ON A CLASS OF SUPERLINEAR (p, q)–LAPLACIAN TYPE EQUATIONS ON $\mathbb{R}^N$

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Abstract. Starting from a new sum decomposition of $W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$ and using a variational approach, we investigate the existence of multiple weak solutions of a (p, q)–Laplacian equation on $\mathbb{R}^N$, for $1 < q < p < N$, with a sign–changing potential and a Carathéodory reaction term satisfying the celebrated Ambrosetti–Rabinowitz condition. Our assumptions are mild and different from those used in related papers and moreover our results improve or complement previous ones for the single p–Laplacian.

1. Introduction

We consider the following nonlinear equation of (p, q)–Laplacian type on $\mathbb{R}^N$:

$$\begin{cases} -\Delta_p u + V(x)|u|^{p-2}u - \Delta_q u + W(x)|u|^{q-2}u \;=\; f(x,u) \quad \text{in } \mathbb{R}^N, \\ u \in W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N), \end{cases} \tag{1.1}$$

where $1 < q < p < N$, $\Delta_r u = \mathrm{div}(|\nabla u|^{r-2}\nabla u)$ for any $1 < r < +\infty$, V, W are potential functions on $\mathbb{R}^N$ and $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a superlinear, but subcritical, nonlinearity (namely, it satisfies the Ambrosetti–Rabinowitz condition (f_2) here below, henceforth simply denoted by (AR)). As it is conceivable, the coexistence of both the p and the q–Laplacian operators calls for a very careful analysis.

Clearly, when $p = q = 2$, the equation in (1.1) turns out to be a semilinear Schrödinger one of the form

$$-\Delta u + U(x)u = f(x,u), \qquad u \in H^1(\mathbb{R}^N), \tag{1.2}$$

which is considered in [7] if U is constant. More in general, problem (1.2) has been widely studied firstly for a constant sign potential U (cf. [5, 23, 27, 28]), later on for sign–changing potentials (cf. [13, 14, 30]). Classical proofs in this last case are based on the fact that the spectrum of the self–adjoint operator $-\Delta + U$ induces a suitable direct sum decomposition of $H^1(\mathbb{R}^N)$.

Since an exhaustive description of the spectrum of the p–Laplace operator is not available for $p \neq 2$, the study of this kind of equations is far more difficult and the linking theorem over cones by Degiovanni and Lancelotti in [12] becomes a keypoint.

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Namely, when only the p–Laplacian operator appears with $p > 1$ but $p \neq 2$, contributions to the case of constant sign potentials can be found in [3, 17, 22], while, based on [12], advances for equations with sign–changing potentials and subcritical $(p-1)$–superlinear f, satisfying a condition complementary to the (AR) one, have been obtained in [18, 21]. Let us point out that, for a class of problems alike this, once we consider the associated energy functional, a typical setting is the Mountain Pass Geometry near zero ensured by the $(p-1)$–superlinearity at 0 and at $+\infty$, while the subcriticality at $+\infty$, plus the (AR) condition, guarantees the boundedness of Palais–Smale sequences, if the loss of compactness is balanced by extra assumptions.

In this paper we look for solutions of (1.1) in the general case $p \neq q$. The interest in this kind of problem is twofold: from one hand since it is quite challenging from an analytical viewpoint and moreover because it has a relevant physical interpretation in applied sciences. In fact, if we denote by u a concentration, the equation derives from a general reaction–diffusion system:

$$u_t \; = \; \mathrm{div}[(|\nabla u|^{p-2}\nabla u) + (|\nabla u|^{q-2}\nabla u)] + \varphi(x,u)$$

which arises not only in physics, bur also in biophysics, plasma physics and chemical reaction design. In most cases, φ is a polynomial with variable coefficients (cf., e.g., [11, 16]).

We are aware of a very few contributions concerning problem (1.1). Superlinear (p,q)–equations without the Ambrosetti–Rabinowitz condition have been studied both in bounded domains (see [24]) and in $\mathbb{R}^N$ but when the weights V and W are continuous, positive and coercive (see [10]). For the unbounded case we refer also to [19] where, taking $V \equiv W \equiv 1$, the set of conditions on f includes (AR) and the Concentration–Compactness Principle is used (cf. also [15, 20]). Finally, we refer to [2] and references therein for the special case of $(p,2)$–equations.

Even if (1.1) has a variational structure, the main problems in the application of classical variational tools are due to the lack both of homogeneity of the (p,q)–Laplacian operator and of compactness of the Sobolev's embeddings on the whole space $\mathbb{R}^N$. Here, we overcome the first defect by looking for a sharp decomposition of the ambient space and the second one by introducing some properties on the weight functions, one of which may change sign.

