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WELLPOSEDNESS OF A NONLINEAR PERIDYNAMIC MODEL

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ABSTRACT. We consider an evolution equation inspired by a model in peridynamics, with a singular pairwise interaction force term, and we give global in time existence, uniqueness and stability results for the Cauchy problem.

1. INTRODUCTION

The exceptional achievements in materials science and the new technological issues ask for a continuous deepening of our understanding of the materials behavior. Since the end of the sixties, the need to enlarge the framework of continuum mechanics in order to keep track of nonlocal effects was recognized by many researchers. More precisely, E. Kröner, D. G. B. Edelen, A. C. Eringen and I. A. Kunin (see [12, 13, 14, 15]) laid the foundations for a comprehensive theoretical treatment of *nonlocal* mechanics.

More recently, in [20] S. A. Silling introduced *peridynamics*, as a nonlocal elasticity theory: a continuum theory avoiding spatial derivatives and incorporating spatial nonlocality (see also [21, 22, 23, 24]). Peridynamics allows to model nonlocal interactions through long-range forces, and it is believed to be suited for the description of a large class of physical phenomena which escape a classical description of continuum mechanics based on partial differential equations. In particular, the theory of peridynamics seems to offer a promising framework to model phenomena such as damage and fracture in solids (see [4, 16, 17]), evolution of phase boundaries in phase transformations, defects, dislocations, etc.

Indeed, one of the advantages of the peridynamic formulation of mechanics is that it maintains its consistency even in presence of fractures, empty zones and discontinuities, while the classical formulations relying on differential equations heavily depend on the smoothness assumptions taken on the model and do not easily adapt to sharp discontinuities.

Besides its mathematical consistency and the flexibility offered in the applications, the peridynamic theory presents the conceptual advantage of focusing on master equations and of taking into account the finite range interactions of material particles at a given distance, thus attempting to describe the physical phenomena from elementary pieces of information. From the point of view of molecular dynamics, the interaction between particles is often referred to with the name of “bond”, which is typically identified in peridynamics by the distance of the particles and their mutual displacements and which produces a contribution in terms of a “force function” (or “force density”). In this formulation, rather than the “contact forces” which

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are taken into account in classical mechanics, the equations are governed by the interactions of a particle with the ones which lie at a finite distance from it, and such maximal distance of interaction is often referred to with the name of “horizon”.

We now introduce the mathematical framework in which we work. Let $\mathcal{U} \subset \mathbb{R}^N$ be the rest configuration of a material body endowed with a mass density $\rho : \mathcal{U} \times [0, T] \rightarrow \mathbb{R}_+$, and let $\mathbf{u} : \mathcal{U} \times [0, T] \rightarrow \mathbb{R}^N$ be the displacement field assigning at the particle having position $\mathbf{x} \in \mathcal{U}$ at time $t = 0$ the new position $\mathbf{x} + \mathbf{u}(\mathbf{x}, t)$ at time t . The crucial assumption of peridynamics relies in postulating the existence of a long range internal force field, in place of the classical contact forces. Therefore the evolution of the material body is ruled by the following nonlocal version of the linear momentum balance:

$$(1.1) \quad \rho(\mathbf{x}, t) \partial_t^2 \mathbf{u}(\mathbf{x}, t) = \int_{V_{\mathbf{x}} \cap \mathcal{U}} \mathbf{f}(\mathbf{x}, \mathbf{x}', \mathbf{u}(\mathbf{x}, t), \mathbf{u}(\mathbf{x}', t), t) d\mathbf{x}' + \mathbf{b}(\mathbf{x}, t),$$

$$(1.2) \quad \mathbf{u}(\cdot, 0) = \mathbf{u}_0, \quad \partial_t \mathbf{u}(\cdot, 0) = \mathbf{v}_0,$$

where $V_{\mathbf{x}} \subset \mathbb{R}^N$ is a measurable subset with

$$(1.3) \quad \mathbf{x} \in V_{\mathbf{x}}, \quad \text{diam}(V_{\mathbf{x}}) \geq \delta > 0$$

and $\mathbf{b}(\mathbf{x}, t)$ represents the external body force field.

Let us make some comments on (1.1). Notice that the internal contact forces, condensed in the Cauchy stress tensor and representing the fundamental concept in classical continuum mechanics, here are replaced by the pairwise force function $\mathbf{f}$ which can be thought as the density of the interaction between the particle at $\mathbf{x}$ and all the particles $\mathbf{x}'$ belonging to the region $V_{\mathbf{x}}$ (one can also assume that $V_{\mathbf{x}} = \mathcal{U}$). Then, whereas in the classical context we have to face with partial differential equations and the evolution problem is an initial boundary value problem, in the present context we have an integro-differential equation.

The fundamental issue in this setting, which constitutes the core of the *mathematical-physics wellposedness*, relies in selecting the force field $\mathbf{f}$ in such a way that it satisfies the general principles of mechanics, to capture the essential features of the material behavior, to deliver a well posed mathematical problem. In this framework, we study the Cauchy problem for an unbounded domain, under general enough assumptions on the force field $\mathbf{f}$.

Due to the balance of linear and angular momentum, the pairwise force function $\mathbf{f}$ has the direction of the vector joining $\mathbf{x} + \mathbf{u}(\mathbf{x}, t)$ to $\mathbf{x}' + \mathbf{u}(\mathbf{x}', t)$, therefore we can write

$$\mathbf{f}(\mathbf{x}, \mathbf{x}', \mathbf{u}(\mathbf{x}, t), \mathbf{u}(\mathbf{x}', t), t) = f(\mathbf{x}, \mathbf{x}', \mathbf{u}(\mathbf{x}, t), \mathbf{u}(\mathbf{x}', t), t) \mathbf{e},$$

$$\text{where } \mathbf{e} = \frac{(\mathbf{x}' + \mathbf{u}(\mathbf{x}', t)) - (\mathbf{x} + \mathbf{u}(\mathbf{x}, t))}{|(\mathbf{x}' + \mathbf{u}(\mathbf{x}', t)) - (\mathbf{x} + \mathbf{u}(\mathbf{x}, t))|}.$$

Furthermore, assuming the invariance with respect to rigid motions and neglecting time dependence for the internal forces, we get

$$(1.4) \quad \mathbf{f}(\mathbf{x}, \mathbf{x}', \mathbf{u}(\mathbf{x}, t), \mathbf{u}(\mathbf{x}', t)) = \mathbf{f}(\mathbf{x}, \mathbf{x}', \mathbf{u}(\mathbf{x}, t) - \mathbf{u}(\mathbf{x}', t))$$

and the Newton law of *actio et reactio* delivers

$$(1.5) \quad \mathbf{f}(\mathbf{x}, \mathbf{x}', -\boldsymbol{\eta}) = -\mathbf{f}(\mathbf{x}, \mathbf{x}', \boldsymbol{\eta}).$$

Coherently with the literature on peridynamics in this section we will often use the notation

$$\boldsymbol{\xi} := \mathbf{x}' - \mathbf{x}, \quad \boldsymbol{\eta} := \mathbf{u}(\mathbf{x}', \cdot) - \mathbf{u}(\mathbf{x}, \cdot).$$

Let us continue by providing some examples of $\mathbf{f}$ in specific cases. For a *Linear Elastic Material*, we have

$$(1.6) \quad \mathbf{f}(\mathbf{x}, \mathbf{x}', \boldsymbol{\eta}) := \mathbf{f}_0(\mathbf{x}, \mathbf{x}') + \mathbf{C}(\mathbf{x}, \mathbf{x}')\boldsymbol{\eta}$$

By recalling (1.4), it is readily seen that the tensor $\mathbf{C}$ takes the form

$$(1.7) \quad \mathbf{C}(\mathbf{x}, \mathbf{x}') := \lambda(|\boldsymbol{\xi}|)\boldsymbol{\xi} \otimes \boldsymbol{\xi}$$

where $\lambda : \mathbb{R}_+ \rightarrow \mathbb{R}$ is a measurable function such that $\lambda(r) = 0$ for $r \geq \delta$ (recall that δ is the quantity introduced in (1.3)). The tensor $\mathbf{C}$ determines the specific material and depends on N and δ .

In the case of a *Nonlinear Elastic Material* one can assume

$$(1.8) \quad \mathbf{f}(\mathbf{x}, \mathbf{x}', \boldsymbol{\eta}) := \begin{cases} \varphi\left(\frac{|\boldsymbol{\xi} + \boldsymbol{\eta}|}{|\boldsymbol{\xi}|} - 1\right) \mathbf{e}, & \text{if } |\boldsymbol{\xi}| < \delta, \\ \mathbf{0}, & \text{if } |\boldsymbol{\xi}| \geq \delta, \end{cases}$$

where the function φ depends on $s := \left(\frac{|\boldsymbol{\xi} + \boldsymbol{\eta}|}{|\boldsymbol{\xi}|} - 1\right)$, which represents the *bond stretch*, i.e. the relative change of the length of a pairwise bond.

The main mathematical problems in peridynamics concern well-posedness and regularity for the integral-differential equation (1.1) under the particular choice of $\mathbf{f}$ related to the material behavior and the analysis of the limit for vanishing nonlocality, i.e. the issue of characterizing the solution as well as the problem as $\delta \rightarrow 0$.

In the linear elastic case, well-posedness and regularity were established in [10], though the problem of the limit as $\delta \rightarrow 0$ is still largely open in its generality (see [16]). In particular, assuming that $\mathbf{u}_0, \mathbf{v}_0 \in L^p(\mathcal{U})$ and $\mathbf{b} \in L^1(0, T; L^p(\mathcal{U}))$, the authors prove the existence of a unique solution $\mathbf{u} \in C^1([0, T]; L^p(\mathcal{U}))$ to the initial-value problem (1.1)-(1.2) in the linear case (1.6).

In the case of nonlinear elasticity, the situation is more involved and at the present time few results are known [5, 6, 7, 8, 9, 10, 18] (the situation is even more complicated in the classical nonlinear elastodynamics). According to the authors of [8], the main known results can be represented by the two theorems below related to the peridynamic operator

$$(K\mathbf{u})(\mathbf{x}) := \int_{\mathcal{U} \cap B_\delta(\mathbf{x})} \mathbf{f}(\mathbf{x}' - \mathbf{x}, \mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x})) d\mathbf{x}',$$

where $B_\delta(\mathbf{x})$ denotes the N -dimensional open ball centered in $\mathbf{x}$ of radius δ .

The first result that we recall is the following:

Theorem 1.1. ([8]) *Let $\mathbf{u}_0, \mathbf{v}_0 \in C(\overline{\mathcal{U}})^N$ and $\mathbf{b} \in C([0, T]; C(\overline{\mathcal{U}})^N)$. Assume that $\mathbf{f} : \overline{B_\delta(\mathbf{0})} \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ is continuous and that there exists a nonnegative function $\ell \in L^1(B_\delta(\mathbf{0}))$ such that for all $\boldsymbol{\xi} \in \mathbb{R}^N$ with $|\boldsymbol{\xi}| \leq \delta$ and $\boldsymbol{\eta}, \boldsymbol{\eta}'$ there holds*

$$|\mathbf{f}(\boldsymbol{\xi}, \boldsymbol{\eta}') - \mathbf{f}(\boldsymbol{\xi}, \boldsymbol{\eta})| \leq \ell(\boldsymbol{\xi})|\boldsymbol{\eta}' - \boldsymbol{\eta}|.$$

Then, the peridynamic operator $K : C(\overline{\mathcal{U}})^N \rightarrow C(\overline{\mathcal{U}})^N$ is well-defined and Lipschitz-continuous, and the initial-value problem (1.1)-(1.2) is globally well-posed and there exists a solution $\mathbf{u} \in C^2([0, T]; C(\overline{\mathcal{U}})^N)$.

Notice that, as remarked by the authors of [8], global Lipschitz-continuity of the pairwise force function with respect to $\boldsymbol{\eta}$ is quite a restrictive assumption since it implies linearly bounded growth.

