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On Hermitian varieties in $\mathrm{PG}(6, q^2)$

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Abstract

In this paper we characterize the non-singular Hermitian variety $\mathcal{H}(6, q^2)$ of $\mathrm{PG}(6, q^2)$, $q \neq 2$ among the irreducible hypersurfaces of degree $q + 1$ in $\mathrm{PG}(6, q^2)$ not containing solids by the number of its points and the existence of a solid S meeting it in $q^4 + q^2 + 1$ points.

Keywords: Unital, Hermitian variety, algebraic hypersurface.

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1 Introduction

The set of all absolute points of a non-degenerate unitary polarity in $\mathrm{PG}(r, q^2)$ determines the Hermitian variety $\mathcal{H}(r, q^2)$. This is a non-singular algebraic hypersurface of degree $q + 1$ in $\mathrm{PG}(r, q^2)$ with a number of remarkable properties, both from the geometrical and the combinatorial point of view; see [6, 16]. In particular, $\mathcal{H}(r, q^2)$ is a 2-character set with respect to the hyperplanes of $\mathrm{PG}(r, q^2)$ and 3-character blocking set with respect to the

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lines of $\mathrm{PG}(r, q^2)$ for $r > 2$. An interesting and widely investigated problem is to provide combinatorial descriptions of $\mathcal{H}(r, q^2)$.

First, we observe that a condition on the number of points and the intersection numbers with hyperplanes is not in general sufficient to characterize Hermitian varieties; see [1, 2]. On the other hand, it is enough to consider in addition the intersection numbers with codimension 2 subspaces in order to get a complete description; see [7].

In general, a hypersurface $\mathcal{H}$ of $\mathrm{PG}(r, q)$ is viewed as a hypersurface over the algebraic closure of $\mathrm{GF}(q)$ and a point of $\mathrm{PG}(r, q^i)$ in $\mathcal{H}$ is called a $\mathrm{GF}(q^i)$-point. A $\mathrm{GF}(q)$-point of $\mathcal{H}$ is also said to be a rational point of $\mathcal{H}$. Throughout this paper, the number of $\mathrm{GF}(q^i)$-points of $\mathcal{H}$ will be denoted by $N_{q^i}(\mathcal{H})$. For simplicity, we shall also use the convention $|\mathcal{H}| = N_q(\mathcal{H})$.

In the present paper, we shall investigate a combinatorial characterization of the Hermitian hypersurface $\mathcal{H}(6, q^2)$ in $\mathrm{PG}(6, q^2)$ among all hypersurfaces of the same degree having also the same number of $\mathrm{GF}(q^2)$-rational points.

More in detail, in [12, 13] it has been proved that if $\mathcal{X}$ is a hypersurface of degree $q+1$ in $\mathrm{PG}(r, q^2)$, $r \geq 3$ odd, with $|\mathcal{X}| = |\mathcal{H}(r, q^2)| = (q^{r+1} + (-1)^r)(q^r - (-1)^r)/(q^2 - 1)$ $\mathrm{GF}(q^2)$-rational points, not containing linear subspaces of dimension greater than $\frac{r-1}{2}$, then $\mathcal{X}$ is a non-singular Hermitian variety of $\mathrm{PG}(r, q^2)$. This result generalizes the characterization of [8] for the Hermitian curve of $\mathrm{PG}(2, q^2)$, $q \neq 2$.

The case where $r > 4$ is even is, in general, currently open. A starting point for a characterization in arbitrary even dimension can be found in [3] where the case of a hypersurface $\mathcal{X}$ of degree $q+1$ in $\mathrm{PG}(4, q^2)$, $q > 3$ is considered. There, it is shown that when $\mathcal{X}$ has the same number of rational points as $\mathcal{H}(4, q^2)$, does not contain any subspaces of dimension greater than 1 and meets at least one plane π in q^2+1 $\mathrm{GF}(q^2)$-rational points, then $\mathcal{X}$ is a Hermitian variety.

In this article we deal with hypersurfaces of degree $q+1$ in $\mathrm{PG}(6, q^2)$ and we prove that a characterization similar to that of [3] holds also in dimension 6. We conjecture that this can be extended to arbitrary even dimension.

Theorem 1.1. *Let S be a hypersurface of $\mathrm{PG}(6, q^2)$, $q > 2$, defined over $\mathrm{GF}(q^2)$, not containing solids. If the degree of S is $q+1$ and the number of its rational points is $q^{11} + q^9 + q^7 + q^4 + q^2 + 1$, then every solid of $\mathrm{PG}(6, q^2)$ meets S in at least $q^4 + q^2 + 1$ rational points. If there is at least a solid Σ_3 such that $|\Sigma_3 \cap S| = q^4 + q^2 + 1$, then S is a non-singular Hermitian variety of $\mathrm{PG}(6, q^2)$.*

Furthermore, we also extend the result of [3] to the case $q = 3$.

2 Preliminaries and notation

In this section we collect some useful information and results that will be crucial to our proof.