Namely, we assume that:

(H_1) the potentials $V, W : \mathbb{R}^N \to \mathbb{R}$ are Lebesgue measurable functions such that

$$\operatorname*{ess\,inf}_{x\in\mathbb{R}^N} V(x) > 0, \qquad \operatorname*{ess\,inf}_{x\in\mathbb{R}^N} W(x) > -\infty \tag{1.3}$$

and

$$\lim_{|x|\to+\infty} \int_{B_1(x)} \frac{1}{V(y)}\,\mathrm{d}y = 0, \qquad \lim_{|x|\to+\infty} \int_{B_1(x)} \frac{1}{W(y)}\,\mathrm{d}y = 0,$$

where $B_1(x) = \{y \in \mathbb{R}^N : |x-y| < 1\}$;

(H_2) $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function (i.e., $f(\cdot,t)$ is measurable in $\mathbb{R}^N$ for all $t \in \mathbb{R}$ and $f(x,\cdot)$ is continuous in $\mathbb{R}$ for a.e. $x \in \mathbb{R}^N$) such that there exist $a > 0$, $s \in \,]p, p^*[$, $\mu > p$, satisfying the following conditions:

(f_1) $\quad |f(x,t)| \;\le\; a(|t|^{s-1} + |t|^{p-1})$ for a.e. $x \in \mathbb{R}^N$, all $t \in \mathbb{R}$,

(f_2) $\quad f(x,t)t \;\ge\; \mu F(x,t) > 0$ for a.e. $x \in \mathbb{R}^N$, all $t \in \mathbb{R} \setminus \{0\}$,

with $F(x,t) := \displaystyle\int_0^t f(x,r)\,\mathrm{d}r$.

Some remarks on the role of the hypoteses are in order. The assumptions in (H_1) were introduced in [6] in the study of the linear Schrödinger equation and then used in [3, 4] for the single p–Laplacian.

Notice that in [27] the existence of a nontrivial solution of (1.2) is shown by using the Mountain Pass Theorem if $U \in C^1(\mathbb{R}^N, \mathbb{R})$ is positive and coercive; later on in [5], by means of the Symmetric Mountain Pass Theorem (cf. [1, Theorem 2.8]), Bartsch and Wang find infinitely many solutions if f is odd in t and U is a positive continuous function such that

$$\text{meas}\left(\left\{x \in \mathbb{R}^N : U(x) \le b\right\}\right) < +\infty \qquad \text{for all } b > 0.$$

As shown in [28, Proposition 3.1], the hypotheses on U both in [5] and in [27] imply that

$$\operatorname*{ess\,inf}_{x\in\mathbb{R}^N} U(x) > 0 \qquad \text{and} \qquad \lim_{|x|\to+\infty} \int_{B_1(x)} \frac{1}{U(y)}\, \mathrm{d}y = 0. \tag{1.4}$$

Therefore, for the (p,q)–equation in (1.1) the assumptions on the weights V and W in (H_1) are weaker than those ones in [10] also because W may change sign.

Now, we can state our main result.

Theorem 1.1. *Assume that (H_1)–(H_2) hold. Then (1.1) has a non–trivial solution. Moreover, if $f(x,\cdot)$ is odd for a.e. $x \in \mathbb{R}^N$, then (1.1) has infinitely many solutions.*

The paper is organized as follows: in Section 2 we introduce the variational setting of our problem and a decomposition for $W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$, then in Section 3 we prove Theorem 1.1.

Notations. Throughout this paper we denote by

- $r^* = \frac{rN}{N-r}$ if $r \in]1, N[$, $r^* = +\infty$ otherwise;
- r' the conjugate exponent of $r \ge 1$, namely $r' = \frac{r}{r-1}$ if $r > 1$ and $r' = +\infty$ if $r = 1$;
- $|\cdot|_r$ the standard norm in the Lebesgue space $L^r(\mathbb{R}^N)$, $1 \le r \le +\infty$;
- $(X, \|\cdot\|_X)$ a Banach space with dual space $(X', \|\cdot\|_{X'})$;
- $B_R = \{u \in X : \|u\|_X < R\}$, $\partial B_R = \{u \in X : \|u\|_X = R\}$ for all $R > 0$.

2. Variational set–up

Let $U : \mathbb{R}^N \to \mathbb{R}$ be a Lebesgue measurable function such that

$$\operatorname*{ess\,inf}_{x\in\mathbb{R}^N} U(x) \; > \; 0. \tag{2.1}$$

Hence, for any $r > 1$ we consider the weighted Sobolev space

$$E_U^r := W_U^{1,r}(\mathbb{R}^N) = \left\{ u \in \; W^{1,r}(\mathbb{R}^N) : \; \int_{\mathbb{R}^N} U(x)|u|^r \mathrm{d}x < +\infty \right\} \tag{2.2}$$

endowed with the norm

$$\|u\|_{r,U} = \left[\int_{\mathbb{R}^N} |\nabla u|^r \mathrm{d}x + \int_{\mathbb{R}^N} U(x)|u|^r \mathrm{d}x \right]^{\frac{1}{r}}. \tag{2.3}$$

The space $(E_U^r, \|\cdot\|_{r,U})$ is a separable and reflexive Banach space; even more, it is a uniformly convex space (cf. [3, Proposition 2.1]).

Before stating a finite dimensional decomposition of E_U^r as in [3, 9], we recall the following compact embedding theorem which plays a crucial role in it (cf. [6, Theorem 3.1]).