The next result that we recall regards the existence of weak solutions. For this, before stating the theorem, we need some definitions. We denote by $(\cdot, \cdot)$ the inner product in $L^2(\mathcal{U})^N$ and we consider a Banach space $X \subset L^2(\mathcal{U})^N$. Moreover, $K : X \rightarrow X^*$ is the energetic extension of the peridynamic operator, namely

$$\langle K\mathbf{w}, \mathbf{z} \rangle := \frac{1}{2} \int_{\mathcal{U} \times \mathcal{U}} a(|\boldsymbol{\xi}|, |\boldsymbol{\eta}|) (\mathbf{w}(\mathbf{x}') - \mathbf{w}(\mathbf{x})) \cdot (\mathbf{z}(\mathbf{x}') - \mathbf{z}(\mathbf{x})) d(\mathbf{x}', \mathbf{x}),$$

for every $\mathbf{w}, \mathbf{z} \in X$.

With this notation, we say that $\mathbf{u} : [0, T] \rightarrow X$ satisfying the initial conditions (1.2) is a weak solution for (1.1) if, for every $\varphi \in C_0^\infty(0, T)$ and every $\mathbf{z} \in X$,

$$- \int_0^T \partial_t \varphi(t) (\partial_t \mathbf{u}(t), \mathbf{z}) dt - \int_0^T \langle K\mathbf{u}(t), \mathbf{z} \rangle \varphi(t) dt = \int_0^T (\mathbf{b}(t), \mathbf{z}) \varphi(t) dt.$$

In the following, the basic function space is the Sobolev-Slobodeckij space $X = W^{\sigma,p}(\mathcal{U})^N$ with $0 < \sigma < 1$ and $2 \leq p < +\infty$, and $C_w([0, T]; X)$ denotes the space of the functions $v : [0, T] \rightarrow X$ which are continuous with respect to the weak convergence in X . In this setting, we have:

Theorem 1.2. ([9]) *Assume that $\mathbf{b} \in L^1(0, T; L^2(\mathcal{U})^N)$, $\mathbf{u}_0 \in W^{\sigma,p}(\mathcal{U})^N$, with $0 < \sigma < 1$ and $2 \leq p < +\infty$, $\mathbf{v}_0 \in L^2(\mathcal{U})^N$, and the pairwise force function $\mathbf{f}$ satisfies suitable growth and regularity conditions.*

Then, there exists a function $\mathbf{u} : [0, T] \rightarrow W^{\sigma,p}(\mathcal{U})^N$ with

$$\mathbf{u} \in C_w([0, T]; W^{\sigma,p}(\mathcal{U})^N), \quad \partial_t \mathbf{u} \in C_w([0, T]; L^2(\mathcal{U})^N), \quad \partial_t^2 \mathbf{u} \in L^1(0, T; (W^{\sigma,p}(\mathcal{U})^N)^*),$$

such that

$$\partial_t^2 \mathbf{u} - K\mathbf{u} = \mathbf{b} \quad \text{in } L^1(0, T; (W^{\sigma,p}(\mathcal{U})^N)^*)$$

and $\mathbf{u}(\cdot, 0) = \mathbf{u}_0$ in $W^{\sigma,p}(\mathcal{U})^N$, $\partial_t \mathbf{u}(\cdot, 0) = \mathbf{v}_0$ in $L^2(\mathcal{U})^N$.

The goal of this paper is to provide:

- **existence** results in the spirit of Theorems 1.1 and 1.2, but which are valid for singular and non-Lipschitz interaction force,
- **uniqueness** and **stability** results.

The complete description of the mathematical setting in which we work will be given in the forthcoming Section 2, but, for simplicity, we mention here that the interaction force we can take into account comprises, among the others, examples of the form

$$\mathbf{f}(\boldsymbol{\xi}, \boldsymbol{\eta}) = \frac{|\boldsymbol{\eta}|^{p-2} \boldsymbol{\eta}}{|\boldsymbol{\xi}|^{N+\alpha p}} + \boldsymbol{\psi}(\boldsymbol{\xi}, \boldsymbol{\eta}),$$

where $p \geq 2$, $\alpha \in (0, 1)$ and $\boldsymbol{\psi}$ plays the role of a “sufficiently smooth perturbation”, e.g.

$$\boldsymbol{\psi}(\boldsymbol{\xi}, \boldsymbol{\eta}) = \sum_{i=1}^N \sin \boldsymbol{\xi}_i \cos \boldsymbol{\eta}_i.$$

In this framework, we establish that

for every initial datum with finite energy (see the forthcoming formula (2.6)), the Cauchy problem admits at least one weak solution whose energy at time t is bounded by the initial one.

A precise statement for this existence result will be given in Theorem 2.4.

In the case $p = 2$, we also provide a uniqueness and stability result. Namely,

if $\mathbf{u}$ and $\tilde{\mathbf{u}}$ are weak solutions with finite energy initial data, then the quantity

$$\|\partial_t \mathbf{u}(\cdot, t) - \partial_t \tilde{\mathbf{u}}(\cdot, t)\|_{L^2(\mathbb{R}^N)}^2 + \int_{\mathbb{R}^N} \int_{B_\delta(\mathbf{0})} \frac{|\mathbf{u}(\mathbf{x}, t) - \mathbf{u}(\mathbf{x} - \mathbf{y}, t) - \tilde{\mathbf{u}}(\mathbf{x}, t) + \tilde{\mathbf{u}}(\mathbf{x} - \mathbf{y}, t)|^2}{|\mathbf{y}|^{N+2\alpha}} d\mathbf{x} d\mathbf{y}$$

is bounded, up to constants, by the same quantities at the initial time, multiplied by an exponential in time.

In particular, if the initial data coincide, the two solutions must coincide as well. A detailed statement for this result will be given in Theorem 2.5.

We think that our theoretic results provide interesting pieces of information on the physical model of peridynamics, since we establish here a mathematically coherent setting, in which solutions are shown to exist and to have universal energy bounds in terms of the initial datum; moreover we show that the solutions are stable with respect to the datum, in the sense that Lebesgue and fractional norms of the evolutions of two different functions are uniformly controlled by the same quantities at the initial time, with at most an exponential growth in time – and this in turn implies that the solution for a given initial datum is unique, with quantitative bounds on how two given solutions separate from each other. Furthermore, we believe that the general mathematical setting in which the wellposedness results are proved would allow to capture a quite general physical phenomenology. It is interesting to point out that the functional spaces in which we work allow, in principle, singular functions (see Appendix B in [3]).

The rest of the paper is organized as follows. In Section 2 we state precisely the problem that we study, the assumptions and the main results. Section 3 is devoted to the proof of the existence of weak solutions. The uniqueness and stability of such solutions is proved in Section 4.

2. STATEMENT OF THE PROBLEM AND MAIN RESULTS

This section is devoted to the rigorous mathematical formulation of the problem, the definition of weak solutions, and the statements of the main results of the paper. To keep the analysis of the mathematical problem clear we assume $\mathbf{b} \equiv \mathbf{0}$.

2.1. Set-up of the problem and assumptions. We consider the Cauchy problem

$$(2.1) \quad \begin{cases} \partial_t^2 \mathbf{u}(\mathbf{x}, t) = (K\mathbf{u}(\cdot, t))(\mathbf{x}), & \mathbf{x} \in \mathbb{R}^N, t > 0, \\ \mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}), \partial_t \mathbf{u}(\mathbf{x}, 0) = \mathbf{v}_0(\mathbf{x}), & \mathbf{x} \in \mathbb{R}^N, \end{cases}$$

where

$$(2.2) \quad (K\mathbf{u})(\mathbf{x}) := \int_{B_\delta(\mathbf{x})} \mathbf{f}(\mathbf{x}' - \mathbf{x}, \mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x})) d\mathbf{x}', \quad \text{for every } \mathbf{x} \in \mathbb{R}^N,$$

for a given $\delta > 0$. Here, the $\mathbb{R}^N$ valued function $\mathbf{f}$ is defined on the set

$$\Omega := (\mathbb{R}^N \setminus \{\mathbf{0}\}) \times \mathbb{R}^N$$

and we assume that

$$(H.1) \quad \mathbf{f} \in C^1(\Omega; \mathbb{R}^N);$$

$$(H.2) \quad \mathbf{f}(-\mathbf{y}, -\mathbf{u}) = -\mathbf{f}(\mathbf{y}, \mathbf{u}), \text{ for every } (\mathbf{y}, \mathbf{u}) \in \Omega \times \mathbb{R}^N;$$

(H.3) the material is hyperelastic, i.e., there exists a function $\Phi \in C^2(\Omega)$ such that

$$\mathbf{f} = \nabla_{\mathbf{u}} \Phi, \quad \Phi(\mathbf{y}, \mathbf{u}) = \kappa \frac{|\mathbf{u}|^p}{|\mathbf{y}|^{N+\alpha p}} + \Psi(\mathbf{y}, \mathbf{u}), \quad \text{for every } (\mathbf{y}, \mathbf{u}) \in \Omega,$$

where κ, p, α are constants such that

$$\kappa > 0, \quad 0 < \alpha < 1, \quad p \geq 2,$$

and

$$\begin{aligned} \Psi(\mathbf{y}, \mathbf{0}) &= 0 \leq \Psi(\mathbf{y}, \mathbf{u}), \\ |\nabla_{\mathbf{u}} \Psi(\mathbf{y}, \mathbf{u})|, |D_{\mathbf{u}}^2 \Psi(\mathbf{y}, \mathbf{u})| &\leq g(\mathbf{y}), \quad \text{for every } (\mathbf{y}, \mathbf{u}) \in \Omega, \end{aligned}$$

for some nonnegative function $g \in L_{\text{loc}}^2(\mathbb{R}^N)$.

We notice that if $\Psi(-\mathbf{y}, -\mathbf{u}) = \Psi(\mathbf{y}, \mathbf{u})$, then (H.2) holds true. In addition, Assumption (H.3) can be easily generalized to the anisotropic case by taking

$$\Phi(\mathbf{y}, \mathbf{u}) := (\mathbb{K} \mathbf{u} \cdot \mathbf{u}) \frac{|\mathbf{u}|^{p-2}}{|\mathbf{y}|^{N+\alpha p}} + \Psi(\mathbf{y}, \mathbf{u}), \quad \text{for every } (\mathbf{y}, \mathbf{u}) \in \Omega \times \mathbb{R}^N,$$

where $\mathbb{K} \in \mathbb{R}^{N \times N}$ is a positive definite matrix. Here, we stick to assumption (H.3) for the sake of simplicity.

We observe that (H.1), (H.2), (H.3) are the only constitutive assumptions characterizing the peridynamic model and they are sufficient to prove global well-posedness of the problem, as it is shown in Theorems 2.4 and 2.5 below. We emphasize that here, in contrast to classical (local) elastodynamics neither polyconvexity or null condition (see [1, 19]) are required to guarantee existence of global (in time) solutions.