A Hermitian variety in $\mathrm{PG}(r, q^2)$ is the algebraic variety of $\mathrm{PG}(r, q^2)$ whose points $\langle v \rangle$ satisfy the equation $\eta(v, v) = 0$ where η is a sesquilinear form $\mathrm{GF}(q^2)^{r+1} \times \mathrm{GF}(q^2)^{r+1} \to \mathrm{GF}(q^2)$. The *radical* of the form η is the vector subspace of $\mathrm{GF}(q^2)^{r+1}$ given by

$$\mathrm{Rad}(\eta) := \{w \in \mathrm{GF}(q^2)^{r+1} : \forall v \in \mathrm{GF}(q^2)^{r+1}, \eta(v, w) = 0\}.$$

The form η is non-degenerate if $\mathrm{Rad}(\eta) = \{\mathbf{0}\}$. If the form η is non-degenerate, then the corresponding Hermitian variety is denoted by $\mathcal{H}(r, q^2)$ and it is a non-singular algebraic variety, of degree $q+1$ containing

$$(q^{r+1}+(-1)^r)(q^r-(-1)^r)/(q^2-1)$$

$\mathrm{GF}(q^2)$-rational points. When η is degenerate we shall call *vertex* R_t of the degenerate Hermitian variety associated to η the projective subspace $R_t := \mathrm{PG}(\mathrm{Rad}(\eta)) := \{\langle w\rangle : w \in \mathrm{Rad}(\eta)\}$ of $\mathrm{PG}(r,q^2)$. A degenerate Hermitian variety can always be described as a cone of vertex R_t and basis a non-degenerate Hermitian variety $\mathcal{H}(r-t,q^2)$ disjoint from R_t where $t = \dim(\mathrm{Rad}(\eta))$ is the vector dimension of the radical of η. In this case we shall write the corresponding variety as $R_t\mathcal{H}(r-t,q^2)$. Indeed,

$$R_t\mathcal{H}(r-t,q^2) := \{X \in \langle P,Q\rangle : P \in R_t, Q \in \mathcal{H}(r-t,q^2)\}.$$

Any line of $\mathrm{PG}(r,q^2)$ meets a Hermitian variety (either degenerate or not) in either $1, q+1$ or q^2+1 points (the latter value only for $r > 2$). The maximal dimension of projective subspaces contained in the non-degenerate Hermitian variety $\mathcal{H}(r,q^2)$ is $(r-2)/2$, if r is even, or $(r-1)/2$, if r is odd. These subspaces of maximal dimension are called *generators* of $\mathcal{H}(r,q^2)$ and the generators of $\mathcal{H}(r,q^2)$ through a point P of $\mathcal{H}(r,q^2)$ span a hyperplane $P^\perp$ of $\mathrm{PG}(r,q^2)$, the *tangent hyperplane* at P.

It is well known that this hyperplane meets $\mathcal{H}(r,q^2)$ in a degenerate Hermitian variety $P\mathcal{H}(r-2,q^2)$, that is in a Hermitian cone having as vertex the point P and as base a non-singular Hermitian variety of $\Theta \cong \mathrm{PG}(r-2,q^2)$ contained in $P^\perp$ with $P \notin \Theta$.

Every hyperplane of $\mathrm{PG}(r,q^2)$ that is not tangent meets $\mathcal{H}(r,q^2)$ in a non-singular Hermitian variety $\mathcal{H}(r-1,q^2)$, and is called a *secant hyperplane* of $\mathcal{H}(r,q^2)$. In particular, a tangent hyperplane contains

$$1+q^2(q^{r-1}+(-1)^r)(q^{r-2}-(-1)^r)/(q^2-1)$$

$\mathrm{GF}(q^2)$-rational points of $\mathcal{H}(r,q^2)$, whereas a secant hyperplane contains

$$(q^r+(-1)^{r-1})(q^{r-1}-(-1)^{r-1})/(q^2-1)$$

$\mathrm{GF}(q^2)$-rational points of $\mathcal{H}(r,q^2)$.

We now recall several results which shall be used in the course of this paper.

Lemma 2.1 ([15]). *Let d be an integer with $1 \le d \le q+1$ and let $\mathcal{C}$ be a curve of degree d in $\mathrm{PG}(2,q)$ defined over $\mathrm{GF}(q)$, which may have $\mathrm{GF}(q)$-linear components. Then the number of its rational points is at most $dq+1$ and $N_q(\mathcal{C}) = dq+1$ if and only if $\mathcal{C}$ is a pencil of d lines of $\mathrm{PG}(2,q)$.*

Lemma 2.2 ([10]). *Let d be an integer with $2 \le d \le q+2$, and $\mathcal{C}$ a curve of degree d in $\mathrm{PG}(2,q)$ defined over $\mathrm{GF}(q)$ without any $\mathrm{GF}(q)$-linear components. Then $N_q(\mathcal{C}) \le (d-1)q+1$, except for a class of plane curves of degree 4 over $\mathrm{GF}(4)$ having 14 rational points.*