Theorem 2.1. *Taking $U : \mathbb{R}^N \to \mathbb{R}$ such that (1.4) holds, we have the following embeddings:*

(i) $(E^r_U, \|\cdot\|_{r,U}) \hookrightarrow (L^s(\mathbb{R}^N), |\cdot|_s)$ *continuously if* $s \in [r, r^*]$;
(ii) $(E^r_U, \|\cdot\|_{r,U}) \hookrightarrow\hookrightarrow (L^s(\mathbb{R}^N), |\cdot|_s)$ *compactly if* $s \in [r, r^*[$.

In order to present the announced decomposition, we recall that if $Y \subseteq X$ is a closed subspace of a Banach space X, a subspace $Z \subseteq X$ is a *topological complement* of Y, briefly $X = Y \oplus Z$, if Z is closed, $Y \cap Z = \{0\}$ and $X = Y + Z$. Then we define $\mathrm{codim} Z = \dim Y$ and every $x \in X$ can be written uniquely as $x = y + z$, with $y \in Y$ and $z \in Z$; moreover, the projection operators onto Y and Z are (linear and) continuous, hence there exists $\beta = \beta(Y, Z) > 0$ such that

$$\|y\| + \|z\| \le \beta \|y + z\|.$$

In general, if $X = Y + Z$ and Y has finite dimension, we say that Z has finite codimension, with $\mathrm{codim} Z \le \dim Y$. We recall also that each closed subspace having finite codimension admits a topological complement (cf., e.g., [8, p. 38]).

Let us define the subset

$$\mathcal{S}_{r,U} = \{v \in E^r_U : |v|_r = 1\},$$

the functional $\Phi_{r,U} : E^r_U \to \mathbb{R}$ by

$$\Phi_{r,U}(u) = \|u\|^r_{r,U}$$

and

$$\eta^1_{r,U} \ = \ \inf_{v \in \mathcal{S}_{r,U}} \Phi_{r,U}(v) \ \ge \ 0.$$

In [3, Section 2] and [9, Section 5], starting from $\eta^1_{r,U}$, it is shown the existence of an increasing diverging sequence $(\eta^k_{r,U})_k$ of positive real numbers, with corresponding functions $(\psi^k_{r,U})_k$ such that $\psi^i_{r,U} \ne \psi^j_{r,U}$ if $i \ne j$. They generate the whole space E^r_U and are such that

$$E^r_U = Y^k_{r,U} \oplus Z^k_{r,U} \quad \text{ for all } k \in \mathbb{N}, \tag{2.4}$$

where $Y^k_{r,U} = \mathrm{span}\{\psi^1_{r,U}, \dots, \psi^k_{r,U}\}$ and its complement $Z^k_{r,U}$ is a closed subspace that can be explicitely described.

Remarkably, for all $k \in \mathbb{N}$ on the infinite dimensional subspace $Z^k_{r,U}$ we have the following inequality:

$$\eta^{k+1}_{r,U} \ |z|^r_r \ \le \ \|z\|^r_{r,U} \quad \text{ for all } z \in Z^k_{r,U} \tag{2.5}$$

(cf. [9, Lemma 5.4]).

Now, let us consider two potentials V and W such that (H_1) holds. By (1.3), condition (2.1) holds with U replaced by V or by $W + \alpha$, by taking $\alpha > 0$ such that

$$\operatorname*{ess\,inf}_{x \in \mathbb{R}^N} (W(x) + \alpha) \ > \ 0.$$

Therefore, taking $p, q \in \,]1, +\infty[$, we can introduce the spaces $(E^p_V, \|\cdot\|_{p,V})$ and $(E^q_{W+\alpha}, \|\cdot\|_{q,W+\alpha})$ defined as in (2.2)–(2.3).

From now on, we set

$$E \ := \ E^p_V \cap E^q_{W+\alpha} \tag{2.6}$$

equipped with the norm

$$\|u\|_E \ := \ \|u\|_{p,V} + \|u\|_{q,W+\alpha}.$$

In particular, by (2.4)–(2.5) applied to $(E^p_V, \|\cdot\|_{p,V})$ and $(E^q_{W+\alpha}, \|\cdot\|_{q,W+\alpha})$, fixing any $k \in \mathbb{N}$ we get

$$E^p_V \;=\; Y^k_{p,V} \oplus Z^k_{p,V} \quad \text{and} \quad E^q_{W+\alpha} \;=\; Y^k_{q,W+\alpha} \oplus Z^k_{q,W+\alpha}.$$

Then, setting

$$Z^k \;:=\; Z^k_{p,V} \cap Z^k_{q,W+\alpha}, \tag{2.7}$$

the following inequalities hold:

$$\eta^{k+1}_{p,V}\; |z|^p_p \;\le\; \|z\|^p_{p,V} \quad \text{and} \quad \eta^{k+1}_{q,W+\alpha}\; |z|^q_q \;\le\; \|z\|^q_{q,W+\alpha} \quad \text{for all } z \in Z^k, \tag{2.8}$$

where

$$\eta^k_{p,V} \nearrow +\infty \quad \text{and} \quad \eta^k_{q,W+\alpha} \nearrow +\infty \qquad \text{as } k \to +\infty. \tag{2.9}$$

Being Z^k a closed subspace of E of finite codimension, then a finite dimensional subspace Y^k of E exists which is its topological complement, i.e.

$$E \;=\; Y^k \oplus Z^k.$$

We highlight a straightforward consequence of Theorem 2.1.