Setting $\mathbf{y} = \mathbf{x}' - \mathbf{x}$ and using (H.2), we will often rewrite the operator K in (2.2) as follows

$$(2.3) \quad (K\mathbf{u})(\mathbf{x}) = - \int_{B_\delta(\mathbf{0})} \mathbf{f}(\mathbf{y}, \mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{x} - \mathbf{y})) d\mathbf{y}, \quad \text{for every } \mathbf{x} \in \mathbb{R}^N.$$

Moreover, due to (H.3), we can also rewrite

$$(2.4) \quad \mathbf{f}(\mathbf{y}, \mathbf{u}) = \kappa p \frac{\mathbf{u} |\mathbf{u}|^{p-2}}{|\mathbf{y}|^{N+\alpha p}} + \nabla_{\mathbf{u}} \Psi(\mathbf{y}, \mathbf{u}), \quad \text{for every } (\mathbf{y}, \mathbf{u}) \in \Omega.$$

As a consequence, in virtue of (2.3) and (2.4), we have that

$$(2.5) \quad \begin{aligned} (K\mathbf{u})(\mathbf{x}) &= - \kappa p \int_{B_\delta(\mathbf{0})} \frac{(\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{x} - \mathbf{y})) |\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{x} - \mathbf{y})|^{p-2}}{|\mathbf{y}|^{N+\alpha p}} d\mathbf{y} \\ &\quad - \int_{B_\delta(\mathbf{0})} \nabla_{\mathbf{u}} \Psi(\mathbf{y}, \mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{x} - \mathbf{y})) d\mathbf{y}. \end{aligned}$$

Furthermore, the energy associated to (2.1) is

$$(2.6) \quad E[\mathbf{u}](t) := \frac{\|\partial_t \mathbf{u}(\cdot, t)\|_{L^2(\mathbb{R}^N)}^2}{2} + \frac{1}{2} \int_{\mathbb{R}^N} \int_{B_\delta(\mathbf{0})} \Phi(\mathbf{y}, \mathbf{u}(\mathbf{x}, t) - \mathbf{u}(\mathbf{x} - \mathbf{y}, t)) d\mathbf{x} d\mathbf{y},$$

see the forthcoming proof of Lemma 3.2 below. Also, the equation in (2.1) is the Euler-Lagrange equation of the action functional

$$(2.7) \quad \mathbf{u} \mapsto \int_0^\infty \int_{\mathbb{R}^N} \left(\frac{(\partial_t \mathbf{u})^2}{2} - \frac{1}{2} \int_{B_\delta(\mathbf{0})} \Phi(\mathbf{y}, \mathbf{u}(\mathbf{x}, t) - \mathbf{u}(\mathbf{x} - \mathbf{y}, t)) d\mathbf{y} \right) dt d\mathbf{x}.$$

2.2. Fractional Sobolev spaces. Let α and p be the parameters introduced in (H.3). We remind that the fractional Sobolev space $W^{\alpha,p}$ is defined through the norm

$$\|\mathbf{u}\|_{W^{\alpha,p}(\mathbb{R}^N;\mathbb{R}^N)} := \left(\int_{\mathbb{R}^N} |\mathbf{u}|^p d\mathbf{x} + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{x} - \mathbf{y})|^p}{|\mathbf{y}|^{N+\alpha p}} d\mathbf{x} d\mathbf{y} \right)^{1/p}$$

(see [2, Section 2]), and the following compact embedding holds

$$W^{\alpha,p}(\mathbb{R}^N;\mathbb{R}^N) \hookrightarrow L_{\text{loc}}^q(\mathbb{R}^N;\mathbb{R}^N), \quad 1 \leq q \leq p,$$

(see [2, Theorem 7.1]). Here and throughout the paper, we use the notation $\hookrightarrow$ to denote a compact embedding.

In this paper, we use a slight modification of the fractional Sobolev space $W^{\alpha,p}$. Namely, we consider the space $\mathcal{W}$ defined through the norm

$$(2.8) \quad \|\mathbf{u}\|_{\mathcal{W}} := \|\mathbf{u}\|_{L^2(\mathbb{R}^N;\mathbb{R}^N)} + \left(\int_{\mathbb{R}^N} \int_{B_\delta(\mathbf{0})} \frac{|\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{x} - \mathbf{y})|^p}{|\mathbf{y}|^{N+\alpha p}} d\mathbf{x} d\mathbf{y} \right)^{1/p}.$$

Then, the following compact embedding can be proved:

Lemma 2.1. $\mathcal{W} \hookrightarrow L_{\text{loc}}^2(\mathbb{R}^N;\mathbb{R}^N)$.

Proof. One can argue as in [2, Theorem 7.1], or make the following observations. We suppose $p > 2$ (the case $p = 2$ following directly from [2, Theorem 7.1]). Fix $\lambda \in (0, \alpha)$ and a bounded domain $\mathcal{K} \subset \mathbb{R}^N$. We write

$$\begin{aligned} N + 2(\alpha - \lambda) &= \frac{2(N + \alpha p)}{p} + \frac{p(N + 2(\alpha - \lambda)) - 2(N + \alpha p)}{p} \\ &= \frac{2(N + \alpha p)}{p} + \frac{(p - 2)N}{p} - 2\lambda. \end{aligned}$$

Therefore, using the Hölder inequality with exponents $\frac{p}{2}$ and $\frac{p}{p-2}$, we find that, for any $\mathbf{z} \in \mathbb{R}^n$,

$$\begin{aligned} \int_{B_{\delta/2}(\mathbf{z})} \int_{B_{\delta/2}(\mathbf{z})} \frac{|\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{y})|^2}{|\mathbf{x} - \mathbf{y}|^{N+2(\alpha-\lambda)}} d\mathbf{x} d\mathbf{y} &= \int_{B_{\delta/2}(\mathbf{z})} \int_{B_{\delta/2}(\mathbf{z})} \frac{|\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{y})|^2}{|\mathbf{x} - \mathbf{y}|^{\frac{2(N+\alpha p)}{p}}} \frac{d\mathbf{x} d\mathbf{y}}{|\mathbf{x} - \mathbf{y}|^{\frac{(p-2)N}{p} - 2\lambda}} \\ &\leq \left(\int_{B_{\delta/2}(\mathbf{z})} \int_{B_{\delta/2}(\mathbf{z})} \frac{|\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{y})|^p}{|\mathbf{x} - \mathbf{y}|^{N+\alpha p}} d\mathbf{x} d\mathbf{y} \right)^{\frac{2}{p}} \left(\int_{B_{\delta/2}(\mathbf{z})} \int_{B_{\delta/2}(\mathbf{z})} \frac{d\mathbf{x} d\mathbf{y}}{|\mathbf{x} - \mathbf{y}|^{N - \frac{2\lambda p}{p-2}}} \right)^{\frac{p-2}{p}} \\ &\leq C \left(\int_{\mathbb{R}^N} \int_{B_\delta(\mathbf{0})} \frac{|\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{x} - \mathbf{y})|^p}{|\mathbf{y}|^{N+\alpha p}} d\mathbf{x} d\mathbf{y} \right)^{\frac{2}{p}}, \end{aligned}$$

for some $C > 0$, possibly depending on N , α , p and λ . Accordingly, if a family of functions is bounded in $\mathcal{W}$, then each component is bounded in $W^{\alpha-\lambda,2}(B_{\delta/2}(\mathbf{z}))$, for any $\mathbf{z} \in \mathbb{R}^N$, and so, by [2, Theorem 7.1], we obtain compactness in $L^2(B_{\delta/2}(\mathbf{z}))$. Arguing component by component and covering $\mathcal{K}$ with a finite number of balls of radius $\delta/2$, we obtain the desired compactness in $L^2(\mathcal{K};\mathbb{R}^N)$. $\square$

Lemma 2.2. For every $\mathbf{u}, \mathbf{v} \in \mathcal{W}$ we have that

$$(2.9) \quad (K\mathbf{u})\mathbf{v} \in L^1(\mathbb{R}^N).$$

Moreover, for every sequence $\{\mathbf{u}_n\}_n \subset \mathcal{W}$ and $\mathbf{u} \in \mathcal{W}$, if

$$(2.10) \quad \mathbf{u}_n \rightharpoonup \mathbf{u} \quad \text{weakly in } \mathcal{W},$$

then

$$(2.11) \quad K\mathbf{u}_n \rightarrow K\mathbf{u} \quad \text{in the sense of distributions on } \mathbb{R}^N,$$

as $n \rightarrow +\infty$.

Proof. We first prove (2.9). To this aim, we let $\mathbf{u}, \mathbf{v} \in \mathcal{W}$. Recalling (2.5) and (H.3), and using the Hölder inequality, we have that

$$\begin{aligned} & \int_{\mathbb{R}^N} (K\mathbf{u})(\mathbf{x})\mathbf{v}(\mathbf{x}) \, d\mathbf{x} \\ &= -\kappa p \int_{\mathbb{R}^N} \int_{B_\delta(\mathbf{0})} \frac{(\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{x} - \mathbf{y}))|\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{x} - \mathbf{y})|^{p-2}}{|\mathbf{y}|^{N+\alpha p}} \mathbf{v}(\mathbf{x}) \, d\mathbf{x} \, d\mathbf{y} \\ & \quad - \int_{\mathbb{R}^N} \int_{B_\delta(\mathbf{0})} \nabla_{\mathbf{u}} \Psi(\mathbf{y}, \mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{x} - \mathbf{y})) \mathbf{v}(\mathbf{x}) \, d\mathbf{x} \, d\mathbf{y} \\ &= -\frac{\kappa p}{2} \int_{\mathbb{R}^N} \int_{B_\delta(\mathbf{0})} \frac{(\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{x} - \mathbf{y}))|\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{x} - \mathbf{y})|^{p-2}}{|\mathbf{y}|^{N+\alpha p}} \mathbf{v}(\mathbf{x}) \, d\mathbf{x} \, d\mathbf{y} \\ & \quad - \frac{\kappa p}{2} \int_{\mathbb{R}^N} \int_{B_\delta(\mathbf{0})} \frac{(\mathbf{u}(\mathbf{z} + \mathbf{y}) - \mathbf{u}(\mathbf{z}))|\mathbf{u}(\mathbf{z} + \mathbf{y}) - \mathbf{u}(\mathbf{z})|^{p-2}}{|\mathbf{y}|^{N+\alpha p}} \mathbf{v}(\mathbf{z} + \mathbf{y}) \, d\mathbf{z} \, d\mathbf{y} \\ & \quad - \int_{\mathbb{R}^N} \int_{B_\delta(\mathbf{0})} \nabla_{\mathbf{u}} \Psi(\mathbf{y}, \mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{x} - \mathbf{y})) \mathbf{v}(\mathbf{x}) \, d\mathbf{x} \, d\mathbf{y} \\ &= -\frac{\kappa p}{2} \int_{\mathbb{R}^N} \int_{B_\delta(\mathbf{0})} \frac{(\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{x} - \mathbf{y}))|\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{x} - \mathbf{y})|^{p-2}}{|\mathbf{y}|^{N+\alpha p}} (\mathbf{v}(\mathbf{x}) - \mathbf{v}(\mathbf{x} - \mathbf{y})) \, d\mathbf{x} \, d\mathbf{y} \\ & \quad - \int_{\mathbb{R}^N} \int_{B_\delta(\mathbf{0})} \nabla_{\mathbf{u}} \Psi(\mathbf{y}, \mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{x} - \mathbf{y})) \mathbf{v}(\mathbf{x}) \, d\mathbf{x} \, d\mathbf{y} \\ &\leq \frac{\kappa p}{2} \int_{\mathbb{R}^N} \int_{B_\delta(\mathbf{0})} \frac{|\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{x} - \mathbf{y})|^{p-1}}{|\mathbf{y}|^{(N+\alpha p)/p'}} \frac{|\mathbf{v}(\mathbf{x}) - \mathbf{v}(\mathbf{x} - \mathbf{y})|}{|\mathbf{y}|^{(N+\alpha p)/p}} \, d\mathbf{x} \, d\mathbf{y} \\ & \quad + \int_{\mathbb{R}^N} \int_{B_\delta(\mathbf{0})} g(\mathbf{y}) |\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{x} - \mathbf{y})| |\mathbf{v}(\mathbf{x})| \, d\mathbf{x} \, d\mathbf{y} \\ &\leq \frac{\kappa p}{2} \left(\int_{\mathbb{R}^N} \int_{B_\delta(\mathbf{0})} \frac{|\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{x} - \mathbf{y})|^p}{|\mathbf{y}|^{N+\alpha p}} \, d\mathbf{x} \, d\mathbf{y} \right)^{1/p'} \left(\int_{\mathbb{R}^N} \int_{B_\delta(\mathbf{0})} \frac{|\mathbf{v}(\mathbf{x}) - \mathbf{v}(\mathbf{x} - \mathbf{y})|^p}{|\mathbf{y}|^{N+\alpha p}} \, d\mathbf{x} \, d\mathbf{y} \right)^{1/p} \\ & \quad + \int_{\mathbb{R}^N} \int_{B_\delta(\mathbf{0})} g(\mathbf{y}) (|\mathbf{u}(\mathbf{x})| + |\mathbf{u}(\mathbf{x} - \mathbf{y})|) |\mathbf{v}(\mathbf{x})| \, d\mathbf{x} \, d\mathbf{y}, \end{aligned}$$

namely

$$\begin{aligned}
(2.12) \quad & \int_{\mathbb{R}^N} (K\mathbf{u})(\mathbf{x})\mathbf{v}(\mathbf{x}) \, d\mathbf{x} \\
& \leq \frac{\kappa p}{2} \left(\int_{\mathbb{R}^N} \int_{B_\delta(\mathbf{0})} \frac{|\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{x} - \mathbf{y})|^p}{|\mathbf{y}|^{N+\alpha p}} \, d\mathbf{x} \, d\mathbf{y} \right)^{1/p'} \times \\
& \quad \times \left(\int_{\mathbb{R}^N} \int_{B_\delta(\mathbf{0})} \frac{|\mathbf{v}(\mathbf{x}) - \mathbf{v}(\mathbf{x} - \mathbf{y})|^p}{|\mathbf{y}|^{N+\alpha p}} \, d\mathbf{x} \, d\mathbf{y} \right)^{1/p} \\
& \quad + \int_{\mathbb{R}^N} \int_{B_\delta(\mathbf{0})} g(\mathbf{y}) (|\mathbf{u}(\mathbf{x})| + |\mathbf{u}(\mathbf{x} - \mathbf{y})|) |\mathbf{v}(\mathbf{x})| \, d\mathbf{x} \, d\mathbf{y},
\end{aligned}$$