Lemma 2.3 ([11]). *Let $\mathcal{S}$ be a surface of degree d in $\mathrm{PG}(3,q)$ over $\mathrm{GF}(q)$. Then*

$$N_q(\mathcal{S}) \le dq^2+q+1$$

Lemma 2.4 ([8]). *Suppose $q \ne 2$. Let $\mathcal{C}$ be a plane curve over $\mathrm{GF}(q^2)$ of degree $q+1$ without $\mathrm{GF}(q^2)$-linear components. If $\mathcal{C}$ has q^3+1 rational points, then $\mathcal{C}$ is a Hermitian curve.*

Lemma 2.5 ([7]). *A subset of points of* $\mathrm{PG}(r, q^2)$ *having the same intersection numbers with respect to hyperplanes and spaces of codimension* 2 *as non-singular Hermitian varieties is a non-singular Hermitian variety of* $\mathrm{PG}(r, q^2)$.

From [9, Theorem 23.5.1, Theorem 23.5.3] we have the following.

Lemma 2.6. *If* $\mathcal{W}$ *is a set of* $q^7 + q^4 + q^2 + 1$ *points of* $\mathrm{PG}(4, q^2)$, $q > 2$ *such that every line of* $\mathrm{PG}(4, q^2)$ *meets* $\mathcal{W}$ *in* $1, q + 1$ *or* $q^2 + 1$ *points, then* $\mathcal{W}$ *is a Hermitian cone with vertex a line and base a unital.*

Finally, we recall that a *blocking set with respect to lines* of $\mathrm{PG}(r, q)$ is a point set which blocks all the lines, i.e., intersects each line of $\mathrm{PG}(r, q)$ in at least one point.

3 Proof of Theorem 1.1

We first provide an estimate on the number of points of a curve of degree $q+1$ in $\mathrm{PG}(2, q^2)$, where q is any prime power.

Lemma 3.1. *Let* $\mathcal{C}$ *be a plane curve over* $\mathrm{GF}(q^2)$*, without* $\mathrm{GF}(q^2)$*-lines as components and of degree* $q + 1$*. If the number of* $\mathrm{GF}(q^2)$*-rational points of* $\mathcal{C}$ *is* $N < q^3 + 1$*, then*

$$N \leq \begin{cases} q^3 - (q^2 - 2) & \text{if } q > 3 \\ 24 & \text{if } q = 3 \\ 8 & \text{if } q = 2. \end{cases} \tag{3.1}$$

Proof. We distinguish the following three cases:

(a) $\mathcal{C}$ has two or more $\mathrm{GF}(q^2)$-components;

(b) $\mathcal{C}$ is irreducible over $\mathrm{GF}(q^2)$, but not absolutely irreducible;

(c) $\mathcal{C}$ is absolutely irreducible.

Suppose first $q \neq 2$.

Case (a) Suppose $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$. Let d_i be the degree of $\mathcal{C}_i$, for each $i = 1, 2$. Hence $d_1 + d_2 = q + 1$. By Lemma 2.2,

$$N \leq N_{q^2}(\mathcal{C}_1) + N_{q^2}(\mathcal{C}_2) \leq [(q + 1) - 2]q^2 + 2 = q^3 - (q^2 - 2)$$

Case (b) Let $\mathcal{C}'$ be an irreducible component of $\mathcal{C}$ over the algebraic closure of $\mathrm{GF}(q^2)$. Let $\mathrm{GF}(q^{2t})$ be the minimum defining field of $\mathcal{C}'$ and σ be the Frobenius morphism of $\mathrm{GF}(q^{2t})$ over $\mathrm{GF}(q^2)$. Then

$$\mathcal{C} = \mathcal{C}' \cup \mathcal{C}'^{\sigma} \cup \mathcal{C}'^{\sigma^2} \cup \ldots \cup \mathcal{C}'^{\sigma^{t-1}},$$

and the degree of $\mathcal{C}'$, say e, satisfies $q + 1 = te$ with $e > 1$. Hence any $\mathrm{GF}(q^2)$-rational point of $\mathcal{C}$ is contained in $\cap_{i=0}^{t-1} \mathcal{C}'^{\sigma^i}$. In particular, $N \leq e^2 \leq (\frac{q+1}{2})^2$ by Bezout's Theorem and $(\frac{q+1}{2})^2 < q^3 - (q^2 - 2)$.

Case (c) Let $\mathcal{C}$ be an absolutely irreducible curve over $\mathrm{GF}(q^2)$ of degree $q + 1$. Either $\mathcal{C}$ has a singular point or not.

In general, an absolutely irreducible plane curve $\mathcal{M}$ over $\mathrm{GF}(q^2)$ is q^2-Frobenius non-classical if for a general point $P(x_0, x_1, x_2)$ of $\mathcal{M}$ the point $P^{q^2} = P^{q^2}(x_0^{q^2}, x_1^{q^2}, x_2^{q^2})$ is on the tangent line to $\mathcal{M}$ at the point P. Otherwise, the curve $\mathcal{M}$ is said to be Frobenius classical. A lower bound of the number of $\mathrm{GF}(q^2)$-points for q^2-Frobenius non-classical curves is given by [4, Corollary 1.4]: for a q^2-Frobenius non-classical curve $\mathcal{C}'$ of degree d, we have $N_{q^2}(\mathcal{C}') \geq d(q^2 - d + 2)$. In particular, if $d = q + 1$, the lower bound is just $q^3 + 1$.