Corollary 2.2. *Assume that (H_1) holds. Then,*

(i) $(E, \|\cdot\|_E) \hookrightarrow (L^s(\mathbb{R}^N), |\cdot|_s)$ *continuously if* $s \in [q, q^*] \cup [p, p^*]$*;*
(ii) $(E, \|\cdot\|_E) \hookrightarrow\!\!\!\!\hookrightarrow (L^s(\mathbb{R}^N), |\cdot|_s)$ *compactly if* $s \in [q, q^*[\,\cup\, [p, p^*[$.

Remark 2.3. In particular, $[q, q^*] \cup [p, p^*] = [q, p^*]$ and $[q, q^*[\,\cup\, [p, p^*[\,=\, [q, p^*[$ if $p \le q^*$.

Remark 2.4. For further use we observe that, defining

$$\|u\|_{\max} := \max\{\|u\|_{p,V}, \|u\|_{q,W+\alpha}\},$$

$\|\cdot\|_E$ and $\|\cdot\|_{\max}$ are equivalent norms, i.e., there exist $c_1, c_2 > 0$ such that

$$c_1\|u\|_{\max} \le \|u\|_E \le c_2\|u\|_{\max} \quad \text{for all } u \in E.$$

As a direct consequence of Corollary 2.2 and [31, Theorem 1.22] we can state the following lemma.

Lemma 2.5. *Assume that (f_1) holds. Then, setting $g : E \to \mathbb{R}$ as*

$$g(u) = \int_{\mathbb{R}^N} F(x,u)\,\mathrm{d}x \quad \text{for all } u \in E,$$

it results that $g \in C^1(E, \mathbb{R})$ with

$$dg(u)[\varphi] \;=\; \int_{\mathbb{R}^N} f(x,u)\varphi\,\mathrm{d}x \quad \text{for all } u,\ \varphi \in E.$$

Moreover, $dg : E \to E'$ is compact.

By Lemma 2.5 and standard variational arguments, the functional $J : E \to \mathbb{R}$, defined as

$$J(u) \;= \frac{1}{p}\int_{\mathbb{R}^N} (|\nabla u|^p + V(x)|u|^p)\,\mathrm{d}x + \frac{1}{q}\int_{\mathbb{R}^N} (|\nabla u|^q + W(x)|u|^q)\,\mathrm{d}x$$
$$- \int_{\mathbb{R}^N} F(x,u)\,\mathrm{d}x,$$

is C^1 with

$$\begin{aligned} \mathrm{d}J(u)[\varphi] \;=\; & \int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u\cdot\nabla\varphi\,\mathrm{d}x + \int_{\mathbb{R}^N} V(x)|u|^{p-2}u\,\varphi\,\mathrm{d}x \\ & + \int_{\mathbb{R}^N} |\nabla u|^{q-2}\nabla u\cdot\nabla\varphi\,\mathrm{d}x + \int_{\mathbb{R}^N} W(x)|u|^{q-2}u\,\varphi\,\mathrm{d}x \\ & - \int_{\mathbb{R}^N} f(x,u)\,\varphi\,\mathrm{d}x \qquad\qquad \text{for all } u,\ \varphi\in E. \end{aligned} \tag{2.10}$$

Hence, its critical points in E are the weak solutions of (1.1).

Let us point out that by (2.3) the functional J can be written as follows:

$$J(u) = \frac{1}{p}\|u\|_{p,V}^p + \frac{1}{q}\|u\|_{q,W+\alpha}^q - \frac{\alpha}{q}|u|_q^q - \int_{\mathbb{R}^N} F(x,u)\,\mathrm{d}x, \qquad u\in E. \tag{2.11}$$

Since we are looking for existence and multiplicity of solutions of (1.1), our aim is to use the Linking Theorem and the Symmetric Mountain Pass Theorem. Here, we recall their statements and the well–known Cerami's variant of the Palais–Smale condition (cf., e.g., [26, 29]).

Definition 2.6. Let $I : X \to \mathbb{R}$ be a C^1 functional on the Banach space $(X, \|\cdot\|_X)$. The functional I satisfies the *Cerami's variant of the Palais–Smale condition*, briefly (CPS), if any sequence $(u_n)_n \subseteq X$ such that

$$(I(u_n))_n \text{ is bounded} \quad \text{and} \quad \lim_{n\to+\infty} \|\mathrm{d}I(u_n)\|_{X'}(1+\|u_n\|_X) = 0 \tag{2.12}$$

converges in X, up to subsequences. We say that $(u_n)_n$ is a (CPS) sequence if it verifies (2.12).

