\mathbb{R}}|\omega||\partial_x^2\omega|\mathrm{d}x\\
&\qquad\le C(T)\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})}+C(T)\left\|\partial_x^2\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\
&2\gamma^2\int_{\mathbb{R}}(\partial_xu_1)^2(\partial_x^2\omega)^2\mathrm{d}x\le C(T)\left\|\partial_x^2\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\
&2\gamma^2\int_{\mathbb{R}}|\partial_x^2u_2||\partial_xu_1+\partial_xu_2)||\omega||\partial_x\omega|\mathrm{d}x\le C(T)\int_{\mathbb{R}}|\omega||\partial_x\omega|\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&\leq C(T)\left\|\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+C(T)\left\|\partial_x\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\
2|\tau|&\int_{\mathbb{R}}|\partial_x u_1||\omega||\partial_x^2\omega|\mathrm{d}x\leq C(T)\int_{\mathbb{R}}|\omega||\partial_x^2\omega|\mathrm{d}x\\
&\leq C(T)\left\|\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+C(T)\left\|\partial_x^2\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\
2|\tau|&\int_{\mathbb{R}}|\partial_x u_1|(\partial_x^2\omega)^2\mathrm{d}x\leq C(T)\left\|\partial_x^2\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\
2|\tau|&\int_{\mathbb{R}}|\partial_x^2 u_2||\omega||\partial_x\omega|\mathrm{d}x\leq C(T)\int_{\mathbb{R}}|\omega||\partial_x\omega|\mathrm{d}x\\
&\leq C(T)\left\|\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+C(T)\left\|\partial_x\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\
2|\tau|&\int_{\mathbb{R}}|\partial_x^2 u_2||\partial_x\omega||\partial_x^2\omega|\mathrm{d}x\leq C(T)\int_{\mathbb{R}}|\partial_x\omega||\partial_x^2\omega|\mathrm{d}x\\
&\leq C(T)\left\|\partial_x\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+C(T)\left\|\partial_x^2\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\
2|\kappa|&\int_{\mathbb{R}}\left[(\partial_x u_1)^2+(\partial_x u_2)^2\right]|\partial_x u_1+\partial_x u_2||\omega||\partial_x\omega|\mathrm{d}x\leq C(T)\int_{\mathbb{R}}|\omega||\partial_x\omega|\mathrm{d}x\\
&\leq C(T)\left\|\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+C(T)\left\|\partial_x\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\
2|\kappa|&\int_{\mathbb{R}}\left[(\partial_x u_1)^2+(\partial_x u_2)^2\right]|\partial_x u_1+\partial_x u_2||\partial_x\omega||\partial_x^2 u|\mathrm{d}x\leq C(T)\int_{\mathbb{R}}|\partial_x\omega||\partial_x^2\omega|\mathrm{d}x\\
&\leq C(T)\left\|\partial_x\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+C(T)\left\|\partial_x^2\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\
2|q|&\int_{\mathbb{R}}|\partial_x u_1+\partial_x u_2||\omega||\partial_x\omega|\mathrm{d}x\leq C(T)\int_{\mathbb{R}}|\omega||\partial_x\omega|\mathrm{d}x\\
&\leq C(T)\left\|\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+C(T)\left\|\partial_x\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\
2|q|&\int_{\mathbb{R}}|\partial_x u_1+\partial_x u_2||\partial_x\omega||\partial_x^2\omega|\mathrm{d}x\leq C(T)\int_{\mathbb{R}}|\partial_x\omega||\partial_x^2\omega|\mathrm{d}x\\
&\leq C(T)\left\|\partial_x\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+C(T)\left\|\partial_x^2\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\
2|\delta|&\int_{\mathbb{R}}|\partial_x u_1||\omega||\partial_x^3\omega|\mathrm{d}x\leq C(T)\int_{\mathbb{R}}|\omega||\partial_x^3\omega|\mathrm{d}x\\
&=\int_{\mathbb{R}}\left|\frac{C(T)\omega}{\beta}\right|\left|\beta\partial_x^3\omega\right|\mathrm{d}x\leq C(T)\left\|\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\frac{\beta^2}{2}\left\|\partial_x^3\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\
2|\delta|&\int_{\mathbb{R}}|\partial_x u_1||\partial_x^2\omega||\partial_x^3\omega|\mathrm{d}x\leq C(T)\int_{\mathbb{R}}|\partial_x^2\omega||\partial_x^3\omega|\mathrm{d}x\\
&=\int_{\mathbb{R}}\left|\frac{C(T)\partial_x^2\omega}{\beta}\right|\left|\beta\partial_x^3\omega\right|\mathrm{d}x\leq C(T)\left\|\partial_x^2\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\frac{\beta^2}{2}\left\|\partial_x^3\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\
2|\delta|&\int_{\mathbb{R}}|\partial_x u_1||\partial_x^2\omega||\partial_x^3\omega|\mathrm{d}x\leq C(T)\int_{\mathbb{R}}|\partial_x^2\omega||\partial_x^3\omega|\mathrm{d}x
\end{aligned}
$$

