MULTIPLICITY AND CONCENTRATION RESULTS FOR A MAGNETIC SCHRÖDINGER EQUATION WITH EXPONENTIAL CRITICAL GROWTH IN $\mathbb{R}^2$

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Abstract. In this paper we study the following nonlinear Schrödinger equation with magnetic field

$$\Big(\frac{\varepsilon}{i}\nabla - A(x)\Big)^2 u + V(x)u = f(|u|^2)u, \quad x \in \mathbb{R}^2,$$

where $\varepsilon > 0$ is a parameter, $V : \mathbb{R}^2 \to \mathbb{R}$ and $A : \mathbb{R}^2 \to \mathbb{R}^2$ are continuous potentials and $f : \mathbb{R} \to \mathbb{R}$ has exponential critical growth. Under a local assumption on the potential V, by variational methods, penalization technique, and Ljusternick-Schnirelmann theory, we prove multiplicity and concentration of solutions for ε small.

Contents

1. Introduction and main results

In this paper, we are concerned with multiplicity and concentration results for the following nonlinear magnetic Schrödinger equation

$$\Big(\frac{\varepsilon}{i}\nabla - A(x)\Big)^2 u + V(x)u = f(|u|^2)u \quad \text{in } \mathbb{R}^2, \tag{1.1}$$

where $u \in H^1(\mathbb{R}^2, \mathbb{C})$, $\varepsilon > 0$ is a parameter, $V : \mathbb{R}^2 \to \mathbb{R}$ is a continuous function, $f : \mathbb{R} \to \mathbb{R}$, and the magnetic potential $A : \mathbb{R}^2 \to \mathbb{R}^2$ is Hölder continuous with exponent $\alpha \in (0, 1]$.

Equation (1.1) arises when one looks for standing wave solutions $\psi(x,t) := e^{-iEt/\hbar}u(x)$, with $E \in \mathbb{R}$, of

$$i\hbar\frac{\partial\psi}{\partial t} = \Big(\frac{\hbar}{i}\nabla - A(x)\Big)^2\psi + U(x)\psi - f(|\psi|^2)\psi \quad \text{in } \mathbb{R}^2 \times \mathbb{R}.$$

From a physical point of view, the existence of such solutions and the study of their shape in the semiclassical limit, namely, as $\hbar \to 0^+$, or, equivalently, as $\varepsilon \to 0^+$ in (1.1), is of the greatest importance, since the transition from Quantum Mechanics to Classical Mechanics can be formally performed by sending the Planck constant $\hbar$ to zero.

For equation (1.1), there is a vast literature concerning the existence and multiplicity of bound state solutions, in particular for the case with $A \equiv 0$. The first result in this direction was given by Floer and Weinstein in [28], where the case $N = 1$ and $f = i_{\mathbb{R}}$ is considered. Later, many authors generalized this

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result to larger values of N, using different methods. In [25], del Pino and Felmer studied existence and concentration of the solutions for the following problem

$$\begin{cases} -\varepsilon^2 \Delta u + V(x)u = f(u) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \ u > 0 \text{ in } \Omega, \end{cases}$$

where Ω is a possibly unbounded domain in $\mathbb{R}^N$, $N \geq 3$, the potential V is locally Hölder continuous, bounded from below away from zero, there exists a bounded open set $\Lambda \subset \Omega$ such that

$$\inf_{x \in \Lambda} V(x) < \min_{x \in \partial\Lambda} V(x),$$

and the nonlinearity f satisfies some subcritical growth conditions. For further results about existence, multiplicity and qualitative properties of semiclassical states with various types of concentration behaviors, which have been established under various assumptions on the potential V and on the nonlinearity f, see [1,3,4,6,8,10,11,16,18,19,24,26,35–37] the references therein (see also [2,7,29] for the fractional case).

On the other hand, also the magnetic nonlinear Schrödinger equation (1.1) has been extensively investigated by many authors applying suitable variational and topological methods (see [5,13–15,20,23,27,32] and references therein). It is well known that the first result involving the magnetic field was obtained by Esteban and Lions [27]. They used the concentration-compactness principle and minimization arguments to obtain solutions for $\varepsilon > 0$ fixed and $N = 2, 3$. In particular, due to our scope, we want to mention [5] where the authors use the penalization method and Ljusternik-Schnirelmann category theory for subcritical nonlinearities and [15] where the existence of a complex solution in presence of a nonlinearity with exponential critical growth in $\mathbb{R}^2$ is proved, and the recent contribution [12] where a multiplicity result for a nonlinear fractional magnetic Schrödinger equation with exponential critical growth in the one-dimensional case is given.

In this paper, motivated by [5,25], we prove multiplicity and concentration of nontrivial solutions for problem (1.1), combining some assumptions on V, the penalization technique by del Pino and Felmer [25] and the Ljusternik-Schnirelmann theory.

Assume that V verifies the following properties:

(V_1) there exists $V_0 > 0$ such that $V(x) \geq V_0$ for all $x \in \mathbb{R}^2$;
(V_2) there exists a bounded open set $\Lambda \subset \mathbb{R}^2$ such that

$$V_0 = \min_{x \in \Lambda} V(x) < \min_{x \in \partial\Lambda} V(x).$$

Observe that

$$M := \{x \in \Lambda : V(x) = V_0\} \neq \emptyset.$$

Moreover, let the nonlinearity f be a C^1-function satisfying:

(f_1) $f(t) = 0$ if $t \leq 0$;
(f_2) there holds

$$\lim_{t \to +\infty} \frac{f(t^2)t}{e^{\alpha t^2}} = \begin{cases} 0, & \text{for } \alpha > 4\pi, \\ +\infty, & \text{for } 0 < \alpha < 4\pi; \end{cases}$$

(f_3) there is a positive constant $\theta > 2$ such that

$$0 < \frac{\theta}{2} F(t) \leq t f(t), \qquad \forall\, t > 0, \quad \text{where } F(t) = \int_0^t f(s)ds;$$

(f_4) there exist two constants $p > 2$ and

$$C_p > \max\left\{ \left[\beta_p \Big(\frac{2\theta}{\theta - 2}\Big) \frac{1}{\min\{1, V_0\}}\right]^{(p-2)/2}, V_0 \left(\frac{p-2}{p}\right)^{\frac{p-2}{2}} S_p^{p/2} \right\} > 0$$

such that

$$f'(t) \geq \frac{p-2}{2} C_p t^{(p-4)/2} \quad \text{for all } t > 0,$$

where

$$\beta_p = \inf_{u \in \tilde{\mathcal{N}}_0} \tilde{I}_0(u), \quad \tilde{I}_0(u) := \frac{1}{2} \int_{\mathbb{R}^2} (|\nabla u|^2 + V_0 |u|^2) dx - \frac{1}{p} \int_{\mathbb{R}^2} |u|^p dx,$$

$$\tilde{\mathcal{N}}_0 := \{u \in H^1(\mathbb{R}^2, \mathbb{R}) \setminus \{0\} : \tilde{I}_0'(u)[u] = 0\},$$

and S_p is the best constant for the Sobolev inequality

$$S_p \Big(\int_{\mathbb{R}^2} |u|^p dx \Big)^{2/p} \leq \int_{\mathbb{R}^2} |(\nabla u|^2 + |u|^2) dx;$$

(f_5) $f'(t) \leq (e^{4\pi t} - 1)$ for any $t \geq 0$.

Observe that assumptions (f_4) and (f_5) imply that there exist two positive constants C_1 and C_2 such that

$$C_1 t^{(p-4)/2} \leq f'(t) \leq C_2 t, \quad \text{as } t \to 0^+$$

and so $p > 6$.

Our main result is

Theorem 1.1. *Assume that V satisfies (V_1), (V_2) and f satisfies (f_1)–(f_5). Then, for any $\delta > 0$ such that*

$$M_\delta := \{x \in \mathbb{R}^2 : dist(x, M) < \delta\} \subset \Lambda,$$

there exists $\varepsilon_\delta > 0$ such that, for any $0 < \varepsilon < \varepsilon_\delta$, problem (1.1) has at least $cat_{M_\delta}(M)$ nontrivial solutions. Moreover, for every sequence $\{\varepsilon_n\}$ such that $\varepsilon_n \to 0^+$ as $n \to +\infty$, if we denote by u_{ε_n} one of these solutions of (1.1) for $\varepsilon = \varepsilon_n$ and $\eta_{\varepsilon_n} \in \mathbb{R}^2$ the global maximum point of $|u_{\varepsilon_n}|$, then

$$\lim_{\varepsilon_n \to 0^+} V(\eta_{\varepsilon_n}) = V_0.$$

It is well known that, when we want to study by variational methods this type of equations in the whole $\mathbb{R}^2$, we meet several difficulties due to the unboundedness of the domain and to the exponential critical growth of the nonlinearity. Moreover, we only know local information on the potential V, and we don't have any condition on V at infinity. Thus we adapt the penalization technique explored in [25]. It consists in making a suitable modification on the nonlinearity f, solving a modified problem and then check that, for ε small enough, the solutions of the modified problem are indeed solutions of the original one. It could be interesting to consider our problem without relying upon condition near 0.

It is worthwhile to remark that in the arguments developed in [25], one of the key points is the existence of estimates involving the L^∞-norm of the solutions of the modified problem. In the the magnetic case, this kind of estimates are more delicate, due also to the fact that we deal with complex valued functions. For subcritical nonlinearities, Alves et al. in [5] obtained L^∞-estimates of the solutions of the modified problem by a different approach, which is based on Moser's iteration method (see [34]) instead of Kato's inequality. Here the problem we deal with has exponential critical growth in $\mathbb{R}^2$, so the method in [5] does not seem fully applicable.

The paper is organized as follows. In Section 2 we introduce the functional setting, give some preliminaries and study the limit problem. In Section 3, we study the modified problem. We prove the Palais-Smale condition for the modified functional and provide some tools which are useful to establish a multiplicity result. In Section 4, we show a multiplicity result for he modified problem. Finally, in Section 5, we complete the paper with the proof of Thereom 1.1.

Notation.

- $C, C_1, C_2, \ldots$ denote positive constants whose exact values are inessential and can change from line to line;
- $B_R(y)$ denotes the open disk centered at $y \in \mathbb{R}^2$ with radius $R > 0$ and $B_R^c(y)$ denotes the complement of $B_R(y)$ in $\mathbb{R}^2$;
- $\|\cdot\|$, $\|\cdot\|_q$, and $\|\cdot\|_{L^\infty(\Omega)}$ denote the usual norms of the spaces $H^1(\mathbb{R}^2,\mathbb{R})$, $L^q(\mathbb{R}^2,\mathbb{R})$, and $L^\infty(\Omega,\mathbb{R})$, respectively, where $\Omega \subset \mathbb{R}^2$, and $\|\cdot\|_{V_0} := (\|\nabla \cdot\|_2 + V_0\|\cdot\|_2)^{1/2}$.

2. The variational framework and the limit problem

In this section, we present the functional spaces that we use, we introduce a *classical* equivalent version of (1.1), we give some useful preliminary remarks, and we study a *limit problem* which will be useful for our arguments.

For $u : \mathbb{R}^2 \to \mathbb{C}$, let us denote by

$$\nabla_A u := \Big(\frac{\nabla}{i} - A\Big)u,$$

and

$$H_A^1(\mathbb{R}^2,\mathbb{C}) := \{u \in L^2(\mathbb{R}^2,\mathbb{C}) : |\nabla_A u| \in L^2(\mathbb{R}^2,\mathbb{R})\}.$$

The space $H_A^1(\mathbb{R}^2,\mathbb{C})$ is an Hilbert space endowed with the scalar product

$$\langle u, v\rangle := \mathrm{Re}\int_{\mathbb{R}^2}\Big(\nabla_A u\overline{\nabla_A v} + u\overline{v}\Big)dx, \quad \text{for any } u, v \in H_A^1(\mathbb{R}^2,\mathbb{C}),$$

where Re and the bar denote the real part of a complex number and the complex conjugation, respectively. Moreover we denote by $\|u\|_A$ the norm induced by this inner product.

On $H_A^1(\mathbb{R}^2,\mathbb{C})$ we will frequently use the following diamagnetic inequality (see e.g. [33, Theorem 7.21])

$$|\nabla_A u(x)| \geq |\nabla|u(x)||. \tag{2.1}$$

Moreover, making a simple change of variables, we can see that (1.1) is equivalent to

$$\Big(\frac{1}{i}\nabla - A_\varepsilon(x)\Big)^2 u + V_\varepsilon(x)u = f(|u|^2)u \quad \text{in } \mathbb{R}^2, \tag{2.2}$$

where $A_\varepsilon(x) = A(\varepsilon x)$ and $V_\varepsilon(x) = V(\varepsilon x)$.

Let H_ε be the Hilbert space obtained as the closure of $C_c^\infty(\mathbb{R}^2,\mathbb{C})$ with respect to the scalar product

$$\langle u, v\rangle_\epsilon := \mathrm{Re}\int_{\mathbb{R}^2}\Big(\nabla_{A_\varepsilon} u\overline{\nabla_{A_\varepsilon} v} + V_\varepsilon(x)u\overline{v}\Big)dx$$

and let us denote by $\|\cdot\|_\varepsilon$ the norm induced by this inner product.

The diamagnetic inequality (2.1) implies that, if $u \in H_{A_\varepsilon}^1(\mathbb{R}^2,\mathbb{C})$, then $|u| \in H^1(\mathbb{R}^2,\mathbb{R})$ and $\|u\| \leq C\|u\|_\varepsilon$. Therefore, the embedding $H_\varepsilon \hookrightarrow L^r(\mathbb{R}^2,\mathbb{C})$ is continuous for $r \geq 2$ and the embedding $H_\varepsilon \hookrightarrow L_{\mathrm{loc}}^r(\mathbb{R}^2,\mathbb{C})$ is compact for $r \geq 1$.