Now, we observe that, for any $\lambda > 0$, $\mathbf{x}, \mathbf{y} \in \mathbb{R}^N$,

$$\begin{aligned}
g(\mathbf{y}) |\mathbf{u}(\mathbf{x} - \mathbf{y})| |\mathbf{v}(\mathbf{x})| &= 2g(\mathbf{y}) \frac{\lambda}{\sqrt{2}} |\mathbf{u}(\mathbf{x} - \mathbf{y})| \frac{1}{\lambda\sqrt{2}} |\mathbf{v}(\mathbf{x})| \\
&\leq \frac{\lambda^2}{2} g(\mathbf{y}) |\mathbf{u}(\mathbf{x} - \mathbf{y})|^2 + \frac{1}{2\lambda^2} g(\mathbf{y}) |\mathbf{v}(\mathbf{x})|^2,
\end{aligned}$$

where we used the Young inequality. Therefore, integrating the last inequality in $\mathbb{R}^N \times B_\delta(\mathbf{0})$, we get

$$\begin{aligned}
& \int_{\mathbb{R}^N} \int_{B_\delta(\mathbf{0})} g(\mathbf{y}) |\mathbf{u}(\mathbf{x} - \mathbf{y})| |\mathbf{v}(\mathbf{x})| \, d\mathbf{x} \, d\mathbf{y} \\
& \leq \frac{\lambda^2}{2} \int_{\mathbb{R}^N} \int_{B_\delta(\mathbf{0})} g(\mathbf{y}) |\mathbf{u}(\mathbf{x} - \mathbf{y})|^2 \, d\mathbf{x} \, d\mathbf{y} + \frac{1}{2\lambda^2} \int_{\mathbb{R}^N} \int_{B_\delta(\mathbf{0})} g(\mathbf{y}) |\mathbf{v}(\mathbf{x})|^2 \, d\mathbf{x} \, d\mathbf{y} \\
& = \frac{\lambda^2}{2} \left(\int_{B_\delta(\mathbf{0})} g(\mathbf{y}) \, d\mathbf{y} \right) \left(\int_{\mathbb{R}^N} |\mathbf{u}(\mathbf{x})|^2 \, d\mathbf{x} \right) + \frac{1}{2\lambda^2} \left(\int_{B_\delta(\mathbf{0})} g(\mathbf{y}) \, d\mathbf{y} \right) \left(\int_{\mathbb{R}^N} |\mathbf{v}(\mathbf{x})|^2 \, d\mathbf{x} \right).
\end{aligned}$$

Hence, the choice

$$\lambda := \left(\frac{\int_{\mathbb{R}^N} |\mathbf{u}(\mathbf{x})|^2 \, d\mathbf{x}}{\int_{\mathbb{R}^N} |\mathbf{v}(\mathbf{x})|^2 \, d\mathbf{x}} \right)^{\frac{1}{4}}$$

allows to state

$$\begin{aligned}
& \int_{\mathbb{R}^N} \int_{B_\delta(\mathbf{0})} g(\mathbf{y}) |\mathbf{u}(\mathbf{x} - \mathbf{y})| |\mathbf{v}(\mathbf{x})| \, d\mathbf{x} \, d\mathbf{y} \\
& \leq \left(\int_{B_\delta(\mathbf{0})} g(\mathbf{y}) \, d\mathbf{y} \right) \left(\int_{\mathbb{R}^N} |\mathbf{u}(\mathbf{x})|^2 \, d\mathbf{x} \right)^{1/2} \left(\int_{\mathbb{R}^N} |\mathbf{v}(\mathbf{x})|^2 \, d\mathbf{x} \right)^{1/2}.
\end{aligned}$$

Plugging this information into (2.12), we conclude that

$$\begin{aligned}
& \int_{\mathbb{R}^N} (K\mathbf{u})(\mathbf{x})\mathbf{v}(\mathbf{x}) \, d\mathbf{x} \\
& \leq \frac{\kappa p}{2} \left(\int_{\mathbb{R}^N} \int_{B_\delta(\mathbf{0})} \frac{|\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{x} - \mathbf{y})|^p}{|\mathbf{y}|^{N+\alpha p}} \, d\mathbf{x} \, d\mathbf{y} \right)^{1/p'} \left(\int_{\mathbb{R}^N} \int_{B_\delta(\mathbf{0})} \frac{|\mathbf{v}(\mathbf{x}) - \mathbf{v}(\mathbf{x} - \mathbf{y})|^p}{|\mathbf{y}|^{N+\alpha p}} \, d\mathbf{x} \, d\mathbf{y} \right)^{1/p}
\end{aligned}$$

$$+ 2 \left(\int_{B_\delta(\mathbf{0})} g(\mathbf{y}) \, d\mathbf{x} \right) \left(\int_{\mathbb{R}^N} |\mathbf{u}(\mathbf{x})|^2 \, d\mathbf{x} \right)^{1/2} \left(\int_{\mathbb{R}^N} |\mathbf{v}(\mathbf{x})|^2 \, d\mathbf{x} \right)^{1/2}.$$

Since $\mathbf{u}, \mathbf{v} \in \mathcal{W}$, this implies (2.9).

Now suppose that (2.10) holds true and we prove (2.11). For this, let $\mathbf{v} \in C^\infty(\mathbb{R}^N; \mathbb{R}^N)$ be such that every component has compact support. For the sake of simplicity, we use the notation

$$(2.13) \quad \begin{aligned} \mathbf{U}(\mathbf{x}, \mathbf{y}) &:= \mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{x} - \mathbf{y}), \\ \mathbf{U}_n(\mathbf{x}, \mathbf{y}) &:= \mathbf{u}_n(\mathbf{x}) - \mathbf{u}_n(\mathbf{x} - \mathbf{y}) \\ \mathbf{V}(\mathbf{x}, \mathbf{y}) &:= \mathbf{v}(\mathbf{x}) - \mathbf{v}(\mathbf{x} - \mathbf{y}). \end{aligned}$$

Arguing as before, we have that

$$(2.14) \quad \begin{aligned} & \int_{\mathbb{R}^N} ((K\mathbf{u})(\mathbf{x}) - (K\mathbf{u}_n)(\mathbf{x})) \mathbf{v}(\mathbf{x}) \, d\mathbf{x} \\ &= - \underbrace{\frac{\kappa p}{2} \int_{\mathbb{R}^N} \int_{B_\delta(\mathbf{0})} \frac{(\mathbf{U}(\mathbf{x}, \mathbf{y}) - \mathbf{U}_n(\mathbf{x}, \mathbf{y})) |\mathbf{U}(\mathbf{x}, \mathbf{y})|^{p-2}}{|\mathbf{y}|^{N+\alpha p}} \mathbf{V}(\mathbf{x}, \mathbf{y}) \, d\mathbf{x} \, d\mathbf{y}}_{\mathcal{A}_n} \\ & \quad - \underbrace{\frac{\kappa p}{2} \int_{\mathbb{R}^N} \int_{B_\delta(\mathbf{0})} \mathbf{U}(\mathbf{x}, \mathbf{y}) \frac{|\mathbf{U}(\mathbf{x}, \mathbf{y})|^{p-2} - |\mathbf{U}_n(\mathbf{x}, \mathbf{y})|^{p-2}}{|\mathbf{y}|^{N+\alpha p}} \mathbf{V}(\mathbf{x}, \mathbf{y}) \, d\mathbf{x} \, d\mathbf{y}}_{\mathcal{B}_n} \\ & \quad - \underbrace{\int_{\mathbb{R}^N} \int_{B_\delta(\mathbf{0})} \left(\nabla_{\mathbf{u}} \Psi(\mathbf{y}, \mathbf{U}(\mathbf{x}, \mathbf{y})) - \nabla_{\mathbf{u}} \Psi(\mathbf{y}, \mathbf{U}_n(\mathbf{x}, \mathbf{y})) \right) \mathbf{v}(\mathbf{x}) \, d\mathbf{x} \, d\mathbf{y}}_{\mathcal{C}_n}. \end{aligned}$$

Now we claim that

$$(2.15) \quad \mathcal{A}_n \rightarrow 0.$$

To check this, we use that each component of $\mathbf{v}$ is compactly supported, and we suppose that the support is contained in some ball $B_R(\mathbf{0})$. Then,

$$\begin{aligned} & \frac{1}{2} \int_{B_{R+\delta}(\mathbf{0})} \int_{B_\delta(\mathbf{0})} |\mathbf{U}_n(\mathbf{x}, \mathbf{y}) - \mathbf{U}(\mathbf{x}, \mathbf{y})|^2 \, d\mathbf{x} \, d\mathbf{y} \\ &= \frac{1}{2} \int_{B_{R+\delta}(\mathbf{0})} \int_{B_\delta(\mathbf{0})} |(\mathbf{u}_n(\mathbf{x}) - \mathbf{u}(\mathbf{x})) - (\mathbf{u}_n(\mathbf{x} - \mathbf{y}) - \mathbf{u}(\mathbf{x} - \mathbf{y}))|^2 \, d\mathbf{x} \, d\mathbf{y} \\ &\leq \int_{B_{R+\delta}(\mathbf{0})} \int_{B_\delta(\mathbf{0})} |\mathbf{u}_n(\mathbf{x}) - \mathbf{u}(\mathbf{x})|^2 \, d\mathbf{x} \, d\mathbf{y} + \int_{B_{R+\delta}(\mathbf{0})} \int_{B_\delta(\mathbf{0})} |\mathbf{u}_n(\mathbf{x} - \mathbf{y}) - \mathbf{u}(\mathbf{x} - \mathbf{y})|^2 \, d\mathbf{x} \, d\mathbf{y}, \end{aligned}$$

which converges to 0 as $n \rightarrow +\infty$, thanks to Lemma 2.1. In particular, up to a subsequence, we can assume that $\mathbf{U}_n \rightarrow \mathbf{U}$ a.e. in $B_{R+\delta}(\mathbf{0}) \times B_\delta(\mathbf{0})$ as $n \rightarrow +\infty$. Consequently,

$$\mathcal{F}_n(x, y) := \frac{\mathbf{U}_n(\mathbf{x}, \mathbf{y}) |\mathbf{U}(\mathbf{x}, \mathbf{y})|^{p-2}}{|\mathbf{y}|^{N+\alpha p}} \mathbf{V}(\mathbf{x}, \mathbf{y}) \rightarrow \mathcal{F}(x, y) := \frac{\mathbf{U}(\mathbf{x}, \mathbf{y}) |\mathbf{U}(\mathbf{x}, \mathbf{y})|^{p-2}}{|\mathbf{y}|^{N+\alpha p}} \mathbf{V}(\mathbf{x}, \mathbf{y})$$

a.e. in $B_{R+\delta}(\mathbf{0}) \times B_\delta(\mathbf{0})$ as $n \rightarrow +\infty$. We also observe that

$$\int_{B_{R+\delta}(\mathbf{0})} \int_{B_\delta(\mathbf{0})} |\mathcal{F}(x, y)| \, d\mathbf{x} \, d\mathbf{y} \leq \int_{B_{R+\delta}(\mathbf{0})} \int_{B_\delta(\mathbf{0})} \frac{|\mathbf{U}(\mathbf{x}, \mathbf{y})|^{p-1}}{|\mathbf{y}|^{N+\alpha p}} |\mathbf{V}(\mathbf{x}, \mathbf{y})| \, d\mathbf{x} \, d\mathbf{y}$$

$$\leq \left(\int_{B_{R+\delta}(\mathbf{0})} \int_{B_\delta(\mathbf{0})} \frac{|\mathbf{U}(\mathbf{x}, \mathbf{y})|^p}{|\mathbf{y}|^{N+\alpha p}} d\mathbf{x} d\mathbf{y} \right)^{\frac{p-1}{p}} \left(\int_{B_{R+\delta}(\mathbf{0})} \int_{B_\delta(\mathbf{0})} \frac{|\mathbf{V}(\mathbf{x}, \mathbf{y})|^p}{|\mathbf{y}|^{N+\alpha p}} d\mathbf{x} d\mathbf{y} \right)^{\frac{1}{p}} < +\infty,$$

where we have used the Hölder Inequality with exponents $\frac{p}{p-1}$ and p . Consequently, $\mathcal{F} \in L^1(B_{R+\delta}(\mathbf{0}) \times B_\delta(\mathbf{0}))$, and therefore $\mathcal{F}$ is finite a.e. in $B_{R+\delta}(\mathbf{0}) \times B_\delta(\mathbf{0})$.