Theorem 2.7. *Consider $\alpha, \beta \in \mathbb{R}$ such that $\alpha < \beta$. Assume that:*

(i) the functional I satisfies (CPS);

(ii) there exist a closed $S \subseteq X$ and $Q \subseteq Y$, being Y a subspace of X, with boundary ∂Q in Y, satisfying:

- *(a) $I(u) \le \alpha$ for all $u \in \partial Q$ and $I(u) \ge \beta$ for all $u \in S$;*
- *(b) S and ∂Q link, i.e. $S \cap \partial Q = \emptyset$ and $\phi(Q) \cap S \ne \emptyset$, for any $\phi \in C(X,X)$ such that $\phi\big|_{\partial Q} = \mathrm{id}$;*
- *(c) $\sup_{u\in Q} I(u) < +\infty$.*

Then, there exists a critical level c of I given by

$$c = \inf_{\phi\in\Gamma} \sup_{u\in Q} I(\phi(u)), \qquad \text{with} \qquad \beta \le c \le \sup_{u\in Q} I(u),$$

where $\Gamma = \left\{\phi \in C(X,X) : \phi\big|_{\partial Q} = \mathrm{id}\right\}$.

Theorem 2.8. *Let X be an infinite dimensional Banach space and let $I \in C^1(X,\mathbb{R})$ be even, satisfy (CPS) and $I(0) = 0$. If $X = Y \oplus Z$, where Y is finite dimensional, and I satisfies*

(i) *there are constants ρ, $\alpha > 0$ such that $I|_{\partial B_\rho \cap Z} \ge \alpha$,*

(ii) *for each finite dimensional subspace $\widetilde{X} \subset X$, there is an $R = R(\widetilde{X}) > 0$ such that $I \le 0$ on $\widetilde{X} \setminus B_R$,*

then I possesses an unbounded sequence of critical values.

3. Proof of Theorem 1.1

From now on, if (H_1) and (f_1) hold, let J be the C^1 functional in (2.11) defined on the Banach space E in (2.6).

Remark 3.1. From (f_2) and direct computations it follows that for all $R_0 > 0$ there exists $a_0 > 0$ such that

$$F(x,t) \;\geq\; a_0|t|^\mu \quad \text{for a.e. } x\in\Omega \text{ if } |t|\geq R_0. \tag{3.1}$$

On the other hand, (f_1) implies that

$$|F(x,t)| \;\leq\; \frac{a}{s}\,|t|^s + \frac{a}{p}\,|t|^p \quad \text{for a.e. } x\in\Omega, \text{ all } t\in\mathbb{R}. \tag{3.2}$$

Hence, if (H_1) holds then it has to be $\mu \leq s$.

In order to apply Theorem 2.8, we need some technical lemmas.

Lemma 3.2. *Assume that (H_1)–(H_2) hold. Then, for any finite dimensional subspace $F\subset E$ it results*

$$\lim_{\substack{v\in F\\ \|v\|_E\to+\infty}} J(v) = -\infty. \tag{3.3}$$

Proof. Arguing by contradiction, we assume that a finite dimensional subspace F of E exists which does not satisfy (3.3). Hence, a sequence $(u_n)_n\subset F$ can be found so that

$$\|u_n\|_E \;\to\; +\infty \qquad \text{as } n\to+\infty \tag{3.4}$$

and for some $M>0$ it is $J(u_n)\geq -M$ for all $n\in\mathbb{N}$. Then, setting $v_n = \frac{u_n}{\|u_n\|_E}$, it follows that $\|v_n\|_E = 1$ and $v_n \rightharpoonup v$ weakly in E (up to subsequences), or better, since $\dim F < +\infty$, $v_n\to v$ strongly in F and almost everywhere in $\mathbb{R}^N$. Thus, $\|v\|_E = 1$ and, setting $A := \{x\in\mathbb{R}^N : v(x)\neq 0\}$, it is meas $A>0$. Hence, for a.e. $x\in A$ it is $\lim_n |v_n(x)| = |v(x)| > 0$, so by (3.4) it follows $|u_n(x)|\to+\infty$. Thus, (3.1) implies that

$$\lim_n \frac{F(x,u_n(x))}{|u_n(x)|^p}|v_n(x)|^p \;=\; +\infty \quad \text{for a.e. } x\in A. \tag{3.5}$$

On the other hand, for n large enough, by standard calculations we get

$$\frac{1}{p}\frac{\|u_n\|^p_{p,V}}{\|u_n\|^p_E} + \frac{1}{q}\frac{\|u_n\|^q_{q,W+\alpha}}{\|u_n\|^p_E} \leq \frac{1}{q}\frac{\|u_n\|^p_{p,V} + \|u_n\|^q_{q,W+\alpha}}{\|u_n\|^p_E} \leq \frac{2}{q}\frac{\|u_n\|^p_E}{\|u_n\|^p_E} = \frac{2}{q}$$