About the nonlinearity, we observe that, by (f_1) and (f_2), fixed $q > 2$, for any $\zeta > 0$ and $\alpha > 4\pi$, there exists a constant $C > 0$, which depends on q, α, ζ, such that

$$f(t) \leq \zeta + Ct^{(q-2)/2}(e^{\alpha t} - 1) \text{ for all } t \geq 0 \tag{2.3}$$

and, using (f_3), we have

$$F(t) \leq \zeta t + Ct^{q/2}(e^{\alpha t} - 1) \text{ for all } t \geq 0. \tag{2.4}$$

Moreover, it is easy to see that, by (2.3) and (2.4),

$$f(t^2)t^2 \leq \zeta t^2 + C|t|^q(e^{\alpha t^2} - 1) \text{ for all } t \in \mathbb{R} \tag{2.5}$$

and

$$F(t^2) \leq \zeta t^2 + C|t|^q(e^{\alpha t^2} - 1) \text{ for all } t \in \mathbb{R}. \tag{2.6}$$

Finally, let us recall the following version of Trudinger-Moser inequality as stated e.g. in [1, Lemma 1.2].

Lemma 2.1. *If $\alpha > 0$ and $u \in H^1(\mathbb{R}^2, \mathbb{R})$, then*

$$\int_{\mathbb{R}^2} (e^{\alpha u^2} - 1)dx < +\infty.$$

Moreover, if $\|\nabla u\|_2^2 \leq 1$, $\|u\|_2 \leq M < +\infty$, and $0 < \alpha < 4\pi$, then there exists a positive constant $C(M, \alpha)$, which depends only on M and α, such that

$$\int_{\mathbb{R}^2} (e^{\alpha u^2} - 1)dx \leq C(M, \alpha).$$

For our scope, we need also to study the following *limit* problem

$$-\Delta u + V_0 u = f(u^2)u, \quad u : \mathbb{R}^2 \to \mathbb{R}, \tag{2.7}$$

whose associated C^1-functional, defined in $H^1(\mathbb{R}^2, \mathbb{R})$, is

$$I_{V_0}(u) := \frac{1}{2}\int_{\mathbb{R}^2} (|\nabla u|^2 + V_0 u^2)dx - \frac{1}{2}\int_{\mathbb{R}^2} F(u^2)dx.$$

Let

$$\mathcal{N}_{V_0} := \{u \in H^1(\mathbb{R}^2, \mathbb{R}) \setminus \{0\} : I'_{V_0}(u)[u] = 0\}$$

and

$$c_{V_0} := \inf_{u \in \mathcal{N}_{V_0}} I_{V_0}(u).$$

By (f_1) and (f_4), for each $u \in H^1(\mathbb{R}^2, \mathbb{R}) \setminus \{0\}$, there is a unique $t(u) > 0$ such that

$$I_{V_0}(t(u)u) = \max_{t \geq 0} I_{V_0}(tu) \quad \text{and} \quad t(u)u \in \mathcal{N}_{V_0}.$$

Then, using the assumptions on f, arguing as in [38, Lemma 4.1 and Theorem 4.2] we have that

$$0 < c_{V_0} = \inf_{u \in H^1(\mathbb{R}^2, \mathbb{R}) \setminus \{0\}} \max_{t \geq 0} I_{V_0}(tu).$$

Moreover, recalling that a positive ground state solution $\omega \in H^1(\mathbb{R}^2, \mathbb{R})$ of (2.7) satisfies $I_{V_0}(\omega) \leq I_{V_0}(v)$ for all positive nontrivial solutions $v \in H^1(\mathbb{R}^2, \mathbb{R})$ of (2.7), by [9, Corollary 1.5], we get

Lemma 2.2. *Problem (2.7) has a positive ground state solution $\omega \in H^1(\mathbb{R}^2, \mathbb{R})$ which is radially symmetric.*

Proof. Let us recall that [9, Corollary 1.5] states that, if $h : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying:

(i) $\lim_{t \to 0^+} h(t)/t = 0$;

(ii) there holds

$$\lim_{t \to +\infty} \frac{h(t)}{e^{\alpha t^2}} = \begin{cases} 0, & \text{for } \alpha > 4\pi, \\ +\infty, & \text{for } 0 < \alpha < 4\pi; \end{cases}$$

(iii) there exist $\lambda > 0$ and $p > 2$ such that $h(t) \geq \lambda t^{p-1}$ for $t \geq 0$ and

$$\lambda > \left(\frac{p-2}{p}\right)^{\frac{p-2}{2}} S_p^{p/2}, \tag{2.8}$$

where S_p is the best Sobolev constant for $S_p\|v\|_p^2 \leq \|v\|^2$;

then

$$-\Delta v + v = h(v) \tag{2.9}$$

has a nontrivial radial positive solution $\hat{v} \in H^1(\mathbb{R}^2,\mathbb{R})$, namely $\mathcal{I}(\hat{v}) \le \mathcal{I}(\tilde{v})$ for every nontrivial positive solution $\tilde{v} \in H^1(\mathbb{R}^2,\mathbb{R})$ of (2.9), where

$$\mathcal{I}(v) := \frac{1}{2}\|v\|^2 - \int_{\mathbb{R}^2} H(v)dx, \quad H(t) := \int_0^t h(s)ds.$$

In particular, condition (2.8) allows to prove that

$$\bar{c} = \inf_{v\in H^1(\mathbb{R}^2,\mathbb{R})\setminus\{0\}} \max_{t\ge 0} \mathcal{I}(tv) < \frac{1}{2}.$$

Now, if we take

$$h(t) := f(t^2)t/V_0 \tag{2.10}$$

in (2.9), we have that (i) and (ii) are easily satisfied, and, using assumption (f_4) we get that (iii) is satisfied for $\lambda = C_p/V_0$. Thus (2.9) with h as in (2.10) admits a positive radial nontrivial ground state solution $\hat{v} \in H^1(\mathbb{R}^2,\mathbb{R})$.
Observe now that, if $\hat{v} \in H^1(\mathbb{R}^2,\mathbb{R})$ is a solution of (2.9) where h is given by (2.10), then $\hat{u} := \hat{v}(\sqrt{V_0}\cdot) \in H^1(\mathbb{R}^2,\mathbb{R})$ is a solution of (2.7) and, since by (2.10),

$$H(t) = \frac{1}{V_0}\int_0^t f(s^2)sds = \frac{1}{2V_0}F(t^2),$$

we have $I_{V_0}(\tilde{u}) = \mathcal{I}(\tilde{v})$. Analogously, if $\tilde{u} \in H^1(\mathbb{R}^2,\mathbb{R})$ is an arbitrary solution of (2.7) and $\tilde{v} := \tilde{u}(\cdot/\sqrt{V_0})$, then $\tilde{v} \in H^1(\mathbb{R}^2,\mathbb{R})$ is a solution of (2.9) and $I_{V_0}(\tilde{u}) = \mathcal{I}(\tilde{v})$.
Hence, if $\hat{v} \in H^1(\mathbb{R}^2,\mathbb{R})$ is a positive radial nontrivial ground state of (2.9), then, if $\omega = \hat{v}(\sqrt{V_0}\cdot)$, $\tilde{u} \in H^1(\mathbb{R}^2,\mathbb{R})$ is an arbitrary solution of (2.7) and $\tilde{v} := \tilde{u}(\cdot/\sqrt{V_0})$, we have

$$I_{V_0}(\omega) = \mathcal{I}(\hat{v}) \le \mathcal{I}(\tilde{v}) = I_{V_0}(\tilde{u})$$

and we conclude. □

Note that, by [39, Proposition 2.1], every radially symmetric ground state solution of (2.7) decays exponentially at infinity with its gradient, and is $C^2(\mathbb{R}^2,\mathbb{R}) \cap L^\infty(\mathbb{R}^2,\mathbb{R})$.

The elements of $\mathcal{N}_{V_0}$ satisfy the following property.

Lemma 2.3. *There exists $K > 0$ such that, for all $u \in \mathcal{N}_{V_0}$, $\|u\|_{V_0} \ge K$.*

Proof. By (2.5), for any $0 < \zeta < V_0/2$ and $\alpha > 4\pi$, we have that there exists $C > 0$ such that, for every $u \in \mathcal{N}_{V_0}$,

$$\int_{\mathbb{R}^2} (|\nabla u|^2 + V_0|u|^2)dx \le \zeta \int_{\mathbb{R}^2} |u|^2 + C\int_{\mathbb{R}^2} |u|^q(e^{\alpha|u|^2} - 1)dx. \tag{2.11}$$

Moreover, by the Hölder inequality it follows

$$\int_{\mathbb{R}^2} |u|^q(e^{\alpha|u|^2} - 1)dx \le \|u\|_{2q}^q \Big(\int_{\mathbb{R}^2} (e^{\alpha|u|^2} - 1)^2 dx\Big)^{1/2} \le C\|u\|_{V_0}^q \Big(\int_{\mathbb{R}^2} (e^{2\alpha|u|^2} - 1)dx\Big)^{1/2} \tag{2.12}$$

where we have used the inequality

$$(e^t - 1)^s \le e^{ts} - 1, \text{ for } s > 1 \text{ and } t \ge 0. \tag{2.13}$$

Now assume by contradiction that there exist a sequence $(u_n) \subset \mathcal{N}_{V_0}$ such that $\|u_n\|_{V_0} \to 0$ as $n \to +\infty$. Then, for n large enough and $\bar{\alpha} \in (0, 4\pi)$, using Lemma 2.1, we get

$$\int_{\mathbb{R}^2} (e^{2\alpha|u_n|^2} - 1)dx \le \int_{\mathbb{R}^2} (e^{2\alpha\|u_n\|_{V_0}^2 \left(\frac{u_n}{\|u_n\|_{V_0}}\right)^2} - 1)dx \le \int_{\mathbb{R}^2} (e^{\bar{\alpha}\left(\frac{u_n}{\|u_n\|_{V_0}}\right)^2} - 1)dx \le C. \tag{2.14}$$

Thus, combining (2.11), (2.12), and (2.14), we reach the contradiction. □

The following lemma is the upper bound estimate of the ground state energy which is important for our arguments.

Lemma 2.4. *The minimax level c_{V_0} verifies*

$$0 < c_{V_0} < \frac{\theta - 2}{2\theta} \min\{1, V_0\}.$$

Proof. Arguing as in [38], we can find that there exists $\omega^* \in H^1(\mathbb{R}^2, \mathbb{R}) \setminus \{0\}$ such that $\tilde{I}_0(\omega^*) = \beta_p$ and $\tilde{I}_0'(\omega^*) = 0$ (see (f_4) for the definitions of $\tilde{I}_0$ and β_p). By the characterization of c_{V_0} given before and by (f_4) we have

$$0 < c_{V_0} \leq \max_{t \geq 0} I_{V_0}(t\omega^*) \leq \max_{t \geq 0} \left\{ \frac{t^2}{2} \|\omega^*\|_{V_0}^2 - \frac{C_p t^p}{p} \|\omega^*\|_p^p \right\} = C_p^{2/(2-p)} \beta_p < \frac{\theta - 2}{2\theta} \min\{1, V_0\}.$$

□

Finally we prove the following useful result.

Lemma 2.5. *Let $(\omega_n) \subset \mathcal{N}_{V_0}$ be a sequence satisfying $I_{V_0}(\omega_n) \to c_{V_0}$. Then (ω_n) is bounded in $H^1(\mathbb{R}^2, \mathbb{R})$ and, up to a subsequence, $\omega_n \rightharpoonup \omega$ in $H^1(\mathbb{R}^2, \mathbb{R})$. Moreover, if $\omega \neq 0$, then $\omega_n \to \omega \in \mathcal{N}_{V_0}$ in $H^1(\mathbb{R}^2, \mathbb{R})$ and ω is a ground state for problem (2.7). If $\omega = 0$, then there exists $(\tilde{y}_n) \subset \mathbb{R}^2$ with $|\tilde{y}_n| \to +\infty$ and $\tilde{\omega} \in \mathcal{N}_{V_0}$ such that, up to a subsequence, $\omega_n(\cdot + \tilde{y}_n) \to \tilde{\omega}$ in $H^1(\mathbb{R}^2, \mathbb{R})$ and $\tilde{\omega}$ is a ground state for problem (2.7).*

Proof. By (f_3) and Lemma 2.4, it follows that

$$\begin{aligned} \frac{\theta - 2}{2\theta} \limsup_n \|\omega_n\|_{V_0}^2 &\leq \limsup_n \left\{ \left(\frac{1}{2} - \frac{1}{\theta} \right) \|\omega_n\|_{V_0}^2 + \int_{\mathbb{R}^2} \left(\frac{1}{\theta} f(\omega_n^2)\omega_n^2 - \frac{1}{2} F(\omega_n^2) \right) dx \right\} \\ &= \limsup_n \left\{ I_{V_0}(\omega_n) - \frac{1}{\theta} I_{V_0}'(\omega_n)[\omega_n] \right\} \\ &= c_{V_0} < \frac{\theta - 2}{2\theta} \min\{1, V_0\}. \end{aligned}$$