Going back to our original curve $\mathcal{C}$, we know that $\mathcal{C}$ is Frobenius classical because $N < q^3 + 1$. Let $F(x, y, z) = 0$ be an equation of $\mathcal{C}$ over $\mathrm{GF}(q^2)$. We consider the curve $\mathcal{D}$ defined by $\frac{\partial F}{\partial x} x^{q^2} + \frac{\partial F}{\partial y} y^{q^2} + \frac{\partial F}{\partial z} z^{q^2} = 0$. Then $\mathcal{C}$ is not a component of $\mathcal{D}$ because $\mathcal{C}$ is Frobenius classical. Furthermore, any $\mathrm{GF}(q^2)$-point P lies on $\mathcal{C} \cap \mathcal{D}$ and the intersection multiplicity of $\mathcal{C}$ and $\mathcal{D}$ at P is at least 2 by Euler's theorem for homogeneous polynomials. Hence by Bézout's theorem, $2N \leq (q + 1)(q^2 + q)$. Hence

$$N \leq \frac{1}{2} q(q + 1)^2.$$

This argument is due to Stöhr and Voloch [18, Theorem 1.1]. This Stöhr and Voloch's bound is lower than the estimate for N in case (a) for $q > 4$ and it is the same for $q = 4$. When $q = 3$ the bound in case (a) is smaller than the Stöhr and Voloch's bound.
Finally, we consider the case $q = 2$. Under this assumption, $\mathcal{C}$ is a cubic curve and neither case (a) nor case (b) might occur. For a degree 3 curve over $\mathrm{GF}(q^2)$ the Stöhr and Voloch's bound is loose, thus we need to change our argument. If $\mathcal{C}$ has a singular point, then $\mathcal{C}$ is a rational curve with a unique singular point. Since the degree of $\mathcal{C}$ is 3, singular points are either cusps or ordinary double points. Hence $N \in \{4, 5, 6\}$. If $\mathcal{C}$ is nonsingular, then it is an elliptic curve and, by the Hasse-Weil bound, see [19], $N \in I$ where $I = \{1, 2, \ldots, 9\}$ and for each number N belonging to I there is an elliptic curve over $\mathrm{GF}(4)$ with N points, from [14, Theorem 4.2]. This completes the proof. □

Henceforth, we shall always suppose $q > 2$ and we denote by $\mathcal{S}$ an algebraic hypersurface of $\mathrm{PG}(6, q^2)$ satisfying the following hypotheses of Theorem 1.1:

(S1) $\mathcal{S}$ is an algebraic hypersurface of degree $q + 1$ defined over $\mathrm{GF}(q^2)$;

(S2) $|\mathcal{S}| = q^{11} + q^9 + q^7 + q^4 + q^2 + 1$;

(S3) $\mathcal{S}$ does not contain projective 3-spaces (solids);

(S4) there exists a solid Σ_3 such that $|\mathcal{S} \cap \Sigma_3| = q^4 + q^2 + 1$.

We first consider the behavior of $\mathcal{S}$ with respect to the lines.

Lemma 3.2. *An algebraic hypersurface $\mathcal{T}$ of degree $q + 1$ in $\mathrm{PG}(r, q^2)$, $q \neq 2$, with $|\mathcal{T}| = |\mathcal{H}(r, q^2)|$ is a blocking set with respect to lines of $\mathrm{PG}(r, q^2)$*

Proof. Suppose on the contrary that there is a line ℓ of $\mathrm{PG}(r, q^2)$ which is disjoint from $\mathcal{T}$. Let α be a plane containing ℓ. The algebraic plane curve $\mathcal{C} = \alpha \cap \mathcal{T}$ of degree $q + 1$ cannot have $\mathrm{GF}(q^2)$-linear components and hence it has at most $q^3 + 1$ points because of Lemma 2.2. If $\mathcal{C}$ had $q^3 + 1$ rational points, then from Lemma 2.4, $\mathcal{C}$ would be a Hermitian curve with an external line, a contradiction since Hermitian curves are blocking sets. Thus $N_{q^2}(\mathcal{C}) \leq q^3$. Since $q > 2$, by Lemma 3.1, $N_{q^2}(\mathcal{C}) < q^3 - 1$ and hence every plane through r meets $\mathcal{T}$ in at most $q^3 - 1$ rational points. Consequently, by considering all planes through r, we can bound the number of rational points of $\mathcal{T}$ by $N_{q^2}(\mathcal{T}) \leq (q^3 - 1)\frac{q^{2r-4} - 1}{q^2 - 1} =$

$q^{2r-3} + \cdots < |\mathcal{H}(r, q^2)|$, which is a contradiction. Therefore there are no external lines to $\mathcal{T}$ and so $\mathcal{T}$ is a blocking set w.r.t. lines of $\mathrm{PG}(r, q^2)$. □

Remark 3.3. The proof of [3, Lemma 3.1] would work perfectly well here under the assumption $q > 3$. The alternative argument of Lemma 3.2 is simpler and also holds for $q = 3$.

