Multiplicity and nondegeneracy of positive solutions to quasilinear equations on compact Riemannian manifolds

MULTIPLICITY AND NONDEGENERACY OF POSITIVE SOLUTIONS TO QUASILINEAR EQUATIONS ON COMPACT RIEMANNIAN MANIFOLDS

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Abstract. We consider a compact, connected, orientable, boundaryless Riemannian manifold (M,g) of class C^∞ where g denotes the metric tensor. Let $n = \dim M \geq 3$. Using Morse techniques, we prove the existence of $2\mathcal{P}_1(M)-1$ non-costant solutions $u \in H^{1,p}(M)$ to the quasilinear problem

$$(P_\epsilon) \begin{cases} -\epsilon^p \, \Delta_{p,g} u + u^{p-1} = u^{q-1} \\ u > 0 \end{cases}$$

for $\varepsilon > 0$ small enough, where $2 \leq p < n$, $p < q < p^*$, $p^* = np/(n-p)$ and $\Delta_{p,g} u = \mathrm{div}_g(|\nabla u|_g^{p-2} \nabla u)$ is the p-laplacian associated to g of u (note that $\Delta_{2,g} = \Delta_g$) and $\mathcal{P}_t(M)$ denotes the Poincaré Polynomial of M. We also establish results of genericity of nondegenerate solutions for the quasilinear elliptic problem (P_ε).

Key words: Quasilinear elliptic equations, Riemannian Manifold, Positive solutions, Morse index, Perturbation results.

2000 Mathematics Subject Classification: 58E05, 35B20, 35J60, 35J70.

1. Introduction

Let (M,g) be a compact, connected, orientable, boundaryless Riemannian manifold of class C^∞ where g denotes the metric tensor. Let $n = \dim M \geq 3$. Let $H^{1,p}(M)$ be the Sobolev space defined as the completion of $C^\infty(M)$ with respect to the norm

$$\|u\|_{H^{1,p}(M)} = \left(\|\nabla u\|^p_{L^p(M)} + \|u\|^p_{L^p(M)} \right)^{\frac{1}{p}} .$$

We look for solutions $u \in H^{1,p}(M)$ to the following problem

$$(P_\epsilon) \begin{cases} -\epsilon^p \, \Delta_{p,g} u + u^{p-1} = u^{q-1} \\ u > 0 \end{cases}$$

where $2 \leq p < n$, $p < q < p^*$, $p^* = np/(n-p)$, $\Delta_{p,g} u = \mathrm{div}_g(|\nabla u|_g^{p-2} \nabla u)$ is the p-laplacian associated to g of u (note that $\Delta_{2,g} = \Delta_g$) and $\varepsilon > 0$.

A large amount of papers is devoted to the study of equation (P_ϵ) in a flat domain of $\mathbb{R}^n$ when $p = 2$. It is shown that the topology of the domain affects the numbers of solutions to (P_ε) for ε small. We limit to quote the papers [2, 5, 6, 7] and references therein. The study of equation (P_ϵ) on a manifold remained open for a long time since it required new ideas for relating the topology and the numbers of solutions to (P_ϵ). It has been understood in the recent paper [4] (see also [17, 23, 19]).

The research of the first and second author is supported by the MIUR project "Variational and topological methods in the study of nonlinear phenomena" (PRIN 2009).

The research of the first author is also supported by Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (INDAM).

In the present paper we are interested to extend the result in [4] to the quasilinear case. Denoting by $cat(M)$ the Lusternick-Schnirelmann of M in itself, we shall establish the following multiplicity result.

Theorem 1.1. *There exists $\epsilon^* > 0$ such that, for any $\epsilon \in (0, \epsilon^*)$, (P_ϵ) has at least $cat(M) + 1$ non-constant, different solutions.*

Moreover using the topological Morse relations, we correlates the topology of the manifold M to the minimum number of solution of (P_ϵ), counted with their multiplicity (the notions of Poincaré Polynomial $\mathcal{P}_t(M)$ and multiplicity are introduced respectively in Definition 6.1 and 6.7). Precisely we state the following result.

Theorem 1.2. *There exists $\epsilon^* > 0$ such that, for any $\epsilon \in (0, \epsilon^*)$, (P_ϵ) has at least $2\,\mathcal{P}_1(M) - 1$ non-constant solutions, possibly counted with their multiplicities.*

We remark that the application of Morse theory allows to obtain a better information on the number of solutions respect to the application of Lusternick-Schnirelmann theory (see Remark 3.7 in [4]). For instance, if M is the n-dimensional torus in $\mathbb{R}^{n+1}$, we derive from Theorem 1.1 the existence of at least $n+2$ solutions, since $cat(M) = n+1$. On the other hand, since $P_t(M) = (t+1)^n$, by Theorem 1.2 we infer the existence of at least $2^{n+1}-1$ solutions, counted with their multiplicities.

Finally, through a deeper look to the notion of multiplicity, we prove the existence of $2\,\mathcal{P}_1(M) - 1$ non-constant, different solutions for quasilinear elliptic problems which are indefinitely close to (P_ϵ). We stress that the interpretation of the multiplicity in the quasilinear case $p > 2$ is not all trivial. Indeed, serious conceptual difficulties arise when one tries to relate topological objects with differential notions and perform Marino-Prodi perturbation type results in a Banach (not Hilbert) setting. We derive the following main result.

Theorem 1.3. *There exists $\epsilon^* > 0$ such that, for any $\epsilon \in (0, \epsilon^*)$, either (P_ϵ) has at least $2\,\mathcal{P}_1(M) - 1$ non-constant, distinct solutions or, if not, for any sequence $\{\alpha_s\}_{s\in\mathbb{N}}$ with $\alpha_s > 0$, $\alpha_s \to 0$, there exists a sequence $\{f_s\}_{s\in\mathbb{N}}$ with $f_s \in C^1(M)$, $\|f_s\|_{C^1(M)} \to 0$ such that problem*

$$(P_s)\begin{cases} -\epsilon^p\, div_g\left(\left(\alpha_s + |\nabla u|_g^2\right)^{(p-2)/2}\nabla u\right) + u^{p-1} = u^{q-1} + f_s \\ u > 0 \end{cases}$$

has at least $2\,\mathcal{P}_1(M) - 1$ non-constant, different solutions, for s large enough. In particular, if $p = 2$, the statement holds also if $\alpha_s \geq 0$.

2. Notations and preliminary remarks

We denote by $B(0, R)$ the ball in $\mathbb{R}^n$ of center 0 and radius R and by $B_g(x, R)$ the ball in M of center x and radius R.

We define a smooth real function χ_R on $\mathbb{R}^+$ such that

$$\chi_R(t) = \begin{cases} 1 & \text{if } 0 \leq t \leq \frac{R}{2} \\ 0 & \text{if } t \geq R \end{cases} \tag{2.1}$$

and $|\chi_R'(t)| \leq \frac{\chi_0}{R}$, with χ_0 positive constant.

We recall some definitions and results about compact connected Riemannian manifolds of class C^∞ (see for example [18]).

Remark 2.1. *On the tangent bundle TM of M the exponential map $\exp : TM \to M$ is defined. This map has the following properties:*

(i) exp *is of class* C^∞*;*
(ii) *there exists a constant* $R > 0$ *such that*

$$\exp_x|_{B(0,R)} : B(0,R) \to B_g(x,R)$$

is a diffeomorphism for all $x \in M$*.*

It is possible to choose an atlas $\mathcal{C}$ on M, whose charts are given by the exponential map (normal coordinates). We denote by $\{\psi_C\}_{C\in\mathcal{C}}$ a partition of unity subordinate to the atlas $\mathcal{C}$. Let g_{x_0} be the Riemannian metric in the normal coordinates of the map $\exp_{x_0}$.

For any $u \in H^{1,p}(M)$ we have that:

$$\|\nabla u\|^p_{L^p(M)} = \int_M |\nabla u(x)|^p_g d\mu_g = \sum_{C\in\mathcal{C}} \int_C \psi_C(x)|\nabla u(x)|^p_g d\mu_g$$
$$= \sum_{C\in\mathcal{C}} \int_{B(0,R)} \psi_C(\exp_{x_C}(z)) \left(g^{ij}_{x_C}(z) \frac{\partial u(\exp_{x_C}(z))}{\partial z_i} \frac{\partial u(\exp_{x_C}(z))}{\partial z_j} \right)^{\frac{p}{2}} |g_{x_C}(z)|^{\frac{1}{2}}\, dz\,,$$

where μ_g denotes the volume form on M associated to the metric and Einstein notation is adopted, that is

$$g^{ij} z_i z_j = \sum_{i,j=1}^n g^{ij} z_i z_j\,,$$

$(g^{ij}_{x_0}(z))$ is the inverse matrix of $g_{x_0}(z)$ and $|g_{x_0}(z)| = \det(g_{x_0}(z))$. In particular we have that $g_{x_0}(0) = \mathrm{Id}$. A similar relation holds for the integration of $|u(x)|^p$.

For convenience we will also write for all $x_0 \in M$ and $z, \xi \in \mathbb{R}^n$

$$|\xi|^2_{g_{x_0}(z)} = g^{ij}_{x_0}(z)\xi_i\xi_j\,. \tag{2.2}$$

Beside the usual norm of $u \in H^{1,p}(M)$, we will consider also the norm

$$\|u\|^p_\epsilon = \frac{1}{\epsilon^n} \int_M \left(\epsilon^p |\nabla u(x)|^p_g + |u(x)|^p\right) d\mu_g\,. \tag{2.3}$$

By the embedding theorem, we assume that M is embedded in $\mathbb{R}^N$, with $N \geq 2n$.

Remark 2.2. *Since* M *is compact, there are three strictly positive constants* h, H *and* $\tilde{h}$ *such that for all* $x \in M$ *and all* $z, \xi \in \mathbb{R}^n$

$$|\exp_x(z) - \exp_x(\xi)|_{\mathbb{R}^N} \leq \tilde{h}|z - \xi|_{\mathbb{R}^n}$$
$$h|\xi|^2_{\mathbb{R}^n} \leq g_x(z)(\xi,\xi) \leq H|\xi|^2_{\mathbb{R}^n}\,.$$

Hence there holds

$$h^n \leq |g_x(z)| \leq H^n\,.$$

Definition 2.3. *We define the radius of topological invariance* $r(M)$ *of* M *as*

$$r(M) := \sup\{\rho > 0 \mid \mathrm{cat}\,(M_\rho) = \mathrm{cat}\,(M)\}\,,$$

where $M_\rho := \{z \in \mathbb{R}^N \mid d(z, M) < \rho\}$.

The solutions to (P_ϵ) are critical points of the C^2 functional $J_\epsilon : H^{1,p}(M) \to \mathbb{R}$, defined by

$$J_\epsilon(u) = \frac{1}{\epsilon^n} \int_M \left(\frac{\epsilon^p}{p} |\nabla u(x)|^p_g + \frac{1}{p}|u(x)|^p - \frac{1}{q}|u^+(x)|^q \right) d\mu_g\,, \tag{2.4}$$

constrained on the Nehari manifold

$$\mathcal{N}_\epsilon = \left\{ u \in H^{1,p}(M) \,\middle|\, u \neq 0 \text{ and} \int_M \left(\epsilon^p |\nabla u|_g^p + |u|^p\right) d\mu_g = \int_M |u^+|^q \, d\mu_g \right\}. \tag{2.5}$$

Remark 2.4. *It is standard that $\mathcal{N}_\epsilon$ is a 1-codimensional submanifold of $H^{1,p}(M)$, as it is C^1-diffeomorphic to*

$$\{u \in H^{1,p}(M) \mid \|u\| = 1\} \setminus \{u \in H^{1,p}(M) \mid u \leq 0 \text{ a.e.}\}.$$

For the proof, see Lemma 2.2 in [6].
In particular, $\mathcal{N}_\epsilon$ is contractible.

Remark 2.5. *It is immediate to see that the only constant critical points of J_ϵ are $u_0 \equiv 0$ and $u_1 \equiv 1$. Furthermore any critical point $u \neq u_0$ of J_ϵ is a solution to (P_ϵ). In fact, denoting by $u^-(x) = \max\{0, -u(x)\}$, it is $\|u^-\|_\epsilon^p = \langle J'_\epsilon(u), u^- \rangle = 0$, so that $u \geq 0$ a.e. in M. Moreover, by classical results (see Theorem 4.1 in* [13]*), we have that $u \in C^1(M)$ and the Strong Maximum Principle (see* [22]*) assures that $u > 0$ in M. In particular there is $\delta_u > 0$ such that $u > \delta_u$ in M.*

We consider also the following functional $J : H^{1,p}(\mathbb{R}^n) \to \mathbb{R}$ defined by

$$J(v) := \int_{\mathbb{R}^n} \left(\frac{1}{p} |\nabla v(z)|^p + \frac{1}{p} |v(z)|^p - \frac{1}{q} |v^+(z)|^q \right) dz \tag{2.6}$$

and the associated Nehari manifold

$$\mathcal{N} = \left\{ v \in H^{1,p}(\mathbb{R}^n) \,\middle|\, v \neq 0 \text{ and} \int_{\mathbb{R}^n} \left(|\nabla v|^p + |v|^p\right) dz = \int_{\mathbb{R}^n} |v^+|^q dz \right\}. \tag{2.7}$$

Let us denote

$$m(J) := \inf\{J(v) \mid v \in \mathcal{N}\}. \tag{2.8}$$

The infimum $m(J)$ is achieved at a positive, spherically symmetric, decreasing function $U(z)$. This function and its first derivatives decay exponentially (see Theorem 3.4 in [12]) and $U \in C^{1,\alpha}(\mathbb{R}^n)$ (see Theorem 1.9 in [15]). We refer to U as a positive ground state solution to

$$-\Delta_p u + u^{p-1} = u^{q-1} \text{ in } \mathbb{R}^n. \tag{2.9}$$

For any $\varepsilon > 0$, the function $U_\epsilon(z) = U\left(\frac{z}{\epsilon}\right)$ satisfies

$$-\epsilon^p \Delta_p u + u^{p-1} = u^{q-1} \text{ in } \mathbb{R}^n.$$

3. The function ϕ_ϵ

Let U be the function defined in Section 2. For any $x_0 \in M$ and $\epsilon > 0$, we consider the function on M

$$W_{x_0,\epsilon}(x) := \begin{cases} U\left(\frac{\exp_{x_0}^{-1}(x)}{\epsilon}\right) - \widetilde{U}_{\frac{R}{\epsilon}} & \text{if } x \in B_g(x_0, R), \\ 0 & \text{otherwise,} \end{cases} \tag{3.1}$$

where R is chosen as in Remark 2.1 (ii) and

$$\widetilde{U}_{\frac{R}{\epsilon}} = U(z) \quad \text{with} \quad z \in \mathbb{R}^n \quad \text{such that} \quad |z| = \frac{R}{\epsilon}.$$

The function $W_{x_0,\epsilon}$ belongs to $H^{1,p}(M)$.

Remark 3.1. *The set $\mathcal{N}_\epsilon$ is a C^1 manifold. Moreover, for all $u \in H^{1,p}(M)$ with $u^+ \neq 0$ there exists a unique $t_\epsilon(u) > 0$ such that $t_\epsilon(u)u \in \mathcal{N}_\epsilon$ and*

$$(t_\epsilon(u))^{q-p} = \frac{\int_M \left(\epsilon^p |\nabla u|_g^p + |u|^p\right) d\mu_g}{\int_M |u^+|^q \, d\mu_g} \tag{3.2}$$

We can define

$$\begin{array}{rccc} \phi_\epsilon & : & M & \longrightarrow & \mathcal{N}_\epsilon \\ & & x_0 & \longmapsto & t_\epsilon(W_{x_0,\epsilon})W_{x_0,\epsilon}\,. \end{array} \tag{3.3}$$

Moreover for any $\delta > 0$ we consider the following subset of $\mathcal{N}_\epsilon$

$$\Sigma_{\epsilon,\delta} := \{u \in \mathcal{N}_\epsilon \mid J_\epsilon(u) < m(J) + \delta\}\,. \tag{3.4}$$

We can derive the following result.

Proposition 3.2. *For any $\epsilon > 0$ the map $\phi_\epsilon : M \to \mathcal{N}_\epsilon$ is continuous. For any $\delta > 0$ there exists $\epsilon_0 = \epsilon_0(\delta) > 0$ such that if $0 < \epsilon < \epsilon_0$*

$$\phi_\epsilon(x_0) \in \Sigma_{\epsilon,\delta}$$

for all $x_0 \in M$.

Proof. **(I)** *The map $\phi_\epsilon : M \to \mathcal{N}_\epsilon$ is continuous.*

By the continuity of $t_\epsilon(u)$ on $H^{1,p}(M)$ (see Remark 3.1), it is enough to prove that

$$\lim_{k\to\infty} \|W_{x_k,\epsilon} - W_{\hat{x},\epsilon}\|_{H^{1,p}(M)} = 0\,.$$

for any sequence $\{x_k\}_{k\in\mathbb{N}}$ in M, converging to $\hat{x}$.

We choose a finite atlas $\mathcal{C}$ for M, which contains the chart with domain $C = B_g(\hat{x}, R)$. The functions $W_{x_k,\epsilon}$ and $W_{\hat{x},\epsilon}$ have support respectively on $B_g(x_k, R)$ and on $B_g(\hat{x}, R)$. Since $x_k \to \hat{x}$ the set $Z_k = [B_g(x_k, R) \setminus B_g(\hat{x}, R)] \cup [B_g(\hat{x}, R) \setminus B_g(x_k, R)]$ is such that $\mu_g(Z_k) \to 0$ as $k \to \infty$. Then we have

$$\int_{Z_k} |\nabla \left(W_{x_k,\epsilon}(x) - W_{\hat{x},\epsilon}(x)\right)|_g^p \, d\mu_g \to 0 \qquad \text{as } k \to \infty\,.$$

We still have to check the integral on $B_g(x_k, R) \cap B_g(\hat{x}, R)$. We write $A_k = \exp_{\hat{x}}^{-1}(B_g(x_k, R) \cap B_g(\hat{x}, R))$ and $\eta_k(z) = \exp_{x_k}^{-1}(\exp_{\hat{x}}(z))$

$$\begin{aligned} \int_{\exp_{\hat{x}}(A_k)} |\nabla \left[W_{x_k,\epsilon}(x) - W_{\hat{x},\epsilon}(x)\right]|_g^p \, d\mu_g &= \int_{A_k} |\nabla \left[U_\epsilon(\eta_k(z)) - U_\epsilon(z)\right]|_{g_{\hat{x}}(z)}^p |g_{\hat{x}}(z)|^{\frac{1}{2}} dz \\ &\leq \frac{H^{\frac{n}{2}}}{h^{\frac{p}{2}}} \int_{A_k} |\nabla \left[U_\epsilon(\eta_k(z)) - U_\epsilon(z)\right]|^p \, dz\,. \end{aligned}$$

Since $\eta_k(z)$ tends point-wise to z and ∇U_ϵ is continuous, $|\nabla[U_\epsilon(\eta_k(z)) - U_\epsilon(z)]|^p$ tends pointwise to zero. Applying Lebesgue theorem, we obtain that

$$\int_M |\nabla \left[W_{x_k,\epsilon}(x) - W_{\hat{x},\epsilon}(x)\right]|_g^p \, d\mu_g \to 0\,.$$

In an analogous way we have that $\|W_{x_k,\epsilon} - W_{\hat{x},\epsilon}\|_{L^p(M)}^p$ tends to zero.