Thus,

$$\limsup_n \|\omega_n\|_{V_0}^2 < 1, \tag{2.15}$$

and for some subsequence, still denoted by (ω_n), we can assume that there exists $\omega \in H^1(\mathbb{R}^2, \mathbb{R})$ such that $\omega_n \rightharpoonup \omega$ in $H^1(\mathbb{R}^2, \mathbb{R})$, $\omega_n \to \omega$ in $L^r_{\text{loc}}(\mathbb{R}^2, \mathbb{R})$, for any $r \geq 1$ and $\omega_n \to \omega$ a.e. in $x \in \mathbb{R}^2$.
Now we divide our study into two cases.
Case 1: $\omega \neq 0$.
Observe that, for every $\phi \in C_c^\infty(\mathbb{R}^2, \mathbb{R})$,

$$f(\omega_n^2)\omega_n \phi \to f(\omega^2)\omega\phi \text{ a.e. in } \mathbb{R}^2 \text{ as } n \to +\infty$$

and that, by (2.3), we have that for any $\zeta > 0$, $q > 2$, and $\alpha > 4\pi$, there exists $C > 0$ such that, for every $\phi \in C_c^\infty(\mathbb{R}^2, \mathbb{R})$,

$$|f(\omega_n^2)\omega_n \phi| \leq \zeta |\omega_n||\phi| + C|\omega_n|^{q-1}(e^{\alpha\omega_n^2} - 1)|\phi|$$

with

$$\zeta|\omega_n||\phi| + C|\omega_n|^{q-1}(e^{\alpha\omega_n^2} - 1)|\phi| \to \zeta|\omega||\phi| + C|\omega|^{q-1}(e^{\alpha\omega^2} - 1)|\phi| \text{ a.e. in } \mathbb{R}^2 \text{ as } n \to +\infty.$$

Moreover, by the Hölder inequality, (2.13), Sobolev inequality, (2.15), and Lemma 2.1, for suitable $r > 1$, $q > 2$, $\alpha > 4\pi$, and $p > 1$, we have that, for all $n \in \mathbb{N}$,

$$\begin{aligned}\int_{\mathbb{R}^2} \Big[|\omega_n|^{q-1}(e^{\alpha\omega_n^2}-1)\Big]^r dx &\le \|\omega_n\|_{pr(q-1)}^{r(q-1)}\Big(\int_{\mathbb{R}^2}(e^{\alpha r p'\omega_n^2}-1)dx\Big)^{1/p'}\\ &\le C\|\omega_n\|_{V_0}^{r(q-1)}\Big(\int_{\mathbb{R}^2}(e^{\alpha r p'\|\omega_n\|_{V_0}^2(|\omega_n|/\|\omega_n\|_{V_0})^2}-1)dx\Big)^{1/p'}\\ &\le C.\end{aligned}$$

Thus

$$|\omega_n|^{q-1}(e^{\alpha|\omega_n|^2}-1) \rightharpoonup |\omega|^{q-1}(e^{\alpha|\omega|^2}-1) \text{ in } L^r(\mathbb{R}^2,\mathbb{R})$$

and, since $|\omega_n| \rightharpoonup |\omega|$ in $L^2(\mathbb{R}^2,\mathbb{R})$ we have that

$$\zeta\int_{\mathbb{R}^2}|\omega_n||\phi|dx + C\int_{\mathbb{R}^2}|\omega_n|^{q-1}(e^{\alpha\omega_n^2}-1)|\phi|dx \to \zeta\int_{\mathbb{R}^2}|\omega||\phi|dx + C\int_{\mathbb{R}^2}|\omega|^{q-1}(e^{\alpha\omega^2}-1)|\phi|dx$$

as $n \to +\infty$.

Hence, a variant of the Lebesgue Dominated Convergence Theorem implies that

$$\int_{\mathbb{R}^2} f(\omega_n^2)\omega_n\phi dx \to \int_{\mathbb{R}^2} f(\omega^2)\omega\phi dx,$$

and so ω is a nontrivial critical point for I_{V_0}.

Since, by the Fatou's Lemma,

$$\int_{\mathbb{R}^2}\Big(\frac{1}{\theta}f(\omega^2)\omega^2 - \frac{1}{2}F(\omega^2)\Big)dx \le \liminf_n \int_{\mathbb{R}^2}\Big(\frac{1}{\theta}f(\omega_n^2)\omega_n^2 - \frac{1}{2}F(\omega_n^2)\Big)dx,$$

we have

$$\begin{aligned} c_{V_0} \le I_{V_0}(\omega) &= I_{V_0}(\omega) - \frac{1}{\theta}I'_{V_0}(\omega)[\omega]\\ &= \Big(\frac{1}{2}-\frac{1}{\theta}\Big)\|\omega\|_{V_0}^2 + \int_{\mathbb{R}^2}\Big(\frac{1}{\theta}f(\omega^2)\omega^2 - \frac{1}{2}F(\omega^2)\Big)dx\\ &\le \liminf_n\Big\{\Big(\frac{1}{2}-\frac{1}{\theta}\Big)\|\omega_n\|_{V_0}^2 + \int_{\mathbb{R}^2}\Big(\frac{1}{\theta}f(\omega_n^2)\omega_n^2 - \frac{1}{2}F(\omega_n^2)\Big)dx\Big\}\\ &= \liminf_n\Big\{I_{V_0}(\omega_n) - \frac{1}{\theta}I'_{V_0}(\omega_n)[\omega_n]\Big\}\\ &= c_{V_0}.\end{aligned}$$

Hence, using again the Fatou's Lemma, we have

$$\begin{aligned} 0 &\le \liminf_n\Big[\Big(\frac{1}{2}-\frac{1}{\theta}\Big)(\|\omega_n\|_{V_0}^2 - \|\omega\|_{V_0}^2)\Big]\\ &\le \limsup_n\Big[\Big(\frac{1}{2}-\frac{1}{\theta}\Big)(\|\omega_n\|_{V_0}^2 - \|\omega\|_{V_0}^2)\Big]\\ &= \limsup_n\Big[I_{V_0}(\omega_n) - c_{V_0} + \int_{\mathbb{R}^2}\Big(\frac{1}{\theta}f(\omega^2)\omega^2 - \frac{1}{2}F(\omega^2)\Big)dx - \int_{\mathbb{R}^2}\Big(\frac{1}{\theta}f(\omega_n^2)\omega_n^2 - \frac{1}{2}F(\omega_n^2)\Big)dx\Big]\\ &= \int_{\mathbb{R}^2}\Big(\frac{1}{\theta}f(\omega^2)\omega^2 - \frac{1}{2}F(\omega^2)\Big)dx - \liminf_n\Big[\int_{\mathbb{R}^2}\Big(\frac{1}{\theta}f(\omega_n^2)\omega_n^2 - \frac{1}{2}F(\omega_n^2)\Big)dx\Big] \le 0,\end{aligned}$$

and we conclude.

Case 2: $\omega = 0$.

We claim that, in this case, there exist $R, \eta > 0$, and $(\tilde{y}_n) \subset \mathbb{R}^2$ such that

$$\lim_n \int_{B_R(\tilde{y}_n)} \omega_n^2 dx \ge \eta. \tag{2.16}$$

Indeed, if this does not hold, for any $R > 0$, one has

$$\lim_n \sup_{y\in\mathbb{R}^2} \int_{B_R(y)} \omega_n^2 dx = 0,$$

and by [30, Chapter 6, Lemma 8.4], for every $\tau > 2$,

$$\lim_n \|\omega_n\|_\tau = 0. \tag{2.17}$$

By (2.5) and the fact that $(\omega_n) \subset \mathcal{N}_{V_0}$, for $0 < \zeta < V_0/2$ and $\alpha > 4\pi$, there exists $C > 0$ such that

$$\int_{\mathbb{R}^2} (|\nabla\omega_n|^2 + V_0\omega_n^2)dx \le \zeta \int_{\mathbb{R}^2} \omega_n^2 + C \int_{\mathbb{R}^2} |\omega_n|^q (e^{\alpha\omega_n^2} - 1)dx. \tag{2.18}$$

In virtue of (2.15), we may choose $r > 1$ and $\alpha > 4\pi$ such that $r\alpha\|\omega_n\|_{V_0}^2 < 4\pi$ for $n \in \mathbb{N}$ large enough. Thus, by the Hölder inequality, inequality (2.13), Lemma 2.1, and (2.17), it follows that

$$\begin{aligned}\int_{\mathbb{R}^2} |\omega_n|^q (e^{\alpha\omega_n^2} - 1)dx &\le \|\omega_n\|_{qr'}^q \Big(\int_{\mathbb{R}^2} (e^{r\alpha\omega_n^2} - 1)dx\Big)^{1/r} \\ &\le \|\omega_n\|_{qr'}^q \Big(\int_{\mathbb{R}^2} (e^{r\alpha\|\omega_n\|_{V_0}^2(\omega_n/\|\omega_n\|_{V_0})^2} - 1)dx\Big)^{1/r} = o_n(1),\end{aligned} \tag{2.19}$$

where r' is the conjugate exponent of r.
Using (2.18) and (2.19), we have that $\omega_n \to 0$ in $H^1(\mathbb{R}^2,\mathbb{R})$ as $n \to +\infty$, and, consequently, $I_{V_0}(\omega_n) \to 0$ as $n \to +\infty$, which is in contradiction with $I_{V_0}(\omega_n) \to c_{V_0} > 0$ as $n \to +\infty$.
By (2.16), we have that $|\tilde{y}_n| \to +\infty$. Otherwise, there exists $\bar{R} > 0$ such that

$$\lim_n \int_{B_{\bar{R}}(0)} \omega_n^2 dx \ge \eta$$

and so $\omega \ne 0$, which is a contradiction.
Since I_{V_0} and the norm $\|\cdot\|_{V_0}$ in $H^1(\mathbb{R}^2,\mathbb{R})$ are invariant by translation, we have

$$I_{V_0}(\omega_n(\cdot + \tilde{y}_n)) \to c_{V_0}.$$

Moreover $\omega_n(\cdot + \tilde{y}_n) \in \mathcal{N}_{V_0}$ and, by (2.15),

$$\limsup_n \|\omega_n(\cdot + \tilde{y}_n)\|_{V_0}^2 < 1.$$

Thus, there exists $\tilde{\omega} \in H^1(\mathbb{R}^2,\mathbb{R})$ with, by (2.16), $\tilde{\omega} \ne 0$, such that

$$\omega_n(\cdot + \tilde{y}_n) \rightharpoonup \tilde{\omega} \quad \text{in} \quad H^1(\mathbb{R}^2,\mathbb{R}).$$

Repeating the same arguments used in Case 1, it is easy to obtain that $\omega_n(\cdot + \tilde{y}_n) \to \tilde{\omega} \in \mathcal{N}_{V_0}$ in $H^1(\mathbb{R}^2,\mathbb{R})$ and $\tilde{\omega}$ is a ground state for problem (2.7). ☐

3. The modified problem

In this section we introduce a modified problem for (2.2) and we show some properties of its functional. As in [25], to study (1.1), or equivalently, (2.2) by variational methods, we modify suitably the nonlinearity f so that, for $\varepsilon > 0$ small enough, the solutions of such modified problem are also solutions of the original one. More precisely, we fix $k > 0$ such that

$$0 < c_{V_0} < \Big(\frac{1}{2} - \frac{1}{\theta} - \frac{1}{2k}\Big) \min\{1, V_0\} < \frac{\theta - 2}{2\theta} \min\{1, V_0\}.$$

By the assumptions on f there exists a unique number $a > 0$ verifying $kf(a) = V_0$, where V_0 is given in (V_1). Hence we consider the function

$$\hat{f}(t) := \begin{cases} f(t), & t \le a, \\ V_0/k, & t > a. \end{cases}$$

As, for instance, in [5], we take $0 < t_a < a < T_a$ and $\vartheta \in C_0^\infty(\mathbb{R}, \mathbb{R})$ such that

(ϑ_1) $\vartheta(t) \le \hat{f}(t)$ for all $t \in [t_a, T_a]$;
(ϑ_2) $\vartheta(t_a) = \hat{f}(t_a)$, $\vartheta(T_a) = \hat{f}(T_a)$, $\vartheta'(t_a) = \hat{f}'(t_a)$, and $\vartheta'(T_a) = \hat{f}'(T_a)$;
(ϑ_3) the map ϑ is increasing in $[t_a, T_a]$.

Using the above functions we can define $\tilde{f} \in C^1(\mathbb{R}, \mathbb{R})$ as follows

$$\tilde{f}(t) := \begin{cases} \hat{f}(t) & \text{if } t \notin [t_a, T_a], \\ \vartheta(t) & \text{if } t \in [t_a, T_a]. \end{cases}$$

Now we introduce the penalized nonlinearity $g : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$

$$g(x,t) := \chi_\Lambda(x) f(t) + (1 - \chi_\Lambda(x))\tilde{f}(t), \tag{3.1}$$

where χ_Λ is the characteristic function on Λ and $G(x,t) := \displaystyle\int_0^t g(x,s)ds$.

In view of (f_1)–(f_5) and (ϑ_1)–(ϑ_3), we have that g is a Carathéodory function satisfying the following properties:

(g_1) $g(x,t) = 0$ for each $t \le 0$;
(g_2) $\lim\limits_{t\to 0^+} g(x,t) = 0$ uniformly in $x \in \mathbb{R}^2$;
(g_3) $g(x,t) \le f(t)$ for all $t \ge 0$ and uniformly in $x \in \mathbb{R}^2$;
(g_4) $0 < \theta G(x,t) \le 2g(x,t)t$, for each $x \in \Lambda$, $t > 0$;
(g_5) $0 < g(x,t) \le V_0/k$, for each $x \in \Lambda^c$, $t > 0$;
(g_6) for each $x \in \Lambda$, the function $t \mapsto g(x,t)$ is strictly increasing in $t \in (0, +\infty)$ and for each $x \in \Lambda^c$, the function $t \mapsto g(x,t)$ strictly is increasing in $(0, t_a)$.