By the previous Lemma and assumptions (S1) and (S2), $\mathcal{S}$ is a blocking set for the lines of $\mathrm{PG}(6, q^2)$ In particular, the intersection of $\mathcal{S}$ with any 3-dimensional subspace Σ of $\mathrm{PG}(6, q^2)$ is also a blocking set with respect to lines of Σ and hence it contains at least $q^4 + q^2 + 1$ $\mathrm{GF}(q^2)$-rational points; see [5].

Lemma 3.4. *Let Σ_3 be a solid of $\mathrm{PG}(6, q^2)$ satisfying condition (S4), that is Σ_3 meets $\mathcal{S}$ in exactly $q^4 + q^2 + 1$ points. Then, $\Pi := \mathcal{S} \cap \Sigma_3$ is a plane.*

Proof. $\mathcal{S} \cap \Sigma_3$ must be a blocking set for the lines of $\mathrm{PG}(3, q^2)$; also it has size $q^4 + q^2 + 1$. It follows from [5] that $\Pi := \mathcal{S} \cap \Sigma_3$ is a plane. □

Lemma 3.5. *Let Σ_3 be a solid of satisfying condition (S4). Then, any 4-dimensional projective space Σ_4 through Σ_3 meets $\mathcal{S}$ in a Hermitian cone with vertex a line and basis a Hermitian curve.*

Proof. Consider all of the $q^6 + q^4 + q^2 + 1$ subspaces $\overline{\Sigma}_3$ of dimension 3 in $\mathrm{PG}(6, q^2)$ containing $\Pi = \mathcal{S} \cap \Sigma_3$.

From Lemma 2.3 and condition (S3) we have $|\overline{\Sigma}_3 \cap \mathcal{S}| \le q^5 + q^4 + q^2 + 1$. Hence,

$$|\mathcal{S}| = (q^7 + 1)(q^4 + q^2 + 1) \le (q^6 + q^4 + q^2)q^5 + q^4 + q^2 + 1 = |\mathcal{S}|.$$

Consequently, $|\overline{\Sigma}_3 \cap \mathcal{S}| = q^5 + q^4 + q^2 + 1$ for all $\overline{\Sigma}_3 \ne \Sigma_3$ such that $\Pi \subset \overline{\Sigma}_3$.

Let $C := \Sigma_4 \cap \mathcal{S}$. Counting the number of rational points of C by considering the intersections with the $q^2 + 1$ subspaces Σ_3' of dimension 3 in Σ_4 containing the plane Π we get

$$|C| = q^2 \cdot q^5 + q^4 + q^2 + 1 = q^7 + q^4 + q^2 + 1.$$

In particular, $C \cap \Sigma_3'$ is a maximal surface of degree $q + 1$; so it must split in $q + 1$ distinct planes through a line of Π; see [17]. So C consists of $q^3 + 1$ distinct planes belonging to distinct q^2 pencils, all containing Π ; denote by $\mathcal{L}$ the family of these planes. Also for each $\Sigma_3' \ne \Sigma_3$, there is a line ℓ' such that all the planes of $\mathcal{L}$ in Σ_3' pass through ℓ'. It is now straightforward to see that any line contained in C must necessarily belong to one of the planes of $\mathcal{L}$ and no plane not in $\mathcal{L}$ is contained in C.

In order to get the result it is now enough to show that a line of Σ_4 meets C in either 1, $q + 1$ or $q^2 + 1$ points. To this purpose, let ℓ be a line of Σ_4 and suppose $\ell \not\subseteq C$. Then, by Bezout's theorem,

$$1 \le |\ell \cap C| \le q + 1.$$

Assume $|\ell \cap C| > 1$. Then we can distinguish two cases:

1. $\ell \cap \Pi \ne \emptyset$. If ℓ and Π are incident, then we can consider the 3-dimensional subspace $\Sigma_3' := \langle \ell, \Pi \rangle$. Then ℓ must meet each plane of $\mathcal{L}$ in Σ_3' in different points (otherwise ℓ passes through the intersection of these planes and then $|\ell \cap C| = 1$). As there are $q + 1$ planes of $\mathcal{L}$ in Σ_3', we have $|\ell \cap C| = q + 1$.

2. $\ell \cap \Pi = \emptyset$. Consider the plane Λ generated by a point $P \in \Pi$ and ℓ. Clearly $\Lambda \notin \mathcal{L}$. The curve $\Lambda \cap S$ has degree $q+1$ by construction, does not contain lines (for otherwise $\Lambda \in \mathcal{L}$) and has q^3+1 $\mathrm{GF}(q^2)$-rational points (by a counting argument). So from Lemma 2.4 it is a Hermitian curve . It follows that ℓ is a $q+1$ secant.