(II) *The limit of $\frac{\epsilon^p}{\epsilon^n} \int_M |\nabla W_{x_0,\epsilon}(x)|_g^p \, d\mu_g$ is $\|\nabla U\|_{L^p(\mathbb{R}^n)}^p$.*

To prove the second statement of this proposition, first we show that

$$\lim_{\epsilon\to 0} \frac{\epsilon^p}{\epsilon^n} \int_M |\nabla W_{x_0,\epsilon}(x)|_g^p \, d\mu_g = \|\nabla U\|_{L^p(\mathbb{R}^n)}^p \tag{3.5}$$

uniformly with respect to $x_0 \in M$.

We evaluate the following:

$$
\begin{aligned}
&\left| \frac{\epsilon^p}{\epsilon^n} \int_M |\nabla W_{x_0,\epsilon}|_g^p \, d\mu_g - \int_{\mathbb{R}^n} |\nabla U|^p \, dz \right| \\
&\quad = \left| \frac{\epsilon^p}{\epsilon^n} \int_{B_g(x_0,R)} \left|\nabla \left[U_\epsilon(\exp_{x_0}^{-1}(x))\right]\right|_g^p d\mu_g - \int_{\mathbb{R}^n} |\nabla U|^p \, dz \right| \\
&\quad = \left| \frac{\epsilon^p}{\epsilon^n} \int_{B(0,R)} |\nabla U_\epsilon(z)|_{g_{x_0}(z)}^p \, |g_{x_0}(z)|^{\frac{1}{2}} dz - \int_{\mathbb{R}^n} |\nabla U|^p \, dz \right| .
\end{aligned}
$$

Changing variables, we obtain

$$
\left| \int_{\mathbb{R}^n} \left[\chi_{B\left(0,\frac{R}{\epsilon}\right)}(z) \left(g_{x_0}^{ij}(\epsilon z) \frac{\partial U}{\partial z_i} \frac{\partial U}{\partial z_j} \right)^{\frac{p}{2}} |g_{x_0}(\epsilon z)|^{\frac{1}{2}} - \left(\delta^{ij} \frac{\partial U}{\partial z_i} \frac{\partial U}{\partial z_j} \right)^{\frac{p}{2}} \right] dz \right| ,
$$

where $\chi_{B\left(0,\frac{R}{\epsilon}\right)}(z)$ denotes the characteristic function of the set $B\left(0,\frac{R}{\epsilon}\right)$ and where δ^{ij} is the Kronecker delta (it takes value 0 for $i \neq j$ and 1 for $i = j$). The previous integral is bounded from above by the sum $I_1 + I_2 + I_3$, with

$$
\begin{aligned}
I_1 &= \int_{\mathbb{R}^n} \chi_{B\left(0,\frac{R}{\epsilon}\right)}(z) \left(g_{x_0}^{ij}(\epsilon z) \frac{\partial U}{\partial z_i} \frac{\partial U}{\partial z_j} \right)^{\frac{p}{2}} \left| |g_{x_0}(\epsilon z)|^{\frac{1}{2}} - 1 \right| dz \, , \\
I_2 &= \int_{\mathbb{R}^n} \chi_{B\left(0,\frac{R}{\epsilon}\right)}(z) \left| \left(g_{x_0}^{ij}(\epsilon z) \frac{\partial U}{\partial z_i} \frac{\partial U}{\partial z_j} \right)^{\frac{p}{2}} - \left(\delta^{ij} \frac{\partial U}{\partial z_i} \frac{\partial U}{\partial z_j} \right)^{\frac{p}{2}} \right| dz \, , \\
I_3 &= \int_{\mathbb{R}^n} \left| \chi_{B\left(0,\frac{R}{\epsilon}\right)}(z) - 1 \right| \left(\delta^{ij} \frac{\partial U}{\partial z_i} \frac{\partial U}{\partial z_j} \right)^{\frac{p}{2}} dz \, .
\end{aligned}
$$

As regards the first term, it can be written

$$
\begin{aligned}
I_1 = &\int_{B(0,T)} \chi_{B\left(0,\frac{R}{\epsilon}\right)}(z) \left(g_{x_0}^{ij}(\epsilon z) \frac{\partial U}{\partial z_i} \frac{\partial U}{\partial z_j} \right)^{\frac{p}{2}} \left| |g_{x_0}(\epsilon z)|^{\frac{1}{2}} - 1 \right| dz \\
&+ \int_{\mathbb{R}^n \setminus B(0,T)} \chi_{B\left(0,\frac{R}{\epsilon}\right)}(z) \left(g_{x_0}^{ij}(\epsilon z) \frac{\partial U}{\partial z_i} \frac{\partial U}{\partial z_j} \right)^{\frac{p}{2}} \left| |g_{x_0}(\epsilon z)|^{\frac{1}{2}} - 1 \right| dz
\end{aligned}
$$

with $T > 0$. It is easy to see that the second addendum vanishes as $T \to \infty$. As regards the first addendum, fixed T, by compactness of the manifold M and regularity of the Riemannian metric g the limit

$$
\lim_{\epsilon \to 0} \left| |g_{x_0}(\epsilon z)|^{\frac{1}{2}} - 1 \right| = 0
$$

holds true uniformly with respect to $x_0 \in M$ and $z \in B(0,T)$. The second term can be written

$$
I_2 = \frac{p}{2} \int_{B\left(0,\frac{R}{\epsilon}\right)} \left[\left(\theta g_{x_0}^{ij}(\epsilon z) + (1-\theta)\delta^{ij} \right) \frac{\partial U}{\partial z_i} \frac{\partial U}{\partial z_j} \right]^{\frac{p-2}{2}} \left[\left| g_{x_0}^{ij}(\epsilon z) - \delta^{ij} \right| \frac{\partial U}{\partial z_i} \frac{\partial U}{\partial z_j} \right] dz
$$

where $\theta = \theta(z) \in [0,1]$. As before, it is possible to split the integral on and outside the ball $B(0,T)$. In $\mathbb{R}^n \setminus B(0,T)$ the integral tends to zero uniformly with respect to $x_0 \in M$, while in $B(0,T)$ $g_{x_0}^{ij}(\epsilon z)$ tends to δ^{ij} uniformly with respect to $x_0 \in M$ for any $1 \leq i,j \leq n$. The third term is independent of x_0 and

$$
I_3 = \int_{\mathbb{R}^n \setminus B\left(0,\frac{R}{\epsilon}\right)} |\nabla U|^p dz
$$

tends to zero. This proves (3.5).

(III) *The limits of* $\frac{1}{\epsilon^n}\int_M |W_{x_0,\epsilon}(x)|^p\, d\mu_g$ *and* $\frac{1}{\epsilon^n}\int_M \left|W^+_{x_0,\epsilon}(x)\right|^q d\mu_g$ *are respectively* $\|U\|^p_{L^p(\mathbb{R}^n)}$ *and* $\|U\|^q_{L^q(\mathbb{R}^n)}$.

As before we consider

$$\begin{aligned}&\left|\frac{1}{\epsilon^n}\int_M |W_{x_0,\epsilon}|^p\, d\mu_g - \int_{\mathbb{R}^n} |U|^p\, dz\right| \\ &\quad = \left|\int_{\mathbb{R}^n}\left[\chi_{B\left(0,\frac{R}{\epsilon}\right)}(z)\left|U(z)-\widetilde{U}_{\frac{R}{\epsilon}}\right|^p |g_{x_0}(\epsilon z)|^{\frac{1}{2}} - |U(z)|^p\right] dz\right|.\end{aligned}$$

Summing and subtracting, one obtains

$$\left|\frac{1}{\epsilon^n}\int_M |W_{x_0,\epsilon}|^p\, d\mu_g - \int_{\mathbb{R}^n} |U|^p\, dz\right| \le I_4 + I_5 + I_6$$

where

$$\begin{aligned}I_4 &= \int_{\mathbb{R}^n}\chi_{B\left(0,\frac{R}{\epsilon}\right)}(z)\left|U(z)-\widetilde{U}_{\frac{R}{\epsilon}}\right|^p \left||g_{x_0}(\epsilon z)|^{\frac{1}{2}}-1\right| dz\,,\\ I_5 &= \int_{\mathbb{R}^n}\chi_{B\left(0,\frac{R}{\epsilon}\right)}(z)\left|\left|U(z)-\widetilde{U}_{\frac{R}{\epsilon}}\right|^p - |U(z)|^p\right| dz\,,\\ I_6 &= \int_{\mathbb{R}^n}\left|\chi_{B\left(0,\frac{R}{\epsilon}\right)}(z)-1\right| |U(z)|^p dz\,.\end{aligned}$$

Analogously to part (II) it is easy to prove that I_4, I_5 and I_6 tend to zero for ϵ tending to zero uniformly with respect to $x_0 \in M$. The proof is the same also for $\frac{1}{\epsilon^n}\int_M \left|W^+_{x_0,\epsilon}(x)\right|^q d\mu_g$. Then we have proved

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon^n}\int_M |W_{x_0,\epsilon}(x)|^p\, d\mu_g = \|U\|^p_{L^p(\mathbb{R}^n)} \tag{3.6}$$

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon^n}\int_M \left|W^+_{x_0,\epsilon}(x)\right|^q d\mu_g = \|U\|^q_{L^q(\mathbb{R}^n)} \tag{3.7}$$

(IV) *The parameter* $t_\epsilon\left(W_{x_0,\epsilon}\right)$ *tends to* 1 *for* ϵ *tending to zero uniformly with respect to* $x_0 \in M$.

By Remark 3.1 and the limits (3.5), (3.6) and (3.7), it is obvious that

$$\lim_{\epsilon\to 0} t_\epsilon(W_{x_0,\epsilon}) = 1.$$

(V) *Conclusion.*

By (II), (III) and (IV) we obtain that $J_\epsilon(\phi_\epsilon(x_0))$ tends to $J(U) = m(J)$ for ϵ tending to zero uniformly with respect to x_0. This completes the proof. □

Remark 3.3. *By the previous proposition, in particular we know that, given* $\delta > 0$, *for any positive* ϵ *sufficiently small* $\Sigma_{\epsilon,\delta}$ *is not empty.*

4. The function β

Given a function $u \in N_\epsilon$, $u \not\equiv 0$, it is possible to define its center of mass $\beta(u) \in \mathbb{R}^N$ by

$$\beta(u) := \frac{\int_M x|u^+(x)|^q\, d\mu_g}{\int_M |u^+(x)|^q\, d\mu_g}\,. \tag{4.1}$$

To prove that $\beta : \Sigma_{\epsilon,\delta} \to M_{r(M)}$ (see Definition 2.3), we use the fact that the functions in $\Sigma_{\epsilon,\delta}$ concentrate for ϵ and δ tending to zero.

First of all we find a positive inferior bound for the functional J_ϵ on the Nehari manifold. Let us denote

$$m_\epsilon = \inf_{u\in\mathcal{N}_\epsilon} J_\epsilon(u)\,. \tag{4.2}$$

It is easy to see that

$$\inf_{u\in\mathcal{N}_\epsilon} \|u\|_{H^{1,p}(M)} > 0$$

and, since the manifold M is compact, that the infimum m_ϵ is achieved.

Lemma 4.1. *There exists a positive constant $K > 0$ such that for any $\epsilon > 0$ the inequality $m_\epsilon \geq K$ holds.*

To prove this lemma we need the following technical lemma, whose proof can be derived, arguing as in Lemma 5.2 in [4].

Lemma 4.2. *For any $r \in (0, r(M))$, there exist constants k_1, $k_2 > 0$ such that for any $u \in H^{1,p}(M)$ and for any $\varepsilon > 0$, there exists $v \in H_0^{1,p}(M_r)$ such that $v|_M \equiv u$ and*

$$\|v\|^p_{H^{1,p}(M_r)} \leq k_1 \|w\|^p_\epsilon\,, \tag{4.3}$$

$$\|v\|^q_{L^q(M_r)} \geq \frac{k_2}{\epsilon^n}\|u\|^q_{L^q(M)}\,. \tag{4.4}$$

Proof of Lemma 4.1. We notice that, for any $u \in \mathcal{N}_\epsilon$, we have

$$J_\varepsilon(u) = \left(\frac{1}{p} - \frac{1}{q}\right)\frac{1}{\epsilon^n}\int_M |u_+|^q\,.$$

Taking into account Remark 3.1, it follows that

$$\begin{aligned} m_\epsilon &= \inf\{\frac{q-p}{pq}\,(t_\varepsilon(w))^q :\ w \in H^{1,p}(M),\ \frac{1}{\epsilon^n}\int_M |w_+|^q = 1\} \\ &= \inf\{\frac{q-p}{pq}\,\|w\|_\varepsilon^{pq/(q-p)} :\ w \in H^{1,p}(M),\ \frac{1}{\epsilon^n}\int_M |w_+|^q = 1\}. \end{aligned}$$

Taking into account Lemma 4.2, we have that for any $w \in H^{1,p}(M)$

$$0 < m(J) \leq \frac{\|v\|^p_{H^1(M_r)}}{\|v\|^p_{L^q(M_r)}} \leq \frac{k_1}{k_2^{p/q}}\frac{\frac{1}{\epsilon^n}\int_M \left(\epsilon^p|\nabla w|_g^p + |w|^p\right)}{(\frac{1}{\epsilon^n}\int_M |w|^q)^{p/q}}.$$

The lemma follows since $\int_M |w|^q \geq \int_M |w_+|^q$. □

In the following lemma for every function $u \in \mathcal{N}_\epsilon$ it is stated the existence of a point in the manifold where u in some sense concentrates.

Lemma 4.3. *Let $\mathcal{C}$ be an atlas for M with open cover given by $B_g(x_\alpha, \frac{R}{2})$, $\alpha = 1, \ldots, k$. There exists a constant $\gamma > 0$ such that for any fixed $\delta > 0$ and for any $0 < \epsilon < \epsilon_0(\delta)$, where ϵ_0 is defined in Proposition 3.2, if $u \in \mathcal{N}_\epsilon$ there exist $\alpha_1 = \alpha_1(u)$ and $\alpha_2 = \alpha_2(u)$ such that*

$$\begin{aligned} &\left(\frac{1}{p} - \frac{1}{q}\right)\frac{1}{\epsilon^n}\int_{B_g\left(x_{\alpha_1}, \frac{R}{2}\right)} \left(\epsilon^p|\nabla u|_g^p + |u|^p\right) d\mu_g \geq \gamma\,, \\ &\left(\frac{1}{p} - \frac{1}{q}\right)\frac{1}{\epsilon^n}\int_{B_g\left(x_{\alpha_2}, \frac{R}{2}\right)} |u^+|^q d\mu_g \geq \gamma\,. \end{aligned} \tag{4.5}$$

Proof. Let u be in $\mathcal{N}_\epsilon$. Let $\{\psi_\alpha\}_{\alpha=1,\dots,k}$ be a partition of unity subordinate to the atlas $\mathcal{C}$. It is possible to write

$$\begin{aligned}
J_\epsilon(u) &= \left[\left(\frac{1}{p}-\frac{1}{q}\right)\|u\|_\epsilon^p\right]^{\frac{1}{2}} (J_\epsilon(u))^{\frac{1}{2}} \\
&= \left[\left(\frac{1}{p}-\frac{1}{q}\right)\frac{1}{\epsilon^n}\sum_{\alpha=1}^k \int_{B_g\left(x_\alpha,\frac{R}{2}\right)} \psi_\alpha(x)\left(\epsilon^p|\nabla u|_g^p+|u|^p\right)d\mu_g\right]^{\frac{1}{2}} (J_\epsilon(u))^{\frac{1}{2}} \\
&\le \left(\frac{1}{p}-\frac{1}{q}\right)^{\frac{1}{2}}\sqrt{k}\max_{1\le\alpha\le k}\left(\frac{1}{\epsilon^n}\int_{B_g\left(x_\alpha,\frac{R}{2}\right)}\left(\epsilon^p|\nabla u|_g^p+|u|^p\right)d\mu_g\right)^{\frac{1}{2}} (J_\epsilon(u))^{\frac{1}{2}}
\end{aligned}$$

By this inequality and Lemma 4.1 we conclude that

$$\max_{1\le\alpha\le k}\left(\frac{1}{p}-\frac{1}{q}\right)\frac{1}{\epsilon^n}\int_{B_g\left(x_\alpha,\frac{R}{2}\right)}\left(\epsilon^p|\nabla u|_g^p+|u|^p\right)d\mu_g \ge \frac{1}{k}J_\epsilon(u)\ge\frac{K}{k}$$

and the first equation in (4.5) is proved. The second equation can be proved in the same way. □

In the following proposition the concentration property is better specified.

Proposition 4.4. *For any $\eta\in(0,1)$ there exists $\delta_1(\eta)<m(J)$ such that, for any $\delta\in(0,\delta_1(\eta))$ there exists $\epsilon_1(\delta)$ such that for any $\epsilon\in(0,\epsilon_1(\delta))$ and for any function $u\in\Sigma_{\epsilon,\delta}$ we can find a point $x_0=x_0(u)\in M$ with the property*

$$\left(\frac{1}{p}-\frac{1}{q}\right)\frac{1}{\epsilon^n}\int_{B_g\left(x_0,\frac{r(M)}{2}\right)}|u^+|^q d\mu_g > \eta\, m(J)\,.$$

Moreover $\epsilon_1(\delta)$ is nondecreasing with respect to δ.

The proof of this proposition needs the following lemmas. We state the following splitting lemma (see Theorem 3.6 [12]).