Then we consider the *modified* problem

$$\Big(\frac{1}{i}\nabla - A_\varepsilon(x)\Big)^2 u + V_\varepsilon(x)u = g(\varepsilon x, |u|^2)u \quad \text{in } \mathbb{R}^2. \tag{3.2}$$

Note that, if u is a solution of problem (3.2) with

$$|u(x)|^2 \le t_a \quad \text{for all } x \in \Lambda_\varepsilon^c, \quad \Lambda_\varepsilon := \{x \in \mathbb{R}^2 : \varepsilon x \in \Lambda\},$$

then u is a solution of problem (2.2).

The functional associated to problem (3.2) is

$$J_\varepsilon(u) := \frac{1}{2}\int_{\mathbb{R}^2} (|\nabla_{A_\varepsilon} u|^2 + V_\varepsilon(x)|u|^2)dx - \frac{1}{2}\int_{\mathbb{R}^2} G(\varepsilon x, |u|^2)dx,$$

defined in H_ε. It is standard to prove that $J_\varepsilon \in C^1(H_\varepsilon, \mathbb{R})$ and its critical points are nontrivial weak solutions of the modified problem (3.2).

Now we show that the functional J_ε satisfies the Mountain Pass Geometry.

Lemma 3.1. *For any fixed $\varepsilon > 0$, the functional J_ε satisfies the following properties:*

(i) there exist $\beta, r > 0$ such that $J_\varepsilon(u) \ge \beta$ if $\|u\|_\varepsilon = r$;
(ii) there exists $e \in H_\varepsilon$ with $\|e\|_\varepsilon > r$ such that $J_\varepsilon(e) < 0$.

Proof. Let us prove (i).
By (g_3) and (2.6), fixed $q > 2$ and $\alpha > 4\pi$, for any $\zeta > 0$ and there exists $C > 0$ such that

$$G(\varepsilon x, |u|^2) \le \zeta|u|^2 + C|u|^q(e^{\alpha|u|^2} - 1) \quad \text{for all } x \in \mathbb{R}^2. \tag{3.3}$$

By the Hölder and Sobolev inequalities and (2.13) it follows

$$\int_{\mathbb{R}^2} |u|^q(e^{\alpha|u|^2} - 1)dx \le \|u\|_{2q}^q \Big(\int_{\mathbb{R}^2} (e^{\alpha|u|^2} - 1)^2 dx\Big)^{1/2} \le C\|u\|_\varepsilon^q \Big(\int_{\mathbb{R}^2} (e^{2\alpha|u|^2} - 1)dx\Big)^{1/2}. \tag{3.4}$$

Now, let us observe that, by the diamagnetic inequality (2.1), if $u \in H_\varepsilon \setminus \{0\}$, it follows that

$$\frac{|u|}{\|u\|_\varepsilon} \in H^1(\mathbb{R}^2, \mathbb{R}), \quad \Big\| \frac{|u|}{\|u\|_\varepsilon} \Big\|_2^2 \leq \frac{1}{V_0}, \quad \Big\| \nabla \frac{|u|}{\|u\|_\varepsilon} \Big\|_2^2 \leq 1.$$

Therefore, if we consider $\|u\|_\varepsilon = r > 0$, for $\alpha r^2 < \pi$, by Lemma 2.1, there exists a constant $C > 0$ such that

$$\int_{\mathbb{R}^2} (e^{2\alpha |u|^2} - 1) dx = \int_{\mathbb{R}^2} (e^{2\alpha r^2 \left(\frac{|u|}{\|u\|_\varepsilon}\right)^2} - 1) dx < \int_{\mathbb{R}^2} (e^{2\pi \left(\frac{|u|}{\|u\|_\varepsilon}\right)^2} - 1) dx \leq C. \tag{3.5}$$

Then, by (3.3), (3.4), and (3.5), for any $\zeta > 0$, there exits $C > 0$ such that

$$J_\varepsilon(u) \geq \frac{1}{2}\Big(1 - \frac{\zeta}{V_0}\Big) r^2 - C r^q$$

for any $u \in H_\varepsilon$ with $\|u\|_\varepsilon = r$ small enough and we can conclude easily since $q > 2$.
To prove (ii), let us fix $\varphi \in C_c^\infty(\mathbb{R}^2, \mathbb{C}) \setminus \{0\}$ with $\mathrm{supp}(\varphi) \subset \Lambda_\varepsilon$. By (3.1) and ($f_4$) we get

$$J_\varepsilon(t\varphi) \leq \frac{t^2}{2} \|\varphi\|_\varepsilon^2 - \frac{C_p}{p} t^p \|\varphi\|_p^p$$

and we can conclude passing to the limit as $t \to +\infty$, being $p > 2$. □

Hence we can define the minimax level

$$c_\varepsilon = \inf_{\gamma \in \Gamma_\varepsilon} \max_{t \in [0,1]} J_\varepsilon(\gamma(t))$$

where

$$\Gamma_\varepsilon = \{\gamma \in C([0,1], H_\varepsilon) : \gamma(0) = 0 \text{ and } J_\varepsilon(\gamma(1)) < 0\}.$$

The following results are important to prove the $(PS)_{c_\varepsilon}$ condition for the functional J_ε.

Lemma 3.2. *Assume that $(u_n) \subset H_\varepsilon$ is a $(PS)_d$ sequence for the functional J_ε. If*

$$0 < d < \min\{1, V_0\} \Big(\frac{1}{2} - \frac{1}{\theta} - \frac{1}{2k}\Big), \tag{3.6}$$

then (u_n) is bounded in H_ε and

$$\limsup_n |||u_n|||^2 < 1.$$

Proof. By (g_4) and (g_5) we have

$$\begin{aligned}
d + o_n(1) + o_n(1)\|u_n\|_\varepsilon &\geq J_\varepsilon(u_n) - \frac{1}{\theta} J_\varepsilon'(u_n)[u_n] \\
&= \Big(\frac{1}{2} - \frac{1}{\theta}\Big)\|u_n\|_\varepsilon^2 + \int_{\mathbb{R}^2} \Big(\frac{1}{\theta} g(\varepsilon x, |u_n|^2)|u_n|^2 - \frac{1}{2} G(\varepsilon x, |u_n|^2)\Big) dx \\
&\geq \Big(\frac{1}{2} - \frac{1}{\theta}\Big)\|u_n\|_\varepsilon^2 + \int_{\Lambda_\varepsilon^c} \Big(\frac{1}{\theta} g(\varepsilon x, |u_n|^2)|u_n|^2 - \frac{1}{2} G(\varepsilon x, |u_n|^2)\Big) dx \\
&\geq \Big(\frac{1}{2} - \frac{1}{\theta}\Big)\|u_n\|_\varepsilon^2 - \frac{1}{2} \int_{\Lambda_\varepsilon^c} G(\varepsilon x, |u_n|^2) dx \\
&\geq \Big(\frac{1}{2} - \frac{1}{\theta}\Big)\|u_n\|_\varepsilon^2 - \frac{1}{2k} \int_{\mathbb{R}^2} V(\varepsilon x)|u_n|^2 dx \\
&\geq \Big(\frac{1}{2} - \frac{1}{\theta} - \frac{1}{2k}\Big)\|u_n\|_\varepsilon^2.
\end{aligned}$$

Thus (u_n) is bounded in H_ε and

$$\Big(\frac{1}{2} - \frac{1}{\theta} - \frac{1}{2k}\Big)\|u_n\|_\varepsilon^2 \leq d + o_n(1).$$

Hence, by (3.6) and the diamagnetic inequality (2.1) we have

$$\min\{1,V_0\}\Big(\frac{1}{2}-\frac{1}{\theta}-\frac{1}{2k}\Big)\limsup_n |||u_n|||^2 \le \Big(\frac{1}{2}-\frac{1}{\theta}-\frac{1}{2k}\Big)\limsup_n \|u_n\|_\varepsilon^2 \le d < \min\{1,V_0\}\Big(\frac{1}{2}-\frac{1}{\theta}-\frac{1}{2k}\Big)$$

and we can conclude. □

The next result is a version of the celebrated Lions Lemma (see e.g. [38]), which is useful in our arguments.

Lemma 3.3. *Let $d>0$ and $(u_n)\subset H_\varepsilon$ be a $(PS)_d$ sequence for J_ε such that $u_n \rightharpoonup 0$ in H_ε as $n\to+\infty$ and $\limsup_n |||u_n||| < 1$. Then, one of the following alternatives occurs:*

(i) $u_n \to 0$ in H_ε as $n\to+\infty$;
(ii) there are a sequence $(y_n)\subset\mathbb{R}^2$ and constants R, $\beta>0$ such that

$$\liminf_n \int_{B_R(y_n)} |u_n|^2 dx \ge \beta.$$

Proof. Assume that (ii) does not hold. Then, for every $R>0$, we have

$$\lim_n \sup_{y\in\mathbb{R}^2}\int_{B_R(y)} |u_n|^2 dx = 0.$$

Being $(|u_n|)$ bounded in $H^1(\mathbb{R}^2)$, by [30, Chapter 6, Lemma 8.4], it follows that $\|u_n\|_\tau \to 0$ as $n\to+\infty$, for any $\tau>2$.
Since, by Lemma 3.2, (u_n) is a bounded $(PS)_d$ sequence for J_ε, then, using (g_3) and (2.5) we have that for any $\zeta>0$ there exists $C>0$ such that

$$\begin{aligned}0 \le \|u_n\|_\varepsilon^2 &= \int_{\mathbb{R}^2} g(\varepsilon x,|u_n|^2)|u_n|^2dx + o_n(1)\\ &\le \zeta\int_{\mathbb{R}^2}|u_n|^2dx + C\int_{\mathbb{R}^2}|u_n|^q(e^{\alpha|u_n|^2}-1)dx + o_n(1)\\ &\le \frac{\zeta}{V_0}\|u_n\|_\varepsilon^2 + C\int_{\mathbb{R}^2}|u_n|^q(e^{\alpha|u_n|^2}-1)dx + o_n(1)\end{aligned}$$

for every $\alpha>4\pi$.
Since $\limsup_n |||u_n||| < 1$, arguing as in the proof of Lemma 3.1, we have that $\|u_n\|_\varepsilon \to 0$ in H_ε and we conclude. □

The following lemma provides a range of levels in which the functional J_ε verifies the Palais-Smale condition.

Lemma 3.4. *The functional J_ε satisfies the $(PS)_d$ condition at any level $0<d<\Big(\frac{1}{2}-\frac{1}{\theta}-\frac{1}{2k}\Big)\min\{1,V_0\}$.*

Proof. Let $(u_n)\subset H_\varepsilon$ be a $(PS)_d$ for J_ε. By Lemma 3.2, (u_n) is bounded in H_ε and $\limsup_n |||u_n||| < 1$. Thus, up to a subsequence, $u_n \rightharpoonup u$ in H_ε and $u_n\to u$ in $L^q_{\rm loc}(\mathbb{R}^2,\mathbb{R})$ for all $q\ge1$ as $n\to+\infty$. Moreover, by (g_3) and (2.3), it follows that, fixed $q>2$, for any $\zeta>0$ and $\alpha>4\pi$, there exists a constant $C>0$, which depends on q, α, ζ, such that for every $\phi\in H_\varepsilon$,

$$\left|\mathrm{Re}\int_{\mathbb{R}^2} g(\varepsilon x,|u_n|^2)u_n\overline{\phi}dx\right| \le \zeta\int_{\mathbb{R}^2}|u_n||\overline{\phi}|dx + C\int_{\mathbb{R}^2}|\overline{\phi}||u_n|^{q-1}(e^{\alpha|u_n|^2}-1)dx.$$

Arguing as in Lemma 2.5, we have

$$\mathrm{Re}\int_{\mathbb{R}^2} g(\varepsilon x,|u_n|^2)u_n\overline{\phi}dx \to \mathrm{Re}\int_{\mathbb{R}^2} g(\varepsilon x,|u|^2)u\overline{\phi}dx.$$

Thus, u is a critical point of J_ε.
Let $R>0$ be such that $\Lambda_\varepsilon \subset B_{R/2}(0)$. We show that for any given $\zeta>0$, for R large enough,

$$\limsup_n \int_{B_R^c(0)} (|\nabla_{A_\varepsilon} u_n|^2 + V_\varepsilon(x)|u_n|^2)dx \le \zeta. \tag{3.7}$$

Let $\phi_R \in C^\infty(\mathbb{R}^2,\mathbb{R})$ be a cut-off function such that

$$\phi_R = 0 \quad x\in B_{R/2}(0), \quad \phi_R = 1 \quad x \in B_R^c(0), \quad 0\le \phi_R \le 1, \quad \text{and} \quad |\nabla \phi_R| \le C/R$$

where $C>0$ is a constant independent of R. Since the sequence $(\phi_R u_n)$ is bounded in H_ε, we have

$$J_\varepsilon'(u_n)(\phi_R u_n) = o_n(1),$$

that is

$$\operatorname{Re}\int_{\mathbb{R}^2} \nabla_{A_\varepsilon} u_n \overline{\nabla_{A_\varepsilon}(\phi_R u_n)}dx + \int_{\mathbb{R}^2} V_\varepsilon(x)|u_n|^2\phi_R dx = \int_{\mathbb{R}^2} g(\varepsilon x, |u_n|^2)|u_n|^2 \phi_R dx + o_n(1).$$

Since $\overline{\nabla_{A_\varepsilon}(u_n\phi_R)} = i\overline{u_n}\nabla\phi_R + \phi_R \overline{\nabla_{A_\varepsilon} u_n}$, using ($g_5$), we have

$$\begin{aligned}\int_{\mathbb{R}^2} (|\nabla_{A_\varepsilon} u_n|^2 + V_\varepsilon(x)|u_n|^2)\phi_R dx &= \int_{\mathbb{R}^2} g(\varepsilon x, |u_n|^2)|u_n|^2\phi_R dx - \operatorname{Re}\int_{\mathbb{R}^2} i\overline{u_n}\nabla_{A_\varepsilon} u_n \nabla\phi_R dx + o_n(1)\\ &\le \frac{1}{k}\int_{\mathbb{R}^2} V_\varepsilon(x)|u_n|^2\phi_R dx - \operatorname{Re}\int_{\mathbb{R}^2} i\overline{u_n}\nabla_{A_\varepsilon} u_n\nabla\phi_R dx + o_n(1).\end{aligned}$$