We claim that

$$(2.16) \quad \mathcal{F}_n \text{ is uniformly integrable.}$$

To prove this, fix $\varepsilon > 0$ and let $\eta_\varepsilon > 0$ be such that for any measurable $E \subset B_{R+\delta}(\mathbf{0}) \times B_\delta(\mathbf{0})$ with measure less than η_ε we have that

$$\iint_E \frac{|\mathbf{V}(\mathbf{x}, \mathbf{y})|^p}{|\mathbf{y}|^{N+\alpha p}} d\mathbf{x} d\mathbf{y} \leq \varepsilon.$$

Then, exploiting the Hölder Inequality with exponents p , $\frac{p}{p-2}$ and p ,

$$\begin{aligned} \iint_E \mathcal{F}_n(\mathbf{x}, \mathbf{y}) d\mathbf{x} d\mathbf{y} &\leq \iint_E \frac{|\mathbf{U}_n(\mathbf{x}, \mathbf{y})|}{|\mathbf{y}|^{\frac{N+\alpha p}{p}}} \cdot \frac{|\mathbf{U}(\mathbf{x}, \mathbf{y})|^{p-2}}{|\mathbf{y}|^{\frac{(N+\alpha p)(p-2)}{p}}} \cdot \frac{|\mathbf{V}(\mathbf{x}, \mathbf{y})|}{|\mathbf{y}|^{\frac{N+\alpha p}{p}}} d\mathbf{x} d\mathbf{y} \\ &\leq \left[\iint_E \frac{|\mathbf{U}_n(\mathbf{x}, \mathbf{y})|^p}{|\mathbf{y}|^{N+\alpha p}} d\mathbf{x} d\mathbf{y} \right]^{\frac{1}{p}} \left[\iint_E \frac{|\mathbf{U}(\mathbf{x}, \mathbf{y})|^p}{|\mathbf{y}|^{N+\alpha p}} d\mathbf{x} d\mathbf{y} \right]^{\frac{p-2}{p}} \left[\iint_E \frac{|\mathbf{V}(\mathbf{x}, \mathbf{y})|^p}{|\mathbf{y}|^{N+\alpha p}} d\mathbf{x} d\mathbf{y} \right]^{\frac{1}{p}} \\ &\leq C \varepsilon^{\frac{1}{p}}, \end{aligned}$$

for some $C > 0$. This establishes (2.16). From it, using Vitali Convergence Theorem, we obtain (2.15).

Similarly, one can prove that

$$\mathcal{B}_n \rightarrow 0 \quad \text{and} \quad \mathcal{C}_n \rightarrow 0$$

as $n \rightarrow +\infty$. Using these pieces of information together with (2.14) we obtain the desired result (2.11). $\square$

2.3. Definition of weak solutions and main results. In order to look for solutions of (2.1), we need to introduce a suitable functional setting. For this, we recall the definition of $\mathcal{W}$ in (2.8) and we denote by $\mathcal{X}$ the functional space defined as follows:

$$(2.17) \quad \mathcal{X} := \left\{ \mathbf{u} : \mathbb{R}^N \times [0, \infty) \rightarrow \mathbb{R}^N; \begin{array}{l} \mathbf{u} \in L^\infty(0, T; \mathcal{W}), T > 0 \\ \partial_t \mathbf{u} \in L^\infty(0, \infty; L^2(\mathbb{R}^N; \mathbb{R}^N)) \end{array} \right\}.$$

With this, we can give the following definition of weak solutions:

Definition 2.3. Let $\mathbf{u} : \mathbb{R}^N \times [0, \infty) \rightarrow \mathbb{R}^N$. We say that $\mathbf{u}$ is a weak solution of the Cauchy problem (2.1) if

(D.1) $\mathbf{u} \in \mathcal{X}$;

(D.2) for every test function $\mathbf{v} \in C^\infty(\mathbb{R}^{N+1}; \mathbb{R}^N)$ such that every component has compact support, it holds that

$$(2.18) \quad \begin{aligned} &\int_0^\infty \int_{\mathbb{R}^N} \left(\mathbf{u}(\mathbf{x}, t) \cdot \partial_t^2 \mathbf{v}(\mathbf{x}, t) - (K\mathbf{u}(\cdot, t))(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x}, t) \right) dt d\mathbf{x} \\ &\quad - \int_{\mathbb{R}^N} \mathbf{v}_0(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x}, 0) d\mathbf{x} + \int_{\mathbb{R}^N} \mathbf{u}_0(\mathbf{x}) \cdot \partial_t \mathbf{v}(\mathbf{x}, 0) d\mathbf{x} = 0. \end{aligned}$$

Recalling also the definition of the energy in (2.6), we can therefore state the main results of this paper:

Theorem 2.4 (Existence). *Let (H.1), (H.2), and (H.3) be satisfied. Then, for every initial datum $(\mathbf{u}_0, \mathbf{v}_0)$ such that*

$$(2.19) \quad \begin{aligned} & \mathbf{u}_0 \in L^2(\mathbb{R}^N; \mathbb{R}^N), \mathbf{v}_0 \in L^2(\mathbb{R}^N; \mathbb{R}^N), \\ & \text{and} \quad \int_{\mathbb{R}^N} \int_{B_\delta(\mathbf{0})} \Phi(\mathbf{y}, \mathbf{u}_0(\mathbf{x}) - \mathbf{u}_0(\mathbf{x} - \mathbf{y})) d\mathbf{x} d\mathbf{y} < \infty, \end{aligned}$$

the Cauchy problem (2.1) admits at least one weak solution in the sense of Definition 2.3 such that

$$(2.20) \quad E[\mathbf{u}](t) \leq E[\mathbf{u}](0), \quad \text{for a.e. } t \geq 0.$$

Theorem 2.5 (Uniqueness and Stability). *Let (H.1), (H.2), and (H.3) be satisfied, and let*

$$(2.21) \quad p = 2.$$

If $\mathbf{u}$ and $\tilde{\mathbf{u}}$ are weak solutions of (2.1) obtained in correspondence of the initial data $(\mathbf{u}_0, \mathbf{v}_0)$ and $(\tilde{\mathbf{u}}_0, \tilde{\mathbf{v}}_0)$, respectively, satisfying (2.19), then the following stability estimate holds true:

$$(2.22) \quad \begin{aligned} & \|\partial_t \mathbf{u}(\cdot, t) - \partial_t \tilde{\mathbf{u}}(\cdot, t)\|_{L^2(\mathbb{R}^N)}^2 \\ & + \kappa \int_{\mathbb{R}^N} \int_{B_\delta(\mathbf{0})} \frac{|\mathbf{u}(\mathbf{x}, t) - \mathbf{u}(\mathbf{x} - \mathbf{y}, t) - \tilde{\mathbf{u}}(\mathbf{x}, t) + \tilde{\mathbf{u}}(\mathbf{x} - \mathbf{y}, t)|^2}{|\mathbf{y}|^{N+2\alpha}} d\mathbf{x} d\mathbf{y} \\ & \leq e^{(\lambda + \frac{1}{\kappa})t} \|\mathbf{v}_0 - \tilde{\mathbf{v}}_0\|_{L^2(\mathbb{R}^N)}^2 \\ & + \kappa e^{(\lambda + \frac{1}{\kappa})t} \int_{\mathbb{R}^N} \int_{B_\delta(\mathbf{0})} \frac{|\mathbf{u}_0(\mathbf{x}) - \mathbf{u}_0(\mathbf{x} - \mathbf{y}) - \tilde{\mathbf{u}}_0(\mathbf{x}) + \tilde{\mathbf{u}}_0(\mathbf{x} - \mathbf{y})|^2}{|\mathbf{y}|^{N+2\alpha}} d\mathbf{x} d\mathbf{y} \end{aligned}$$

for every $t > 0$, where

$$\lambda := \int_{B_\delta(\mathbf{0})} g^2(\mathbf{y}) |\mathbf{y}|^{N+2\alpha} d\mathbf{y},$$

and κ is the one appearing in (H.3).

In the forthcoming Sections 3 and 4 we will give the proofs of Theorems 2.4 and 2.5, respectively.

3. PROOF OF THEOREM 2.4

In this section we prove Theorem 2.4. The arguments rely on the compactness of the solutions of suitable approximations of (2.1).

More precisely, let $\varepsilon > 0$ and

$$(3.1) \quad \mathbf{u}_\varepsilon \in H^3(\mathbb{R}^N \times (0, T); \mathbb{R}^N) \cap L^\infty(0, T; H^4(\mathbb{R}^N; \mathbb{R}^N)), \quad T > 0,$$

be the unique smooth solution of the fourth order problem

$$(3.2) \quad \begin{cases} \partial_t^2 \mathbf{u}_\varepsilon(\mathbf{x}, t) = (K \mathbf{u}_\varepsilon(\cdot, t))(\mathbf{x}) - \varepsilon \Delta^2 \mathbf{u}_\varepsilon, & \mathbf{x} \in \mathbb{R}^N, t > 0, \\ \mathbf{u}_\varepsilon(\mathbf{x}, 0) = \mathbf{u}_{0,\varepsilon}(\mathbf{x}), & \mathbf{x} \in \mathbb{R}^N, \\ \partial_t \mathbf{u}_\varepsilon(\mathbf{x}, 0) = \mathbf{v}_{0,\varepsilon}(\mathbf{x}), & \mathbf{x} \in \mathbb{R}^N, \end{cases}$$

where $\mathbf{u}_{0,\varepsilon}$ and $\mathbf{v}_{0,\varepsilon}$ are smooth approximations of $\mathbf{u}_0$ and $\mathbf{v}_0$, respectively, such that

$$\begin{aligned}
 (3.3) \quad & \mathbf{u}_{0,\varepsilon} \in H^4\mathcal{B}(\mathbb{R}^N; \mathbb{R}^N), \quad \mathbf{v}_{0,\varepsilon} \in H^2(\mathbb{R}^N; \mathbb{R}^N), \quad \text{for any } \varepsilon > 0, \\
 & \mathbf{u}_{0,\varepsilon}, \mathbf{v}_{0,\varepsilon} \in C^\infty(\mathbb{R}^N; \mathbb{R}^N), \quad \text{for any } \varepsilon > 0, \\
 & \mathbf{u}_{0,\varepsilon} \rightarrow \mathbf{u}_0, \quad \mathbf{v}_{0,\varepsilon} \rightarrow \mathbf{v}_0 \quad \text{a.e. in } \mathbb{R}^N \text{ and in } L^2(\mathbb{R}^N; \mathbb{R}^N) \text{ as } \varepsilon \rightarrow 0, \\
 & \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^N} \int_{B_\delta(\mathbf{0})} \Phi(\mathbf{y}, \mathbf{u}_{0,\varepsilon}(\mathbf{x}) - \mathbf{u}_{0,\varepsilon}(\mathbf{x} - \mathbf{y})) \, d\mathbf{x} \, d\mathbf{y} \\
 & \quad = \int_{\mathbb{R}^N} \int_{B_\delta(\mathbf{0})} \Phi(\mathbf{y}, \mathbf{u}_0(\mathbf{x}) - \mathbf{u}_0(\mathbf{x} - \mathbf{y})) \, d\mathbf{x} \, d\mathbf{y}, \\
 & \lim_{\varepsilon \rightarrow 0} \sqrt{\varepsilon} \|\Delta \mathbf{u}_{0,\varepsilon}\|_{L^2(\mathbb{R}^N)} = 0.
 \end{aligned}$$

The well-posedness of (3.2), hence the existence of smooth solutions for that problem follows by classical semigroup based arguments, see e.g. [11].