Lemma 4.5. *Let $\{v_k\}_{k\in\mathbb{N}}\subset\mathcal{N}$ be a sequence such that:*

$$\begin{array}{ll} J(v_k)\to m(J) & \text{as } k\to\infty\,, \\ J'(v_k)\to 0 \text{ in } (H^{1,p}(\mathbb{R}^n))^* & \text{as } k\to+\infty\,. \end{array}$$

Then

- *either $\{v_k\}_{k\in\mathbb{N}}$ converges strongly in $H^{1,p}(\mathbb{R}^n)$ to a positive ground state solution of (2.9) or*
- *there exist a sequence of points $\{y_k\}_{k\in\mathbb{N}}\subset\mathbb{R}^n$, with $|y_k|\to+\infty$ as $k\to+\infty$, a positive ground state solution U of (2.9) and a sequence $\{v_k^0\}_{k\in\mathbb{N}}$ such that, up to subsequence:*

  (i) $v_k(z)=v_k^0(z)+U(z-y_k)$ *for any* $z\in\mathbb{R}^n$ *;*
  (ii) $v_k^0\to 0$ *strongly in* $H^{1,p}(\mathbb{R}^n)$, *as* $k\to+\infty$.

Lemma 4.6. *Let $\{\epsilon_k\}_{k\in\mathbb{N}}$ and $\{\delta_k\}_{k\in\mathbb{N}}$ be two positive sequences tending to zero for k tending to infinity. For any $k\in\mathbb{N}$ let u_k be a function in $\Sigma_{\epsilon_k,\delta_k}$ such that for any $u\in T_{u_k}\Sigma_{\epsilon_k,\delta_k}$*

$$|J'_{\epsilon_k}(u_k)(u)|=o(1)\|u\|_{\epsilon_k}\,,$$

where $\|\cdot\|_\epsilon$ is defined in (2.3). There exist a sequence $\{x_k\}_{k\in\mathbb{N}}$ of points in M and a sequence of functions $\{w_k\}_{k\in\mathbb{N}}$ on $\mathbb{R}^n$, defined as

$$w_k(z) = u_k(\exp_{x_k}(\epsilon_k z))\chi_{\frac{R}{\epsilon_k}}(|z|)\,, \tag{4.6}$$

such that the following properties hold:

(i) *There exists $w \in H^{1,p}(\mathbb{R}^n)$ such that, up to a subsequence, $\{w_k\}_{k\in\mathbb{N}}$ tends to w weakly in $H^{1,p}(\mathbb{R}^n)$ and strongly in $L^q_{loc}(\mathbb{R}^n)$.*
(ii) *The function w is a weak solution of $-\Delta_p w + |w|^{p-2}w = (w^+)^{q-1}$ on $\mathbb{R}^n$.*
(iii) *The function w is a positive ground state solution.*
(iv) $\lim_{k\to\infty} J_{\epsilon_k}(u_k) = m(J)$.

Proof. To get started we consider x_k to be points in M such that u_k has the property (4.5). We will be more precise in point (iii).
(i) It is sufficient to prove that the sequence $\{w_k\}_{k\in\mathbb{N}}$ is bounded in $H^{1,p}(\mathbb{R}^n)$. We write:

$$\begin{aligned}
\|w_k\|^p_{H^{1,p}(\mathbb{R}^n)} =& \int_{B\left(0,\frac{R}{\epsilon_k}\right)} \left(|\nabla w_k(z)|^p + |w_k(z)|^p\right) dz \\
\leq& C\int_{B\left(0,\frac{R}{\epsilon_k}\right)} |\nabla[u_k(\exp_{x_k}(\epsilon_k z))]|^p \left[\chi_{\frac{R}{\epsilon_k}}(|z|)\right]^p dz \\
&+ C\int_{B\left(0,\frac{R}{\epsilon_k}\right)} \left[\left|\chi'_{\frac{R}{\epsilon_k}}(|z|)\right|^p + \left|\chi_{\frac{R}{\epsilon_k}}(|z|)\right|^p\right] |u_k(\exp_{x_k}(\epsilon_k z))|^p dz \\
=& I_1 + I_2\,.
\end{aligned}$$

We consider the following inequality:

$$\begin{aligned}
\frac{\epsilon_k^p}{\epsilon_k^n}\int_M |\nabla u_k|_g^p d\mu_g \geq& \frac{\epsilon_k^p}{\epsilon_k^n}\int_{B_g(x_k,R)} |\nabla u_k|^p_g d\mu_g \\
=& \frac{\epsilon_k^p}{\epsilon_k^n}\int_{B(0,R)} |\nabla u_k(\exp_{x_k}(z))|^p_{g_{x_k}(z)} |g_{x_k}(z)|^{\frac12} dz \\
=& \int_{B\left(0,\frac{R}{\epsilon_k}\right)} |\nabla u_k(\exp_{x_k}(\epsilon_k z))|^p_{g_{x_k}(\epsilon_k z)} |g_{x_k}(\epsilon_k z)|^{\frac12} dz \\
\geq& \frac{h^{\frac n2}}{H^{\frac p2}}\int_{B\left(0,\frac{R}{\epsilon_k}\right)} |\nabla u_k(\exp_{x_k}(\epsilon_k z))|^p dz \geq \frac{h^{\frac n2}}{CH^{\frac p2}}\, I_1\,.
\end{aligned} \tag{4.7}$$

Moreover for k sufficiently big the following inequality holds

$$\begin{aligned}
I_2 \leq& C\left(\frac{\chi_0^p\epsilon_k^p}{R^p} + 1\right)\int_{B\left(0,\frac{R}{\epsilon_k}\right)} |u_k(\exp_{x_k}(\epsilon_k z))|^p dz \\
\leq& \frac{2C}{\epsilon_k^n}\int_{B(0,R)} |u_k(\exp_{x_k}(z))|^p dz \\
\leq& \frac{2C}{h^{\frac n2}\epsilon_k^n}\int_{B_g(x_k,R)} |u_k(x)|^p d\mu_g\,,
\end{aligned} \tag{4.8}$$

with C a positive constant. By (4.7) and (4.8), we have that

$$I_1 + I_2 \leq \frac{C_1}{\epsilon_k^n}\int_M \left(\epsilon_k^p|\nabla u_k|^p_g + |u_k(x)|^p\right) d\mu_g \leq C_2 J_{\epsilon_k}(u_k) \leq C_2(m(J)+1)\,,$$

where C_1, C_2 are positive constants and k is sufficiently big.

(ii) First of all we prove that for any $\xi \in C_0^\infty(\mathbb{R}^n)$ $J'(w_k)(\xi) = o(1)\|\xi\|_{H^{1,p}(\mathbb{R}^n)}$ for k tending to infinity. For k sufficiently large and for any z in the support of ξ Ξ we have $\chi_{\frac{R}{\epsilon_k}}(|z|) = 1$, so $J'(w_k)(\xi) = J'(u_k(\exp_{x_k}(\epsilon_k z))(\xi)$. Now we define the function ξ_k in $H^{1,p}(M)$ as follows:

$$\xi_k(x) = \begin{cases} \xi\left(\frac{\exp_{x_k}^{-1}(x)}{\epsilon_k}\right) & \forall x \in B_g(x_k, R)\,, \\ 0 & \text{otherwise.} \end{cases}$$

Then we want to write

$$J'(w_k)(\xi) = J'_{\epsilon_k}(u_k)(\xi_k) + E_k\,,$$

where E_k is an error. Now, if $\xi_k \in T_{u_k}\Sigma_{\epsilon_k,\delta_k}$, by hypothesis

$$\left|J'_{\epsilon_k}(u_k)(\xi_k)\right| = o(1)\|\xi_k\|_{\epsilon_k} = o(1)\|\xi\|_{H^{1,p}(\mathbb{R}^n)}.$$

If $\xi_k \notin T_{u_k}\Sigma_{\epsilon_k,\delta_k}$, then it is easy to show that $\xi_k + \lambda_k u_k \in T_{u_k}\Sigma_{\epsilon_k,\delta_k}$ with

$$\begin{aligned}\lambda_k =& \frac{1}{(q-p)\|u_k\|_{\epsilon_k}^p}\frac{1}{\epsilon_k^n}\int_M \big(p\epsilon_k^p|\nabla u_k|_g^{p-2} g_{x_k}(\nabla u_k, \nabla\xi_k) + p|u_k|^{p-2}u_k\xi_k \\ &-q|u_k^+|^{q-1}\xi_k\big)\,d\mu_g\,.\end{aligned}$$

Then we have

$$\begin{aligned}\left|J'_{\epsilon_k}(u_k)(\xi_k)\right| &= \left|J'_{\epsilon_k}(u_k)(\xi_k + \lambda_k u_k)\right| = o(1)\|\xi_k + \lambda_k u_k\|_{\epsilon_k} \\ &= o(1)(\|\xi_k\|_{\epsilon_k} + |\lambda_k|\|u_k\|_{\epsilon_k})\,.\end{aligned}$$

Since $\|u_k\|_{\epsilon_k}$ is bounded from above and from below away from zero (recall Lemma 4.1),

$$\begin{aligned}|\lambda_k| &\le \frac{C}{\|u_k\|_{\epsilon_k}^p}\left(\|u_k\|_{\epsilon_k}^{p-1}\|\xi_k\|_{\epsilon_k} + \|u_k\|_{\epsilon_k}^{\frac{p(q-1)}{q}}\frac{1}{\epsilon_k^{\frac{n}{q}}}\|\xi_k\|_{L^q(M)}\right) \\ &\le C'\left(\|\xi_k\|_{\epsilon_k} + \frac{1}{\epsilon_k^{\frac{n}{q}}}\|\xi_k\|_{L^q(M)}\right),\end{aligned}$$

where C and C' are positive constants. By the fact that

$$\|\xi_k\|_{\epsilon_k} \le C''\|\xi\|_{H^{1,p}(\mathbb{R}^n)} \quad \text{and} \quad \frac{1}{\epsilon_k^{\frac{n}{q}}}\|\xi_k\|_{L^q(M)} \le C''\|\xi\|_{H^{1,p}(\mathbb{R}^n)}$$

for a suitable positive constant C'', we can conclude that

$$\left|J'_{\epsilon_k}(u_k)(\xi_k)\right| = o(1)\|\xi\|_{H^{1,p}(\mathbb{R}^n)}\,.$$

Now we have to check the error: we can write

$$|E_k| \le |E_{1,k}| + |E_{2,k}| + |E_{3,k}|\,,$$

where

$$\begin{aligned}E_{1,k} &= \int_{\mathbb{R}^n} |\nabla w_k|^{p-2}\nabla w_k \cdot \nabla\xi\,dz - \frac{1}{\epsilon_k^n}\int_M \epsilon_k^p|\nabla u_k|_g^{p-2} g_{x_k}(\nabla u_k, \nabla\xi_k)\,d\mu_g\,, \\ E_{2,k} &= \int_{\mathbb{R}^n} |w_k|^{p-2}w_k\xi\,dz - \frac{1}{\epsilon_k^n}\int_M |u_k|^{p-2}u_k\xi_k\,d\mu_g\,, \\ E_{3,k} &= \int_{\mathbb{R}^n} |w_k^+|^{q-1}\xi\,dz - \frac{1}{\epsilon_k^n}\int_M |u_k^+|^{q-1}\xi_k\,d\mu_g\,.\end{aligned}$$

Let $\hat{u}_k(z) = u_k(\exp_{x_k}(\epsilon_k z))$, then for the first term we have

$$\begin{aligned}
|E_{1,k}| \leq & \int_\Xi \left| (|\nabla \hat{u}_k|^{p-2}\delta^{ij} - |\nabla \hat{u}_k|_g^{p-2} g^{ij}_{x_k}(\epsilon_k z)|g_{x_k}(\epsilon_k z)|^{\frac{1}{2}}) \frac{\partial \hat{u}_k}{\partial z_i}\frac{\partial \xi}{\partial z_j}\right| dz \\
\leq & \int_\Xi \left|(|\nabla \hat{u}_k|^{p-2} - |\nabla \hat{u}_k|_g^{p-2})\nabla \hat{u}_k \cdot \xi\right| dz \\
& + \int_\Xi |\nabla \hat{u}_k|_g^{p-2} \left|(\delta^{ij} - g^{ij}_{x_k}(\epsilon_k z)|g_{x_k}(\epsilon_k z)|^{\frac{1}{2}}) \frac{\partial \hat{u}_k}{\partial z_i}\frac{\partial \xi}{\partial z_j}\right| dz \\
= & \frac{p-2}{2}\int_\Xi |\nabla \hat{u}_k|^{p-4}_{\theta_k \operatorname{Id} + (1-\theta_k)g} \left|(g^{ij}_{x_k}(\epsilon_k z) - \delta^{ij})\frac{\partial \hat{u}_k}{\partial z_i}\frac{\partial \hat{u}_k}{\partial z_j}\right| |\nabla \hat{u}_k \cdot \nabla \xi| \, dz \\
& + \int_\Xi |\nabla \hat{u}_k|_g^{p-2} \left|(\delta^{ij} - g^{ij}_{x_k}(\epsilon_k z)|g_{x_k}(\epsilon_k z)|^{\frac{1}{2}}) \frac{\partial \hat{u}_k}{\partial z_i}\frac{\partial \xi}{\partial z_j}\right| dz \,,
\end{aligned}$$

where $\theta_k = \theta_k(z)$ is in $(0,1)$. The limits

$$\lim_{k\to\infty} |g^{ij}_{x_k}(\epsilon_k z) - \delta^{ij}| = 0\,, \quad \lim_{k\to\infty} |\delta^{ij} - g^{ij}_{x_k}(\epsilon_k z)|g_{x_k}(\epsilon_k z)|^{\frac{1}{2}}| = 0$$

are uniform with respect to $z \in \Xi$. Since there exists a positive constant C such that

$$\begin{aligned}
&\int_\Xi |\nabla \hat{u}_k|^{p-4}_{\theta_k \operatorname{Id} + (1-\theta_k)g} \left|\frac{\partial \hat{u}_k}{\partial z_i}\frac{\partial \hat{u}_k}{\partial z_j}\right| |\nabla \hat{u}_k \cdot \nabla \xi| \, dz \leq C\|\hat{u}_k\|^{p-1}_{H_1^p(\Xi)}\|\xi\|_{H_1^p(\Xi)}\,, \\
&\int_\Xi |\nabla \hat{u}_k|_g^{p-2} \left|\frac{\partial \hat{u}_k}{\partial z_i}\frac{\partial \xi}{\partial z_j}\right| dz \leq C\|\hat{u}_k\|^{p-1}_{H_1^p(\Xi)}\|\xi\|_{H_1^p(\Xi)}
\end{aligned}$$

and $\hat{u}_k$ is bounded in $H_1^p(\Xi)$, we conclude that $|E_{1,k}| = o(1)\|\xi\|_{H^{1,p}(\mathbb{R}^n)}$. Similar arguments give that also $|E_{2,k}|$ and $|E_{3,k}|$ are $o(1)\|\xi\|_{H^{1,p}(\mathbb{R}^n)}$.

Our second and last step is to prove that for any $\xi \in C_0^\infty(\mathbb{R}^n)$ $J'(w_k)(\xi)$ tends to $J'(w)(\xi)$ for k tending to infinity. By Lemma 3.1 in [21], we have that

$$\left\{\begin{array}{ll} \nabla w_k \to \nabla w & \text{a.e. in } \mathbb{R}^n \\ |\nabla w_k|^{p-2}\nabla w_k \rightharpoonup |\nabla w|^{p-2}\nabla w & \text{weakly in } L^{\frac{p}{p-1}}(\mathbb{R}^n) \text{ and a.e. in } \mathbb{R}^n \end{array}\right. \tag{4.9}$$

as $k \to \infty$. Therefore $\int_{\mathbb{R}^n} |\nabla w_k|^{p-2}\nabla w_k \cdot \nabla \xi \, dz$ tends to $\int_{\mathbb{R}^n} |\nabla w|^{p-2}\nabla w \cdot \nabla \xi \, dz$. By the mean value theorem there exists a function $\theta(z)$ with values in $(0,1)$ such that

$$\begin{aligned}
&\left||w_k|^{p-2}w_k - |w_k^+|^{q-1} - |w|^{p-2}w + |w^+|^{q-1}\right| |\xi| \\
&= \left|(p-1)|\theta w_k + (1-\theta)w|^{p-2} - (q-1)|(\theta w_k + (1-\theta)w)^+|^{q-2}\right| |w_k - w||\xi|.
\end{aligned}$$

Integrating on $\mathbb{R}^n$ the previous quantity, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^n} \left||w_k|^{p-2}w_k - |w_k^+|^{q-1} - |w|^{p-2}w + |w^+|^{q-1}\right| |\xi| \\
&\leq (p-1)\int_{\mathbb{R}^n} |\theta w_k + (1-\theta)w|^{p-2}|w_k - w||\xi| \, dz \\
&\quad + (q-1)\int_{\mathbb{R}^n} |(\theta w_k + (1-\theta)w)^+|^{q-2}|w_k - w||\xi| \, dz\,.
\end{aligned}$$

By Hölder inequality the righthand side is bounded from above by

$$\begin{aligned}
&(p-1)\|\theta w_k + (1-\theta)w\|^{p-2}_{L^p(\Xi)}\|w_k - w\|_{L^p(\Xi)}\|\xi\|_{L^p(\Xi)} \\
&+ (q-1)\|\theta w_k + (1-\theta)w\|^{q-2}_{L^q(\Xi)}\|w_k - w\|_{L^q(\Xi)}\|\xi\|_{L^q(\Xi)}\,,
\end{aligned}$$

where $\|w_k - w\|_{L^p(\Xi)}$ and $\|w_k - w\|_{L^q(\Xi)}$ tend to zero by (i). Besides the sequences $\|\theta w_k + (1-\theta)w\|_{L^p(\Xi)}^{p-2}$ and $\|\theta w_k + (1-\theta)w\|_{L^q(\Xi)}^{q-2}$ are bounded.
(iii) Let $t_k = t(w_k)$ be the multiplier for the problem on $\mathbb{R}^n$ defined analogously to Remark 3.1. First of all we prove that there exist $0 < t_1 \le t_2$ such that $t_1 \le t_k \le t_2$ for all $k \in \mathbb{N}$. Since

$$t_k^{q-p} = \frac{\|w_k\|_{H^{1,p}(\mathbb{R}^n)}^p}{\|w_k^+\|_{L^q(\mathbb{R}^n)}^q},$$

we estimate

$$\begin{aligned}
\|w_k\|_{H^{1,p}(\mathbb{R}^n)}^p &\ge \frac{h^{\frac{p}{2}}\epsilon_k^p}{H^{\frac{n}{2}}\epsilon_k^n}\int_{B_g\left(x_k,\frac{R}{2}\right)} |\nabla u_k|_g^p \, d\mu_g + \frac{1}{H^{\frac{n}{2}}\epsilon_k^n}\int_{B_g\left(x_k,\frac{R}{2}\right)} |u_k|^p \, d\mu_g \\
&\ge \frac{\min\{h^{\frac{p}{2}},1\}}{H^{\frac{n}{2}}}\frac{1}{\epsilon_k^n}\int_{B_g\left(x_k,\frac{R}{2}\right)} \left(\epsilon_k^p|\nabla u_k|_g^p + |u_k|^p\right) d\mu_g \\
&\ge \frac{\min\{h^{\frac{p}{2}},1\}pq}{H^{\frac{n}{2}}(q-p)}\gamma, \\
\|w_k\|_{H^{1,p}(\mathbb{R}^n)}^p &\le \frac{H^{\frac{p}{2}}+1}{h^{\frac{n}{2}}}\|u_k\|_{\epsilon_k}^p = \frac{(H^{\frac{p}{2}}+1)pq}{h^{\frac{n}{2}}(q-p)}J_{\epsilon_k}(u_k) \\
&\le \frac{(H^{\frac{p}{2}}+1)pq(m(J)+1)}{h^{\frac{n}{2}}(q-p)}, \\
\|w_k^+\|_{L^q(\mathbb{R}^n)}^q &\ge \frac{1}{H^{\frac{n}{2}}\epsilon_k^n}\int_{B_g\left(x_k,\frac{R}{2}\right)} |u_k^+|^q \, d\mu_g \ge \frac{pq}{H^{\frac{n}{2}}(q-p)}\gamma, \\
\|w_k^+\|_{L^q(\mathbb{R}^n)}^q &\le \frac{1}{h^{\frac{n}{2}}\epsilon_k^n}\|u_k^+\|_{L^q(M)}^q = \frac{pq}{h^{\frac{n}{2}}(q-p)}J_{\epsilon_k}(u_k) \le \frac{pq(m(J)+1)}{h^{\frac{n}{2}}(q-p)},
\end{aligned}$$

where we have used both equations in (4.5). So we consider

$$\begin{aligned}
t_1 &= \left(\frac{\min\{h^{\frac{p}{2}},1\}h^{\frac{n}{2}}\gamma}{H^{\frac{n}{2}}(m(J)+1)}\right)^{\frac{1}{q-p}}, \\
t_2 &= \left(\frac{(H^{\frac{p}{2}}+1)(m(J)+1)H^{\frac{n}{2}}}{h^{\frac{n}{2}}\gamma}\right)^{\frac{1}{q-p}}.
\end{aligned}$$

By the boundedness of $\{t_k\}_{k\in\mathbb{N}}$ we conclude that, up to subsequences, t_k converges to $\bar{t}$.