By the definition of ϕ_R, the Hölder inequality and the boundedness of (u_n) in H_ε, we obtain

$$\left(1-\frac{1}{k}\right)\int_{\mathbb{R}^2} (|\nabla_{A_\varepsilon} u_n|^2 + V_\varepsilon(x)|u_n|^2)\phi_R dx \le \frac{C}{R}\|u_n\|_2 \|\nabla_{A_\varepsilon} u_n\|_2 + o_n(1) \le \frac{C_1}{R} + o_n(1)$$

and so we can reach our claim.
Since $u_n \to u$ in $L^r_{\mathrm{loc}}(\mathbb{R}^2)$, for all $r\ge 1$, up to a subsequence, we have that

$$|u_n| \to |u| \text{ a.e. in } \mathbb{R}^2 \text{ as } n\to+\infty.$$

Then

$$g(\varepsilon x, |u_n|^2)|u_n|^2 \to g(\varepsilon x, |u|^2)|u|^2 \text{ a.e. in } \mathbb{R}^2 \text{ as } n\to+\infty.$$

Moreover, $|u_n|\to|u|$ in $L^r_{\mathrm{loc}}(\mathbb{R}^2)$ for all $r\ge1$.
Let

$$P(x,t) := g(\varepsilon x, t^2)t \quad \text{and} \quad Q(t) := e^{\alpha t^2} - 1, \quad t\in\mathbb{R},$$

where $\alpha > 4\pi$ with $\alpha |||u_n||| < 4\pi$ for n large. Using (g_3) and (f_2), it is easy to see that

$$\lim_{t\to+\infty} \frac{P(x,t)}{Q(t)} = 0 \quad \text{uniformly for } x\in\mathbb{R}^2$$

and, by Lemma 2.1,

$$\sup_n \int_{\mathbb{R}^2} Q(|u_n|)dx \le C.$$

Then [17, Theorem A.I] implies

$$\lim_n \int_{B_R(0)} \left| g(\varepsilon x, |u_n|^2)|u_n|^2 - g(\varepsilon x, |u|^2)|u|^2\right| dx = 0.$$

Moreover, by (g_5) and (3.7) we have

$$\limsup_n \int_{B_R^c(0)} \left| g(\varepsilon x, |u_n|^2)|u_n|^2 - g(\varepsilon x, |u|^2)|u|^2\right| dx \le \limsup_n \frac{2}{k}\int_{B_R^c(0)} (|\nabla_{A_\varepsilon} u_n|^2 + V(\varepsilon x)|u_n|^2)dx < \frac{2\zeta}{k}$$

for every $\zeta > 0$.
Hence

$$\int_{\mathbb{R}^2} g(\varepsilon x, |u_n|^2)|u_n|^2 dx \to \int_{\mathbb{R}^2} g(\varepsilon x, |u|^2)|u|^2 dx \text{ as } n \to +\infty.$$

Finally, since $J'_\varepsilon(u) = 0$, we have

$$o_n(1) = J'_\varepsilon(u_n)[u_n] = \|u_n\|_\varepsilon^2 - \int_{\mathbb{R}^2} g(\varepsilon x, |u_n|^2)|u_n|^2 dx = \|u_n\|_\varepsilon^2 - \|u\|_\varepsilon^2 + o_n(1).$$

Thus, the sequence (u_n) strong converges to u in H_ε. □

Since we would like to find multiple solutions of the functional J_ε, it is natural to consider it constrained to the Nehari manifold associated to our problem, that is

$$\mathcal{N}_\varepsilon := \{u \in H_\varepsilon \backslash \{0\} : J'_\varepsilon(u)[u] = 0\}.$$

In virtue of (g_6), it can be shown that for any $u \in H_\varepsilon \setminus \{0\}$, there exists a unique $t_\varepsilon > 0$ such that

$$\max_{t \geq 0} J_\varepsilon(tu) = J_\varepsilon(t_\varepsilon u)$$

and $t_\varepsilon u \in \mathcal{N}_\varepsilon$. Thus, c_ε can be characterized as follows

$$c_\varepsilon = \inf_{u \in H_\varepsilon \backslash \{0\}} \sup_{t \geq 0} J_\varepsilon(tu) = \inf_{u \in \mathcal{N}_\varepsilon} J_\varepsilon(u).$$

Moreover, arguing as in Lemma 2.3, we also have that there exists $\gamma > 0$, which is independent of $\varepsilon > 0$, such that

$$\|u\|_\varepsilon \geq \gamma > 0, \quad \text{for each } u \in \mathcal{N}_\varepsilon. \tag{3.8}$$

Now we show that $\mathcal{N}_\varepsilon$ is a natural constraint, namely that the constrained critical points of the functional J_ε on $\mathcal{N}_\varepsilon$ are the critical points of J_ε in H_ε. First we prove the following property.

Proposition 3.5. *The functional J_ε restricted to $\mathcal{N}_\varepsilon$ satisfies the $(PS)_d$ condition at any level* $0 < d < \left(\frac{1}{2} - \frac{1}{\theta} - \frac{1}{2k}\right) \min\{1, V_0\}$.

Proof. Let $(u_n) \subset \mathcal{N}_\varepsilon$ be a $(PS)_d$ sequence of J_ε restricted to $\mathcal{N}_\varepsilon$. Then, $J_\varepsilon(u_n) \to d$ as $n \to +\infty$ and there exists $(\lambda_n) \subset \mathbb{R}$ such that

$$J'_\varepsilon(u_n) = \lambda_n T'_\varepsilon(u_n) + o_n(1), \tag{3.9}$$

where $T_\varepsilon : H_\varepsilon \to \mathbb{R}$ is defined as

$$T_\varepsilon(u) := \|u\|_\varepsilon^2 - \int_{\mathbb{R}^2} g(\varepsilon x, |u|^2)|u|^2 dx.$$

Observe that, arguing as in Lemma 3.2, we get that (u_n) is bounded in H_ε and $\limsup_n |||u_n|||^2 < 1$. Note that, using the definition of g, the monotonicity of ϑ, and (f_4), we obtain

$$\begin{aligned} T'_\varepsilon(u_n)[u_n] &= -2\int_{\mathbb{R}^2} g'(\varepsilon x, |u_n|^2)|u_n|^4 dx \leq -2\int_{\Lambda_\varepsilon \cup \{|u_n|^2 < t_a\}} f'(|u_n|^2)|u_n|^4 dx \\ &\leq -(p-2)C_p \int_{\Lambda_\varepsilon \cup \{|u_n|^2 < t_a\}} |u_n|^p dx \leq -(p-2)C_p \int_{\Lambda_\varepsilon} |u_n|^p dx. \end{aligned}$$

Thus, up to a subsequence, we may assume that $T'_\varepsilon(u_n)[u_n] \to \varsigma \leq 0$.
Let us prove that $\varsigma \neq 0$. Indeed, if $\varsigma = 0$, then

$$o_n(1) = |T'_\varepsilon(u_n)[u_n]| \geq C \int_{\Lambda_\varepsilon} |u_n|^p dx.$$

Thus we obtain that $u_n \to 0$ in $L^p(\Lambda_\varepsilon, \mathbb{C})$, and by interpolation, we also have $u_n \to 0$ in $L^\tau(\Lambda_\varepsilon, \mathbb{C})$, for all $\tau \geq 1$. Moreover, arguing as in Lemma 3.2, we have that $|||u_n||| < 1$ for n large. Hence, from $J'_\varepsilon(u_n)[u_n] = 0$, ($g_3$), ($g_5$), (2.5), the Hölder inequality and Lemma 2.1, we conclude that

$$\begin{aligned}
\|u_n\|_\varepsilon^2 &= \int_{\mathbb{R}^2} g(\varepsilon x, |u_n|^2)|u_n|^2 dx \leq \int_{\Lambda_\varepsilon} f(|u_n|^2)|u_n|^2 dx + \frac{1}{k}\int_{\Lambda_\varepsilon^c} V(\varepsilon x)|u_n|^2 dx \\
&\leq \zeta \int_{\Lambda_\varepsilon} |u_n|^2 dx + C\int_{\Lambda_\varepsilon} |u_n|^q (e^{\alpha |u_n|^2} - 1) dx + \frac{1}{k}\int_{\Lambda_\varepsilon^c} V(\varepsilon x)|u_n|^2 dx \\
&= \frac{1}{k}\int_{\Lambda_\varepsilon^c} V(\varepsilon x)|u_n|^2 dx + o_n(1),
\end{aligned}$$

which implies that $u_n \to 0$ in H_ε. This is a contradiction with (3.8). Therefore, $\varsigma < 0$ and by (3.9) we deduce that $\lambda_n = o_n(1)$.

On the other hand, since, by the definition of g and (f_5), for every $\phi \in H_\varepsilon$ we have that

$$\begin{aligned}
\int_{\mathbb{R}^2} g'(\varepsilon x, |u_n|^2)|u_n|^3|\phi| dx &= \int_{\Lambda_\varepsilon} f'(|u_n|^2)|u_n|^3|\phi| dx + \int_{\Lambda_\varepsilon^c} \tilde{f}'(|u_n|^2)|u_n|^3|\phi| dx \\
&\leq \int_{\Lambda_\varepsilon} (e^{4\pi|u_n|^2} - 1)|u_n|^3|\phi| dx + \int_{\Lambda_\varepsilon^c \cap \{|u_n|^2 \leq T_a\}} \tilde{f}'(|u_n|^2)|u_n|^3|\phi| dx \\
&\leq \int_{\mathbb{R}^2} (e^{4\pi|u_n|^2} - 1)|u_n|^3|\phi| dx + C\int_{\mathbb{R}^2} |u_n|^3|\phi| dx,
\end{aligned}$$

using (g_3), (2.3), the fact that $\limsup_n |||u_n||| < 1$, the Hölder and Sobolev inequalities, for every $\phi \in H_\varepsilon$, we obtain

$$\begin{aligned}
|T'_\varepsilon(u_n)[\phi]| &\leq 2\|u_n\|_\varepsilon \|\phi\|_\varepsilon + 2\int_{\mathbb{R}^2} g(\varepsilon x, |u_n|^2)|u_n||\phi| dx + 2\int_{\mathbb{R}^2} g'(\varepsilon x, |u_n|^2)||u_n|^3|\phi| dx \\
&\leq C\Bigg[\|u_n\|_\varepsilon \|\phi\|_\varepsilon + \int_{\mathbb{R}^2} |u_n|^{q-1}(e^{\alpha|u_n|^2} - 1)|\phi| dx + \int_{\mathbb{R}^2} (e^{4\pi|u_n|^2} - 1)|u_n|^3|\phi| dx \\
&\qquad + \int_{\mathbb{R}^2} |u_n|^3|\phi| dx\Bigg] \\
&\leq C(\|u_n\|_\varepsilon + \|u_n\|_\varepsilon^{q-1} + \|u_n\|_\varepsilon^3)\|\phi\|_\varepsilon.
\end{aligned}$$

Then, the boundedness of (u_n) implies the boundedness of $T'_\varepsilon(u_n)$ and so, by (3.9), we can infer that $J'_\varepsilon(u_n) = o_n(1)$, that is (u_n) is a $(PS)_d$ sequence for J_ε. Hence, we apply Lemma 3.4 to conclude. $\square$

As a consequence we get

Corollary 3.6. *The constrained critical points of the functional J_ε on $\mathcal{N}_\varepsilon$ are critical points of J_ε in H_ε.*

4. Multiple solutions for the modified problem

In this section, we prove a multiplicity result for the modified problem (3.2) using the Ljusternik-Schnirelmann category theory. In order to get it, we first provide some useful preliminaries.

Let $\delta > 0$ be such that $M_\delta \subset \Lambda$, $\omega \in H^1(\mathbb{R}^2, \mathbb{R})$ be a positive ground state solution of the limit problem (2.7), and $\eta \in C^\infty(\mathbb{R}^+, [0, 1])$ be a nonincreasing cut-off function defined in $[0, +\infty)$ such that $\eta(t) = 1$ if $0 \leq t \leq \delta/2$ and $\eta(t) = 0$ if $t \geq \delta$.

For any $y \in M$, let us introduce the function

$$\Psi_{\varepsilon,y}(x) := \eta(|\varepsilon x - y|)\omega\Big(\frac{\varepsilon x - y}{\varepsilon}\Big)\exp\Big(i\tau_y\Big(\frac{\varepsilon x - y}{\varepsilon}\Big)\Big),$$

where

$$\tau_y(x) := A_1(y)x_1 + A_2(y)x_2.$$

Let $t_\varepsilon > 0$ be the unique positive number such that

$$\max_{t\geq 0} J_\varepsilon(t\Psi_{\varepsilon,y}) = J_\varepsilon(t_\varepsilon \Psi_{\varepsilon,y}).$$

Note that $t_\varepsilon \Psi_{\varepsilon,y} \in \mathcal{N}_\varepsilon$.
Let us define $\Phi_\varepsilon : M \to \mathcal{N}_\varepsilon$ as

$$\Phi_\varepsilon(y) := t_\varepsilon \Psi_{\varepsilon,y}.$$

By construction, $\Phi_\varepsilon(y)$ has compact support for any $y \in M$.
Moreover, the energy of the above functions has the following behavior as $\varepsilon \to 0^+$.