(without loss of generality, by (3.4) we assume $\|u_n\|_E\geq 1$ for all $n\in\mathbb{N}$). Then, by (2.11) and the Fatou's Lemma (let us recall that (f_2) implies $F(x,t)\geq 0$ for a.e. $x\in\mathbb{R}^N$ and all $t\in\mathbb{R}$) we infer that

$$\begin{aligned}
0 \;&=\; \lim_n \frac{-M}{\|u_n\|^p_E} \leq \limsup_n \frac{J(u_n)}{\|u_n\|^p_E} \;\leq\; \limsup_n\left(\frac{2}{q} - \int_{\mathbb{R}^N}\frac{F(x,u_n(x))}{\|u_n\|^p_E}\,\mathrm{d}x\right)\\
&= \frac{2}{q} - \liminf_n \int_{\mathbb{R}^N}\frac{F(x,u_n(x))}{\|u_n\|^p_E}\,\mathrm{d}x \;\leq\; \frac{2}{q} - \int_{\mathbb{R}^N}\liminf_n \frac{F(x,u_n(x))}{|u_n(x)|^p}|v_n(x)|^p\,\mathrm{d}x\\
&= \frac{2}{q} - \int_{\mathbb{R}^N}\lim_n \frac{F(x,u_n(x))}{|u_n(x)|^p}|v_n(x)|^p\,\mathrm{d}x;
\end{aligned}$$

whence,

$$\int_{\mathbb{R}^N}\lim_n \frac{F(x,u_n(x))}{|u_n(x)|^p}|v_n(x)|^p\,\mathrm{d}x \;\leq\; \frac{2}{q}, \tag{3.6}$$

in contradiction with (3.5) as meas $A > 0$. □

Lemma 3.3. *Assume that* (H_1) *and* (f_1) *hold. Then, for any* $k \in \mathbb{N}$ *large enough there exist* $\rho, c > 0$ *such that*

$$J(u) \geq c \quad \text{for all } u \in \partial B_\rho \cap Z^k.$$

Proof. By (3.2) and (2.11) we have that

$$J(u) \geq \frac{1}{p}\|u\|_{p,V}^p + \frac{1}{q}\|u\|_{q,W+\alpha}^q - \frac{\alpha}{q}|u|_q^q - \frac{a}{p}|u|_p^p - \frac{a}{s}|u|_s^s.$$

Fixing any $k \in \mathbb{N}$, by applying inequality (2.8) and by Corollary 2.2 a constant $a_1 > 0$ exists such that

$$J(u) \ \geq \ \frac{1}{p}\|u\|_{p,V}^p + \frac{1}{q}\|u\|_{q,W+\alpha}^q - \frac{\alpha}{q\eta_{q,W+\alpha}^{k+1}}\|u\|_{q,W+\alpha}^q - \frac{a}{p\eta_{p,V}^{k+1}}\|u\|_{p,V}^p - a_1\|u\|_E^s$$

for all $u \in Z^k$ (cf. (2.7)). Hence, by (2.9) we can take k large enough such that $1 - \frac{\alpha}{\eta_{q,W+\alpha}^{k+1}} > 0$ and $1 - \frac{a}{\eta_{p,V}^{k+1}} > 0$, so by standard calculations, taking

$$m_k = \min\left\{1 - \frac{\alpha}{\eta_{q,W+\alpha}^{k+1}}, 1 - \frac{a}{\eta_{p,V}^{k+1}}\right\} \ > \ 0,$$

we get

$$\begin{aligned} J(u) &\geq \frac{1}{p}\left(1 - \frac{a}{\eta_{p,V}^{k+1}}\right)\|u\|_{p,V}^p + \frac{1}{p}\left(1 - \frac{\alpha}{\eta_{q,W+\alpha}^{k+1}}\right)\|u\|_{q,W+\alpha}^q - a_1\|u\|_E^s \\ &\geq \frac{m_k}{p}\left(\|u\|_{p,V}^p + \|u\|_{q,W+\alpha}^q\right) - a_1\|u\|_E^s \end{aligned}$$

for all $u \in Z^k$. Now, since for $\|u\|_E < 1$ we have $\|u\|_{q,W+\alpha} < 1$, it results

$$\|u\|_E^p \leq 2^{p-1}(\|u\|_{p,V}^p + \|u\|_{q,W+\alpha}^q),$$

then we infer

$$J(u) \ \geq \ \frac{m_k}{2^{p-1}p}\ \|u\|_E^p - a_1\|u\|_E^s,$$

so, being $s > p$, the proof is complete once we fix $\|u\|_E = \rho$ small enough. □

Lemma 3.4. *Assume that hypotheses* (H_1)*–*(H_2) *hold. Then, the functional* J *satisfies the* (CPS) *condition in* E.

Proof. Let $(u_n)_n$ be a (CPS) sequence, hence (2.12) holds with I replaced by J. Then, for some $\gamma > 0$ and for n large enough we have

$$J(u_n) - \frac{1}{\mu}\, dJ(u_n)[u_n] \ \leq \ \gamma.$$

Hence, by (2.10)–(2.11) we get that

$$\begin{aligned} \gamma \ &\geq \ \left(\frac{1}{p} - \frac{1}{\mu}\right)\|u_n\|_{p,V}^p + \left(\frac{1}{q} - \frac{1}{\mu}\right)\|u_n\|_{q,W+\alpha}^q - \alpha\left(\frac{1}{q} - \frac{1}{\mu}\right)|u_n|_q^q \\ &\quad + \int_{\mathbb{R}^N}\left(\frac{1}{\mu}f(x,u_n)u_n - F(x,u_n)\right)\mathrm{d}x, \end{aligned}$$

thus (f_2) implies

$$\gamma \ \geq \ \left(\frac{1}{p} - \frac{1}{\mu}\right)\|u_n\|_{p,V}^p + \left(\frac{1}{q} - \frac{1}{\mu}\right)\|u_n\|_{q,W+\alpha}^q - \alpha\left(\frac{1}{q} - \frac{1}{\mu}\right)|u_n|_q^q. \tag{3.7}$$