If $\xi \in C_0^\infty(\mathbb{R}^n)$, it is easy to see that $|J'(t_k w_k)(\xi)| = o(1)\|\xi\|_{H^{1,p}(\mathbb{R}^n)}$. We can then apply the Splitting Lemma 4.5 to the sequence $\{W_k = t_k w_k\}_{k\in\mathbb{N}}$.

In the first case we have that $\{t_k w_k\}_{k\in\mathbb{N}}$ strongly converges to a positive ground state solution $\bar{w}$ in $H^{1,p}(\mathbb{R}^n)$. It is easy to show that $\{t_k w_k\}_{k\in\mathbb{N}}$ strongly converges to $\bar{t}w$. Therefore we conclude that $\bar{w} = \bar{t}w$, and thus $w \neq 0$. Since both w and $\bar{w}$ belong to $\mathcal{N}$, we infer $\bar{t} = 1$ and (iii) follows.

Otherwise, there exist a sequence of points $\{y_k\}_{k\in\mathbb{N}}$, with $|y_k| \to +\infty$, a sequence of functions $\{w_k^0\}_{k\in\mathbb{N}}$ and a ground state solution U of $-\Delta_p w + |w|^{p-2}w = (w^+)^{q-1}$ on $\mathbb{R}^n$ such that, up to a subsequence, $W_k(z) = t_k w_k(z) = w_k^0(z) + U(z-y_k)$ where $\{w_k^0\}_{k\in\mathbb{N}}$ tends strongly to zero in $H^{1,p}(\mathbb{R}^n)$.

We can consider three different cases:

(a) $\lim_{k\to\infty} |y_k| - \frac{R}{\epsilon_k} \geq 2T > 0$ for some $T > 0$;
(b) $\lim_{k\to\infty} |y_k| - \frac{R}{\epsilon_k} = 0$;
(c) $\lim_{k\to\infty} \frac{R}{\epsilon_k} - |y_k| \geq 2T > 0$ for some $T > 0$.

(a) Since by definition $W_k \equiv 0$ in $\mathbb{R}^n \setminus B\left(0, \frac{R}{\epsilon_k}\right)$, we have that $w_k^0(z) = -U(z - y_k)$ for any $z \in \mathbb{R}^n \setminus B\left(0, \frac{R}{\epsilon_k}\right)$. Then we have

$$\begin{aligned}
&\int_{\mathbb{R}^n \setminus B\left(0, \frac{R}{\epsilon_k}\right)} \left(|\nabla w_k^0(z)|^p + |w_k^0(z)|^p\right) dz \\
&= \int_{\mathbb{R}^n \setminus B\left(0, \frac{R}{\epsilon_k}\right)} \left(|\nabla U(z - y_k)|^p + |U(z - y_k)|^p\right) dz \\
&\geq \int_{B(y_k, T)} \left(|\nabla U(z - y_k)|^p + |U(z - y_k)|^p\right) dz \\
&= \int_{B(y_k, T)} \left(|\nabla U(z - y_k)|^p + |U(z - y_k)|^p\right) dz + o(1) \\
&= \int_{B(0,T)} \left(|\nabla w(z)|^p + |w(z)|^p\right) dz + o(1) > 0
\end{aligned}$$

and this is in contradiction with the fact that w_k^0 tends strongly to zero.

(b) If $\lim_{k\to\infty} |y_k| - \frac{R}{\epsilon_k} = 0$, let $\pi(y_k)$ denote the projection of y_k onto the sphere centered in the origin with radius $\frac{R}{\epsilon_k}$. Then

$$\begin{aligned}
&\int_{\mathbb{R}^n \setminus B\left(0, \frac{R}{\epsilon_k}\right)} \left(|\nabla w_k^0(z)|^p + |w_k^0(z)|^p\right) dz \\
&= \int_{\mathbb{R}^n \setminus B\left(0, \frac{R}{\epsilon_k}\right)} \left(|\nabla U(z - y_k)|^p + |U(z - y_k)|^p\right) dz \\
&\geq \int_{\left\{z \in B(y_k, T) \;\middle|\; |z| \geq \frac{R}{\epsilon_k}\right\}} \left(|\nabla U(z - y_k)|^p + |U(z - y_k)|^p\right) dz + o(1) \\
&= \int_{\left\{z \in B(O, T) \;\middle|\; |y_k| \geq \frac{R}{\epsilon_k}\right\}} \left(|\nabla w(z)|^p + |w(z)|^p\right) dz + o(1) \\
&= \int_{\left\{z \in B(0, T) \;\middle|\; |z + \pi(y_k)| \geq \frac{R}{\epsilon_k}\right\}} \left(|\nabla U(z)|^p + |U(z)|^p\right) dz + o(1) \\
&= \int_{\left\{z \in B(0, T) \;\middle|\; z \cdot \frac{(y_k)}{|y_k|} \geq 0\right\}} \left(|\nabla U(z)|^p + |U(z)|^p\right) dz + o(1) \\
&\geq \min_{\zeta \in S^n} \int_{\{z \in B(0,T) \mid z \cdot \zeta \geq 0\}} \left(|\nabla U(z)|^p + |U(z)|^p\right) dz + o(1) \\
&= C + o(1) > 0\,,
\end{aligned}$$

where S^n is the unit sphere in $\mathbb{R}^n$, $z \cdot \zeta$ is the scalar product in $\mathbb{R}^n$ and C is a positive constant. The previous quantity is greater than $\frac{C}{2}$ for k sufficiently large, which is a contradiction.

(c) Finally, if $\lim_{k\to+\infty} \frac{R}{\epsilon_k} - |y_k| \geq 2T > 0$, for k sufficiently large $B(y_k,T)$ is contained in $B\left(0, \frac{R}{\epsilon_k}\right)$. There holds

$$\left(\frac{1}{p}-\frac{1}{q}\right)\int_{B(y_k,T)} |U(z-y_k)|^q dz$$
$$= \left(\frac{1}{p}-\frac{1}{q}\right)\int_{B(0,T)} |U(z)|^q dz = \gamma_0 > 0\,.$$

We consider the new sequence of points

$$\tilde{x}_k = \exp_{x_k}(\epsilon_k y_k) \in B_g(x_k, R)\,.$$

For k sufficiently large, if we set $U(\tilde{x}_k) = \exp_{x_k}(\epsilon_k B(y_k,T))$, which is a neighborhood of $\tilde{x}_k$, then

$$\frac{1}{\epsilon_k^n}\int_{U(\tilde{x}_k)} |u_k^+|^q d\mu_g = \frac{1}{\epsilon_k^n}\int_{\epsilon_k B(y_k,T)} |u_k^+(\exp_{x_k}(z))|^q |g_{x_k}(z)|^{\frac{1}{2}} dz$$
$$\geq h^{\frac{n}{2}}\int_{B(y_k,T)} |w_k^+(z)|^q dz\,.$$

Since $t_k \in [t_1, t_2]$,

$$\int_{B(y_k,T)} |w_k^+(z)|^q dz \geq \frac{1}{t_2^q}\int_{B(y_k,T)} |t_k w_k^+(z)|^q dz\,.$$

By the Splitting Lemma 4.5 we have

$$\int_{B(y_k,T)} |W_k^+(z)|^q dz = \int_{B(y_k,T)} \left|\left(w_k^0(z) + U(z-y_k)\right)^+\right|^q dz$$
$$= \int_{B(y_k,T)} |U(z-y_k)|^q\, dz + o(1)$$
$$= \int_{B(0,T)} |U(z)|^q dz + o(1) = \frac{pq}{q-p}\gamma_0 + o(1)\,.$$

So we have proved that for any k sufficiently large

$$\frac{1}{\epsilon_k^n}\int_{U(\tilde{x}_k)} |u_k^+|^q d\mu_g > \widetilde{\gamma}_0 > 0\,. \tag{4.10}$$

By definition, for k big enough $U(\tilde{x}_k)$ is contained in $B_g(\tilde{x}_k, R)$ and so we can substitute x_k by $\tilde{x}_k$ and w_k by $\widetilde{w}_k$, defined as in (4.6) with the new choice of points. Steps (i) and (ii) are independent of x_k (provided w_k is not identically zero) and so $\widetilde{w}_k$ tends weakly to a weak solution $\widetilde{w}$. It is possible to see that there exists $\widetilde{T} > 0$ such that $U(\tilde{x}_k) \subset B_g(\tilde{x}_k, \epsilon_k\widetilde{T})$ for any k. Then we have

$$\int_{B(0,\widetilde{T})} \left|\widetilde{w}_k^+(z)\right|^q dz \geq \frac{1}{H^{\frac{n}{2}}\epsilon_k^n}\int_{B_g(\tilde{x}_k,\epsilon_k\widetilde{T})} |u_k^+(x)|^q d\mu_g$$
$$\geq \frac{1}{H^{\frac{n}{2}}\epsilon_k^n}\int_{U(\tilde{x}_k)} |u_k^+(x)|^q d\mu_g\,.$$

By (4.10) and by the strong convergence of $\widetilde{w}_k$ to $\widetilde{w}$ in $L^q(B(0,\widetilde{T}))$, we conclude that

$$\int_{B(0,\widetilde{T})} \left|\widetilde{w}^+(z)\right|^q dz \geq \frac{\widetilde{\gamma}_0}{H^{\frac{n}{2}}}$$

and so $\widetilde{w} \not\equiv 0$ and $\widetilde{w} \in \mathcal{N}$.

From now we write as before w_k instead of $\tilde{w}_k$, x_k instead of $\tilde{x}_k$ and w instead of $\tilde{w}$. Finally the last step is to show that $J(w) = m(J)$. Let us consider the following inequalities

$$\begin{aligned} m(J) + \delta_k \geq J_{\epsilon_k}(u_k) &= \frac{1}{\epsilon_k^n}\left(\frac{1}{p} - \frac{1}{q}\right)\|u_k^+\|^q_{L^q(M)} \\ &\geq \left(\frac{1}{p} - \frac{1}{q}\right)\int_{\mathbb{R}^n} |w_k^+|^q |g_{x_k}(\epsilon_k z)|^{\frac{1}{2}}\, dz\,. \end{aligned} \tag{4.11}$$

We define the sequence of functions in $L^q(\mathbb{R}^n)$:

$$F_k(z) = \left(\frac{1}{p} - \frac{1}{q}\right)^{\frac{1}{q}} |w_k^+(z)||g_{x_k}(\epsilon_k z)|^{\frac{1}{2q}}\,.$$

By (4.11) this sequence is bounded in $L^q(\mathbb{R}^n)$ and there exists a weak limit $F \in L^q(\mathbb{R}^n)$. We prove that

$$F(z) = \left(\frac{1}{p} - \frac{1}{q}\right)^{\frac{1}{q}} |w^+(z)|\,. \tag{4.12}$$

Let ξ be in $C_0^\infty(\mathbb{R}^n)$. There holds

$$F_k(z)\xi(z) \to \left(\frac{1}{p} - \frac{1}{q}\right)^{\frac{1}{q}} |w^+(z)|\xi(z)$$

for almost every $z \in \Xi$, the support of ξ. We can now apply Lebesgue theorem. In fact, there holds

$$|F_k(z)|\,|\xi(z)| \leq H^{\frac{n}{2q}}\left(\frac{1}{p} - \frac{1}{q}\right)^{\frac{1}{q}} |w_k^+(z)||\xi(z)|$$

and, since w_k converges strongly to w in $L^q(\Xi)$, there exists $W \in L^q(\Xi)$ such that for all k $|w_k(z)| \leq W(z)$ almost everywhere and $|F_k(z)|\,|\xi(z)| \leq H^{\frac{n}{2q}}\left(\frac{1}{p} - \frac{1}{q}\right)^{\frac{1}{q}} W(z)|\xi(z)| \in L^q(\Xi)$. So (4.12) is proved. By weak lower semicontinuity of the norm

$$\|F\|^q_{L^q(\mathbb{R}^n)} \leq \liminf_{k\to\infty} \|F_k\|^q_{L^q(\mathbb{R}^n)}\,,$$

that is

$$\left(\frac{1}{p} - \frac{1}{q}\right)\int_{\mathbb{R}^n} |w^+|^q dz \leq \liminf_{k\to\infty}\left(\frac{1}{p} - \frac{1}{q}\right)\int_{\mathbb{R}^n} |w_k^+|^q |g_{x_k}(\epsilon_k z)|^{\frac{1}{2}}\, dz\,.$$

By this inequality and (4.11) we conclude that

$$\begin{aligned} m(J) = \lim_{k\to\infty} m(J) + \delta_k &\geq \lim_{k\to\infty} J_{\epsilon_k}(u_k) \\ &\geq \liminf_{k\to\infty}\left(\frac{1}{p} - \frac{1}{q}\right)\int_{\mathbb{R}^n} |w_k^+(z)|^q |g_{x_k}(\epsilon_k z)|^{\frac{1}{2}}\, dz \\ &\geq \left(\frac{1}{p} - \frac{1}{q}\right)\int_{\mathbb{R}^n} |w^+(z)|^q dz \geq m(J)\,. \end{aligned} \tag{4.13}$$

The equality is immediate from (4.13). □

We recall here Ekeland Principle (see for instance [14]).

Definition 4.7. *Let X be a complete metric space and $\Psi : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semi-continuous function on X, bounded from below. Given $\eta > 0$ and $\bar{u} \in X$ such that*

$$\Psi(\bar{u}) < \inf_{u \in X} \Psi(u) + \frac{\eta}{2},$$

for all $\lambda > 0$ there exists $u_\lambda \in X$ such that

$$\Psi(u_\lambda) < \Psi(\bar{u}), \qquad d(u_\lambda, \bar{u}) < \lambda$$

and for all $u \neq u_\lambda$ it holds

$$\Psi(u_\lambda) < \Psi(u) + \frac{\eta}{\lambda} d(u_\lambda, u).$$

Remark 4.8.

(1) *We apply Lemma 4.6 when u_k is a minimum solution $u_k \in \mathcal{N}_{\epsilon_k}$, $J_{\epsilon_k}(u_k) = m_{\epsilon_k}$. By (iv) we have $\lim_{k\to\infty} m_{\epsilon_k} = m(J)$. In particular there exists a nondecreasing function*

$$\delta \in (0, +\infty) \mapsto \epsilon_1(\delta) \in (0, +\infty)$$

such that if $\epsilon \in \big(0, \epsilon_1(\delta)\big)$, then $|m_\epsilon - m(J)| < \delta$.