Lemma 4.1. *The limit*

$$\lim_{\varepsilon\to 0^+} J_\varepsilon(\Phi_\varepsilon(y)) = c_{V_0}$$

holds uniformly in $y \in M$.

Proof. Assume by contradiction that the statement is false. Then there exist $\delta_0 > 0$, $(y_n) \subset M$ and $\varepsilon_n \to 0^+$ satisfying

$$\Big| J_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) - c_{V_0} \Big| \geq \delta_0.$$

For simplicity, we write Φ_n, Ψ_n and t_n for $\Phi_{\varepsilon_n}(y_n)$, Ψ_{ε_n,y_n} and t_{ε_n}, respectively.
We can check that

$$\|\Psi_n\|^2_{\varepsilon_n} \to \int_{\mathbb{R}^2} (|\nabla\omega|^2 + V_0\omega^2)dx \text{ as } n \to +\infty. \tag{4.1}$$

Indeed, by a change of variable of $z = (\varepsilon_n x - y_n)/\varepsilon_n$, the Lebesgue Dominated Convergence Theorem, the continuity of V and $y_n \in M \subset \Lambda$(which is bounded), we deduce that

$$\int_{\mathbb{R}^2} V(\varepsilon_n x)|\Psi_n|^2 dx = \int_{\mathbb{R}^2} V(\varepsilon_n z + y_n)|\eta(|\varepsilon_n z|)\omega(z)|^2 dx \to V_0 \int_{\mathbb{R}^2} \omega^2 dx \text{ as } n \to +\infty.$$

Moreover, by the same change of variable $z = (\varepsilon_n x - y_n)/\varepsilon_n$, we also have

$$\begin{aligned}
\int_{\mathbb{R}^2} |\nabla_{A_{\varepsilon_n}} \Psi_n|^2 dx = {} & \varepsilon_n^2 \int_{\mathbb{R}^2} |\eta'(|\varepsilon_n z|)\omega(z)|^2 dz + \int_{\mathbb{R}^2} |\eta(|\varepsilon_n z|)\nabla\omega(z)|^2 dz \\
& + \int_{\mathbb{R}^2} \Big|\eta(|\varepsilon_n z|)\Big(A(y_n) - A(\varepsilon_n z + y_n)\Big)\omega(z)\Big|^2 dz \\
& + 2\varepsilon_n \int_{\mathbb{R}^2} \eta(|\varepsilon_n z|)\eta'(|\varepsilon_n z|)\omega(z)\nabla\omega(z) \cdot \frac{z}{|z|} dz.
\end{aligned}$$

It is clear that

$$\lim_n \int_{\mathbb{R}^2} |\eta(|\varepsilon_n z|)\nabla\omega(z)|^2 dz = \int_{\mathbb{R}^2} |\nabla\omega(z)|^2 dz.$$

Moreover, using the definition of η, the Hölder continuity with exponent $\alpha \in (0,1]$ of A, the exponential decay of ω, and the Lebesgue Dominated Convergence Theorem, we can infer

$$\int_{\mathbb{R}^2} |\eta'(|\varepsilon_n z|)\omega(z)|^2 dz = o_n(1),$$

$$\int_{\mathbb{R}^2} |\eta(|\varepsilon_n z|)\eta'(|\varepsilon_n z|)\omega(z)\nabla\omega(z)| dz = o_n(1),$$

and

$$\int_{\mathbb{R}^2} \Big|\eta(|\varepsilon_n z|)\Big(A(y_n) - A(\varepsilon_n z + y_n)\Big)\omega(z)\Big|^2 dz \leq C\varepsilon_n^{2\alpha} \int_{|\varepsilon_n z|\leq\delta} \omega^2(z)|z|^{2\alpha} dz = o_n(1),$$

obtaining (4.1).
On the other hand, since $J'_{\varepsilon_n}(t_n\Psi_n)(t_n\Psi_n) = 0$, by the change of variables $z = (\varepsilon_n x - y_n)/\varepsilon_n$, observe that, if $z \in B_{\delta/\varepsilon_n}(0)$, then $\varepsilon_n z + y_n \in B_\delta(y_n) \subset M_\delta \subset \Lambda$, we have

$$\begin{aligned}
\|\Psi_n\|^2_{\varepsilon_n} &= \int_{\mathbb{R}^2} g(\varepsilon_n z + y_n, t_n^2\eta^2(|\varepsilon_n z|)\omega^2(z))\eta^2(|\varepsilon_n z|)\omega^2(z)dz \\
&= \int_{\mathbb{R}^2} f(t_n^2\eta^2(|\varepsilon_n z|)\omega^2(z))\eta^2(|\varepsilon_n z|)\omega^2(z)dz \\
&\geq \int_{B_{\delta/(2\varepsilon_n)}(0)} f(t_n^2\omega^2(z))\omega^2(z)dz \\
&\geq \int_{B_{\delta/2}(0)} f(t_n^2\omega^2(z))\omega^2(z)dz \\
&\geq f(t_n^2\gamma^2)\int_{B_{\delta/2}(0)} \omega^2(z)dz
\end{aligned}$$

for all n large enough and where $\gamma = \min\{\omega(z) : |z| \leq \delta/2\}$.
If $t_n \to +\infty$, by (f_4) we deduce that $\|\Psi_n\|^2_{\varepsilon_n} \to +\infty$ which contradicts (4.1).
Therefore, up to a subsequence, we may assume that $t_n \to t_0 \geq 0$.
If $t_n \to 0$, using the fact that f is increasing and the Lebesgue Dominated Convergence Theorem, we obtain that

$$\|\Psi_n\|^2_{\varepsilon_n} = \int_{\mathbb{R}^2} f(t_n^2\eta^2(|\varepsilon_n z|)\omega^2(z))\eta^2(|\varepsilon_n z|)\omega^2(z)dz \to 0, \text{ as } n \to +\infty,$$

which contradicts (4.1). Thus, we have $t_0 > 0$ and

$$\int_{\mathbb{R}^2}(|\nabla\omega|^2 + V_0\omega^2)dx = \int_{\mathbb{R}^2} f(t_0\omega^2)\omega^2 dx,$$

so that $t_0\omega \in \mathcal{N}_{V_0}$. Since $\omega \in \mathcal{N}_{V_0}$, we obtain that $t_0 = 1$ and so, using the Lebesgue Dominated Convergence Theorem, we get

$$\lim_n \int_{\mathbb{R}^2} F(|t_n\Psi_n|^2)dx = \int_{\mathbb{R}^2} F(\omega^2)dx.$$

Hence

$$\lim_n J_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) = I_{V_0}(\omega) = c_{V_0}$$

which is a contradiction and conclude. □

Now we define the barycenter map.
Let $\rho > 0$ be such that $M_\delta \subset B_\rho$ and consider $\Upsilon : \mathbb{R}^2 \to \mathbb{R}^2$ defined by setting

$$\Upsilon(x) := \begin{cases} x, & \text{if } |x| < \rho, \\ \rho x/|x|, & \text{if } |x| \geq \rho. \end{cases}$$

The barycenter map $\beta_\varepsilon : \mathcal{N}_\varepsilon \to \mathbb{R}^2$ is defined by

$$\beta_\varepsilon(u) := \frac{1}{\|u\|_2^2}\int_{\mathbb{R}^2} \Upsilon(\varepsilon x)|u(x)|^2 dx.$$

We have

Lemma 4.2. *The limit*

$$\lim_{\varepsilon\to 0^+} \beta_\varepsilon(\Phi_\varepsilon(y)) = y$$

holds uniformly in $y \in M$.

Proof. Assume by contradiction that there exists $\kappa > 0$, $(y_n) \subset M$ and $\varepsilon_n \to 0$ such that

$$|\beta_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) - y_n| \geq \kappa. \tag{4.2}$$

Using the change of variable $z = (\varepsilon_n x - y_n)/\varepsilon_n$, we can see that

$$\beta_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) = y_n + \frac{\displaystyle\int_{\mathbb{R}^2} (\Upsilon(\varepsilon_n z + y_n) - y_n)\eta^2(|\varepsilon_n z|)\omega^2(z)dz}{\displaystyle\int_{\mathbb{R}^2} \eta^2(|\varepsilon_n z|)\omega^2(z)dz}.$$

Taking into account $(y_n) \subset M \subset M_\delta \subset B_\rho$ and the Lebesgue Dominated Convergence Theorem, we can obtain that

$$|\beta_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) - y_n| = o_n(1),$$

which contradicts (4.2). □

Now, we prove the following useful compactness result.

Proposition 4.3. *Let $\varepsilon_n \to 0^+$ and $(u_n) \subset \mathcal{N}_{\varepsilon_n}$ be such that $J_{\varepsilon_n}(u_n) \to c_{V_0}$. Then there exists $(\tilde{y}_n) \subset \mathbb{R}^2$ such that the sequence $(|v_n|) \subset H^1(\mathbb{R}^2, \mathbb{R})$, where $v_n(x) := u_n(x + \tilde{y}_n)$, has a convergent subsequence in $H^1(\mathbb{R}^2, \mathbb{R})$. Moreover, up to a subsequence, $y_n := \varepsilon_n \tilde{y}_n \to y \in M$ as $n \to +\infty$.*

Proof. Since $J'_{\varepsilon_n}(u_n)[u_n] = 0$ and $J_{\varepsilon_n}(u_n) \to c_{V_0}$, arguing as in the proof of Lemma 3.2, using Lemma 2.4, we can prove that there exists $C > 0$ such that $\|u_n\|_{\varepsilon_n} \leq C$ for all $n \in \mathbb{N}$ and $\limsup_n \||u_n|\| < 1$. Arguing as in the proof of Lemma 3.3 and recalling that $c_{V_0} > 0$, we have that there exist a sequence $(\tilde{y}_n) \subset \mathbb{R}^2$ and constants R, $\beta > 0$ such that

$$\liminf_n \int_{B_R(\tilde{y}_n)} |u_n|^2 dx \geq \beta. \tag{4.3}$$

Now, let us consider the sequence $(|v_n|) \subset H^1(\mathbb{R}^2, \mathbb{R})$, where $v_n(x) := u_n(x + \tilde{y}_n)$.
By the diamagnetic inequality (2.1), we get that $(|v_n|)$ is bounded in $H^1(\mathbb{R}^2, \mathbb{R})$, and using (4.3), we may assume that $|v_n| \rightharpoonup v$ in $H^1(\mathbb{R}^2, \mathbb{R})$ for some $v \neq 0$.
Let now $t_n > 0$ be such that $\tilde{v}_n := t_n|v_n| \in \mathcal{N}_{V_0}$, and set $y_n := \varepsilon_n \tilde{y}_n$.
By the diamagnetic inequality (2.1), we have

$$c_{V_0} \leq I_{V_0}(\tilde{v}_n) \leq \max_{t \geq 0} J_{\varepsilon_n}(tu_n) = J_{\varepsilon_n}(u_n) = c_{V_0} + o_n(1),$$

which yields $I_{V_0}(\tilde{v}_n) \to c_{V_0}$ as $n \to +\infty$.
Since the sequences $(|v_n|)$ and $(\tilde{v}_n)$ are bounded in $H^1(\mathbb{R}^2, \mathbb{R})$ and $|v_n| \not\to 0$ in $H^1(\mathbb{R}^2, \mathbb{R})$, then (t_n) is also bounded and so, up to a subsequence, we may assume that $t_n \to t_0 \geq 0$.
We claim that $t_0 > 0$. Indeed, if $t_0 = 0$, then, since $(|v_n|)$ is bounded, we have $\tilde{v}_n \to 0$ in $H^1(\mathbb{R}^2, \mathbb{R})$, that is $I_{V_0}(\tilde{v}_n) \to 0$, which contradicts $c_{V_0} > 0$.
Thus, up to a subsequence, we may assume that $\tilde{v}_n \rightharpoonup \tilde{v} := t_0 v \neq 0$ in $H^1(\mathbb{R}^2, \mathbb{R})$, and, by Lemma 2.5, we can deduce that $\tilde{v}_n \to \tilde{v}$ in $H^1(\mathbb{R}^2, \mathbb{R})$, which gives $|v_n| \to v$ in $H^1(\mathbb{R}^2, \mathbb{R})$.
Now we show the final part, namely that (y_n) has a subsequence such that $y_n \to y \in M$.
Assume by contradiction that (y_n) is not bounded and so, up to a subsequence, $|y_n| \to +\infty$ as $n \to +\infty$. Choose $R > 0$ such that $\Lambda \subset B_R(0)$. Then for n large enough, we have $|y_n| > 2R$, and, for any $x \in B_{R/\varepsilon_n}(0)$,

$$|\varepsilon_n x + y_n| \geq |y_n| - \varepsilon_n |x| > R.$$

Since $u_n \in \mathcal{N}_{\varepsilon_n}$, using ($V_1$) and the diamagnetic inequality (2.1), we get that

$$\begin{aligned}\int_{\mathbb{R}^2}(|\nabla|v_n||^2 + V_0|v_n|^2)dx &\leq \int_{\mathbb{R}^2} g(\varepsilon_n x + y_n, |v_n|^2)|v_n|^2 dx \\ &\leq \int_{B_{R/\varepsilon_n}(0)} \tilde{f}(|v_n|^2)|v_n|^2 dx + \int_{B^c_{R/\varepsilon_n}(0)} f(|v_n|^2)|v_n|^2 dx.\end{aligned} \tag{4.4}$$