We can now apply Lemma 2.6 to see that C is a Hermitian cone with vertex a line. □

Lemma 3.6. *Let Σ_3 be a space satisfying condition (S4) and take Σ_5 to be a 5-dimensional projective space with $\Sigma_3 \subseteq \Sigma_5$. Then $\mathcal{S} \cap \Sigma_5$ is a Hermitian cone with vertex a point and basis a Hermitian hypersurface $\mathcal{H}(4, q^2)$.*

Proof. Let

$$\Sigma_4 := \Sigma_4^1, \Sigma_4^2, \ldots, \Sigma_4^{q^2+1}$$

be the 4-spaces through Σ_3 contained in Σ_5. Put $C_i := \Sigma_4^i \cap \mathcal{S}$, for all $i \in \{1, \ldots, q^2+1\}$ and $\Pi = \Sigma_3 \cap C_1$. From Lemma 3.5 C_i is a Hermitian cone with vertex a line, say ℓ_i. Furthermore $\Pi \subseteq \Sigma_3 \subseteq \Sigma_4^i$ where Π is a plane. Choose a plane $\Pi' \subseteq \Sigma_4^1$ such that $m := \Pi' \cap C_1$ is a line m incident with Π but not contained in it. Let $P_1 := m \cap \Pi$. It is straightforward to see that in Σ_4^1 there are exactly 1 plane through m which is a (q^4+q^2+1)-secant, q^4 planes which are (q^3+q^2+1)-secant and q^2 planes which are (q^2+1)-secant. Also P_1 belongs to the line ℓ_1. There are now two cases to consider:

(a) There is a plane $\Pi'' \neq \Pi'$ not contained in Σ_4^i for all $i = 1, \ldots, q^2+1$ with $m \subseteq \Pi'' \subseteq S \cap \Sigma_5$.

We first show that the vertices of the cones C_i are all concurrent. Consider $m_i := \Pi'' \cap \Sigma_4^i$. Then $\{m_i : i = 1, \ldots, q^2+1\}$ consists of q^2+1 lines (including m) all through P_1. Observe that for all i, the line m_i meets the vertex ℓ_i of the cone C_i in $P_i \in \Pi$. This forces $P_1 = P_2 = \cdots = P_{q^2+1}$. So $P_1 \in \ell_1, \ldots, \ell_{q^2+1}$.

Now let $\overline{\Sigma}_4$ be a 4-dimensional space in Σ_5 with $P_1 \notin \overline{\Sigma}_4$; in particular $\Pi \not\subseteq \overline{\Sigma}_4$. Put also $\overline{\Sigma}_3 := \Sigma_4^1 \cap \overline{\Sigma}_4$. Clearly, $r := \overline{\Sigma}_3 \cap \Pi$ is a line and $P_1 \notin r$. So $\overline{\Sigma}_3 \cap \mathcal{S}$ cannot be the union of $q+1$ planes, since if this were to be the case, these planes would have to pass through the vertex ℓ_1. It follows that $\overline{\Sigma}_3 \cap \mathcal{S}$ must be a Hermitian cone with vertex a point and basis a Hermitian curve. Let $\mathcal{W} := \overline{\Sigma}_4 \cap \mathcal{S}$. The intersection $\mathcal{W} \cap \Sigma_4^i$, as i varies, is a Hermitian cone with basis a Hermitian curve, so, the points of $\mathcal{W}$ are

$$|\mathcal{W}| = (q^2+1)q^5 + q^2 + 1 = (q^2+1)(q^5+1);$$

in particular, $\mathcal{W}$ is a hypersurface of $\overline{\Sigma}_4$ of degree $q+1$ such that there exists a plane of $\overline{\Sigma}_4$ meeting $\mathcal{W}$ in just one line (such planes exist in $\overline{\Sigma}_3$). Also suppose $\mathcal{W}$ to contain planes and let $\Pi''' \subseteq \mathcal{W}$ be such a plane. Since $\Sigma_4^i \cap \mathcal{W}$ does not contain planes, all Σ_4^i meet Π''' in a line t_i. Also Π''' must be contained in $\bigcup_{i=1}^{q^2+1} t_i$. This implies that the set $\{t_i\}_{i=1,\ldots,q^2+1}$ consists of q^2+1 lines through a point $P \in \Pi \setminus \{P_1\}$.

Furthermore each line t_i passing through P must meet the radical line ℓ_i of the Hermitian cone $\mathcal{S} \cap \Sigma_4^i$ and this forces P to coincide with P_1, a contradiction. It follows that $\mathcal{W}$ does not contain planes.

So by the characterization of $\mathcal{H}(4, q^2)$ of [3] we have that $\mathcal{W}$ is a Hermitian variety $\mathcal{H}(4, q^2)$.