We claim that $(u_n)_n$ is bounded in E. In fact, arguing by contradiction, we assume that the limit (3.4) holds up to subsequences. Setting $v_n = \frac{u_n}{\|u_n\|_E}$ it follows $\|v_n\|_E = 1$ for all $n \in \mathbb{N}$ large enough, $v_n \rightharpoonup v$ weakly in E and, by Corollary 2.2, if $r \in [q, q^*[\cup [p, p^*[$ then $v_n \to v$ strongly in $L^r(\mathbb{R}^N)$ and $v_n(x) \to v(x)$ for a.e. $x \in \mathbb{R}^N$. If $v \neq 0$, setting $A := \{x \in \mathbb{R}^N : v(x) \neq 0\}$, we have that meas $A > 0$; thus, since

$$\lim_n \frac{J(u_n)}{\|u_n\|_E^p} = 0,$$

we can reason as in the proof of Lemma 3.2 so that (3.6) still holds and we get a contradiction.

On the other hand, if $v = 0$ by Remark 2.4 and (3.4) also $\|u_n\|_{\max} \to +\infty$ and, up to subsequences, it may be either

$$\|u_n\|_{\max} = \|u_n\|_{p,V} \to +\infty \tag{3.8}$$

or

$$\|u_n\|_{\max} = \|u_n\|_{q,W+\alpha} \to +\infty. \tag{3.9}$$

If (3.8) holds, dividing (3.7) by $\|u_n\|_{p,V}^p$ we obtain

$$\frac{\gamma}{\|u_n\|_{p,V}^p} \geq \left(\frac{1}{p} - \frac{1}{\mu}\right) - \alpha\left(\frac{1}{q} - \frac{1}{\mu}\right)\frac{|u_n|_q^q}{\|u_n\|_{p,V}^p}, \tag{3.10}$$

where

$$\frac{|u_n|_q^q}{\|u_n\|_{p,V}^p} = |v_n|_q^q \frac{\|u_n\|_E^p}{\|u_n\|_{p,V}^p} \frac{1}{\|u_n\|_E^{p-q}}.$$

Hence, by Remark 2.4, (3.8) and since $v_n \to 0$ in $L^q(\mathbb{R}^N)$, (3.10) yields to a contradiction. Therefore, $(\|u_n\|_{p,V})_n$ must be bounded and (3.9) holds. Thus, dividing (3.7) by $\|u_n\|_{q,W+\alpha}^q$, we have that

$$\frac{\gamma}{\|u_n\|_{q,W+\alpha}^q} \geq \left(\frac{1}{q} - \frac{1}{\mu}\right) - \alpha\left(\frac{1}{q} - \frac{1}{\mu}\right)\frac{|u_n|_q^q}{\|u_n\|_{q,W+\alpha}^q}.$$

In this case, we observe that

$$\frac{|u_n|_q^q}{\|u_n\|_{q,W+\alpha}^q} = |v_n|_q^q \frac{\|u_n\|_E^q}{\|u_n\|_{q,W+\alpha}^q},$$

therefore we get a contradiction by (3.9) and again Remark 2.4 as $v_n \to 0$ in $L^q(\mathbb{R}^N)$. Hence, also $(\|u_n\|_{q,W+\alpha})_n$ must be bounded and the boundedness of $(u_n)_n$ in E holds.

Next, let $\bar{u} \in E$ be such that $u_n \rightharpoonup \bar{u}$ weakly in E, up to subsequences. We prove that $u_n \to \bar{u}$ strongly in E, too. To this aim, let us observe that Corollary 2.2 and the Hölder inequality imply that

$$\lim_n \int_{\mathbb{R}^N} |u_n|^{q-2} u_n |u_n - \bar{u}| \, \mathrm{d}x \;=\; 0. \tag{3.11}$$

Moreover, by (f_1) and again the Hölder inequality we get that

$$\int_{\mathbb{R}^N} |f(x, u_n)||u_n - \bar{u}| \, \mathrm{d}x \;\leq\; a|u_n|_s^{s-1}|u_n - \bar{u}|_s + a|u_n|_p^{p-1}|u_n - \bar{u}|_p,$$

therefore Corollary 2.2 implies

$$\lim_n \int_{\mathbb{R}^N} |f(x, u_n)||u_n - \bar{u}| \, \mathrm{d}x \;=\; 0. \tag{3.12}$$