Since $|v_n| \to v$ in $H^1(\mathbb{R}^2, \mathbb{R})$ and $\tilde{f}(t) \leq V_0/k$, we can see that (4.4) yields

$$\min\Big\{1, V_0\Big(1 - \frac{1}{k}\Big)\Big\} \int_{\mathbb{R}^2}(|\nabla|v_n||^2 + |v_n|^2)dx = o_n(1),$$

that is $|v_n| \to 0$ in $H^1(\mathbb{R}^2, \mathbb{R})$, which contradicts to $v \not\equiv 0$.
Therefore, we may assume that $y_n \to y_0 \in \mathbb{R}^2$.
Assume by contradiction that $y_0 \notin \overline{\Lambda}$. Then there exists $r > 0$ such that for every n large enough we have that $|y_n - y_0| < r$ and $B_{2r}(y_0) \subset \overline{\Lambda}^c$. Then, if $x \in B_{r/\varepsilon_n}(0)$, we have that $|\varepsilon_n x + y_n - y_0| < 2r$ so that $\varepsilon_n x + y_n \in \overline{\Lambda}^c$ and so, arguing as before, we reach a contradiction.
Thus, $y_0 \in \overline{\Lambda}$.
To prove that $V(y_0) = V_0$, we suppose by contradiction that $V(y_0) > V_0$. Using the Fatou's lemma, the change of variable $z = x + \tilde{y}_n$ and $\max_{t\geq 0} J_{\varepsilon_n}(tu_n) = J_{\varepsilon_n}(u_n)$, we obtain

$$\begin{aligned}c_{V_0} = I_{V_0}(\tilde{v}) &< \frac{1}{2}\int_{\mathbb{R}^2}(|\nabla \tilde{v}|^2 + V(y_0)|\tilde{v}|^2)dx - \frac{1}{2}\int_{\mathbb{R}^2} F(|\tilde{v}|^2)dx \\ &\leq \liminf_n \Big(\frac{1}{2}\int_{\mathbb{R}^2}(|\nabla \tilde{v}_n|^2 + V(\varepsilon_n x + y_n)|\tilde{v}_n|^2)dx - \frac{1}{2}\int_{\mathbb{R}^2} F(|\tilde{v}_n|^2)dx\Big) \\ &= \liminf_n \Big(\frac{t_n^2}{2}\int_{\mathbb{R}^2}(|\nabla|u_n||^2 + V(\varepsilon_n z)|u_n|^2)dz - \frac{1}{2}\int_{\mathbb{R}^2} F(|t_n u_n|^2)dz\Big) \\ &\leq \liminf_n J_{\varepsilon_n}(t_n u_n) \leq \liminf_n J_{\varepsilon_n}(u_n) = c_{V_0}\end{aligned}$$

which is impossible and we conclude. □

Let now

$$\tilde{\mathcal{N}}_\varepsilon := \{u \in \mathcal{N}_\varepsilon : J_\varepsilon(u) \leq c_{V_0} + h(\varepsilon)\},$$

where $h : \mathbb{R}^+ \to \mathbb{R}^+$, $h(\varepsilon) \to 0$ as $\varepsilon \to 0^+$.
Fixed $y \in M$, since, by Lemma 4.1, $|J_\varepsilon(\Phi_\varepsilon(y)) - c_{V_0}| \to 0$ as $\varepsilon \to 0^+$, we get that $\tilde{\mathcal{N}}_\varepsilon \neq \emptyset$ for any $\varepsilon > 0$ small enough.

We have the following relation between $\tilde{\mathcal{N}}_\varepsilon$ and the barycenter map.

Lemma 4.4. *We have*

$$\lim_{\varepsilon\to 0^+} \sup_{u\in\tilde{\mathcal{N}}_\varepsilon} \operatorname{dist}(\beta_\varepsilon(u), M_\delta) = 0.$$

Proof. Let $\varepsilon_n \to 0^+$ as $n \to +\infty$. For any $n \in \mathbb{N}$, there exists $u_n \in \tilde{\mathcal{N}}_{\varepsilon_n}$ such that

$$\sup_{u\in\tilde{\mathcal{N}}_{\varepsilon_n}} \inf_{y\in M_\delta} |\beta_{\varepsilon_n}(u) - y| = \inf_{y\in M_\delta} |\beta_{\varepsilon_n}(u_n) - y| + o_n(1).$$

Therefore, it is enough to prove that there exists $(y_n) \subset M_\delta$ such that

$$\lim_n |\beta_{\varepsilon_n}(u_n) - y_n| = 0.$$

By the diamagnetic inequality (2.1), we can see that $I_{V_0}(t|u_n|) \leq J_{\varepsilon_n}(tu_n)$ for any $t \geq 0$. Therefore, recalling that $(u_n) \subset \tilde{\mathcal{N}}_{\varepsilon_n} \subset \mathcal{N}_{\varepsilon_n}$, we can deduce that

$$c_{V_0} \leq \max_{t\geq 0} I_{V_0}(t|u_n|) \leq \max_{t\geq 0} J_{\varepsilon_n}(tu_n) = J_{\varepsilon_n}(u_n) \leq c_{V_0} + h(\varepsilon_n) \tag{4.5}$$

which implies that $J_{\varepsilon_n}(u_n) \to c_{V_0}$ as $n \to +\infty$.
Then, Proposition 4.3 implies that there exists $(\tilde{y}_n) \subset \mathbb{R}^2$ such that $y_n = \varepsilon_n \tilde{y}_n \in M_\delta$ for n large enough. Thus, making the change of variable $z = x - \tilde{y}_n$, we get

$$\beta_{\varepsilon_n}(u_n) = y_n + \frac{\int_{\mathbb{R}^2} (\Upsilon(\varepsilon_n z + y_n) - y_n) |u_n(z + \tilde{y}_n)|^2 dz}{\int_{\mathbb{R}^2} |u_n(z + \tilde{y}_n)|^2 dz}.$$

Since, up to a subsequence, $|u_n|(\cdot + \tilde{y}_n)$ converges strongly in $H^1(\mathbb{R}^2, \mathbb{R})$ and $\varepsilon_n z + y_n \to y \in M$ for any $z \in \mathbb{R}^2$, we conclude. □

Finally, we present a relation between the topology of M and the number of solutions of the modified problem (3.2).

Theorem 4.5. *For any $\delta > 0$ such that $M_\delta \subset \Lambda$, there exists $\tilde{\varepsilon}_\delta > 0$ such that, for any $\varepsilon \in (0, \tilde{\varepsilon}_\delta)$, problem (3.2) has at least $cat_{M_\delta}(M)$ nontrivial solutions.*

Proof. Given $\delta > 0$, by Lemma 4.1, Lemma 4.2, and Lemma 4.4, and arguing as in [21, Section 6], we can find $\tilde{\varepsilon}_\delta > 0$ such that for any $\varepsilon \in (0, \tilde{\varepsilon}_\delta)$, the following diagram

$$M \xrightarrow{\Phi_\varepsilon} \tilde{\mathcal{N}}_\varepsilon \xrightarrow{\beta_\varepsilon} M_\delta$$

is well defined and $\beta_\varepsilon \circ \Phi_\varepsilon$ is homotopically equivalent to the embedding $\iota : M \to M_\delta$. Thus, [16, Lemma 4.3] (see also [22, Lemma 2.2]) implies that

$$\mathrm{cat}_{\tilde{\mathcal{N}}_\varepsilon}(\tilde{\mathcal{N}}_\varepsilon) \geq \mathrm{cat}_{M_\delta}(M).$$

By Proposition 3.5, we have also that J_ε satisfies the Palais-Smale condition on $\tilde{\mathcal{N}}_\varepsilon$ (taking $\tilde{\varepsilon}_\delta$ smaller if necessary). Hence, by the Ljusternik-Schnirelmann theory for C^1 functionals (see [38, Theorem 5.20]), we get at least $\mathrm{cat}_{M_\delta}(M)$ critical points of J_ε restricted to $\mathcal{N}_\varepsilon$ which are, by Corollary 3.6, critical points for J_ε in $\tilde{\mathcal{N}}_\varepsilon$. □

5. Proof of Theorem 1.1

In this section we prove our main result. The idea is to show that the solutions u_ε obtained in Theorem 4.5 satisfy

$$|u_\varepsilon(x)|^2 \leq t_a \text{ for } x \in \Lambda_\varepsilon^c$$

for ε small. The key ingredient is the following result.

Lemma 5.1. *Let $\varepsilon_n \to 0^+$ and $u_n \in \tilde{\mathcal{N}}_{\varepsilon_n}$ be a solution of problem (3.2) for $\varepsilon = \varepsilon_n$. Then $J_{\varepsilon_n}(u_n) \to c_{V_0}$. Moreover, there exists $(\tilde{y}_n) \subset \mathbb{R}^2$ such that, if $v_n(x) := u_n(x + \tilde{y}_n)$, we have that $(|v_n|)$ is bounded in $L^\infty(\mathbb{R}^2, \mathbb{R})$ and*

$$\lim_{|x| \to +\infty} |v_n(x)| = 0 \quad \textit{uniformly in } n \in \mathbb{N}.$$

Proof. Since $J_{\varepsilon_n}(u_n) \leq c_{V_0} + h(\varepsilon_n)$ with $\lim_n h(\varepsilon_n) = 0$, we can argue as in the proof of Lemma 4.4 (see (4.5)) to conclude that $J_{\varepsilon_n}(u_n) \to c_{V_0}$.
Thus, by Proposition 4.3, we obtain the existence of a sequence $(\tilde{y}_n) \subset \mathbb{R}^2$ such that $(|v_n|) \subset H^1(\mathbb{R}^2, \mathbb{R})$, where $v_n(x) := u_n(x + \tilde{y}_n)$, has a convergent subsequence in $H^1(\mathbb{R}^2, \mathbb{R})$. Moreover, up to a subsequence, $y_n := \varepsilon_n \tilde{y}_n \to y \in M$ as $n \to +\infty$.
For any $R > 0$ and $0 < r \leq R/2$, let $\eta \in C^\infty(\mathbb{R}^2)$, $0 \leq \eta \leq 1$ with $\eta(x) = 1$ if $|x| \geq R$ and $\eta(x) = 0$ if $|x| \leq R - r$ and $|\nabla \eta| \leq 2/r$.
For each $n \in \mathbb{N}$ and $L > 0$, we consider the functions

$$v_{L,n}(x) := \begin{cases} |v_n(x)| & \text{if } |v_n(x)| \leq L, \\ L & \text{if } |v_n(x)| > L, \end{cases} \qquad z_{L,n} := \eta^2 v_{L,n}^{2(\beta-1)} v_n, \quad \text{and} \quad w_{L,n} := \eta v_{L,n}^{\beta-1} |v_n|,$$

where $\beta > 1$ will be determined later.

Since, by the diamagnetic inequality (2.1) we have that

$$\begin{aligned}\mathrm{Re}(\nabla_{A_{\varepsilon_n}(\cdot+\tilde{y}_n)}v_n \cdot \overline{\nabla_{A_{\varepsilon_n}(\cdot+\tilde{y}_n)}z_{L,n}}) &= \eta^2 v_{L,n}^{2(\beta-1)}|\nabla_{A_{\varepsilon_n}(\cdot+\tilde{y}_n)}v_n|^2 + \mathrm{Re}(\nabla v_n\overline{v_n})\nabla(\eta^2 v_{L,n}^{2(\beta-1)})\\ &= \eta^2 v_{L,n}^{2(\beta-1)}|\nabla_{A_{\varepsilon_n}(\cdot+\tilde{y}_n)}v_n|^2 + |v_n|\nabla|v_n|\nabla(\eta^2 v_{L,n}^{2(\beta-1)})\\ &\geq \eta^2 v_{L,n}^{2(\beta-1)}|\nabla|v_n||^2 + 2\eta\nabla\eta v_{L,n}^{2(\beta-1)}|v_n|\nabla|v_n|,\end{aligned}$$

using also the fact that u_n is a solution of problem (3.2) for $\varepsilon = \varepsilon_n$, the Young inequality (with $\tau > 0$), (g_3), (2.5), for $\alpha > 4\pi$ and for a fixed $q > 2$, given $0 < \zeta < V_0$, there exists $C > 0$ such that

$$\begin{aligned}\int_{\mathbb{R}^2}|\nabla|v_n||^2\eta^2 v_{L,n}^{2(\beta-1)}dx &\leq \int_{\mathbb{R}^2}|\nabla|v_n||^2\eta^2 v_{L,n}^{2(\beta-1)}dx + 2\int_{\mathbb{R}^2}\eta\nabla\eta v_{L,n}^{2(\beta-1)}|v_n|\nabla|v_n|dx\\ &\quad+\int_{\mathbb{R}^2}V(\varepsilon_n x+\varepsilon_n\tilde{y}_n)\eta^2 v_{L,n}^{2(\beta-1)}|v_n|^2dx\\ &\quad+2\int_{\mathbb{R}^2}\eta|\nabla\eta|v_{L,n}^{2(\beta-1)}|v_n||\nabla|v_n|| - \zeta\int_{\mathbb{R}^2}\eta^2 v_{L,n}^{2(\beta-1)}|v_n|^2dx\\ &\leq \mathrm{Re}\int_{\mathbb{R}^2}(\nabla_{A_{\varepsilon_n}(\cdot+\tilde{y}_n)}v_n\cdot\overline{\nabla_{A_{\varepsilon_n}(\cdot+\tilde{y}_n)}z_{L,n}})dx\\ &\quad+\mathrm{Re}\int_{\mathbb{R}^2}V(\varepsilon_n x+\varepsilon_n\tilde{y}_n)v_n\overline{z_{L,n}}dx\\ &\quad+\tau\int_{\mathbb{R}^2}|\nabla|v_n||^2\eta^2 v_{L,n}^{2(\beta-1)}dx + \frac{1}{\tau}\int_{\mathbb{R}^2}|\nabla\eta|^2 v_{L,n}^{2(\beta-1)}|v_n|^2dx\\ &\quad-\zeta\int_{\mathbb{R}^2}\eta^2 v_{L,n}^{2(\beta-1)}|v_n|^2dx\\ &= \int_{\mathbb{R}^2}g(\varepsilon_n x+\varepsilon_n\tilde{y}_n,|v_n|^2)\eta^2 v_{L,n}^{2(\beta-1)}|v_n|^2dx\\ &\quad+\tau\int_{\mathbb{R}^2}|\nabla|v_n||^2\eta^2 v_{L,n}^{2(\beta-1)}dx + \frac{4}{\tau r^2}\int_{R-r\leq|x|\leq R}v_{L,n}^{2(\beta-1)}|v_n|^2dx\\ &\quad-\zeta\int_{\mathbb{R}^2}\eta^2 v_{L,n}^{2(\beta-1)}|v_n|^2dx\\ &\leq C\int_{\mathbb{R}^2}|v_n|^q(e^{\alpha|v_n|^2}-1)\eta^2 v_{L,n}^{2(\beta-1)}dx\\ &\quad+\tau\int_{\mathbb{R}^2}|\nabla|v_n||^2\eta^2 v_{L,n}^{2(\beta-1)}dx + \frac{4}{\tau r^2}\int_{R-r\leq|x|\leq R}|v_n|^{2\beta}dx.\end{aligned} \tag{5.1}$$