Recalling the notation introduced in Subsections 2.2 and 2.3, the main compactness result of this section is the following:

Lemma 3.1. *Let (H.1), (H.2) and (H.3) be satisfied. Then, there exist a sequence $\{\varepsilon_k\}_k \subset (0, \infty)$ and a function $\mathbf{u} \in \mathcal{X}$ such that, as $k \rightarrow \infty$,*

$$(3.4) \quad \mathbf{u}_{\varepsilon_k} \rightarrow \mathbf{u} \quad \text{a.e. in } \mathbb{R}^N \times [0, \infty) \text{ and in } L^2_{\text{loc}}(\mathbb{R}^N \times (0, \infty); \mathbb{R}^N),$$

$$(3.5) \quad \partial_t \mathbf{u}_{\varepsilon_k} \rightharpoonup \partial_t \mathbf{u} \quad \text{in } L^r(0, T; L^2(\mathbb{R}^N; \mathbb{R}^N)), \text{ for any } 1 \leq r < \infty, \text{ and } T > 0,$$

$$(3.6) \quad \mathbf{u}_{\varepsilon_k} \rightarrow \mathbf{u} \quad \text{a.e. in } \mathbb{R}^N \times [0, \infty) \text{ and in } L^r(0, T; \mathcal{W}), \text{ for any } 1 \leq r < \infty \text{ and } T > 0,$$

$$(3.7) \quad \mathbf{u} \text{ is a weak solution of (3.2) in the sense of Definition 2.3.}$$

In order to prove Lemma 3.1 we need the following preliminary results:

Lemma 3.2 (Energy estimate). *Let (H.1), (H.2) and (H.3) be satisfied. Then, the following formula holds true:*

$$\begin{aligned}
 (3.8) \quad & \frac{\|\partial_t \mathbf{u}_\varepsilon(\cdot, t)\|_{L^2(\mathbb{R}^N)}^2 + \varepsilon \|\Delta \mathbf{u}_\varepsilon(\cdot, t)\|_{L^2(\mathbb{R}^N)}^2}{2} \\
 & \quad + \int_{\mathbb{R}^N} \int_{B_\delta(\mathbf{0})} \Phi(\mathbf{y}, \mathbf{u}_\varepsilon(\mathbf{x}, t) - \mathbf{u}_\varepsilon(\mathbf{x} - \mathbf{y}, t)) \, d\mathbf{x} \, d\mathbf{y} \\
 & = \frac{\|\mathbf{v}_{0,\varepsilon}\|_{L^2(\mathbb{R}^N)}^2 + \varepsilon \|\Delta \mathbf{u}_{0,\varepsilon}\|_{L^2(\mathbb{R}^N)}^2}{2} \\
 & \quad + \frac{1}{2} \int_{\mathbb{R}^N} \int_{B_\delta(\mathbf{0})} \Phi(\mathbf{y}, \mathbf{u}_{0,\varepsilon}(\mathbf{x}) - \mathbf{u}_{0,\varepsilon}(\mathbf{x} - \mathbf{y})) \, d\mathbf{x} \, d\mathbf{y} \leq C,
 \end{aligned}$$

for every $t \geq 0$ and for some constant $C > 0$ independent on ε .

Proof. Multiplying the equation in (3.2) by $\partial_t \mathbf{u}_\varepsilon$, integrating over $\mathbb{R}^N$ and recalling (2.2), we get

$$\begin{aligned}
 (3.9) \quad 0 &= \int_{\mathbb{R}^N} \partial_t^2 \mathbf{u}_\varepsilon \partial_t \mathbf{u}_\varepsilon \, d\mathbf{x} - \int_{\mathbb{R}^N} (K \mathbf{u}_\varepsilon(\cdot, t))(\mathbf{x}) \partial_t \mathbf{u}_\varepsilon \, d\mathbf{x} + \varepsilon \int_{\mathbb{R}^N} \Delta^2 \mathbf{u}_\varepsilon \partial_t \mathbf{u}_\varepsilon \, d\mathbf{x} \\
 &= \int_{\mathbb{R}^N} \partial_t^2 \mathbf{u}_\varepsilon \partial_t \mathbf{u}_\varepsilon \, d\mathbf{x} - \int_{\mathbb{R}^N} (K \mathbf{u}_\varepsilon(\cdot, t))(\mathbf{x}) \partial_t \mathbf{u}_\varepsilon \, d\mathbf{x} + \varepsilon \int_{\mathbb{R}^N} \Delta \mathbf{u}_\varepsilon \partial_t \Delta \mathbf{u}_\varepsilon \, d\mathbf{x} \\
 &= \frac{d}{dt} \int_{\mathbb{R}^N} \frac{|\partial_t \mathbf{u}_\varepsilon|^2 + \varepsilon |\Delta \mathbf{u}_\varepsilon|^2}{2} \, d\mathbf{x} \\
 &\quad + \int_{\mathbb{R}^N} \int_{B_\delta(\mathbf{0})} \mathbf{f}(\mathbf{y}, \mathbf{u}_\varepsilon(\mathbf{x}, t) - \mathbf{u}_\varepsilon(\mathbf{x} - \mathbf{y}, t)) \partial_t \mathbf{u}_\varepsilon(\mathbf{x}, t) \, d\mathbf{x} \, d\mathbf{y}.
 \end{aligned}$$

Now we use (H.2) to see that

$$\begin{aligned}
 &\int_{\mathbb{R}^N} \int_{B_\delta(\mathbf{0})} \mathbf{f}(\mathbf{y}, \mathbf{u}_\varepsilon(\mathbf{x}, t) - \mathbf{u}_\varepsilon(\mathbf{x} - \mathbf{y}, t)) \partial_t \mathbf{u}_\varepsilon(\mathbf{x}, t) \, d\mathbf{x} \, d\mathbf{y} \\
 &= \frac{1}{2} \int_{\mathbb{R}^N} \int_{B_\delta(\mathbf{0})} \mathbf{f}(\mathbf{y}, \mathbf{u}_\varepsilon(\mathbf{x}, t) - \mathbf{u}_\varepsilon(\mathbf{x} - \mathbf{y}, t)) \partial_t \mathbf{u}_\varepsilon(\mathbf{x}, t) \, d\mathbf{x} \, d\mathbf{y} \\
 &\quad + \frac{1}{2} \int_{B_\delta(\mathbf{0})} \left(\int_{\mathbb{R}^N} \mathbf{f}(\mathbf{y}, \mathbf{u}_\varepsilon(\mathbf{x}, t) - \mathbf{u}_\varepsilon(\mathbf{x} - \mathbf{y}, t)) \partial_t \mathbf{u}_\varepsilon(\mathbf{x}, t) \, d\mathbf{x} \right) \, d\mathbf{y} \\
 &= \frac{1}{2} \int_{\mathbb{R}^N} \int_{B_\delta(\mathbf{0})} \mathbf{f}(\mathbf{y}, \mathbf{u}_\varepsilon(\mathbf{x}, t) - \mathbf{u}_\varepsilon(\mathbf{x} - \mathbf{y}, t)) \partial_t \mathbf{u}_\varepsilon(\mathbf{x}, t) \, d\mathbf{x} \, d\mathbf{y} \\
 &\quad + \frac{1}{2} \int_{B_\delta(\mathbf{0})} \left(\int_{\mathbb{R}^N} \mathbf{f}(\mathbf{y}, \mathbf{u}_\varepsilon(\mathbf{z} + \mathbf{y}, t) - \mathbf{u}_\varepsilon(\mathbf{z}, t)) \partial_t \mathbf{u}_\varepsilon(\mathbf{z} + \mathbf{y}, t) \, d\mathbf{z} \right) \, d\mathbf{y} \\
 &= \frac{1}{2} \int_{\mathbb{R}^N} \int_{B_\delta(\mathbf{0})} \mathbf{f}(\mathbf{y}, \mathbf{u}_\varepsilon(\mathbf{x}, t) - \mathbf{u}_\varepsilon(\mathbf{x} - \mathbf{y}, t)) \partial_t \mathbf{u}_\varepsilon(\mathbf{x}, t) \, d\mathbf{x} \, d\mathbf{y} \\
 &\quad + \frac{1}{2} \int_{\mathbb{R}^N} \left(\int_{B_\delta(\mathbf{0})} \mathbf{f}(-\mathbf{y}, \mathbf{u}_\varepsilon(\mathbf{z} - \mathbf{y}, t) - \mathbf{u}_\varepsilon(\mathbf{z}, t)) \partial_t \mathbf{u}_\varepsilon(\mathbf{z} - \mathbf{y}, t) \, d\mathbf{y} \right) \, d\mathbf{z} \\
 &= \frac{1}{2} \int_{\mathbb{R}^N} \int_{B_\delta(\mathbf{0})} \mathbf{f}(\mathbf{y}, \mathbf{u}_\varepsilon(\mathbf{x}, t) - \mathbf{u}_\varepsilon(\mathbf{x} - \mathbf{y}, t)) \partial_t \mathbf{u}_\varepsilon(\mathbf{x}, t) \, d\mathbf{x} \, d\mathbf{y} \\
 &\quad - \frac{1}{2} \int_{\mathbb{R}^N} \left(\int_{B_\delta(\mathbf{0})} \mathbf{f}(\mathbf{y}, \mathbf{u}_\varepsilon(\mathbf{z}, t) - \mathbf{u}_\varepsilon(\mathbf{z} - \mathbf{y}, t)) \partial_t \mathbf{u}_\varepsilon(\mathbf{z} - \mathbf{y}, t) \, d\mathbf{y} \right) \, d\mathbf{z} \\
 &= \frac{1}{2} \int_{\mathbb{R}^N} \int_{B_\delta(\mathbf{0})} \mathbf{f}(\mathbf{y}, \mathbf{u}_\varepsilon(\mathbf{x}, t) - \mathbf{u}_\varepsilon(\mathbf{x} - \mathbf{y}, t)) \left(\partial_t \mathbf{u}_\varepsilon(\mathbf{x}, t) - \partial_t \mathbf{u}_\varepsilon(\mathbf{x} - \mathbf{y}, t) \right) \, d\mathbf{x} \, d\mathbf{y}.
 \end{aligned}$$

Plugging this information into (3.9), we conclude that

$$\begin{aligned}
 0 &= \frac{d}{dt} \int_{\mathbb{R}^N} \frac{|\partial_t \mathbf{u}_\varepsilon|^2 + \varepsilon |\Delta \mathbf{u}_\varepsilon|^2}{2} \, d\mathbf{x} \\
 &\quad + \frac{1}{2} \int_{\mathbb{R}^N} \int_{B_\delta(\mathbf{0})} \mathbf{f}(\mathbf{y}, \mathbf{u}_\varepsilon(\mathbf{x}, t) - \mathbf{u}_\varepsilon(\mathbf{x} - \mathbf{y}, t)) \left(\partial_t \mathbf{u}_\varepsilon(\mathbf{x}, t) - \partial_t \mathbf{u}_\varepsilon(\mathbf{x} - \mathbf{y}, t) \right) \, d\mathbf{x} \, d\mathbf{y}.
 \end{aligned}$$

As a consequence, using **(H.3)**,

$$\begin{aligned} 0 &= \frac{d}{dt} \int_{\mathbb{R}^N} \frac{|\partial_t \mathbf{u}_\varepsilon|^2 + \varepsilon |\Delta \mathbf{u}_\varepsilon|^2}{2} d\mathbf{x} \\ &\quad + \frac{1}{2} \int_{\mathbb{R}^N} \int_{B_\delta(\mathbf{0})} \nabla_u \Phi(\mathbf{y}, \mathbf{u}_\varepsilon(\mathbf{x}, t) - \mathbf{u}_\varepsilon(\mathbf{x} - \mathbf{y}, t)) \left(\partial_t \mathbf{u}_\varepsilon(\mathbf{x}, t) - \partial_t \mathbf{u}_\varepsilon(\mathbf{x} - \mathbf{y}, t) \right) d\mathbf{x} d\mathbf{y} \\ &= \frac{d}{dt} \left[\int_{\mathbb{R}^N} \frac{|\partial_t \mathbf{u}_\varepsilon|^2 + \varepsilon |\Delta \mathbf{u}_\varepsilon|^2}{2} d\mathbf{x} + \frac{1}{2} \int_{\mathbb{R}^N} \int_{B_\delta(\mathbf{0})} \Phi(\mathbf{y}, \mathbf{u}_\varepsilon(\mathbf{x}, t) - \mathbf{u}_\varepsilon(\mathbf{x} - \mathbf{y}, t)) d\mathbf{x} d\mathbf{y} \right]. \end{aligned}$$

Hence, an integration over $(0, t)$ gives the desired equality in (3.8). Furthermore, the boundedness of the quantity in (3.8) follows from the convergence assumptions in (3.3). The proof of Lemma 3.2 is thus complete. $\square$

Lemma 3.3 (*L^2 -estimate*). *Let **(H.1)**, **(H.2)**, and **(H.3)** be satisfied. Then, the following estimate holds true:*

$$(3.10) \quad \|\mathbf{u}_\varepsilon(\cdot, t)\|_{L^2(\mathbb{R}^N)} \leq C(1 + t),$$

for every $t \geq 0$ and for some constant $C > 0$ independent on ε .