We also have that $|\mathcal{S}\cap\Sigma_5| = |P_1\mathcal{H}(4,q^2)|$. Let now r be any line of $\mathcal{H}(4,q^2) = \mathcal{S}\cap\overline{\Sigma}_4$ and let Θ be the plane $\langle r, P_1\rangle$. The plane Θ meets Σ_4^i in a line $q_i \subseteq \mathcal{S}$ for each $i = 1,\dots,q^2+1$ and these lines are concurrent in P_1. It follows that all the points of Θ are in S. This completes the proof for the current case and shows that $\mathcal{S}\cap\Sigma_5$ is a Hermitian cone $P_1\mathcal{H}(4,q^2)$.

(b) All planes Π'' with $m \subseteq \Pi'' \subseteq \mathcal{S}\cap\Sigma_5$ are contained in Σ_4^i for some $i = 1,\dots,q^2+1$. We claim that this case cannot happen. We can suppose without loss of generality $m\cap\ell_1 = P_1$ and $P_1 \notin \ell_i$ for all $i = 2,\dots,q^2+1$. Since the intersection of the subspaces Σ_4^i is Σ_3, there is exactly one plane through m in Σ_5 which is (q^4+q^2+1)-secant, namely the plane $\langle\ell_1, m\rangle$. Furthermore, in Σ_4^1 there are q^4 planes through m which are (q^3+q^2+1)-secant and q^2 planes which are (q^2+1)-secant. We can provide an upper bound to the points of $\mathcal{S}\cap\Sigma_5$ by counting the number of points of $\mathcal{S}\cap\Sigma_5$ on planes in Σ_5 through m and observing that a plane through m not in Σ_5 and not contained in $\mathcal{S}$ has at most q^3+q^2+1 points in common with $\mathcal{S}\cap\Sigma_5$. So

$$|\mathcal{S}\cap\Sigma_5| \le q^6\cdot q^3 + q^7 + q^4 + q^2 + 1.$$

As $|\mathcal{S}\cap\Sigma_5| = q^9+q^7+q^4+q^2+1$, all planes through m which are neither (q^4+q^2+1)-secant nor (q^2+1)-secant are (q^3+q^2+1)-secant. That is to say that all of these planes meet $\mathcal{S}$ in a curve of degree $q+1$ which must split into $q+1$ lines through a point because of Lemma 2.1.

Take now $P_2 \in \Sigma_4^2\cap\mathcal{S}$ and consider the plane $\Xi := \langle m, P_2\rangle$. The line $\langle P_1, P_2\rangle$ is contained in Σ_4^2; so it must be a $(q+1)$-secant, as it does not meet the vertex line ℓ_2 of C_2 in Σ_4^2. Now, Ξ meets every of Σ_4^i for $i = 2,\dots,q^2+1$ in a line through P_1 which is either a 1-secant or a $q+1$-secant; so

$$|\mathcal{S}\cap\Xi| \le q^2(q) + q^2 + 1 = q^3 + q^2 + 1.$$

It follows that $|\mathcal{S}\cap\Xi| = q^3+q^2+1$ and $\mathcal{S}\cap\Xi$ is a set of $q+1$ lines all through the point P_1. This contradicts our previous construction.

□

Lemma 3.7. *Every hyperplane of* $\mathrm{PG}(6,q^2)$ *meets* $\mathcal{S}$ *either in a non-singular Hermitian variety* $\mathcal{H}(5,q^2)$ *or in a cone with vertex a point over a Hermitian hypersurface* $\mathcal{H}(4,q^2)$.

Proof. Let Σ_3 be a solid satisfying condition (S4). Denote by Λ a hyperplane of $\mathrm{PG}(6,q^2)$. If Λ contains Σ_3 then, from Lemma 3.6 it follows that $\Lambda\cap\mathcal{S}$ is a Hermitian cone $P\mathcal{H}(4,q^2)$.

Now assume that Λ does not contain Σ_3. Denote by S_5^j, with $j = 1,\dots,q^2+1$ the q^2+1 hyperplanes through Σ_4^1, where as before, Σ_4^1 is a 4-space containing Σ_3. By Lemma 3.6 again we get that $S_5^j\cap\mathcal{S} = P^j\mathcal{H}(4,q^2)$. We count the number of rational points of $\Lambda\cap\mathcal{S}$ by studying the intersections of $S_5^j\cap\mathcal{S}$ with Λ for all $j\in\{1,\dots,q^2+1\}$. Setting $\mathcal{W}_j := S_5^j\cap\mathcal{S}\cap\Lambda$, $\Omega := \Sigma_4^1\cap\mathcal{S}\cap\Lambda$ then

$$|\mathcal{S}\cap\Lambda| = \sum_j |\mathcal{W}_j\setminus\Omega| + |\Omega|.$$

If Π is a plane of Λ then Ω consists of $q+1$ planes of a pencil. Otherwise let m be the line in which Λ meets the plane Π. Then Ω is either a Hermitian cone $P_0\mathcal{H}(2,q^2)$, or $q+1$

planes of a pencil, according as the vertex $P^j \in \Pi$ is an external point with respect to m or not.