(2) *Applying Ekeland principle for $X = \Sigma_{\epsilon,\delta}$, with $\epsilon \le \epsilon_1(\delta)$ as in (1), we obtain that for all $\bar{u} \in \Sigma_{\epsilon,\delta}$ there exists $u_\delta \in \Sigma_{\epsilon,\delta}$ such that*

$$J_\epsilon(u_\delta) < J_\epsilon(\bar{u}), \qquad \|u_\delta - \bar{u}\|_\epsilon < 4\sqrt{\delta}$$

and for all $u \in T\Sigma_{\epsilon,\delta}$

$$|J'_\epsilon(u_\delta)(u)| < \sqrt{\delta}\|u\|_\epsilon. \tag{4.14}$$

Proof of Proposition 4.4. By contradiction we assume that there exist $\eta_0 \in (0, 1)$, two positive sequences $\{\delta_k\}_{k\in\mathbb{N}}$, $\{\epsilon_k\}_{k\in\mathbb{N}}$ tending to zero as k tends to infinity and a sequence of functions $\{u_k\}_{k\in\mathbb{N}}$, with $u_k \in \Sigma_{\epsilon_k,\delta_k}$, such that for any $x \in M$

$$\left(\frac{1}{p} - \frac{1}{q}\right) \frac{1}{\epsilon_k^n} \int_{B_g\left(x, \frac{r(M)}{2}\right)} |u_k^+|^q d\mu_g \le \eta_0\, m(J). \tag{4.15}$$

By Ekeland principle for any k we can consider $\tilde{u}_k$ as in 2 of Remark 4.8. Property (4.15) becomes

$$\left(\frac{1}{p} - \frac{1}{q}\right) \frac{1}{\epsilon_k^n} \int_{B_g\left(x, \frac{r(M)}{2}\right)} |\tilde{u}_k^+|^q d\mu_g \le \eta_1\, m(J) \tag{4.16}$$

with η_1 still in $(0, 1)$. To prove this we have to evaluate the difference

$$\frac{1}{\epsilon_k^n} \int_{B_g\left(x, \frac{r(M)}{2}\right)} \big||\tilde{u}_k^+|^q - |u_k^+|^q\big|\, d\mu_g,$$

which by mean value theorem can be written

$$\frac{q}{\epsilon_k^n} \int_B \left|(u_k^*)^+\right|^{q-1} |\tilde{u}_k - u_k|\, d\mu_g, \tag{4.17}$$

where B is $B_g\left(x, \frac{r(M)}{2}\right)$ and $u_k^*(x) = \theta(x)\tilde{u}_k(x) + (1 - \theta(x))u_k(x)$ for a suitable function $\theta(x)$ with values in $(0, 1)$. By Hölder inequality (4.17) is bounded from above by

$$q\left(\frac{1}{\epsilon_k^n} \int_B |(u_k^*)^+|^q d\mu_g\right)^{\frac{q-1}{q}} \left(\frac{1}{\epsilon_k^n} \int_B |\tilde{u}_k - u_k|^q d\mu_g\right)^{\frac{1}{q}}. \tag{4.18}$$

We prove that the first factor in (4.18) is bounded and the second one is infinitesimal. In fact, for positive constants C, C', C'', we have

$$\begin{aligned}\frac{1}{\epsilon_k^n}\int_B |(u_k^*)^+|^q d\mu_g &\leq \frac{1}{\epsilon_k^n}\int_B |\tilde{u}_k^+ + u_k^+|^q d\mu_g \leq \frac{C}{\epsilon_k^n}\int_B \left(|\tilde{u}_k^+|^q + |u_k^+|^q\right) d\mu_g \\ &\leq C'\left(J_{\epsilon_k}(\tilde{u}_k) + J_{\epsilon_k}(u_k)\right) \leq C''\,.\end{aligned}$$

For the second factor, if it is zero for infinitely many k, we finished. Otherwise, on each chart with domain $B_g(x_\alpha, R)$ overlapping B, we consider the functions on $\mathbb{R}^n$

$$v_{\alpha,k}(z) = \left(\tilde{u}_k(\exp_{x_\alpha}(\epsilon_k z)) - u_k(\exp_{x_\alpha}(\epsilon_k z))\right)\chi_{\frac{2R}{\epsilon_k}}(z)\,.$$

Then we have

$$\begin{aligned}&\frac{1}{\epsilon_k^n}\int_{B_g(x_\alpha,R)} |\tilde{u}_k - u_k|^q d\mu_g \leq \frac{H^{\frac{n}{2}}}{\epsilon_k^n}\int_{B(0,R)} |\tilde{u}_k(\exp_{x_\alpha}(y)) - u_k(\exp_{x_\alpha}(y))|^q dy \\ &\quad \leq H^{\frac{n}{2}}\int_{\mathbb{R}^n} |v_{\alpha,k}(z)|^q dz\,.\end{aligned}$$

As it is written in the proof of (iii), Lemma 4.6, there exist $0 < t_1 \leq t_2$ such that $t_1 \leq t_k, t_k' \leq t_2$, with $t_k v_{k,\alpha} \in \mathcal{N}$ and $-t_k' v_{k,\alpha} \in \mathcal{N}$, so

$$\begin{aligned}\int_{\mathbb{R}^n} |v_{\alpha,k}(z)|^q dz &= \int_{\mathbb{R}^n}\left(|(v_{\alpha,k})^+(z)|^q + |(v_{\alpha,k})^-(z)|^q\right) dz \\ &\leq \frac{1}{t_1^q}\int_{\mathbb{R}^n}\left(|t_k(v_{\alpha,k})^+(z)|^q + |t_k'(-v_{\alpha,k})^+(z)|^q\right) dz \\ &= \frac{1}{t_1^q}\int_{\mathbb{R}^n}\left(|t_k\nabla v_{\alpha,k}(z)|^p + |t_k v_{\alpha,k}(z)|^p + |t_k'\nabla v_{\alpha,k}(z)|^p + |t_k' v_{\alpha,k}(z)|^p\right) dz \\ &\leq \frac{2t_2^p}{t_1^q}\int_{\mathbb{R}^n}\left(|\nabla v_{\alpha,k}(z)|^p + |v_{\alpha,k}(z)|^p\right) dz \\ &\leq \frac{2t_2^p H^{\frac{p}{2}}}{t_1^q h^{\frac{n}{2}}}\|\tilde{u}_k - u_k\|_{\epsilon_k}^p \leq \frac{2^{2p+1}t_2^p H^{\frac{p}{2}}}{t_1^q h^{\frac{n}{2}}}\delta_k^{\frac{p}{2}}\end{aligned}$$

and this proves that the second factor in (4.18) is infinitesimal.

We apply Lemma 4.6 to the sequences $\{\delta_k\}_{k\in\mathbb{N}}$, $\{\epsilon_k\}_{k\in\mathbb{N}}$ and $\{\tilde{u}_k\}_{k\in\mathbb{N}}$, obtaining a sequence of functions on $\mathbb{R}^n$ $\{w_k\}_{k\in\mathbb{N}}$. Let w be the weak limit in $H^{1,p}(\mathbb{R}^n)$ of w_k. Let η_2 be a constant in $(0,1)$ such that $\eta_2 > \frac{1+\eta_1}{2}$. Since $J(w) = m(J)$, there exists $T > 0$ such that

$$\left(\frac{1}{p} - \frac{1}{q}\right)\int_{B(0,T)} |w^+(z)|^q dz \geq \eta_2 m(J)\,. \tag{4.19}$$

On the other hand, up to a subsequence, we have

$$\begin{aligned}\left(\frac{1}{p} - \frac{1}{q}\right)\int_{B(0,T)} |w^+|^q dz &= \lim_{k\to\infty}\left(\frac{1}{p} - \frac{1}{q}\right)\int_{B(0,T)} |w_k^+|^q dz \\ &= \lim_{k\to\infty}\left(\frac{1}{p} - \frac{1}{q}\right)\frac{1}{\epsilon_k^n}\int_{B(0,\epsilon_k T)}\left|\left(\tilde{u}_k \circ \exp_{x_k}\right)^+\right|^q dz\,.\end{aligned} \tag{4.20}$$

By compactness the sequence $\{x_k\}_{k\in\mathbb{N}}$ converges (up to a subsequence) to $\bar{x}$ and for any $z \in B(0,T)$ the limit of $|g_{x_k}(\epsilon_k z)|^{\frac{1}{2}}$ for k tending to infinity is $|g_{\bar{x}}(0)|^{\frac{1}{2}} = 1$.

Since $\frac{2\eta_1}{1+\eta_1} \in (0,1)$, for k sufficiently big for any $y \in B(0, \epsilon_k T)$ we have $|g_{x_k}(y)|^{\frac{1}{2}} > \frac{2\eta_1}{1+\eta_1}$. So the last limit in (4.20) is less than

$$\begin{aligned}
&\frac{1+\eta_1}{2\eta_1} \lim_{k\to\infty} \left(\frac{1}{p} - \frac{1}{q}\right) \frac{1}{\epsilon_k^n} \int_{B(0,\epsilon_k T)} \left| \left(\tilde{u}_k \circ \exp_{x_k}\right)^+ \right|^q |g_{x_k}(z)|^{\frac{1}{2}} dz \\
&\qquad = \frac{1+\eta_1}{2\eta_1} \lim_{k\to\infty} \left(\frac{1}{p} - \frac{1}{q}\right) \frac{1}{\epsilon_k^n} \int_{B_g(x_k,\epsilon_k T)} \left|\tilde{u}_k^+\right|^q d\mu_g \le \frac{1+\eta_1}{2} m(J)\,,
\end{aligned}$$

where we have used property (4.16). By this inequality together with (4.20) and (4.19) we get $\eta_2 \le \frac{1+\eta_1}{2}$ which is in contradiction with the choice of η_2. $\square$

It is now possible to prove the following proposition:

Proposition 4.9. *There exists $\bar{\delta} \in (0, m(J))$ such that for any $\delta \in (0, \bar{\delta})$ there exists $\epsilon_1 = \epsilon_1(\delta) > 0$ such that for any $\epsilon \in (0, \epsilon_1(\delta))$ and $u \in \Sigma_{\epsilon,\delta}$ the barycenter $\beta(u)$ is in $M_{r(M)}$. Moreover $\delta \in (0, \bar{\delta}) \mapsto \epsilon_1(\delta)$ is a nondecreasing function.*

Proof. Denoting by D be the diameter of the manifold M, it is natural to assume that $r(M) < D$. Let $\bar{\eta} = 1 - \frac{r(M)}{4D} \in (0,1)$ and $\bar{\delta} = \min\{\delta_1(\bar{\eta}),\ r(M)m(J)/4D\}$, where $\delta_1(\bar{\eta})$ is defined by Proposition 4.4. So, in particular, $\bar{\delta} \le \delta_1(\bar{\eta}) < m(J)$ and Proposition 4.4 defines $\epsilon_1(\delta)$, which is nondecreasing with respect to δ. For any $\delta \in (0, \bar{\delta})$, $\epsilon \in (0, \epsilon_1(\delta))$ and $u \in \Sigma_{\epsilon,\delta}$, there exists a point x_0 such that

$$\left(\frac{1}{p} - \frac{1}{q}\right) \frac{1}{\epsilon^n} \int_{B_g\left(x_0, \frac{r(M)}{2}\right)} |u^+|^q d\mu_g > \bar{\eta}\, m(J)\,.$$

Since $u \in \Sigma_{\epsilon,\delta}$ we also have

$$\left(\frac{1}{p} - \frac{1}{q}\right) \frac{1}{\epsilon^n} \int_M |u^+|^q d\mu_g \le m(J) + \delta\,.$$

By the previous inequalities we have then

$$\frac{\int_{B_g\left(x_0, \frac{r(M)}{2}\right)} |u^+(x)|^q \, d\mu_g}{\int_M |u^+(x)|^q d\mu_g} > \frac{\bar{\eta}}{1 + \frac{\delta}{m(J)}}\,.$$

We can now esteem

$$\begin{aligned}
|\beta(u) - x_0| &= \left| \frac{\int_M (x - x_0)|u^+(x)|^q \, d\mu_g}{\int_M |u^+(x)|^q \, d\mu_g} \right| \\
&\le \left| \frac{\int_{B_g\left(x_0, \frac{r(M)}{2}\right)} (x - x_0)|u^+(x)|^q \, d\mu_g}{\int_M |u^+(x)|^q \, d\mu_g} \right| + \left| \frac{\int_{M \setminus B_g\left(x_0, \frac{r(M)}{2}\right)} (x - x_0)|u^+(x)|^q \, d\mu_g}{\int_M |u^+(x)|^q \, d\mu_g} \right| \\
&\le \frac{r(M)}{2} + D\left(1 - \frac{\bar{\eta}}{1 + \frac{\delta}{m(J)}}\right) < r(M).
\end{aligned}$$

$\square$

5. The function I_ϵ

We prove now that the composition I_ϵ of ϕ_ϵ and β is well defined and homotopic to the identity on M:

Proposition 5.1. *There exists $\epsilon_2 > 0$ such that for any $\epsilon \in (0, \epsilon_2)$ the composition*

$$I_\epsilon = \beta \circ \phi_\epsilon : M \to M_{r(M)}$$

is well defined and homotopic to the identity on M.

Proof. By Proposition 3.2 and 4.9, I_ϵ is well defined, choosing ϵ suitably small.

Let us consider the function $H : [0,1] \times M \to M_{r(M)}$, defined by $H(t,x) = tI_\epsilon(x) + (1-t)x$. This function is a homotopy if for any $t \in [0,1]$ $H(t,x) \in M_{r(M)}$. It is enough to prove that for any $x_0 \in M$ $|I_\epsilon(x_0) - x_0| < r(M)$. Since the support of $\phi_\epsilon(x_0)$ is contained in $B_g(x_0, R)$, taking account of Remark 2.2 we have

$$\begin{aligned} |I_\epsilon(x_0) - x_0| &= \frac{\int_M (x - x_0)\,(\phi_\epsilon(x_0)(x))^q d\mu_g}{\int_M (\phi_\epsilon(x_0)(x))^q d\mu_g} \\ &= \frac{\int_{B(0,\frac{R}{\epsilon})} (exp_{x_0}(\epsilon z) - exp_{x_0}(0))(U(z) - \widetilde{U}_{\frac{R}{\epsilon}})^q |g_{x_0}(\epsilon z)|^{\frac{1}{2}} dz}{\int_{B(0,\frac{R}{\epsilon})} (U(z) - \widetilde{U}_{\frac{R}{\epsilon}})^q |g_{x_0}(\epsilon z)|^{\frac{1}{2}} dz} \\ &\le \frac{\tilde{h} H^{\frac{n}{2}} \epsilon \int_{B(0,\frac{R}{\epsilon})} |z| (U(z) - \widetilde{U}_{\frac{R}{\epsilon}})^q dz}{h^{\frac{n}{2}} \int_{B(0,\frac{R}{\epsilon})} (U(z) - \widetilde{U}_{\frac{R}{\epsilon}})^q dz} \end{aligned}$$

where $\widetilde{U}_{\frac{R}{\epsilon}}$ is the value $U(z)$ for any $z \in \mathbb{R}^n$ such that $|z| = \frac{R}{\epsilon}$. Since U decays exponentially, we conclude that there exists a positive constant C_1 such that $|I_\epsilon(x_0) - x_0| \le C_1 \epsilon$. □

We immediately infer the following lemma.

Corollary 5.2. *There exists $\bar{\delta} > 0$ such that for each $\delta \in (0, \bar{\delta})$, there exists $\bar{\epsilon}(\delta) > 0$ such that*

$$cat(M) \le cat(\Sigma_{\epsilon,\delta}).$$

Proof of Theorem 1.1. Since M is a compact manifold, it is standard to prove that for any $\varepsilon > 0$ the functional J_ε satisfies the $(P.S.)$ condition on $\mathcal{N}_\varepsilon$. The existence of $cat(M)$ critical point of J with energy less then $m(J)+\delta$ follows from Corollary 5.2 and classical results in Lusternick- Schnirelmann theory. The existence of a critical point u of J with $m(J) + \delta < J_\varepsilon(u) < c$ can be derived arguing as in section 6 of [4] (see also [7]). We remain to prove that the found solutions are not constant. This follows from the fact that the only constant solution to (P_ε) is $\bar{u} = 1$, for which $J_\varepsilon(\bar{u}) = \frac{q-p}{qp} \frac{\mu_g(M)}{\varepsilon^n} \to +\infty$, as $\varepsilon \to 0^+$. □

Proposition 5.1 has another immediate consequence.

Corollary 5.3. *There exists $\bar{\delta} > 0$ such that for each $\delta \in (0, \bar{\delta})$, there exists $\bar{\epsilon}(\delta) > 0$ such that*

$$\dim H^k(\Sigma_{\epsilon,\delta}) \ge \dim H^k(M).$$

Moreover $\delta \in (0, \bar{\delta}) \mapsto \bar{\epsilon}(\delta)$ *is a nondecreasing function.*

Proof. Let $\bar{\delta}$ and $\epsilon_1(\delta)$ be defined by Proposition 4.9. The thesis follows from Proposition 5.1, choosing $\bar{\epsilon}(\delta) = \min\{\epsilon_1(\delta), \epsilon_2\}$. □

Remark 5.4. *Following the notations of the previous corollary, we can assume that for any $\delta \in (0, \bar{\delta})$ and for any $\epsilon \in (0, \bar{\epsilon}(\delta))$ there exists $\delta' \in [\delta, \bar{\delta})$ such that $m(J) + \delta'$ is a regular value for J_ϵ and, moreover, ϵ belongs also to $(0, \bar{\epsilon}(\delta'))$, due to the monotonicity of function $\bar{\epsilon}$.*
In fact, if not, there is $\tilde{\delta} \in (0, \bar{\delta})$ and $\tilde{\epsilon} \in (0, \bar{\epsilon}(\tilde{\delta}))$ such that $m(J) + \delta'$ is critical for any $\delta' \in [\tilde{\delta}, \bar{\delta})$, so that, considering also Remark 2.5, the corresponding problem (P_ϵ) has infinitely many solutions and Theorems 1.2-1.3 are both proved.

6. Morse Polynomials

Definition 6.1. *Let $\mathbb{K}$ be a field and X a topological space. For any $B \subset A \subset X$, we denote $\mathcal{P}_t(A, B)$ the Poincaré polynomial of the topological pair (A, B), defined by*

$$\mathcal{P}_t(A, B) = \sum_{k=0}^{+\infty} \dim H^k(A, B)\, t^k.$$

where $H^k(A, B)$ stands for the k-th Alexander-Spanier relative cohomology group of (A, B), with coefficient in $\mathbb{K}$; we also set $H^k(A) = H^k(A, \emptyset)$ and $\mathcal{P}_t(A) = \mathcal{P}_t(A, \emptyset)$ is called the Poincaré polynomial of A.

In what follows, if $a \le b$, we denote by
${J_\epsilon}^b = \{u \in H^{1,p}(M) \mid J_\epsilon(u) \le b\}$
$J_a^b = \{u \in H^{1,p}(M) \mid a \le J_\epsilon(u) \le b\}$, where we skip ϵ for simplicity
$\mathrm{int}(J_a^b) = \{u \in H^{1,p}(M) \mid a < J_\epsilon(u) < b\}$.

Lemma 6.2. *$J_\epsilon^{-1}\{a\} \setminus \mathcal{N}_\epsilon$ is a deformation retract of $J_a^b \setminus \mathcal{N}_\epsilon$, for any $a > 0$ and $b \ge a$. In particular, if $a \in (0, m_\epsilon)$ and $b \ge a$, then $J_\epsilon^{-1}\{a\}$ is a deformation retract of $J_a^b \setminus \mathcal{N}_\epsilon$.*

Proof. Let $D = H^{1,p}(M) \setminus \big(\mathcal{N}_\epsilon \cup \{0\}\big)$, $C = \{u \in H^{1,p}(M) \setminus \{0\} \mid u \le 0\ a.e.\}$.
For any $u \in C$, the function

$$f : [0, +\infty) \to \mathbb{R} \quad f(t) = J_\epsilon(tu)$$

is strictly increasing, while, if $u \in D \setminus C$, f has exactly one maximum point θ_u and $\theta_u \neq 1$, since $u \notin \mathcal{N}_\epsilon$. So we also define the sets $A = \{u \in D \setminus C \mid \theta_u < 1\}$ and $B = \{u \in D \setminus C \mid \theta_u > 1\}$. It is apparent that $J_a^b \setminus \mathcal{N}_\epsilon \subset (A \cup B \cup C)$.

If $u \in J_a^b \cap A$, let $\delta(u)$ be the only value $t \ge 1$ such that $J_\epsilon(tu) = a$.

If $u \in J_a^b \cap (B \cup C)$, let $\delta(u)$ be defined as the only $t \in (0, 1]$ such that $J_\epsilon(tu) = a$.