Proof. We observe that

$$\begin{aligned} (3.11) \quad |\mathbf{u}_\varepsilon(\mathbf{x}, t)| &\leq |\mathbf{u}_{0,\varepsilon}(\mathbf{x})| + \int_0^t |\partial_t \mathbf{u}_\varepsilon(\mathbf{x}, s)| ds \\ &\leq |\mathbf{u}_{0,\varepsilon}(\mathbf{x})| + \sqrt{t} \sqrt{\int_0^t |\partial_t \mathbf{u}_\varepsilon(\mathbf{x}, s)|^2 ds}. \end{aligned}$$

Taking the square of both sides of (3.11) and integrating over $\mathbb{R}^N$, we have that

$$\begin{aligned} \int_{\mathbb{R}^N} |\mathbf{u}_\varepsilon(\mathbf{x}, t)|^2 d\mathbf{x} &\leq 2 \int_{\mathbb{R}^N} |\mathbf{u}_{0,\varepsilon}(\mathbf{x})|^2 d\mathbf{x} + 2t \int_0^t \int_{\mathbb{R}^N} |\partial_t \mathbf{u}_\varepsilon(\mathbf{x}, s)|^2 ds d\mathbf{x} \\ &\leq 2 \|\mathbf{u}_{0,\varepsilon}\|_{L^2(\mathbb{R}^N)}^2 + 2t^2 \sup_{t \geq 0} \|\partial_t \mathbf{u}_\varepsilon(\cdot, t)\|_{L^2(\mathbb{R}^N)}^2. \end{aligned}$$

Therefore, the desired estimate follows from (3.3) and (3.8). $\square$

Proof of Lemma 3.1. We notice that, by virtue of Lemma 3.2,

$$(3.12) \quad \{\partial_t \mathbf{u}_\varepsilon\}_\varepsilon \text{ is a bounded sequence in } L^\infty(0, \infty; L^2(\mathbb{R}^N; \mathbb{R}^N)).$$

Furthermore, using again Lemmas 3.2 and 3.3 and assumption **(H.3)** we obtain that

$$(3.13) \quad \{\mathbf{u}_\varepsilon\}_\varepsilon \text{ is a bounded sequence in } L^\infty(0, T; \mathcal{W}) \text{ for every } T > 0.$$

Therefore, by Lemma 2.1, we have that there exist a subsequence $\{\mathbf{u}_{\varepsilon_k}\}_k$ and a function $\mathbf{u} \in L^2_{\text{loc}}(\mathbb{R}^N; \mathbb{R}^N)$ such that (3.4) holds true.

As a matter of fact, by virtue of (3.12) and (3.13) we have that $\mathbf{u} \in L^\infty(0, T; \mathcal{W})$ for every $T > 0$, and $\{\partial_t \mathbf{u}_\varepsilon\}_\varepsilon \in L^\infty(0, \infty; L^2(\mathbb{R}^N; \mathbb{R}^N))$. That is, recalling the definition of $\mathcal{X}$ in (2.17), the function $\mathbf{u} \in \mathcal{X}$. Thus, condition **(D.1)** in Definition 2.3 holds true for $\mathbf{u}$.

Hence, to prove (3.7), we now focus on proving that $\mathbf{u}$ satisfies **(D.2)** in Definition 2.3. To this aim, let $\mathbf{v} \in C^\infty(\mathbb{R}^{N+1}; \mathbb{R}^N)$ be a given test function such that every component has

compact support. Multiplying (3.2) by $\mathbf{v}$ and integrating over $(0, \infty) \times \mathbb{R}^N$, we get

$$\begin{aligned}
 (3.14) \quad & \int_0^\infty \int_{\mathbb{R}^N} (\mathbf{u}_{\varepsilon_k}(\mathbf{x}, t) \cdot \partial_t^2 \mathbf{v}(\mathbf{x}, t) - (K \mathbf{u}_{\varepsilon_k}(\cdot, t))(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x}, t)) dt d\mathbf{x} \\
 & - \int_{\mathbb{R}^N} \mathbf{v}_{0, \varepsilon_k}(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x}, 0) d\mathbf{x} + \int_{\mathbb{R}^N} \mathbf{u}_{0, \varepsilon_k}(\mathbf{x}) \cdot \partial_t \mathbf{v}(\mathbf{x}, 0) d\mathbf{x} \\
 & = -\varepsilon_k \int_0^\infty \int_{\mathbb{R}^N} \mathbf{u}_{\varepsilon_k}(\mathbf{x}, t) \cdot \Delta^2 \mathbf{v}(\mathbf{x}, t) dt d\mathbf{x}.
 \end{aligned}$$

Hence, sending $k \rightarrow \infty$ in (3.14) and using Lemma 2.2 and formula (3.3), we obtain **(D.2)**.

Therefore, $\mathbf{u}$ is a weak solution of (3.2) in the sense of Definition 2.3. This proves (3.7), and so the proof of Lemma 3.1 is complete. $\square$

Proof of Theorem 2.4. Thanks to Lemma 3.1, in order to complete the proof of Theorem 2.4, it remains to prove that $\mathbf{u}$ satisfies (2.20). For this, we use **(H.3)** and (3.8) to see that

$$\begin{aligned}
 (3.15) \quad & \frac{\|\partial_t \mathbf{u}_{\varepsilon_k}(\cdot, t)\|_{L^2(\mathbb{R}^N)}^2}{2} + \frac{\kappa}{2} \int_{\mathbb{R}^N} \int_{B_\delta(\mathbf{0})} \frac{|\mathbf{u}_{\varepsilon_k}(\mathbf{x}) - \mathbf{u}_{\varepsilon_k}(\mathbf{x} - \mathbf{y})|^p}{|\mathbf{y}|^{N+\alpha p}} d\mathbf{x} d\mathbf{y} \\
 & + \frac{1}{2} \int_{B_R(\mathbf{0})} \int_{B_\delta(\mathbf{0})} \Psi(\mathbf{y}, \mathbf{u}_{\varepsilon_k}(\mathbf{x}, t) - \mathbf{u}_{\varepsilon_k}(\mathbf{x} - \mathbf{y}, t)) d\mathbf{x} d\mathbf{y} \\
 & \leq \frac{\|\mathbf{v}_{0, \varepsilon_k}\|_{L^2(\mathbb{R}^N)}^2 + \varepsilon_k \|\Delta \mathbf{u}_{0, \varepsilon_k}\|_{L^2(\mathbb{R}^N)}^2}{2} \\
 & + \frac{1}{2} \int_{\mathbb{R}^N} \int_{B_\delta(\mathbf{0})} \Phi(\mathbf{y}, \mathbf{u}_{0, \varepsilon_k}(\mathbf{x}) - \mathbf{u}_{0, \varepsilon_k}(\mathbf{x} - \mathbf{y})) d\mathbf{x} d\mathbf{y},
 \end{aligned}$$

for every $R > 0$. Now, making use of (3.3) and recalling (2.6), we can say that

$$\begin{aligned}
 (3.16) \quad & \lim_k \left(\frac{\|\mathbf{v}_{0, \varepsilon_k}\|_{L^2(\mathbb{R}^N)}^2 + \varepsilon_k \|\Delta \mathbf{u}_{0, \varepsilon_k}\|_{L^2(\mathbb{R}^N)}^2}{2} \right. \\
 & \left. + \int_{\mathbb{R}^N} \int_{B_\delta(\mathbf{0})} \Phi(\mathbf{y}, \mathbf{u}_{0, \varepsilon_k}(\mathbf{x}) - \mathbf{u}_{0, \varepsilon_k}(\mathbf{x} - \mathbf{y})) d\mathbf{x} d\mathbf{y} \right) \\
 & = \frac{\|\partial_t \mathbf{u}(\cdot, t)\|_{L^2(\mathbb{R}^N)}^2}{2} + \frac{1}{2} \int_{\mathbb{R}^N} \int_{B_\delta(\mathbf{0})} \Phi(\mathbf{y}, \mathbf{u}(\mathbf{x}, t) - \mathbf{u}(\mathbf{x} - \mathbf{y}, t)) d\mathbf{x} d\mathbf{y} = E[\mathbf{u}](0).
 \end{aligned}$$

Moreover, the Dominated Convergence Theorem, **(H.3)** and (3.4) give that, for a.e. $t > 0$,

$$\begin{aligned}
 (3.17) \quad & \lim_k \|\partial_t \mathbf{u}_{\varepsilon_k}(\cdot, t)\|_{L^2(\mathbb{R}^N)}^2 \geq \|\partial_t \mathbf{u}(\cdot, t)\|_{L^2(\mathbb{R}^N)}^2, \\
 & \lim_k \int_{B_R(\mathbf{0})} \int_{B_\delta(\mathbf{0})} \Psi(\mathbf{y}, \mathbf{u}_{\varepsilon_k}(\mathbf{x}, t) - \mathbf{u}_{\varepsilon_k}(\mathbf{x} - \mathbf{y}, t)) d\mathbf{x} d\mathbf{y} \\
 & = \int_{B_R(\mathbf{0})} \int_{B_\delta(\mathbf{0})} \Psi(\mathbf{y}, \mathbf{u}(\mathbf{x}, t) - \mathbf{u}(\mathbf{x} - \mathbf{y}, t)) d\mathbf{x} d\mathbf{y}, \\
 & \liminf_k \int_{\mathbb{R}^N} \int_{B_\delta(\mathbf{0})} \frac{|\mathbf{u}_{\varepsilon_k}(\mathbf{x}) - \mathbf{u}_{\varepsilon_k}(\mathbf{x} - \mathbf{y})|^p}{|\mathbf{y}|^{N+\alpha p}} d\mathbf{x} d\mathbf{y} \\
 & \geq \int_{\mathbb{R}^N} \int_{B_\delta(\mathbf{0})} \frac{|\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{x} - \mathbf{y})|^p}{|\mathbf{y}|^{N+\alpha p}} d\mathbf{x} d\mathbf{y}.
 \end{aligned}$$

Therefore, sending $k \rightarrow \infty$ in (3.15) we get

$$(3.18) \quad \begin{aligned} & \frac{\|\partial_t \mathbf{u}(\cdot, t)\|_{L^2(\mathbb{R}^N)}^2}{2} + \frac{\kappa}{2} \int_{\mathbb{R}^N} \int_{B_\delta(\mathbf{0})} \frac{|\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{x} - \mathbf{y})|^p}{|\mathbf{y}|^{N+\alpha p}} d\mathbf{x} d\mathbf{y} \\ & + \frac{1}{2} \int_{B_R(\mathbf{0})} \int_{B_\delta(\mathbf{0})} \Psi(\mathbf{y}, \mathbf{u}(\mathbf{x}, t) - \mathbf{u}(\mathbf{x} - \mathbf{y}, t)) d\mathbf{x} d\mathbf{y} \leq E[\mathbf{u}](0). \end{aligned}$$

Sending $R \rightarrow \infty$ in (3.18) and recalling again (2.6), we gain (2.20), as desired. $\square$

4. UNIQUENESS, STABILITY AND PROOF OF THEOREM 2.5

This section is devoted to the proof of Theorem 2.5.