Thus, by (2.12), (3.11) and (3.12) we infer in particular that

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla(u_n - \bar{u})\, \mathrm{d}x + \int_{\mathbb{R}^N} V(x)|u_n|^{p-2}u_n\,(u_n - \bar{u})\, \mathrm{d}x \\ + \int_{\mathbb{R}^N} |\nabla u_n|^{q-2}\nabla u_n \cdot \nabla(u_n - \bar{u})\, \mathrm{d}x \\ + \int_{\mathbb{R}^N} (W(x)+\alpha)|u_n|^{q-2}u_n\,(u_n - \bar{u})\, \mathrm{d}x \;=\; o(1),\end{aligned} \tag{3.13}$$

while the weak convergence of $u_n \rightharpoonup \bar{u}$ implies that

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla \bar{u}|^{q-2}\nabla \bar{u} \cdot \nabla(u_n - \bar{u})\, \mathrm{d}x + \int_{\mathbb{R}^N} (W(x)+\alpha)|\bar{u}|^{q-2}\bar{u}\,(u_n - \bar{u})\, \mathrm{d}x \\ + \int_{\mathbb{R}^N} V(x)|\bar{u}|^{p-2}\bar{u}\,(u_n - \bar{u})\, \mathrm{d}x \;=\; o(1),\end{aligned} \tag{3.14}$$

where $o(1) \to 0$ as $n \to +\infty$.
On the other hand, by [25, Lemma 6.3] (see also [25, Example 6.4]) the following operators are monotone on E:

$$u \mapsto \Delta_p u, \quad u \mapsto \Delta_q u, \quad u \mapsto V(x)|u|^{p-2}u, \quad u \mapsto (W(x)+\alpha)|u|^{q-2}u;$$

hence, it follows

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla(u_n - \bar{u})\, \mathrm{d}x \;\le\; \int_{\mathbb{R}^N} |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla(u_n - \bar{u})\, \mathrm{d}x \\ + \int_{\mathbb{R}^N} V(x)|u_n|^{p-2}u_n\,(u_n - \bar{u})\, \mathrm{d}x + \int_{\mathbb{R}^N} |\nabla u_n|^{q-2}\nabla u_n \cdot \nabla(u_n - \bar{u})\, \mathrm{d}x \\ + \int_{\mathbb{R}^N} (W(x)+\alpha)|u_n|^{q-2}u_n\,(u_n - \bar{u})\, \mathrm{d}x - \int_{\mathbb{R}^N} |\nabla \bar{u}|^{q-2}\nabla \bar{u} \cdot \nabla(u_n - \bar{u})\, \mathrm{d}x \\ - \int_{\mathbb{R}^N} (W(x)+\alpha)|\bar{u}|^{q-2}\bar{u}\,(u_n - \bar{u})\, \mathrm{d}x - \int_{\mathbb{R}^N} V(x)|\bar{u}|^{p-2}\bar{u}\,(u_n - \bar{u})\, \mathrm{d}x.\end{aligned}$$

Therefore, by (3.13) and (3.14) it follows that

$$\limsup_n \int_{\mathbb{R}^N} |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla(u_n - \bar{u})\, \mathrm{d}x \le 0. \tag{3.15}$$

Similar arguments yield also the following estimates:

$$\limsup_n \int_{\mathbb{R}^N} |\nabla u_n|^{q-2}\nabla u_n \cdot \nabla(u_n - \bar{u})\, \mathrm{d}x \le 0, \tag{3.16}$$

$$\limsup_n \int_{\mathbb{R}^N} V(x)|u_n|^{p-2}u_n(u_n - \bar{u})\, \mathrm{d}x \le 0 \tag{3.17}$$

and

$$\limsup_n \int_{\mathbb{R}^N} (W(x)+\alpha)|u_n|^{q-2}u_n(u_n - \bar{u})\, \mathrm{d}x \le 0. \tag{3.18}$$

At last, let us recall that by [8, Proposition 3.20] it is also $u_n \rightharpoonup \bar{u}$ weakly both in $E_{p,V}$ and in $E_{q,W+\alpha}$. Whence, by (3.15) and (3.17) firstly, (3.16) and (3.18) then, it follows that, up to subsequences, it is

$$\lim_n u_n = \bar{u} \quad \text{in } E_{p,V} \qquad \text{and} \qquad \lim_n u_n = \bar{u} \quad \text{in } E_{q,W+\alpha}$$

which implies

$$\lim_n u_n = \bar{u} \quad \text{in } E$$

and the proof is complete. □

Proof of Theorem 1.1. By Lemmas 3.2, 3.3 and 3.4 it follows that for k large enough some constants ρ, c, R_1, $R_2 > 0$ exist such that

$$R_2 > \rho, \quad \inf_{u\in S} J(u) \geq c > 0, \quad \sup_{u\in\partial Q} J(u) \leq 0,$$

where $S = \partial B_\rho \cap Z^k$ and

$$Q = \{u + te \in E : u \in Y^k, \ e \in Z^k, \ \|u\|_E \leq R_1, \ t \in [0, R_2]\}.$$

Then, as S and ∂Q link, by Theorem 2.7 problem (1.1) has a nontrivial solution. Furthermore, if $f(x, \cdot)$ is odd for a.e. $x \in \mathbb{R}^N$, then the same lemmas allow one to apply Theorem 2.8; whence, (1.1) admits infinitely many nontrivial weak solutions. □

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