In this way

$$\forall t \in [0, 1], \forall u \in J_a^b \setminus \mathcal{N}_\epsilon \quad (t\delta(u) + 1 - t)\, u \in J_a^b \setminus \mathcal{N}_\epsilon.$$

The function $\delta : J_a^b \setminus \mathcal{N}_\epsilon \to \mathbb{R}$ is continuous. In fact, let $F : (0, +\infty) \times H^{1,p}(M) \to \mathbb{R}$ be defined by $F(t, u) = J_\epsilon(tu) - a$. For any $u_0 \in J_a^b \setminus \mathcal{N}_\epsilon$, $F(\delta(u_0), u_0) = 0$ and, as $\delta(u_0)u_0 \notin \mathcal{N}_\epsilon$,

$$\frac{\partial F}{\partial t}(\delta(u_0), u_0) = \langle {J_\epsilon}'(\delta(u_0)u_0), u_0\rangle \neq 0$$

so, by the Implicit Function Theorem, δ is continuous.

Now let $H : [0, 1] \times \big(J_a^b \setminus \mathcal{N}_\epsilon\big) \to H^{1,p}(M)$ be defined by $H(t, u) = \big(t\,\delta(u) + 1 - t\big)u$. The proof is completed, as we see immediately that:

- H is continuous;

- $H(0,u) = u \quad \forall\, u;$
- $J_\epsilon(H(1,u)) = a \quad \forall\, u;$
- $H(t,u) \in J_a^b \setminus \mathcal{N}_\epsilon \quad \forall\, t,\ \forall\, u;$
- $H(t,u) = u \quad \forall\, t,\ \forall\, u \in J_\epsilon^{-1}\{a\} \setminus \mathcal{N}_\epsilon.$

□

Proposition 6.3. *If $a \in (0, m_\epsilon)$ and $b \geq a$ is a noncritical level for J_ϵ, then*

$$\mathcal{P}_t\left({J_\epsilon}^b, {J_\epsilon}^a\right) = t\,\mathcal{P}_t\left(J_a^b \cap \mathcal{N}_\epsilon\right).$$

Proof. We recall (see Lemma 5.3 in [6]) that if $\mathcal{M}$ is a manifold, $\mathcal{N} \subset \mathcal{M}$ a closed oriented submanifold of codimension d and W is a subset of $\mathcal{N}$ closed in $\mathcal{N}$, then

$$\mathcal{P}_t(\mathcal{M}, \mathcal{M} \setminus W) = t^d\ \mathcal{P}_t(\mathcal{N}, \mathcal{N} \setminus W).$$

Taking account of Remark 2.4, if we set $\mathcal{M} = \operatorname{int}(J_a^b)$, $\mathcal{N} = \mathcal{M} \cap \mathcal{N}_\epsilon$ and $W = \mathcal{N}$, the previous equality gives

$$\mathcal{P}_t\left(\operatorname{int}(J_a^b), \operatorname{int}(J_a^b) \setminus \mathcal{N}_\epsilon\right) = t\ \mathcal{P}_t\left(\operatorname{int}(J_a^b) \cap \mathcal{N}_\epsilon\right)$$

hence, as a and b are not critical values for J_ϵ, we have

$$\mathcal{P}_t\left(J_a^b, J_a^b \setminus \mathcal{N}_\epsilon\right) = t\ \mathcal{P}_t\left(J_a^b \cap \mathcal{N}_\epsilon\right). \tag{6.1}$$

Since by excision

$$\mathcal{P}_t\left({J_\epsilon}^b, {J_\epsilon}^a\right) = \mathcal{P}_t\left(J_a^b, J_\epsilon^{-1}\{a\}\right),$$

the assert comes by (6.1) and Lemma 6.2.

□

Proposition 6.4. *There exist a, b, $\hat{\epsilon} > 0$ such that, for any $\epsilon \in (0, \hat{\epsilon})$, $a \in (0, m_\epsilon)$, $b > m_\epsilon$ and*

$$\mathcal{P}_t(J_\epsilon^b, J_\epsilon^a) = t\mathcal{P}_t(M) + t\mathcal{Z}_\epsilon(t) \tag{6.2}$$

$$\mathcal{P}_t(H^{1,p}(M), J_\epsilon^b) = t^2\big(\mathcal{P}_t(M) - 1 + \mathcal{Z}_\epsilon(t)\big) \tag{6.3}$$

where $\mathcal{Z}_\epsilon(t)$ is a polynomial with nonnegative integer coefficients.

Proof. Following the notations of Corollary 5.3, fix $\hat{\delta} \in (0, \bar{\delta})$ and $\hat{\epsilon} = \bar{\epsilon}(\hat{\delta})$. Let $a \in (0, m_\epsilon)$. By Remark 5.4, for any $\epsilon \in (0, \hat{\epsilon})$, there is $\delta' \in (\hat{\delta}, \bar{\delta})$ such that $b = m(J) + \delta'$, is a regular value for J_ϵ. In this way (6.2) follows from Proposition 6.3 and Corollary 5.3.
Moreover, as $\mathcal{N}_\epsilon$ is contractible (see Remark 2.4), Proposition 6.3 gives also

$$\mathcal{P}_t(H^{1,p}(M), J_\epsilon^a) = t\mathcal{P}_t(\mathcal{N}_\epsilon) = t \tag{6.4}$$

Combining this relation with the exactness of sequence

$$H^{k-1}(H^{1,p}(M), J_\epsilon^a) \to H^{k-1}(J_\epsilon^b, J_\epsilon^a) \to H^k(H^{1,p}(M), J_\epsilon^b) \to H^k(H^{1,p}(M), J_\epsilon^a)$$

we have that

$$H^0(H^{1,p}(M), J_\epsilon^b) \simeq \{0\} \tag{6.5}$$

and

$$\dim H^k(H^{1,p}(M),J^b_\epsilon)=\dim H^{k-1}(J^b_\epsilon,J^a_\epsilon)\quad\forall k\geq 3. \tag{6.6}$$

Moreover, writing the previous exact sequence for $k=1,2$, we have

$$\begin{aligned}&H^0(J^b_\epsilon,J^a_\epsilon)\to H^1(H^{1,p}(M),J^b_\epsilon)\xrightarrow{j^1}H^1(H^{1,p}(M),J^a_\epsilon)\xrightarrow{i^1}\\&\xrightarrow{i^1}H^1(J^b_\epsilon,J^a_\epsilon)\to H^2(H^{1,p}(M),J^b_\epsilon)\to H^2(H^{1,p}(M),J^a_\epsilon).\end{aligned}$$

By Proposition 6.3 $H^0(J^b_\epsilon,J^a_\epsilon)\simeq H^{-1}\big((J_\epsilon)^b_a\cap\mathcal{N}_\epsilon\big)\simeq\{0\}$, so that j^1 is injective. We will exploit the geometry of functional J_ϵ to show that i^1 is injective.
First note that $H^1(H^{1,p}(M))\simeq\{0\}$, so, taking account of (6.4), by the exact sequence

$$H^0(H^{1,p}(M),J^a_\epsilon)\to H^0(H^{1,p}(M))\to H^0(J^a_\epsilon)\to H^1(H^{1,p}(M),J^a_\epsilon)\to H^1(H^{1,p}(M))$$

we infer that $\dim H^0(J^a_\epsilon)=\dim H^0(H^{1,p}(M))+\dim H^1(H^{1,p}(M),J^a_\epsilon)=2$. This shows that J^a_ϵ has exactly two connected components, one bounded containing $u_0\equiv 0$ and the other unbounded. From the geometry of J_ϵ there is a path $\bar\sigma$ in J^b_ϵ whose end points belong to the two different connected components of J^a_ϵ.
Let us choose a co-chain $\bar c\in Z^1(H^{1,p}(M),J^a_\epsilon)$ such that $[\bar\sigma,\bar c]=1$. Denoting by $\bar x$ the cohomology class of $\bar c$ in $H^1(H^{1,p}(M),J^a_\epsilon)$ and by $\bar x_b=i^1(\bar x)$, we have that $\bar x_b$ is the cohomology class of $\bar c_b=S^1(i)(\bar c)$, where $S^1(i):Z^1(H^{1,p}(M),J^a_\epsilon)\to Z^1(J^b_\epsilon,J^a_\epsilon)$ is the homomorphism induced by the injection $i:(J^b_\epsilon,J^a_\epsilon)\to(H^{1,p}(M),J^a_\epsilon)$.
So, in particular,

$$[\bar\sigma,\bar c_b]=[\bar\sigma,S^1(i)(\bar c)]=[i\circ\bar\sigma,\bar c]=[\bar\sigma,\bar c]=1$$

which implies that $\bar c_b\notin B^1(J^b_\epsilon,J^a_\epsilon)$, so that $\bar x_b\neq 0$. Hence i^1 is injective, as $\dim H^1(H^{1,p}(M),J^a_\epsilon)=1$ and there is $\bar x\in H^1(H^{1,p}(M),J^a_\epsilon)$ such that $i^1(\bar x)=\bar x_b\neq 0$. Consequently we have

$$H^1(H^{1,p}(M),J^b_\epsilon)\simeq\{0\}\simeq H^0(J^b_\epsilon,J^a_\epsilon). \tag{6.7}$$

Moreover we infer that $H^1(J^b_\epsilon,J^a_\epsilon)\simeq H^1(H^{1,p}(M),J^a_\epsilon)\oplus H^2(H^{1,p}(M),J^b_\epsilon)$ so that

$$\dim H^2(H^{1,p}(M),J^b_\epsilon)=\dim H^1(J^b_\epsilon,J^a_\epsilon)-1. \tag{6.8}$$

(6.5), (6.6), (6.7) and (6.8) can be written as

$$\mathcal{P}_t(H^{1,p}(M),J^b_\epsilon)=t\mathcal{P}_t(J^b_\epsilon,J^a_\epsilon)-t^2$$

which, together to (6.2), proves (6.3). □

We need to recall some useful definitions and results (cf. [8, 9]).

Definition 6.5. *Let X be a Banach space and f be a C^2 functional on X. If u is a critical point of f, the Morse index of f in u is the supremum of the dimensions of the subspaces of X on which $f''(u)$ is negative definite. It is denoted by $m(f,u)$. Moreover, the large Morse index of f in u is the sum of $m(f,u)$ and the dimension of the kernel of $f''(u)$. It is denoted by $m^*(f,u)$.*

Definition 6.6. *Let X be a Banach space and f be a C^1 functional on X. Let $\mathbb{K}$ be a field. Let u be a critical point of f, $c = f(u)$, and U be a neighborhood of u. We call*

$$C_q(f,u) = H^q(f^c \cap U, (f^c \setminus \{u\}) \cap U)$$

the q-th critical group of f at u, $q = 0,1,2,\dots$, where $f^c = \{v \in X \,:\, f(v) \le c\}$, $H^q(A,B)$ stands for the q-th Alexander-Spanier cohomology group of the pair (A,B) with coefficients in $\mathbb{K}$. By the excision property of the singular cohomology theory the critical groups do not depend on a special choice of the neighborhood U.

Definition 6.7. *We denote $\mathcal{P}_t(f,u)$ the Morse polynomial of f in u, defined by*

$$\mathcal{P}_t(f,u) = \sum_{k=0}^{+\infty} \dim C_k(f,u)\, t^k.$$

We call the multiplicity of u the number $\mathcal{P}_1(f,u) \in \mathbb{N} \cup \{+\infty\}$.

In order to obtain a multiplicity result of solutions to problem (P_ε) via Morse relations, we recall an abstract theorem, proved in [10] (see also [3] and [8]).

Theorem 6.8. *Let A be a open subset of a Banach space X. Let f be a C^1 functional on A and $u \in A$ be an isolated critical point of f. Assume that there exists an open neighborhood U of u such that $\overline{U} \subset A$, u is the only critical point of f in $\overline{U}$ and f satisfies the Palais–Smale condition in $\overline{U}$.*
Then there exists $\bar{\mu} > 0$ such that, for any $g \in C^1(A,\mathbb{R})$ such that

- $\|f - g\|_{C^1(A)} < \bar{\mu}$,
- *g satisfies the Palais–Smale condition in $\overline{U}$,*
- *g has a finite number $\{u_1, u_2, \dots, u_m\}$ of critical points in U,*

we have

$$\sum_{j=1}^{m} \mathcal{P}_t(g,u_j) = \mathcal{P}_t(f,u) + (1+t)Q(t),$$

where $Q(t)$ is a formal series with coefficients in $\mathbb{N} \cup \{+\infty\}$.

Proof of Theorem 1.2. Let $\hat{\epsilon}$ be as required by Proposition 6.4. From (6.2) and (6.3) we have that, for any $\epsilon \in (0,\hat{\epsilon})$, J_ϵ has at least $2\mathcal{P}_1(M) - 1$ critical points, if they are counted with their multiplicity.

From Morse relations (see Theorem 3.4 [8]) we get

$$\sum_{a<J_\epsilon(u)<b} P_1(J_\varepsilon,u) = \mathcal{P}_1(M) + \mathcal{Z}_\epsilon(1) + 2\bar{Q}_\epsilon(1) \ge \mathcal{P}_1(M)$$

$$\sum_{J_\epsilon(u)>b} P_1(J_\varepsilon,u) = \mathcal{P}_1(M) - 1 + \mathcal{Z}_\epsilon(1) + 2\hat{Q}_\epsilon(1) \ge \mathcal{P}_1(M) - 1$$

where $\bar{Q}_\epsilon(t)$ and $\hat{Q}_\epsilon(t)$ are suitable formal series with coefficients in $\mathbb{N} \cup \{+\infty\}$. Moreover it is useful to remark that

$$\sum_{J_\epsilon(u)>a} \sum_{q=0}^{+\infty} \dim C_q(J_\epsilon,u) t^q = (t+t^2)\mathcal{P}_t(M) - t^2 + (1+t)S_\epsilon(t) \tag{6.9}$$

where $S_\epsilon(t) = \bar{Q}_\epsilon(t) + \hat{Q}_\epsilon(t) + t\mathcal{Z}_\epsilon(t)$ and $\mathcal{Z}_\epsilon(t)$ is defined by Proposition 6.4.
Taking account of Remark 2.5, as $J_\epsilon(0) = 0 < a$, we infer immediately that (P_ϵ) has at least $2\mathcal{P}_1(M) - 1$ positive solutions, if counted with their multiplicity. In order to prove that these solutions are not constant, i.e. different from $u_1 \equiv 1$, we must deal with two cases separately.
When $p > 2$, we choose $\epsilon^* = \hat{\epsilon}$. As $\langle J''_\epsilon(u_1)v, v\rangle = \frac{p-q}{\epsilon^n}\int_M v^2 \, d\mu_g$, we have that $J''_\epsilon(u_1)$ is negative on $H^{1,p}(M)$, so by Theorem 3.1 in [20], it is clear that $C_q(J_\epsilon, u_1) = 0$ for any q in $\mathbb{N}$. This means that u_1 does not appear in (6.9), hence the $2\mathcal{P}_1(M) - 1$ solutions given by this equation are surely non-constant.
When $p = 2$, let us consider the nondecreasing sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ of the eigenvalues of $-\Delta_g$ in M, where it is known that $\lambda_0 = 0 < \lambda_1$ and $\lambda_n \to +\infty$. In this case we choose $\epsilon^* = \min\left\{\hat{\epsilon}, \sqrt{\frac{q-2}{\lambda_{n+2}}}\right\}$. Considering that in this case

$$\langle J''_\epsilon(u_1)v, v\rangle = \frac{1}{\epsilon^n}\left(\int_M \epsilon^2|\nabla v|^2 \, d\mu_g + (2-q)\int_M v^2 \, d\mu_g\right)$$

if $\epsilon \in (0, \epsilon^*)$ then $m(J_\epsilon, u_1) \geq n + 3$ so that

$$C_q(J_\epsilon, u_1) = 0 \qquad \forall q \leq n+2. \tag{6.10}$$

As the Poincaré polynomial $\mathcal{P}_t(M)$ has no terms with an order higher than n, from equations (6.9) and (6.10) we infer that (P_ϵ) has at least $2\mathcal{P}_1(M)-1$ non-costant solutions. □

Remark 6.9. *When $p = 2$, Theorem 1.2 can be slightly improved, as there is a decreasing infinitesimal sequence $\{\mu_i\}_{i\in\mathbb{N}}$ in $(0, \epsilon^*)$ such that if $\epsilon \neq \mu_i$ for any $i \in \mathbb{N}$, then (P_ϵ) has at least $2\mathcal{P}_1(M)$ non-costant solutions. Indeed, following the notations of the previous proof, let $\mu_i = \sqrt{\frac{q-2}{\lambda_{n+2+i}}}$. If $\epsilon \in (\mu_{i+1}, \mu_i)$, then u_1 is a nondegenerate critical point and, from classical Morse theory in Hilbert spaces,*

$$\sum_{q=0}^{+\infty} \dim C_q(J_\epsilon, u_1)t^q = t^{m(J_\epsilon, u_1)},$$

where the Morse index $m(J_\epsilon, u_1) = n + 3 + i \geq n + 3$. As $(t + t^2)\mathcal{P}_t(M)$ has no term with order higher than $n+2$, by (6.9) we infer that $S_\epsilon(t) \neq 0$, thus there is at least one more solution $\bar{u}$ of (P_ϵ), such that $C_q(J_\epsilon, \bar{u}) \neq 0$ for $q = m(J_\epsilon, u_1) + 1$ or $q = m(J_\epsilon, u_1) - 1$.

7. Regularity of ψ

In all this section we consider solutions $u \in H^{1,p}(M)$ to a quasilinear equation of the form

$$-\mathrm{div}_g\left(\left(\alpha + |\nabla u|_g^2\right)^{(p-2)/2}\nabla u\right) = h(x, u)$$

where $0 < \alpha$, $2 \leq p < n$ and $h : M \times \mathbb{R} \to \mathbb{R}$ satisfies the assumption:

- (h) for any $s \in \mathbb{R}$, $h(\cdot, s)$ is continuous on M, $h(x, \cdot)$ is C^1 on $\mathbb{R}$ and $\forall (x, s) \in M \times \mathbb{R}$, $|D_s h(x, s)| \leq c_1|s|^{p^*-2} + c_2$, with c_1, c_2 positive constants, $p^* = pn/(n-p)$.

Naturally, any solution $u \in H^{1,p}(M)$ corresponds to a critical point of the Euler functional $I_\alpha : H^{1,p}(M) \to \mathbb{R}$, defined by

$$I_\alpha(u) = \int_M \frac{1}{p}\left(\alpha + |\nabla u(x)|_g^2\right)^{\frac{p}{2}}\, d\mu_g \; - \int_M H(x, u(x))\, d\mu_g \tag{7.1}$$

where $H(x,s) = \int_0^s h(x,t)\,dt$.