$$= \int_{\mathbb{R}} \left| \frac{C(T)\partial_x^2\omega}{\beta} \right| \left| \beta\partial_x^3\omega \right| \mathrm{d}x \leq C(T) \left\| \partial_x^2\omega(t,\cdot) \right\|^2_{L^2(\mathbb{R})} + \frac{\beta^2}{2} \left\| \partial_x^3\omega(t,\cdot) \right\|^2_{L^2(\mathbb{R})},$$

$$2|\delta| \int_{\mathbb{R}} |\partial_x^3 u_2||\omega||\partial_x\omega|\mathrm{d}x \leq C(T) \int_{\mathbb{R}} |\omega||\partial_x\omega|\mathrm{d}x$$
$$\leq C(T) \|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} + C(T) \|\partial_x\omega(t,\cdot)\|^2_{L^2(\mathbb{R})},$$

$$2|\delta| \int_{\mathbb{R}} |\partial_x^3 u_2||\partial_x\omega||\partial_x^2\omega|\mathrm{d}x \leq C(T) \int_{\mathbb{R}} |\partial_x\omega||\partial_x^2\omega|\mathrm{d}x$$
$$\leq C(T) \|\partial_x\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} + C(T) \left\|\partial_x^2\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.$$

It follows from (3.6) that

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} \|\omega(t,\cdot)\|^2_{H^1(\mathbb{R})} + 2\beta^2 \left\|\partial_x^2\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \beta^2 \left\|\partial_x^3\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\ &\quad \leq C(T) \|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} + C(T) \|\partial_x\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} + C(T) \left\|\partial_x^2\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})}. \end{aligned} \tag{3.7}$$

Observe that

$$C(T) \left\|\partial_x^2\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})} = C(T) \int_{\mathbb{R}} \partial_x^2\omega\partial_x^2\omega\mathrm{d}x = -C(T) \int_{\mathbb{R}} \partial_x\omega\partial_x^3\omega\mathrm{d}x.$$

Therefore, by the Young inequality,

$$C(T) \left\|\partial_x^2\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \leq \int_{\mathbb{R}} \left| \frac{C(T)\partial_x\omega}{\beta} \right| \left| \beta\partial_x^3\omega \right| \mathrm{d}x$$
$$\leq C(T) \|\partial_x\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{\beta^2}{2} \left\|\partial_x^3\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.$$

Consequently, by (3.7),

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} \|\omega(t,\cdot)\|^2_{H^1(\mathbb{R})} + 2\beta^2 \left\|\partial_x^2\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{\beta^2}{2} \left\|\partial_x^3\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\ &\quad \leq C(T) \|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} + C(T) \|\partial_x\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} \leq C(T) \|\omega(t,\cdot)\|^2_{H^1(\mathbb{R})}. \end{aligned}$$