In the former case $\mathcal{W}_j$ is a non singular Hermitian variety $\mathcal{H}(4, q^2)$ and thus $|\mathcal{S} \cap \Lambda| = (q^2+1)(q^7) + q^5 + q^2 + 1 = q^9 + q^7 + q^5 + q^2 + 1$.

In the case in which Ω consists of $q+1$ planes of a pencil then $\mathcal{W}_j$ is either a $P_0\mathcal{H}(3, q^2)$ or a Hermitian cone with vertex a line ℓ and basis a Hermitian curve $\mathcal{H}(2, q^2)$.

If there is at least one index j such that $\mathcal{W}_j = \ell\mathcal{H}(2, q^2)$, then there must be a 3-dimensional space Σ'_3 of $S_5^j \cap \Lambda$ meeting $\mathcal{S}$ in a generator. Hence, from Lemma 3.6 we get that $\mathcal{S} \cap \Lambda$ is a Hermitian cone $P'\mathcal{H}(4, q^2)$.

Assume that for all $j \in \{1, \dots, q^2+1\}$, $\mathcal{W}_j$ is a $P_0\mathcal{H}(3, q^2)$. In this case

$$|\mathcal{S} \cap \Lambda| = (q^2+1)q^7 + (q+1)q^4 + q^2 + 1 = q^9 + q^7 + q^5 + q^4 + q^2 + 1 = |\mathcal{H}(5, q^2)|.$$

We are going to prove that the intersection numbers of $\mathcal{S}$ with hyperplanes are only two that is $q^9 + q^7 + q^5 + q^4 + q^2 + 1$ or $q^9 + q^7 + q^4 + q^2 + 1$.

Denote by x_i the number of hyperplanes meeting $\mathcal{S}$ in i rational points with $i \in \{q^9 + q^7 + q^4 + q^2 + 1, q^9 + q^7 + q^5 + q^2 + 1, q^9 + q^7 + q^5 + q^4 + q^2 + 1\}$. Double counting arguments give the following equations for the integers x_i:

$$\begin{cases} \sum_i x_i = q^{12} + q^{10} + q^8 + q^6 + q^4 + q^2 + 1 \\ \sum_i i x_i = |\mathcal{S}|(q^{10} + q^8 + q^6 + q^4 + q^2 + 1) \\ \sum_{i=1} i(i-1)x_i = |\mathcal{S}|(|\mathcal{S}| - 1)(q^8 + q^6 + q^4 + q^2 + 1). \end{cases} \tag{3.2}$$

Solving (3.2) we obtain $x_{q^9+q^7+q^5+q^2+1} = 0$. In the case in which $|\mathcal{S} \cap \Lambda| = |\mathcal{H}(5, q^2)|$, since $\mathcal{S} \cap \Lambda$ is an algebraic hypersurface of degree $q+1$ not containing 3-spaces, from [19, Theorem 4.1] we get that $\mathcal{S} \cap \Lambda$ is a Hermitian variety $\mathcal{H}(5, q^2)$ and this completes the proof. □

Proof of Theorem 1.1. The first part of Theorem 1.1 follows from Lemma 3.4. From Lemma 3.7, $\mathcal{S}$ has the same intersection numbers with respect to hyperplanes and 4-spaces as a non-singular Hermitian variety of $\mathrm{PG}(6, q^2)$, hence Lemma 2.5 applies and $\mathcal{S}$ turns out to be a $\mathcal{H}(6, q^2)$. □

Remark 3.8. The characterization of the non-singular Hermitian variety $\mathcal{H}(4, q^2)$ given in [3] is based on the property that a given hypersurface is a blocking set with respect to lines of $\mathrm{PG}(4, q^2)$, see [3, Lemma 3.1]. This lemma holds when $q > 3$. Since Lemma 3.2 extends the same property to the case $q = 3$ it follows that the result stated in [3] is also valid in $\mathrm{PG}(4, 3^2)$.

4 Conjecture

We propose a conjecture for the general $2n$-dimensional case.

Let $\mathcal{S}$ be a hypersurface of $\mathrm{PG}(2d, q^2)$*, $q > 2$, defined over* $\mathrm{GF}(q^2)$*, not containing d-dimensional projective subspaces. If the degree of $\mathcal{S}$ is $q+1$ and the number of its rational points is $|\mathcal{H}(2d, q^2)|$, then every d-dimensional subspace of* $\mathrm{PG}(2d, q^2)$ *meets $\mathcal{S}$ in at least $\theta_{q^2}(d-1) := (q^{2d-2} - 1)/(q^2 - 1)$ rational points. If there is at least a d-dimensional* *subspace* Σ_d *such that* $|\Sigma_d \cap \mathcal{S}| = |\mathrm{PG}(d-1, q^2)|$, *then* $\mathcal{S}$ *is a non-singular Hermitian variety of* $\mathrm{PG}(2d, q^2)$.

Lemma 3.1 and Lemma 3.2 can be a starting point for the proof of this conjecture since from them we get that $\mathcal{S}$ is a blocking set with respect to lines of $\mathrm{PG}(2d, q^2)$.

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