We recall this crucial result proved in [13].

Theorem 7.1. *Let u_0 be a critical point of the functional I_α such that $I''_\alpha(u_0)$ is injective. Then $m(I_\alpha, u_0)$ is finite and*

$$C_q(I_\alpha, u_0) \cong \mathbb{K}, \quad \text{if } q = m(I_\alpha, u_0),$$
$$C_q(I_\alpha, u_0) = \{0\}, \quad \text{if } q \neq m(I_\alpha, u_0).$$

This theorem, extending a classical result in Hilbert spaces, suggests the following definition.

Definition 7.2. *A critical point u_0 of I_α is said nondegenerate if $I''_\alpha(u_0)$ is injective.*

In the following of this section we want to complete some results proved in [13]. Let us fix an isolated critical point u_0 of I_α. In [13] (see Proposition 4.7) the following result is proved.

Proposition 7.3. *There are V and W subspaces of $H^{1,p}(M)$, $r > 0$ and $\rho \in (0, r)$ such that*

(1) $H^{1,p}(M) = V \oplus W$
(2) *$V \subset C^1(M)$ is finite dimensional*
(3) *V and W are orthogonal in $L^2(M)$*
(4) *for each v in $V \cap \overline{B}_\rho(0)$, there exists one and only one $\overline{w} \in W \cap B_r(0)$ such that for any $z \in W \cap \overline{B}_r(0)$ we have*

$$I_\alpha(u_0 + v + \overline{w}) \leq I_\alpha(u_0 + v + z)$$

moreover $\overline{w}$ is the only element of $W \cap \overline{B}_r(0)$ such that

$$\langle I'_\alpha(u_0 + v + \overline{w}), z\rangle = 0 \qquad \forall z \in W \tag{7.2}$$

(5) *for any $v \in V$, $z \in W$*

$$\langle I''_\alpha(u_0)v, z\rangle = 0.$$

Remark 7.4. *If (1), (2) and (3) of the previous proposition hold when replacing V with a new subspace $\bar{V}$ and W with $\bar{W}$ such that $\bar{W} \subset W$, then even (4) still holds.*

From the previous proposition, we can define the map

$$\psi : V \cap \overline{B}_\rho(0) \to W \cap B_r(0) \tag{7.3}$$

where $\psi(v)$ is the unique minimum point of the function $w \in W \cap \overline{B}_\rho(0) \mapsto I_\alpha(u_0 + v + w)$.

In this way, reasoning as in Section 4 of [13], we have the following result.

Remark 7.5. *There exist $R_0,\ C_0,\ \mu > 0$ such that*

- *if* $z \in B_{R_0}(u_0)$ *and* $\langle I'_\alpha(z), w\rangle = 0$ *for any* $w \in W$*, then* $z \in C^{1,\beta}(M)$ *and* $\|z\|_{C^{1,\beta}} \leq C_0$*, with* $\beta \in]0,1[$*;*
- *setting* $\tilde{K} = \{z \in B_{R_0}(u_0) \cap C^{1,\beta}(M) \mid \|z\|_{C^{1,\beta}} \leq C_0\}$*, there is* $\mu > 0$ *such that, if* $z \in \tilde{K}$*, then* $\langle I''_\alpha(z)w, w\rangle \geq \mu\|w\|^2_{H^{1,2}(M)}$ *for any* $w \in W$*;*
- $\tilde{K}$ *is convex and* $u_0 + v + \psi(v) \in \tilde{K}$*, for any* $v \in V \cap \overline{B}_\rho(0)$*.*

We begin to derive the following lemma.

Lemma 7.6. *The map* $\psi : V \cap \overline{B}_\rho(0) \to W$ *is Lipschitz continuous with respect to the norm* $\|\cdot\|_{H^{1,2}(M)}$ *on* W*.*

Proof. Let $v, z \in V \cap \overline{B}_\rho(0)$. We evaluate

$$\begin{aligned} 0 &= \langle I'_\alpha(u_0 + v + \psi(v)), \psi(v) - \psi(z)\rangle - \langle I'_\alpha(u_0 + z + \psi(z)), \psi(v) - \psi(z)\rangle \\ &= \langle I''_\alpha(u_0 + tv + t\psi(v) + (1-t)z + (1-t)\psi(z))(\psi(v) - \psi(z)), v - z + \psi(v) - \psi(z)\rangle \end{aligned}$$

for a suitable $t \in (0,1)$. By Remark 7.5 we have that $u_t = u_0 + tv + t\psi(v) + (1-t)z + (1-t)\psi(z) \in \tilde{K}$, so that

$$\begin{aligned} \|\psi(v) - \psi(z)\|^2_{H^{1,2}(M)} &\leq 1/\mu\langle I''_\alpha(u_t)(\psi(v) - \psi(z)), \psi(v) - \psi(z)\rangle \\ &= -1/\mu\langle I''_\alpha(u_t)(\psi(v) - \psi(z)), v - z\rangle \leq K\|v - z\|\,\|\psi(v) - \psi(z)\|_{H^{1,2}(M)}. \end{aligned}$$

Hence we have

$$\|\psi(v) - \psi(z)\|_{H^{1,2}(M)} \leq K\|v - z\|$$

where K is a positive constant. ∎

If $u \in \tilde{K}$, we can extend $I'_\alpha(u)$ to $H^{1,2}(M)$ by defining $A_1(u) : H^{1,2}(M) \to \mathbb{R}$

$$\begin{aligned} \langle A_1(u), z\rangle = &\int_M \left(\alpha + |\nabla u(x)|_g^2\right)^{\frac{p-2}{2}} (\nabla u(x)|\nabla z(x))_g \, d\mu_g \\ &- \int_M h\,(x, u(x))\, z(x) \, d\mu_g; \end{aligned}$$

for any $z \in H^{1,2}(M)$.
Analogously we can extend $I''_\alpha(u)$ by defining $A_2(u) : H^{1,2}(M) \times H^{1,2}(M) \to \mathbb{R}$

$$\begin{aligned} \langle A_2(u)z, \theta\rangle = &\int_M \left(\alpha + |\nabla u(x)|_g^2\right)^{\frac{p-2}{2}} (\nabla z(x)|\nabla \theta(x))_g \, d\mu_g \\ &+ (p-2)\int_M \left(\alpha + |\nabla u(x)|_g^2\right)^{\frac{p-4}{2}} (\nabla u(x)|\nabla z(x))_g (\nabla u(x)|\nabla \theta(x))_g \, d\mu_g \\ &- \int_M D_s h(x, u(x)) z(x)\theta(x) \, d\mu_g \end{aligned}$$

for any $z, \theta \in H^{1,2}(M)$.

We now denote by H^+ the orthogonal of V in $H^{1,2}(M)$ according to the scalar product in $L^2(M)$, so that H^+ is the completion of W in $H^{1,2}(M)$.
It is easy to see that $A_1(u)$ is linear, $A_2(u)$ is bilinear and symmetric, both are continuous and the following result holds.

Lemma 7.7. *It results that*

(1) *if* $v \in V \cap \overline{B}_\rho(0)$ *then* $\langle A_1(u_0 + v + \psi(v)), z\rangle = 0$ *for any* $z \in H^+$*;*
(2) *there is* $\mu > 0$ *such that* $\langle A_2(u)z, z\rangle \geq \mu\|z\|^2_{H^{1,2}(M)}$ *for any* $u \in \tilde{K}$ *and* $z \in H^+$*;*

(3) *if* $u_1, u_2 \in \tilde{K}$ *and* $z \in H^{1,2}(M)$, *the real function* $g : (0,1) \to \mathbb{R}$ *defined by* $g(t) = \langle A_1(tu_1+(1-t)u_2), z\rangle$ *is* C^1 *and* $g'(t) = \langle A_2(tu_1+(1-t)u_2)z, u_1-u_2\rangle$.

In what follows, we prove directly that the map ψ is C^1 with respect to the norm $\|\cdot\|_{H^{1,2}(M)}$ on W. The same argument can be also performed for problems defined on domains of $\mathbb{R}^n$ instead of on manifolds. We also precise that in Lemma 2.2 of [11] - which will be completely stated and proved again in next Theorem 7.12 - the C^1 regularity of the map ψ is already stated. However, even if the statement is true, that proof does not work, since it relies on the introduction of a penalized functional, which is not C^2 on the Hilbert space (see, for instance, Proposition 2.8, Chapter 1 in [1]).

Theorem 7.8. *ψ is a C^1 map with respect to the $\|\cdot\|_{H^{1,2}(M)}$ norm on W.*

Proof. We begin to prove that ψ is differentiable with respect to the $\|\cdot\|_{H^{1,2}(M)}$ norm on W. Let us consider $\bar{v} \in V \cap \overline{B}_\rho(0)$. Setting $\bar{u} = u_0 + \bar{v} + \psi(\bar{v})$, by (2) of Lemma 7.7 we have that $L_{\bar{u}} : H^+ \to (H^+)^*$ defined by $\langle L_{\bar{u}}(z), \theta\rangle = \langle A_2(\bar{u})z, \theta\rangle$ is a linear and continuous isomorphism. Moreover, for any $h \in V$, $\langle A_2(\bar{u})\cdot, h\rangle$ belongs to $(H^+)^*$. We denote by $B_{\bar{u}}(h) = L_{\bar{u}}^{-1}(\langle A_2(\bar{u})\cdot, h\rangle)$, so that $B_{\bar{u}}(h)$ is the only element of H^+ verifying the equality

$$\langle A_2(\bar{u})z, B_{\bar{u}}(h)\rangle = \langle A_2(\bar{u})z, h\rangle \quad \forall z \in H^+. \tag{7.4}$$

It is obvious that $B_{\bar{u}} : V \to H^+$ is linear, moreover it is also continuous, as

$$\|B_{\bar{u}}(h)\|_{H^{1,2}(M)} \le \|L_{\bar{u}}^{-1}\| \sup_{z\in H,^+ \|z\|_{H^{1,2}(M)}=1} |\langle A_2(\bar{u})z, h\rangle| \le C\|h\|. \tag{7.5}$$

If we show that

$$\lim_{h\to 0} \frac{\|\psi(\bar{v}+h) - \psi(\bar{v}) + B_{\bar{u}}(h)\|_{H^{1,2}(M)}}{\|h\|} = 0$$

then the differentiability of ψ is proved, being $\psi'(\bar{v}) = -B_{\bar{u}}$.

Let us choose $h \in V$, $h \neq 0$ such that $\bar{v} + h \in V \cap B_\rho(0)$.
Denoting by $z_h = \psi(\bar{v}+h) - \psi(\bar{v}) + B_{\bar{u}}(h) \in H^+$, by Lemma 7.7 and (7.4) we have, for a suitable $t \in (0,1)$, that

$$\begin{aligned} 0 \; &= \langle A_1(u_0 + \bar{v} + h + \psi(\bar{v}+h)), z_h\rangle - \langle A_1(u_0 + \bar{v} + \psi(\bar{v})), z_h\rangle \\ &= \langle A_2(u_0 + \bar{v} + th + t\psi(\bar{v}+h) + (1-t)\psi(\bar{v}))z_h, h\rangle \\ &+\langle A_2(u_0 + \bar{v} + th + t\psi(\bar{v}+h) + (1-t)\psi(\bar{v}))z_h, z_h\rangle \\ &-\langle A_2(u_0 + \bar{v} + th + t\psi(\bar{v}+h) + (1-t)\psi(\bar{v}))z_h, B_{\bar{u}}(h)\rangle \\ &+\langle A_2(\bar{u})z_h, B_{\bar{u}}(h)\rangle - \langle A_2(\bar{u})z_h, h\rangle. \end{aligned} \tag{7.6}$$

In what follows we denote by $u_{t_h} = u_0 + \bar{v} + th + t\psi(\bar{v}+h) + (1-t)\psi(\bar{v})$.

The Ascoli-Arzelà Theorem assures that $\lim\limits_{h\to 0} \|u_{t_h} - \bar{u}\|_{C^1} = 0$, so

$$|\langle A_2(u_{t_h})z, \theta\rangle - \langle A_2(\bar{u})z, \theta\rangle| \le o(h)\|z\|_{H^{1,2}(M)}\|\theta\|_{H^{1,2}(M)} \qquad \forall\, z, \theta \in H^{1,2}(M) \tag{7.7}$$

Therefore (7.6), (7.7) and (7.5), taking account of Lemma 7.7, give

$$\begin{aligned} &\mu\|z_h\|^2_{H^{1,2}(M)} \le \langle A_2(u_{t_h})z_h, z_h\rangle \\ &\le |\langle \big(A_2(u_{t_h}) - A_2(\bar{u})\big)z_h, B_{\bar{u}}(h)\rangle| + |\langle \big(A_2(\bar{u}) - A_2(u_{t_h})\big)z_h, h\rangle| \\ &\quad \le o(h)\,\|z_h\|_{H^{1,2}(M)}(C+1)\|h\|_{H^{1,2}(M)} \end{aligned}$$

so that

$$\lim_{h\to 0}\frac{\|z_h\|_{H^{1,2}(M)}}{\|h\|}=0.$$

Now we recognize that ψ is C^1. We consider a sequence $\{v_n\}_{n\in\mathbb{N}}\subset V$ such that $v_n\to\bar v$ with $\bar v\in V$, as $n\to+\infty$. Let us denote $u_n=u_0+v_n+\psi(v_n)$ and $L_{u_n}=A_2(u_n)_{|_{H^+}}:H^+\to(H^+)^*$ the linear isomorphism. It results that

$$\begin{aligned}\|\psi'(v_n)-\psi'(\bar v)\| &= \sup_{\|h\|=1}\|\psi'(v_n)h-\psi'(\bar v)h\|_{H^{1,2}(M)}\\
&\le \sup_{h\in V,\|h\|=1}\|L_{u_n}^{-1}(\langle A_2(u_n)\cdot,h\rangle)-L_{\bar u}^{-1}(\langle A_2(\bar u)\cdot,h\rangle)\|_{H^{1,2}(M)}\\
&\le \sup_{h\in V,\|h\|=1}\|L_{u_n}^{-1}(\langle A_2(u_n)\cdot,h\rangle)-L_{u_n}^{-1}(\langle A_2(\bar u)\cdot,h\rangle)\|_{H^{1,2}(M)}\\
&\quad+\sup_{h\in V,\|h\|=1}\|L_{u_n}^{-1}(\langle A_2(\bar u)\cdot,h\rangle)-L_{\bar u}^{-1}(\langle A_2(\bar u)\cdot,h\rangle)\|_{H^{1,2}(M)}\\
&\le \|L_{u_n}^{-1}\|\sup_{h\in V,\|h\|=1}\ \sup_{z\in W,\|z\|=1}|\langle\big(A_2(u_n)-A_2(\bar u)\big)z,h\rangle|\\
&\quad+\|L_{u_n}^{-1}-L_{\bar u}^{-1}\|\sup_{h\in V,\|h\|=1}\ \sup_{z\in W,\|z\|=1}|\langle A_2(\bar u)z,h\rangle|\end{aligned}$$

which tends to zero as $n\to+\infty$, as u_n tends to $\bar u$ and ∇u_n tends to $\nabla\bar u$ uniformly in M, as $n\to+\infty$. □

Remark 7.9. *We notice that $\psi'(0)=0$. In fact, by (5) of Proposition 7.3 we have that, for each $h\in V$, $\langle A_2(u_0)\cdot,h\rangle=0$ on H^+ and so, from the previous proof,*

$$\psi'(0)(h)=-L_{u_0}^{-1}\big(\langle A_2(u_0)\cdot,h\rangle\big)=0.$$

Lemma 7.10. *Let $H{=}V$ or $H{=}H^+$. The function $B_H:V\cap\overline{B}_\rho(0)\to H^*$ defined by*

$$\langle B_H(v),z\rangle=\langle A_1\left(u_0+v+\psi(v)\right),z\rangle\quad\forall v\in V\cap\overline{B}_\rho(0),z\in H,$$

is C^1 and

$$\langle B_H'(v)h,z\rangle=\langle A_2(u_0+v+\psi(v))(h+\psi'(v)h),z\rangle \tag{7.8}$$

for any $v\in V\cap\overline{B}_\rho(0)$, $h\in V$, $z\in H$.

Proof. Let us consider $v\in V\cap\overline{B}_\rho(0)$ and $h\in V$, such that $v+h\in V\cap\overline{B}_\rho(0)$. Denoting $\omega_h\equiv\psi(v+h)-\psi(v)$, we have, for a suitable $t\in(0,1)$,

$$\begin{aligned}
&\|B_H(v+h) - B_H(v) - \langle A_2(u_0+v+\psi(v))(h+\psi'(v)h), \cdot\rangle\| \\
&\quad = \sup_{z\in H, \|z\|=1} |\langle A_1(u_0+v+h+\psi(v+h)), z\rangle - \langle A_1(u_0+v+\psi(v)), z\rangle \\
&\qquad\qquad - \langle A_2(u_0+v+\psi(v))(h+\psi'(v)h), z\rangle| \\
&\quad = \sup_{z\in H, \|z\|=1} |\langle A_2(u_0+v+th+\psi(v)+t\omega_h)z, h\rangle \\
&\qquad\qquad + \langle A_2(u_0+v+th+\psi(v)+t\omega_h)z, \omega_h\rangle \\
&\qquad\qquad - \langle A_2(u_0+v+\psi(v))(h+\psi'(v)h), z\rangle| \\
&\quad \leq \sup_{z\in H, \|z\|=1} |\langle (A_2(u_0+v+th+\psi(v)+t\omega_h) - A_2(u_0+v+\psi(v))z, h\rangle| \\
&\qquad\qquad + |\langle A_2(u_0+v+th+\psi(v)+t\omega_h) - A_2(u_0+v+\psi(v))\omega_h, z\rangle| \\
&\qquad\qquad + |\langle A_2(u_0+v+\psi(v))\, z, (\psi(v+h) - \psi(v) - \psi'(v)h)\rangle|
\end{aligned}$$

From the above inequality we immediately derive

$$\lim_{\|h\|\to 0} \frac{\|B_H(v+h) - B_H(v) - \langle A_2(u_0+v+\psi(v))(h+\psi'(v)h), \cdot\rangle\|}{\|h\|} = 0.$$

In order to prove continuity of B'_H, let us consider a sequence $\{v_n\}_{n\in\mathbb{N}} \subset \big(V \cap \overline{B}_\rho(0)\big)$ such that $v_n \to \bar{v}$. Reasoning as in (7.7), we have that

$$\begin{aligned}
&|\langle B'_H(v_n)h, z\rangle - \langle B'_H(\bar{v})h, z\rangle| \\
&= |\langle A_2(u_0+v_n+\psi(v_n))(h+\psi'(v_n)h), z\rangle - \langle A_2(u_0+\bar{v}+\psi(\bar{v}))(h+\psi'(\bar{v})h), z\rangle| \\
&\leq |\langle \big(A_2(u_0+v_n+\psi(v_n)) - A_2(u_0+\bar{v}+\psi(\bar{v}))\big)h, z\rangle| \\
&+ |\langle \big(A_2(u_0+v_n+\psi(v_n)) - A_2(u_0+\bar{v}+\psi(\bar{v}))\big)\psi'(v_n)h, z\rangle| \\
&+ |\langle A_2(u_0+\bar{v}+\psi(\bar{v}))\big)z, \psi'(v_n)h - \psi'(\bar{v})h\rangle| \leq o(n)\, \|h\|\, \|z\|.
\end{aligned}$$

Therefore

$$\lim_{n\to\infty} \|B'_H(v_n) - B'_H(\bar{v})\| = 0$$

$\square$

Proposition 7.11. *For any $v \in V \cap \overline{B}_\rho(0)$ and $h \in V$*

$$\psi'(v)h \in \big(C^1(M) \cap H^+\big) \subset W. \tag{7.9}$$

Proof. Using the notations of Lemma 7.10, where $H = H^+$, from (1) of Lemma 7.7 $B_{H^+} : V \cap \overline{B}_\rho(0) \to (H^+)^*$ is constantly equal to zero, so that (7.8) gives

$$\langle A_2(u_0+v+\psi(v))(h+\psi'(v)h), z\rangle = 0, \quad \forall\, v \in V \cap \overline{B}_\rho(0),\ h \in V,\ z \in H^+. \tag{7.10}$$

Since $u_0 + v + \psi(v) \in C^{1,\beta}(M)$, we derive that $h + \psi'(v)h$ belongs to $C^{1,\eta}(M)$ for some $\eta \in (0,1)$ (see [16]), and, as $V \subset C^1(M)$ and $\psi'(v)h \in H^+$, (7.9) is proved.