The Gronwall Lemma and (3.2) gives

$$\begin{aligned} &\|\omega(t,\cdot)\|^2_{H^1(\mathbb{R})} + \beta^2 e^{C(T)t} \int_0^t e^{-C(T)s} \left\|\partial_x^2\omega(s,\cdot)\right\|^2_{L^2(\mathbb{R})} \mathrm{d}s \\ &\quad + \frac{\beta^2 e^{C(T)t}}{2} \int_0^t e^{-C(T)s} \left\|\partial_x^3\omega(s,\cdot)\right\|^2_{L^2(\mathbb{R})} \mathrm{d}s \leq e^{C(T)t} \|\omega_0\|_{H^1(\mathbb{R})}. \end{aligned} \tag{3.8}$$

(1.11) follows from (3.1) and (3.8). □

Lemma 3.2. *Fix $T > 0$. Under Assumptions (1.2) and (1.12), there exists a unique solution u of (1.1), such that (1.13) and (1.14) hold.*

Proof. We begin by observing that, since $\delta = 0$, (1.1) reads

$$\begin{cases} \partial_t u + \alpha\partial_x^2 u + \beta^2\partial_x^4 u - \gamma^2(\partial_x u)^2\partial_x^2 u + \kappa(\partial_x u)^4 + q(\partial_x u)^2 = 0, & t > 0, \quad x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{3.9}$$

Thanks to Lemmas 2.1, (2.2), (2.4), (2.5), (2.6) and the Cauchy–Kovalevskaya Theorem [58], we have that u is solution of (3.9) and (1.13) holds.

We prove (1.14). Let u_1 and u_2 be two solutions of (3.9), which satisfy (1.13), that is

$$\begin{cases}\partial_t u_1+\alpha\partial_x^2 u_1+\beta^2\partial_x^4 u_1-\gamma^2(\partial_x u_1)^2\partial_x^2 u_1+\kappa(\partial_x u_1)^4+q(\partial_x u_1)^2=0, & t>0, \quad x\in\mathbb{R},\\ u_1(0,x)=u_{1,0}(x), & x\in\mathbb{R},\end{cases}$$

$$\begin{cases}\partial_t u_2+\alpha\partial_x^2 u_2+\beta^2\partial_x^4 u_2-\gamma^2(\partial_x u_2)^2\partial_x^2 u_2+\kappa(\partial_x u_2)^4+q(\partial_x u_2)^2=0, & t>0, \quad x\in\mathbb{R},\\ u_1(0,x)=u_{1,0}(x), & x\in\mathbb{R},\end{cases}$$

Then, the function ω, defined in (3.1), is the solution of the following Cauchy problem:

$$\begin{cases}\partial_t\omega+\alpha\partial_x^2\omega+\beta^2\partial_x^4\omega-\gamma^2\left[(\partial_x u_1)^2\partial_x^2 u_1-(\partial_x u_2)^2\partial_x^2 u_2\right]\\ \quad+\tau\left[\partial_x u_1\partial_x^2 u_1-\partial_x u_2\partial_x^2 u_2\right]\\ \quad+\kappa\left[(\partial_x u_1)^4-(\partial_x u_2)^4\right]\\ \quad+q\left[(\partial_x u_1)^2-(\partial_x u_2)^2\right]=0, & t>0, \quad x\in\mathbb{R},\\ \omega(0,x)=u_{1,0}(x)-u_{2,0}(x), & x\in\mathbb{R}.\end{cases} \tag{3.10}$$