Proof of Theorem 2.5. Let $\mathbf{u}$ and $\tilde{\mathbf{u}}$ be two weak solutions of (2.1) according to Definition 2.3, and define

$$(4.1) \quad \mathbf{w} := \mathbf{u} - \tilde{\mathbf{u}}.$$

We claim that $\mathbf{w}$ solves the equation

$$(4.2) \quad \begin{aligned} \partial_t^2 \mathbf{w}(\mathbf{x}, t) = & -2\kappa \int_{B_\delta(\mathbf{0})} \frac{\mathbf{w}(\mathbf{x}, t) - \mathbf{w}(\mathbf{x} - \mathbf{y}, t)}{|\mathbf{y}|^{N+2\alpha}} d\mathbf{y} \\ & - \int_{B_\delta(\mathbf{0})} \int_0^1 \mathcal{F}(\theta, \mathbf{x}, \mathbf{y}, t) (\mathbf{w}(\mathbf{x}, t) - \mathbf{w}(\mathbf{x} - \mathbf{y}, t)) d\mathbf{y} d\theta. \end{aligned}$$

where

$$(4.3) \quad \mathcal{F}(\theta, \mathbf{x}, \mathbf{y}, t) := D_{\mathbf{u}}^2 \Psi(\mathbf{y}, \theta(\mathbf{u}(\mathbf{x}, t) - \mathbf{u}(\mathbf{x} - \mathbf{y}, t)) + (1 - \theta)(\tilde{\mathbf{u}}(\mathbf{x}, t) - \tilde{\mathbf{u}}(\mathbf{x} - \mathbf{y}, t))).$$

To prove (4.2), we recall (2.21) and we use (2.1), (2.2), ((H.2)) and (H.3) to see that

$$\begin{aligned} \partial_t^2 \mathbf{w}(\mathbf{x}, t) &= \partial_t^2 \mathbf{u}(\mathbf{x}, t) - \partial_t^2 \tilde{\mathbf{u}}(\mathbf{x}, t) \\ &= (K\mathbf{u}(\cdot, t))(\mathbf{x}) - (K\tilde{\mathbf{u}}(\cdot, t))(\mathbf{x}) \\ &= \int_{B_\delta(\mathbf{0})} \left(\mathbf{f}(-\mathbf{y}, \mathbf{u}(\mathbf{x} - \mathbf{y}, t) - \mathbf{u}(\mathbf{x}, t)) - \mathbf{f}(-\mathbf{y}, \tilde{\mathbf{u}}(\mathbf{x} - \mathbf{y}, t) - \tilde{\mathbf{u}}(\mathbf{x}, t)) \right) d\mathbf{y} \\ &= - \int_{B_\delta(\mathbf{0})} \left(\mathbf{f}(\mathbf{y}, \mathbf{u}(\mathbf{x}, t) - \mathbf{u}(\mathbf{x} - \mathbf{y}, t)) - \mathbf{f}(\mathbf{y}, \tilde{\mathbf{u}}(\mathbf{x}, t) - \tilde{\mathbf{u}}(\mathbf{x} - \mathbf{y}, t)) \right) d\mathbf{y} \\ &= -2\kappa \int_{B_\delta(\mathbf{0})} \frac{\mathbf{w}(\mathbf{x}, t) - \mathbf{w}(\mathbf{x} - \mathbf{y}, t)}{|\mathbf{y}|^{N+2\alpha}} d\mathbf{y} \\ &\quad - \int_{B_\delta(\mathbf{0})} \left(\nabla_{\mathbf{u}} \Psi(\mathbf{y}, \mathbf{u}(\mathbf{x}, t) - \mathbf{u}(\mathbf{x} - \mathbf{y}, t)) - \nabla_{\mathbf{u}} \Psi(\mathbf{y}, \tilde{\mathbf{u}}(\mathbf{x}, t) - \tilde{\mathbf{u}}(\mathbf{x} - \mathbf{y}, t)) \right) d\mathbf{y} \\ &= -2\kappa \int_{B_\delta(\mathbf{0})} \frac{\mathbf{w}(\mathbf{x}, t) - \mathbf{w}(\mathbf{x} - \mathbf{y}, t)}{|\mathbf{y}|^{N+2\alpha}} d\mathbf{y} \\ &\quad - \int_{B_\delta(\mathbf{0})} \int_0^1 \mathcal{F}(\theta, \mathbf{x}, \mathbf{y}, t) (\mathbf{w}(\mathbf{x}, t) - \mathbf{w}(\mathbf{x} - \mathbf{y}, t)) d\mathbf{y} d\theta, \end{aligned}$$

where $\mathcal{F}$ is defined in (4.3). This proves (4.2).

Now, we multiply (4.2) by $\partial_t \mathbf{w}$ and we integrate over $\mathbb{R}^N$. In this way, making again use of (H.2) and (H.3), we get

$$\begin{aligned}
0 &= \int_{\mathbb{R}^N} \partial_t^2 \mathbf{w} \partial_t \mathbf{w} \, d\mathbf{x} + 2\kappa \int_{\mathbb{R}^N} \int_{B_\delta(\mathbf{0})} \frac{\mathbf{w}(\mathbf{x}, t) - \mathbf{w}(\mathbf{x} - \mathbf{y}, t)}{|\mathbf{y}|^{N+2\alpha}} \partial_t \mathbf{w}(\mathbf{x}, t) \, d\mathbf{x} \, d\mathbf{y} \\
&\quad + \int_{\mathbb{R}^N} \int_{B_\delta(\mathbf{0})} \int_0^1 \mathcal{F}(\theta, \mathbf{x}, \mathbf{y}, t) (\mathbf{w}(\mathbf{x}, t) - \mathbf{w}(\mathbf{x} - \mathbf{y}, t)) \partial_t \mathbf{w}(\mathbf{x}, t) \, d\mathbf{x} \, d\mathbf{y} \, d\theta \\
&= \int_{\mathbb{R}^N} \partial_t^2 \mathbf{w} \partial_t \mathbf{w} \, d\mathbf{x} + \kappa \int_{\mathbb{R}^N} \int_{B_\delta(\mathbf{0})} \frac{\mathbf{w}(\mathbf{x}, t) - \mathbf{w}(\mathbf{x} - \mathbf{y}, t)}{|\mathbf{y}|^{N+2\alpha}} \partial_t (\mathbf{w}(\mathbf{x}, t) - \mathbf{w}(\mathbf{x} - \mathbf{y}, t)) \, d\mathbf{x} \, d\mathbf{y} \\
&\quad + \int_{\mathbb{R}^N} \int_{B_\delta(\mathbf{0})} \int_0^1 \mathcal{F}(\theta, \mathbf{x}, \mathbf{y}, t) (\mathbf{w}(\mathbf{x}, t) - \mathbf{w}(\mathbf{x} - \mathbf{y}, t)) \partial_t \mathbf{w}(\mathbf{x}, t) \, d\mathbf{x} \, d\mathbf{y} \, d\theta \\
&= \frac{d}{dt} \int_{\mathbb{R}^N} \frac{|\partial_t \mathbf{w}|^2}{2} \, d\mathbf{x} + \frac{\kappa}{2} \frac{d}{dt} \int_{\mathbb{R}^N} \int_{B_\delta(\mathbf{0})} \frac{|\mathbf{w}(\mathbf{x}, t) - \mathbf{w}(\mathbf{x} - \mathbf{y}, t)|^2}{|\mathbf{y}|^{N+2\alpha}} \, d\mathbf{x} \, d\mathbf{y} \\
&\quad + \int_{\mathbb{R}^N} \int_{B_\delta(\mathbf{0})} \int_0^1 \mathcal{F}(\theta, \mathbf{x}, \mathbf{y}, t) (\mathbf{w}(\mathbf{x}, t) - \mathbf{w}(\mathbf{x} - \mathbf{y}, t)) \partial_t \mathbf{w}(\mathbf{x}, t) \, d\mathbf{x} \, d\mathbf{y} \, d\theta.
\end{aligned}$$

Therefore, thanks to (H.3),

$$\begin{aligned}
&\frac{d}{dt} \left(\int_{\mathbb{R}^N} \frac{|\partial_t \mathbf{w}|^2}{2} \, d\mathbf{x} + \frac{\kappa}{2} \int_{\mathbb{R}^N} \int_{B_\delta(\mathbf{0})} \frac{|\mathbf{w}(\mathbf{x}, t) - \mathbf{w}(\mathbf{x} - \mathbf{y}, t)|^2}{|\mathbf{y}|^{N+2\alpha}} \, d\mathbf{x} \, d\mathbf{y} \right) \\
&= - \int_{\mathbb{R}^N} \int_{B_\delta(\mathbf{0})} \int_0^1 \mathcal{F}(\theta, \mathbf{x}, \mathbf{y}, t) (\mathbf{w}(\mathbf{x}, t) - \mathbf{w}(\mathbf{x} - \mathbf{y}, t)) \partial_t \mathbf{w}(\mathbf{x}, t) \, d\mathbf{x} \, d\mathbf{y} \, d\theta \\
&\leq \int_{\mathbb{R}^N} \int_{B_\delta(\mathbf{0})} g(\mathbf{y}) |\mathbf{w}(\mathbf{x}, t) - \mathbf{w}(\mathbf{x} - \mathbf{y}, t)| |\partial_t \mathbf{w}(\mathbf{x}, t)| \, d\mathbf{x} \, d\mathbf{y} \\
&\leq \frac{1}{2} \underbrace{\left(\int_{B_\delta(\mathbf{0})} g^2(\mathbf{y}) |\mathbf{y}|^{N+2\alpha} \, d\mathbf{y} \right)}_{\lambda} \int_{\mathbb{R}^N} |\partial_t \mathbf{w}(\mathbf{x}, t)|^2 \, d\mathbf{x} \\
&\quad + \frac{1}{2} \int_{\mathbb{R}^N} \int_{B_\delta(\mathbf{0})} \frac{|\mathbf{w}(\mathbf{x}, t) - \mathbf{w}(\mathbf{x} - \mathbf{y}, t)|^2}{|\mathbf{y}|^{N+2\alpha}} \, d\mathbf{x} \, d\mathbf{y} \\
&\leq \left(\lambda + \frac{1}{\kappa} \right) \left(\int_{\mathbb{R}^N} \frac{|\partial_t \mathbf{w}(\mathbf{x}, t)|^2}{2} \, d\mathbf{x} + \frac{\kappa}{2} \int_{\mathbb{R}^N} \int_{B_\delta(\mathbf{0})} \frac{|\mathbf{w}(\mathbf{x}, t) - \mathbf{w}(\mathbf{x} - \mathbf{y}, t)|^2}{|\mathbf{y}|^{N+2\alpha}} \, d\mathbf{x} \, d\mathbf{y} \right).
\end{aligned}$$

Hence, applying the Gronwall Lemma, we conclude that

$$\begin{aligned}
&\int_{\mathbb{R}^N} \frac{|\partial_t \mathbf{w}(\mathbf{x}, t)|^2}{2} \, d\mathbf{x} + \frac{\kappa}{2} \int_{\mathbb{R}^N} \int_{B_\delta(\mathbf{0})} \frac{|\mathbf{w}(\mathbf{x}, t) - \mathbf{w}(\mathbf{x} - \mathbf{y}, t)|^2}{|\mathbf{y}|^{N+2\alpha}} \, d\mathbf{x} \, d\mathbf{y} \\
&\leq e^{(\lambda + \frac{1}{\kappa})t} \left(\int_{\mathbb{R}^N} \frac{|\partial_t \mathbf{w}(\mathbf{x}, 0)|^2}{2} \, d\mathbf{x} + \frac{\kappa}{2} \int_{\mathbb{R}^N} \int_{B_\delta(\mathbf{0})} \frac{|\mathbf{w}(\mathbf{x}, 0) - \mathbf{w}(\mathbf{x} - \mathbf{y}, 0)|^2}{|\mathbf{y}|^{N+2\alpha}} \, d\mathbf{x} \, d\mathbf{y} \right).
\end{aligned}$$

Consequently, recalling (4.1), we obtain (2.22), as desired. $\square$

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