Hence, choosing $\tau > 0$ sufficiently small, we get

$$\int_{\mathbb{R}^2}|\nabla|v_n||^2\eta^2 v_{L,n}^{2(\beta-1)} \leq C\Big[\int_{|x|\geq R-r}|v_n|^{q+2(\beta-1)}(e^{\alpha|v_n|^2}-1)dx + \frac{1}{r^2}\int_{R-r\leq|x|\leq R}|v_n|^{2\beta}dx\Big]. \tag{5.2}$$

Moreover, arguing similarly to (5.1), we can conclude that

$$\int_{\mathbb{R}^2}\eta^2 v_{L,n}^{2(\beta-1)}|v_n|^2dx \leq C\Big[\int_{|x|\geq R-r}|v_n|^{q+2(\beta-1)}(e^{\alpha|v_n|^2}-1)dx + \frac{1}{r^2}\int_{R-r\leq|x|\leq R}|v_n|^{2\beta}dx\Big]. \tag{5.3}$$

On the other hand, using the Sobolev embedding, (5.2), (5.3), the Hölder inequality with $t, \sigma, \tau > 1$, $1/\sigma + 1/\tau = 1/t$, $\sigma(q-2) \geq 2$, and (2.13), we have

$$
\begin{aligned}
\|w_{L,n}\|_q^2 &\leq C \int_{\mathbb{R}^2} (|\nabla w_{L,n}|^2 + |w_{L,n}|^2) dx \\
&\leq C \Big(\int_{\mathbb{R}^2} |\nabla \eta|^2 |v_n|^{2\beta} dx + \beta^2 \int_{\mathbb{R}^2} \eta^2 v_{L,n}^{2(\beta-1)} |\nabla |v_n||^2 dx + \int_{\mathbb{R}^2} \eta^2 v_{L,n}^{2(\beta-1)} |v_n|^2 dx \Big) \\
&\leq C\beta^2 \Big(\frac{1}{r^2} \int_{R-r \leq |x| \leq R} |v_n|^{2\beta} dx + \int_{|x| \geq R-r} |v_n|^{q+2(\beta-1)} (e^{\alpha |v_n|^2} - 1) dx \Big) \\
&\leq C\beta^2 \left[\frac{R^{2/t}}{r^2} + \Big(\int_{|x| \geq R-r} |v_n|^{\sigma(q-2)} dx \Big)^{1/\sigma} \Big(\int_{\mathbb{R}^2} (e^{\tau\alpha |v_n|^2} - 1) dx \Big)^{1/\tau} \right] \\
&\qquad \Big(\int_{|x| \geq R-r} |v_n|^{2\beta t/(t-1)} dx \Big)^{(t-1)/t}.
\end{aligned} \tag{5.4}
$$

Since $(|v_n|)$ is convergent in $H^1(\mathbb{R}^2, \mathbb{R})$, there exists $h \in H^1(\mathbb{R}^2, \mathbb{R})$ such that, for all $n \in \mathbb{N}$, $|v_n(x)| \leq h(x)$ a.e. in $\mathbb{R}^2$. So, using Lemma 2.1, for all $\tau > 1$ and $\alpha > 4\pi$, we know that

$$
\int_{\mathbb{R}^2} (e^{\tau\alpha |v_n|^2} - 1) dx \leq \int_{\mathbb{R}^2} (e^{\tau\alpha h^2} - 1) dx < +\infty. \tag{5.5}
$$

By (5.4) and (5.5), it follows that

$$
\Big(\int_{|x| \geq R} v_{L,n}^{q\beta} dx \Big)^{2/q} \leq \|w_{L,n}\|_q^2 \leq C\beta^2 \Big(1 + \frac{R^{2/t}}{r^2} \Big) \Big(\int_{|x| \geq R-r} |v_n|^{2\beta t/(t-1)} \Big)^{(t-1)/t}
$$

and, applying the Fatou's Lemma as $L \to +\infty$, we obtain

$$
\Big(\int_{|x| \geq R} |v_n|^{q\beta} dx \Big)^{2/q} \leq C\beta^2 \Big(1 + \frac{R^{2/t}}{r^2} \Big) \Big(\int_{|x| \geq R-r} |v_n|^{2\beta t/(t-1)} \Big)^{(t-1)/t}.
$$

Arguing as in [31], if we take $\zeta := \frac{q(t-1)}{2t}$, $\beta := \zeta^m$, with $m \in \mathbb{N}^*$, and $s := \frac{2t}{t-1}$, we obtain

$$
\Big(\int_{|x| \geq R} |v_n|^{s\zeta^{m+1}} dx \Big)^{1/(s\zeta^{m+1})} \leq C^{\zeta^{-m}} \zeta^{m\zeta^{-m}} \Big(1 + \frac{R^{2/t}}{r^2} \Big)^{1/(2\zeta^m)} \Big(\int_{|x| \geq R-r} |v_n|^{s\zeta^m} \Big)^{1/(s\zeta^m)}
$$

for every $m \in \mathbb{N}^*$. Then, for $r = r_m := R/2^m$, $m \in \mathbb{N}^*$, using also that $2/t < 2$, we get

$$
\begin{aligned}
\Big(\int_{|x| \geq R} |v_n|^{s\zeta^{m+1}} dx \Big)^{1/(s\zeta^{m+1})} &\leq \Big(\int_{|x| \geq R - r_{m+1}} |v_n|^{s\zeta^{m+1}} dx \Big)^{1/(s\zeta^{m+1})} \\
&\leq C^{\sum_{i=1}^m \zeta^{-i}} \zeta^{\sum_{i=1}^m i\zeta^{-i}} \exp\Big(\sum_{i=1}^m \frac{\ln(1 + 2^{2(i+1)})}{2\zeta^i} \Big) \Big(\int_{|x| \geq R/2} |v_n|^{s\zeta} dx \Big)^{1/(s\zeta)}.
\end{aligned}
$$

Hence, passing to the limit as $m \to +\infty$ in the last inequality, we obtain

$$
\|v_n\|_{L^\infty(B_R^c(0))} \leq C \Big(\int_{|x| \geq R} |v_n|^q dx \Big)^{1/q}. \tag{5.6}
$$

For $x_0 \in \mathbb{R}^2$, we can use the same argument taking $\eta \in C_0^\infty(\mathbb{R}^2, [0,1])$ with $\eta(x) = 1$ if $|x - x_0| \leq \tilde{\rho}$, $\eta(x) = 0$ if $|x - x_0| > 2\rho$, with $\tilde{\rho} < \rho$, and $|\nabla \eta| \leq 2/\tilde{\rho}$, to prove that

$$
\|v_n\|_{L^\infty(\overline{B_{2\rho}(x_0)})} \leq C \Big(\int_{|x| \leq 2\rho} |v_n|^q dx \Big)^{1/q}. \tag{5.7}
$$

Thus, by (5.6), (5.7), and using a standard covering argument and the boundedness of $(|v_n|)$ in $L^q(\mathbb{R}^2,\mathbb{R})$, it follows that

$$\|v_n\|_\infty \le C.$$

Now, we use again the convergence of $(|v_n|)$ in $H^1(\mathbb{R}^2,\mathbb{R})$ on the right side of (5.6) to get

$$\lim_{|x|\to+\infty} |v_n| = 0 \quad \text{uniformly in } n\in\mathbb{N}$$

and the proof is complete. □

Now, we are ready to give the proof of Theorem 1.1.

Proof of Theorem 1.1. Let $\delta>0$ be such that $M_\delta\subset\Lambda$. We want to show that there exists $\hat\varepsilon_\delta>0$ such that for any $\varepsilon\in(0,\hat\varepsilon_\delta)$ and any $u_\varepsilon\in\tilde{\mathcal{N}}_\varepsilon$ solution of problem (3.2), it holds

$$\|u_\varepsilon\|^2_{L^\infty(\Lambda^c_\varepsilon)} \le t_a. \tag{5.8}$$

We argue by contradiction and assume that there is a sequence $\varepsilon_n\to0$ such that for every n there exists $u_n\in\tilde{\mathcal{N}}_{\varepsilon_n}$ which satisfies $J'_{\varepsilon_n}(u_n)=0$ and

$$\|u_n\|^2_{L^\infty(\Lambda^c_{\varepsilon_n})} > t_a. \tag{5.9}$$

As in Lemma 5.1, we have that $J_{\varepsilon_n}(u_n)\to c_{V_0}$, and therefore we can use Proposition 4.3 to obtain a sequence $(\tilde y_n)\subset\mathbb{R}^2$ such that $y_n:=\varepsilon_n\tilde y_n\to y_0$ for some $y_0\in M$. Then, we can find $r>0$, such that $B_r(y_n)\subset\Lambda$, and so $B_{r/\varepsilon_n}(\tilde y_n)\subset\Lambda_{\varepsilon_n}$ for all n large enough.
Using Lemma 5.1, there exists $R>0$ such that $|v_n|^2\le t_a$ in $B^c_R(0)$ and n large enough, where $v_n=u_n(\cdot+\tilde y_n)$. Hence $|u_n|^2\le t_a$ in $B^c_R(\tilde y_n)$ and n large enough. Moreover, if n is so large that $r/\varepsilon_n>R$, then $\Lambda^c_{\varepsilon_n}\subset B^c_{r/\varepsilon_n}(\tilde y_n)\subset B^c_R(\tilde y_n)$, which gives $|u_n|^2\le t_a$ for any $x\in\Lambda^c_{\varepsilon_n}$. This contradicts (5.9) and proves the claim.
Let now $\varepsilon_\delta:=\min\{\hat\varepsilon_\delta,\tilde\varepsilon_\delta\}$, where $\tilde\varepsilon_\delta>0$ is given by Theorem 4.5. Then we have $\mathrm{cat}_{M_\delta}(M)$ nontrivial solutions to problem (3.2). If $u_\varepsilon\in\tilde{\mathcal{N}}_\varepsilon$ is one of these solutions, then, by (5.8) and the definition of g, we conclude that u_ε is also a solution to problem (2.2).
Finally, we study the behavior of the maximum points of $|\hat u_\varepsilon|$, where $\hat u_\varepsilon(x):=u_\varepsilon(x/\varepsilon)$ is a solution to problem (1.1), as $\varepsilon\to0^+$.
Take $\varepsilon_n\to0^+$ and the sequence (u_n) where each u_n is a solution of (3.2) for $\varepsilon=\varepsilon_n$. In view of ($g_2$), there exists $\gamma\in(0,t_a)$ such that

$$g(\varepsilon x,t^2)t^2\le\frac{V_0}{2}t^2,\quad \text{for all } x\in\mathbb{R}^2,\ |t|\le\gamma.$$

Arguiguing as above we can take $R>0$ such that, for n large enough,

$$\|u_n\|_{L^\infty(B^c_R(\tilde y_n))}<\gamma. \tag{5.10}$$

Up to a subsequence, we may also assume that for n large enough

$$\|u_n\|_{L^\infty(B_R(\tilde y_n))}\ge\gamma. \tag{5.11}$$

Indeed, if (5.11) does not hold, up to a subsequence, if necessary, we have $\|u_n\|_\infty<\gamma$. Thus, since $J'_{\varepsilon_n}(u_{\varepsilon_n})=0$, using ($g_5$) and the diamagnetic inequality (2.1) that

$$\int_{\mathbb{R}^2}(|\nabla|u_n||^2+V_0|u_n|^2)dx\le\int_{\mathbb{R}^2}g(\varepsilon_n x,|u_n|^2)|u_n|^2dx\le\frac{V_0}{k}\int_{\mathbb{R}^2}|u_n|^2dx$$

and, being $k>1$, $\|u_n\|=0$, which is a contradiction.
Taking into account (5.10) and (5.11), we can infer that the global maximum points p_n of $|u_{\varepsilon_n}|$ belongs to $B_R(\tilde y_n)$, that is $p_n=q_n+\tilde y_n$ for some $q_n\in B_R$. Recalling that the associated solution of problem

(1.1) is $\hat{u}_n(x) = u_n(x/\varepsilon_n)$, we can see that a maximum point η_{ε_n} of $|\hat{u}_n|$ is $\eta_{\varepsilon_n} = \varepsilon_n \tilde{y}_n + \varepsilon_n q_n$. Since $q_n \in B_R$, $\varepsilon_n \tilde{y}_n \to y_0$ and $V(y_0) = V_0$, the continuity of V allows to conclude that

$$\lim_n V(\eta_{\varepsilon_n}) = V_0.$$

□

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