Arguing as in Lemma 3.1, (3.2) is equivalent the following one:

$$\begin{aligned}&\partial_t\omega+\alpha\partial_x^2\omega+\beta^2\partial_x^4\omega-\gamma^2(\partial_x u_1)^2\partial_x^2\omega-\gamma^2\partial_x^2 u_2(\partial_x u_1+\partial_x u_2)\partial_x\omega\\ &\qquad+\tau\partial_x u_1\partial_x^2\omega+\tau\partial_x^2 u_2\partial_x\omega+\kappa\left[(\partial_x u_1)^2+(\partial_x u_2)^2\right](\partial_x u_1+\partial_x u_2)\partial_x\omega\\ &\qquad+q(\partial_x u_1+\partial_x u_2)\partial_x\omega=0\end{aligned} \tag{3.11}$$

Observe that, since $u_1,\ u_2\in H^3(\mathbb{R})$, for every $0\le t\le T$, we have that

$$\|\partial_x u_1\|_{L^\infty((0,T)\times\mathbb{R})},\ \|\partial_x u_2\|_{L^\infty((0,T)\times\mathbb{R})},\ \left\|\partial_x^2 u_2\right\|_{L^\infty((0,T)\times\mathbb{R})}\le C(T). \tag{3.12}$$

Therefore, by (3.12),

$$\begin{aligned}|\partial_x^2 u_2||\partial_x u_1+\partial_x u_2|&\le C(T),\\ \left[(\partial_x u_1)^2+(\partial_x u_2)^2\right]|\partial_x u_1+\partial_x u_2|&\le C(T),\\ |\partial_x u_1+\partial_x u_2|&\le C(T).\end{aligned} \tag{3.13}$$

Since

$$\begin{aligned}2\alpha\int_{\mathbb{R}}\omega\partial_x^2\omega\mathrm{d}x&=-2\alpha\|\partial_x\omega(t,\cdot)\|^2_{L^2(\mathbb{R})},\\ \beta^2\int_{\mathbb{R}}\omega\partial_x^4\omega\mathrm{d}x&=2\beta^2\left\|\partial_x^4\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\end{aligned}$$

multiplying (3.11) by 2ω, an integration on $\mathbb{R}$ gives

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})}+2\beta^2\left\|\partial_x^2\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ &\quad=2\alpha\|\partial_x\omega(t,\cdot)\|^2_{L^2(\mathbb{R})}+2\gamma^2\int_{\mathbb{R}}(\partial_x u_1)^2\omega\partial_x^2\omega\mathrm{d}x\\ &\qquad+2\gamma^2\int_{\mathbb{R}}\partial_x^2 u_2(\partial_x u_1+\partial_x u_2)\omega\partial_x\omega\mathrm{d}x\\ &\qquad-2\tau\int_{\mathbb{R}}\partial_x u_1\omega\partial_x^2\omega\mathrm{d}x-2\tau\int_{\mathbb{R}}\partial_x^2 u_2\omega\partial_x\omega\mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&- 2\kappa \int_{\mathbb{R}} \left[(\partial_x u_1)^2 + (\partial_x u_2)^2\right] (\partial_x u_1 + \partial_x u_2)\omega\partial_x\omega \mathrm{d}x \\
&- 2q \int_{\mathbb{R}} (\partial_x u_1 + \partial_x u_2)\omega\partial_x\omega \mathrm{d}x.
\end{aligned} \tag{3.14}
$$