$\square$

Now we can derive the following regularity result.

Theorem 7.12. *Let I_α be defined by (7.1), where $\alpha > 0$, $p \in [2, n)$ and h satisfies assumption (h), let V, W satisfy (1), (2) and (3) of Proposition 7.3 and ψ be defined by (7.3). The map $\varphi : V \cap \overline{B}_\rho(0) \to \mathbb{R}$ defined by $\varphi(v) = I_\alpha(u_0 + v + \psi(v))$ is C^2 and, for any $v \in V \cap \overline{B}_\rho(0)$ and $z, h \in V$*

$$\langle \varphi'(v), z\rangle = \langle I'_\alpha(u_0 + v + \psi(v)), z\rangle \tag{7.11}$$

$$\langle \varphi''(v)h, z\rangle = \langle I''_\alpha(u_0 + v + \psi(v))(h + \psi'(v)h), z\rangle \tag{7.12}$$

(7.13) $\quad$ *$\varphi''(v)$ is an isomorphism if and only if $I''_\alpha(u_0 + v + \psi(v))$ is injective.*

Proof. Let us choose $v \in V \cap \overline{B}_\rho(0)$ and $h \in V$, such that $v + h \in V \cap \overline{B}_\rho(0)$. Denoting $\omega_h \equiv \psi(v + h) - \psi(v)$, we have, for suitable $t, s, \tau \in (0, 1)$

$$\begin{aligned}
\varphi(v+h) &- \varphi(v) - \langle I'_\alpha(u_0 + v + \psi(v)), h\rangle \\
=&\langle I'_\alpha(u_0 + v + th + \psi(v) + t(\psi(v+h) - \psi(v))) - I'_\alpha(u_0 + v + \psi(v)), h\rangle \\
&+ \langle I'_\alpha(u_0 + v + th + \psi(v) + t\omega_h) - I'_\alpha(u_0 + v + \psi(v)), \omega_h\rangle \\
=&t\langle A_2(u_0 + v + sth + \psi(v) + st\omega_h)h, h\rangle \\
&+ t\langle A_2(u_0 + v + sth + \psi(v) + st\omega_h)\omega_h, h\rangle \\
&+ t\langle A_2(u_0 + v + \tau th + \psi(v) + \tau t\omega_h)\omega_h, h\rangle \\
&+ t\langle A_2(u_0 + v + \tau th + \psi(v) + \tau t\omega_h)\omega_h, \omega_h\rangle.
\end{aligned}$$

We infer that

$$\begin{aligned}
&\frac{|\varphi(v+h) - \varphi(v) - \langle I'_\alpha(u_0 + v + \psi(v)), h\rangle|}{\|h\|} \\
&\le \frac{|\langle A_2(u_0 + v + \psi(v) + sth + st\omega_h)h, h\rangle|}{\|h\|} \\
&+ \frac{|\langle A_2(u_0 + v + \psi(v) + sth + st\omega_h)\omega_h, h\rangle|}{\|h\|} \\
&+ \frac{|\langle A_2(u_0 + v + \psi(v) + \tau th + \tau t\omega_h)\omega_h, h\rangle|}{\|h\|} \\
&+ \frac{|\langle A_2(u_0 + v + \psi(v) + \tau th + \tau t\omega_h)\omega_h, \omega_h\rangle|}{\|h\|}
\end{aligned}$$

which tends to zero as $\|h\| \to 0$, proving (7.11). Moreover, by Lemma 7.10, where $H = V$, we immediately see that $\varphi' = B_V$, φ is C^2 and

$$\langle \varphi''(v)h, z\rangle = \langle A_2(u_0 + v + \psi(v))(h + \psi'(v)h), z\rangle \quad \forall\, h, z \in V. \tag{7.14}$$

Proposition 7.11 assures that any $h + \psi'(v)h \in H^{1,p}(M)$ which, together with (7.14), gives (7.12).

In order to prove (7.13), we fix $v \in V \cap \overline{B}_\rho(0)$ and suppose that $\varphi''(v)$ is an isomorphism. By way of contradiction, if $I''_\alpha(u_0 + v + \psi(v))$ is not injective, there exists $\bar{z} \in H^{1,p}(M) \setminus \{0\}$ such that

$$\langle I''_\alpha(u_0 + v + \psi(v))z, \bar{z}\rangle = 0, \quad \forall\, z \in H^{1,p}(M)$$

Writing $\bar z = \bar v + \bar w$, where $\bar v \in V$ and $\bar w \in W$, by (7.12) and (7.10) we infer

$$\begin{aligned}\langle \varphi''(v)h, \bar v\rangle &= \langle I''_\alpha(u_0 + v + \psi(v))(h + \psi'(v)h), \bar v\rangle \\ &= \langle I''_\alpha(u_0 + v + \psi(v))(h + \psi'(v)h), \bar z\rangle = 0, \qquad \forall\, h \in V\end{aligned}$$

so that $\bar v = 0$ and $\bar z \in W$. By Remark 7.5, $\bar z = 0$ which is absurd.

On the other side, if $I''_\alpha(u_0 + v + \psi(v))$ is injective but $\varphi''(v)$ is not, there is $\bar v \in V \setminus \{0\}$ such that

$$\langle I''_\alpha(u_0 + v + \psi(v))(\bar v + \psi'(v)\bar v), h\rangle = \langle \varphi''(v)\bar v, h\rangle = 0, \qquad \forall\, h \in V$$

which, by (7.10), gives

$$\langle I''_\alpha(u_0 + v + \psi(v))(\bar v + \psi'(v)\bar v), z\rangle = 0, \qquad \forall\, z \in H^{1,2}(M).$$

As $I''_\alpha(u_0 + v + \psi(v))$ is injective, this means that $\bar v + \psi'(v)\bar v = 0$, so also $\bar v = 0$ which is again a contradiction. □

Corollary 7.13. *Any nondegenerate critical point is isolated.*

Proof. If u_0 is nondegenerate, $I''_\alpha(u_0)$ is injective and (7.13) assures that $\varphi''(0)$ is an isomorphism. As V is finite dimensional, this implies that 0 is an isolated critical point for φ and, by (7.11), u_0 is an isolated critical point for I_α. □

8. Morse Theory and interpretation of multiplicity

For any $\epsilon > 0$ and $\alpha \geq 0$, we introduce the C^2 functional $T_{\alpha,\epsilon} : H^{1,p}(M) \to \mathbb{R}$ defined by

$$T_{\alpha,\epsilon}(u) = \frac{1}{\epsilon^n}\int_M \left(\frac{\epsilon^p}{p}\left(\alpha + |\nabla u(x)|_g^2\right)^{p/2} + \frac{1}{p}|u(x)|^p - \frac{1}{q}|u^+(x)|^q \right) d\mu_g\,.$$

Moreover, for any given $T_{\alpha,\epsilon}$ and any $f \in C^1(M)$ we define $I_{\alpha,\epsilon,f} : H^{1,p}(M) \to \mathbb{R}$ by

$$I_{\alpha,\epsilon,f}(u) = T_{\alpha,\epsilon}(u) - \int_M f u\; d\mu_g\,.$$

We remark that, for any bounded subset A of $H^{1,p}(M)$,

$$\lim_{\alpha\to 0} \|T_{\alpha,\epsilon} - J_\epsilon\|_{C^1(A)} = 0 \qquad \text{and} \qquad \lim_{\|f\|_{C^1(M)} \to 0} \|I_{\alpha,\epsilon,f} - T_{\alpha,\epsilon}\|_{C^1(A)} = 0.$$

Moreover, if $\alpha > 0$ then $T_{\alpha,\epsilon}$ and $I_{\alpha,\epsilon,f}$ satisfy the hypothesis required in the previous section, for any $f \in C^1(M)$ and $\epsilon > 0$.

Proof of Theorem 1.3. Let ϵ^* be as required by Theorem 1.2 and $\epsilon \in (0, \epsilon^*)$, so that (P_ϵ) has at least $2\,\mathcal{P}_1(M) - 1$ non-constant solutions, possibly counted with their multiplicities. If these solutions are distinct, then the theorem is proved. Otherwise J_ϵ has finite number h of isolated critical points $\tilde u_1, \tilde u_1, \ldots \tilde u_h$ having multiplicities $\tilde m_1, \tilde m_1, \ldots \tilde m_h$, where $1 \leq h < 2\,\mathcal{P}_1(M) - 1$ and, by Theorem 1.2,

$$\sum_{j=1}^{h} \tilde m_j \geq 2\,\mathcal{P}_1(M) - 1.$$

Let $\delta > 0$ be such that $B_\delta(\tilde{u}_1), B_\delta(\tilde{u}_2), \dots B_\delta(\tilde{u}_h)$ are pairwise disjoint, hence introduce the open set A defined by

$$A = \bigcup_{j=1}^{h} B_\delta(\tilde{u}_j)\,. \tag{8.1}$$

Let $\{\alpha_s\}_{s\in\mathbb{N}}$ be a sequence such that $\alpha_s > 0$ and $\alpha_s \to 0$. If $T_{\alpha_s,\epsilon}$ has at least $2\,\mathcal{P}_1(M) - 1$ distinct critical points in A, then we just choose $f_s = 0$, otherwise $T_{\alpha_s,\epsilon}$ has $k < 2\,\mathcal{P}_1(M) - 1$ isolated critical points $u_1, \dots u_k$, having multiplicities $m_1, \dots m_k$. For simplicity, we omit the dependence from s of k, u_i and their related objects. If s is sufficiently large, by Theorem 6.8, $k \geq h$ and

$$\sum_{i=1}^{k} m_i \geq \sum_{j=1}^{h} \tilde{m}_j \geq 2\,\mathcal{P}_1(M) - 1. \tag{8.2}$$

For any $i = 1, \dots k$, let V_i and W_i be the subspaces of $H^{1,p}(M)$ defined by Proposition 7.3 applied to $T_{\alpha_s,\epsilon}$ and u_i. Setting $V = V_1 + V_2 + \dots V_k$ and $W = \bigcap_{i=1}^{k} W_i$, (4) of Proposition 7.3 still holds for $T_{\alpha_s,\epsilon}$ and u_i replacing V_i with V and W_i with W, as underlined by Remark 7.4. In particular there are $r > 0$ and $\rho \in (0, r)$ such that, for any $i = 1, \dots k$ and $v \in V \cap B_\rho(0)$, there exists one and only one $w_i = \psi_i(v) \in W \cap B_r(0)$ which verifies

$$\langle T'_{\alpha_s,\epsilon}(u_i + v + w_i), z\rangle = 0 \quad \forall z \in W. \tag{8.3}$$

Moreover r and ρ can be chosen so that, setting $U_i = u_i + (V \cap B_\rho(0)) + (W \cap B_r(0))$, we have $U_i \cap U_j = \emptyset$ if $i \neq j$ and $\displaystyle\bigcup_{i=1}^{k} \overline{U}_i \subset A$, where A is the bounded open set defined by (8.1).

Now we consider the k functionals $\varphi_i : V \cap B_\rho(0) \to \mathbb{R}$ defined by

$$\varphi_i(v) = T_{\alpha_s,\epsilon}\left(u + v + \psi_i(v)\right).$$

Let $\{e_1, \dots e_l\}$ be an L^2-orthonormal basis of V, where $l = \dim V$. For any $v' \in V'$ we introduce the functional $L_{v'} : H^{1,p}(M) \to \mathbb{R}$ defined by

$$L_{v'}(u) = \int_M f_{v'} u \, d\mu_g\,, \qquad \text{where} \qquad f_{v'} = \sum_{j=1}^{l} \langle v', e_k\rangle e_k\,.$$

For any $i = 1, \dots k$, let μ_i be defined by Theorem 6.8 relatively to $T_{\alpha_s,\epsilon}, u_i$, A and U_i and $\mu = \min\{\mu_1, \dots \mu_k\}$. Let $\gamma > 0$ be such that $\|L_{v'}\|_{C^1(A)} < \mu/k$ if $v' \in V'$ and $\|v'\|_{V'} \leq \gamma$. Denoting by $\gamma_1 = \min\{\gamma, 1/n\}$, by Sard's Lemma there exists $v'_1 \in V'$ such that $\|v'_1\|_{V'} < \gamma_1$ and if $\varphi'_1(v) = v'_1$ then $\varphi''_1(v)$ is an isomorphism. Moreover there is $\beta_1 > 0$ such that if $v' \in V'$, $\|v'\|_{V'} \leq \beta_1$ and $\varphi'_1(v) = v'_1 + v'$ then $\varphi''_1(v)$ is an isomorphism.
Analogously, for $i = 2, \dots k$, there exist $\beta_i > 0$, $\gamma_i = \min\{\gamma_1, \frac{\beta_1}{k-1}, \dots \frac{\beta_{i-1}}{k-i+1}\}$ and $v'_i \in V'$ such that $\|v'_i\|_{V'} < \gamma_i$ and

$$v' \in V', \|v'\|_{V'} \leq \beta_i,\ \varphi'_i(v) = v'_1 + \dots v'_i + v' \ \Rightarrow\ \varphi''_i(v) \text{ is an isomorphism.} \tag{8.4}$$

So we can choose

$$f_s = \sum_{i=1}^{k} \sum_{j=1}^{l} \langle v_i', e_j \rangle e_j = \sum_{i=1}^{k} f_{v_i'}.$$

We immediately see that $f_s \in C^1(M)$, $\lim_{s\to\infty} \|f_s\|_{C^1(M)} = 0$ and solutions to

$$(P_s) \begin{cases} -\epsilon^p \operatorname{div}_g \left(\left(\alpha_s + |\nabla u|_g^2\right)^{(p-2)/2} \nabla u \right) + u^{p-1} = u^{q-1} + f_s \\ u > 0 \end{cases}$$

are critical points of the functional

$$T_s = u \in H^{1,p}(M) \mapsto T_{\alpha_s,\epsilon}(u) - \int_M f_s u \; d\mu_g \, .$$

Moreover we will see that any critical point of T_s belonging to $\bigcup_{i=1}^{k} U_i$ is nondegenerate. In fact, we start by observing that

$$(8.5) \qquad \int_M f_s w \; d\mu_g = 0 \quad \forall w \in W \quad \text{and} \quad \int_M f_s v \; d\mu_g = \sum_{i=1}^{k} \langle v_i', v \rangle \quad \forall v \in V.$$

Denoting by $K_i = \{u \in U_i \mid T_s'(u) = 0\}$ and fixed $\bar{u} \in K_i$, there is $(\bar{v}, \bar{w}) \in V \times W$ such that $u = u_i + \bar{v} + \bar{w}$ and, by (8.5),

$$\langle T'_{\alpha_s,\epsilon}(\bar{u}), w \rangle = \langle T_s'(\bar{u}), w \rangle = 0 \quad \forall w \in W$$

so (8.3) assures that $\bar{w} = \psi_i(\bar{v})$.
Applying Theorem 7.12 to φ_i we see that $\varphi_i'(\bar{v}) = v_1' + \dots v_i' + v_{i+1}' + \dots v_k'$, hence considering that $\|v_{i+1}' + \dots v_k'\|_{V'} < \beta_i$, by (8.4) $\varphi_i''(\bar{v})$ is an isomorphism so that by (7.13) $T_s''(\bar{u}) = T''_{\alpha_s,\epsilon}(\bar{u})$ is injective, that is $\bar{u}$ is nondegenerate. As all the critical points of T_s in $\bigcup_{i=1}^{k} U_i$ are nondegenerate, by Definition 6.7 and Theorem 7.1

$$\mathcal{P}_1(u, T_s) = 1 \quad \forall u \in \bigcup_{i=1}^{k} K_i.$$

Moreover $\|T_s - T_{\alpha_s,\epsilon}\|_{C^1(A)} < \mu$ and (8.2) holds, so Theorem 6.8 gives

$$\sum_{u \in \bigcup_{i=1}^{k} K_i} \mathcal{P}_1(u, T_s) \geq \sum_{i=1}^{k} m_i \geq 2\,\mathcal{P}_1(M) - 1$$

which means that T_s has at least $2\,\mathcal{P}_1(M) - 1$ distinct critical points in A. In order to prove that they correspond to non-constant solutions to (P_s), we note that, for any $j = 1, \dots h$, there are at least $\tilde{m}_j$ of these $2\,\mathcal{P}_1(M) - 1$ critical points of T_s which belong to $B_\delta(\tilde{u}_j)$. Let us denote them by $u_s^{j,1}, \dots u_s^{j,\tilde{m}_j}$. As $u_s^{j,l}$ are uniformly bounded in $C^1(M)$ (see [16]), we have that $u_s^{j,l} \to \tilde{u}_j$ in $C^1(M)$, for any $l = 1, \dots \tilde{m}_j$. If s is sufficiently large, any $u_s^{j,l}$ is positive and non-constant, so the proof is completed. □

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