Due to (3.12), (3.13) and the Young inequality,

$$
\begin{aligned}
&2\gamma^2 \int_{\mathbb{R}} (\partial_x u_1)^2 |\omega||\partial_x^2\omega| \mathrm{d}x \le C(T) \int_{\mathbb{R}} |\omega||\partial_x^2\omega| \mathrm{d}x \\
&\quad = \int_{\mathbb{R}} \left|\frac{C(T)\omega}{\beta}\right| \left|\beta\partial_x^2\omega\right| \mathrm{d}x \le C(T) \left\|\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{\beta^2}{2} \left\|\partial_x^2\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})}, \\
&2\gamma^2 \int_{\mathbb{R}} |\partial_x^2 u_2||\partial_x u_1 + \partial_x u_2||\omega||\partial_x\omega| \mathrm{d}x \le C(T) \int_{\mathbb{R}} |\omega||\partial_x\omega| \mathrm{d}x \\
&\quad \le C(T) \left\|\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + C(T) \left\|\partial_x\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})}, \\
&2|\tau| \int_{\mathbb{R}} |\partial_x u_1||\omega||\partial_x^2\omega| \mathrm{d}x \le C(T) \int_{\mathbb{R}} |\omega||\partial_x^2\omega| \mathrm{d}x \\
&\quad = \int_{\mathbb{R}} \left|\frac{C(T)\omega}{\beta}\right| \left|\beta\partial_x^2\omega\right| \mathrm{d}x \le C(T) \left\|\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{\beta^2}{2} \left\|\partial_x^2\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})}, \\
&2|\tau| \int_{\mathbb{R}} |\partial_x^2 u_2||\omega||\partial_x\omega| \mathrm{d}x \le C(T) \int_{\mathbb{R}} |\omega||\partial_x\omega| \mathrm{d}x \le C(T) \left\|\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + C(T) \left\|\partial_x\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})}, \\
&2|\kappa| \int_{\mathbb{R}} \left[(\partial_x u_1)^2 + (\partial_x u_2)^2\right] |\partial_x u_1 + \partial_x u_2|\omega\partial_x\omega \mathrm{d}x \le C(T) \int_{\mathbb{R}} |\omega||\partial_x\omega| \mathrm{d}x \\
&\quad \le C(T) \left\|\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + C(T) \left\|\partial_x\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})}, \\
&|2q| \int_{\mathbb{R}} |\partial_x u_1 + \partial_x u_2||\omega||\partial_x\omega| \mathrm{d}x \le C(T) \int_{\mathbb{R}} |\omega\partial_x\omega| \mathrm{d}x \\
&\quad \le C(T) \left\|\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + C(T) \left\|\partial_x\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.
\end{aligned}
$$

It follows from (3.14) that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \left\|\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \beta^2 \left\|\partial_x^2\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\
&\quad \le C(T) \left\|\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + C(T) \left\|\partial_x\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.
\end{aligned} \tag{3.15}
$$

Observe that

$$
C(T) \left\|\partial_x\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})} = C(T) \int_{\mathbb{R}} \partial_x\omega\partial_x\omega \mathrm{d}x = -C(T) \int_{\mathbb{R}} \omega\partial_x^2\omega \mathrm{d}x.
$$

Therefore, by the Young inequality,

$$
\begin{aligned}
C(T) \left\|\partial_x\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})} &\le \int_{\mathbb{R}} \left|\frac{C(T)\omega}{\beta}\right| \left|\beta\partial_x^2\omega\right| \mathrm{d}x \\
&\le C(T) \left\|\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{\beta^2}{2} \left\|\partial_x^2\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.
\end{aligned}
$$

Consequently, by (3.15),

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})}+\frac{\beta^2}{2}\left\|\partial_x^2\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\le C(T)\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})}$$

The Gronwall Lemma and (3.10) gives

$$\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})}+\frac{\beta^2e^{C(T)t}}{2}\int_0^t e^{-C(T)s}\left\|\partial_x^2\omega(s,\cdot)\right\|^2_{L^2(\mathbb{R})}\mathrm{d}s\le e^{C(T)t}\|\omega_0\|_{L^2(\mathbb{R})} \tag{3.16}$$

(1.14) follows from (3.1) and (3.16). □

Lemma 3.3. *Fix $T>0$. Under Assumptions (1.2) and (1.15), there exists a solution u of (1.1), such that (1.16) holds.*

Proof. Let $T>0$. Thanks to Lemmas 2.1, (2.2), (2.4), (2.5) and the Cauchy–Kovalevskaya Theorem [58], we have that u is solution of (3.9) and (1.13) holds. □

Proof of Theorem 1.1. Theorem 1.1 follows from Lemmas 3.1, 3.2 and 3.3. □

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