Characterizing Hermitian varieties in three- and four-dimensional projective spaces

CHARACTERIZING HERMITIAN VARIETIES IN 3- AND 4-DIMENSIONAL PROJECTIVE SPACES

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Abstract

We characterize Hermitian cones among the surfaces of degree $q+1$ of $\mathrm{PG}(3,q^2)$ by their intersection numbers with planes. We then use this result and provide a characterization of non-singular Hermitian varieties of $\mathrm{PG}(4,q^2)$, among quasi-Hermitian ones.

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1. Introduction

An *m-character set* in the projective space $\mathrm{PG}(n,q)$, q any prime power, is a set of points of $\mathrm{PG}(n,q)$ with the property that the intersection number with any hyperplane only takes m values, where m is a positive integer.

A non-singular Hermitian variety $\mathcal{H}(n,q^2)$ of $\mathrm{PG}(n,q^2)$ is a remarkable example of a two-character set, precisely a set of $(q^{n+1}+(-1)^n)(q^n-(-1)^n)/(q^2-1)$ points of $\mathrm{PG}(n,q)$ with the property that a hyperplane Π meets it in either

$$(q^n+(-1)^{n-1})(q^{(n-1)}-(-1)^{(n-1)})/(q^2-1)$$

points, in case Π is a non-tangent hyperplane to $\mathcal{H}(n,q^2)$ or,

$$1+q^2(q^{n-1}+(-1)^n)(q^{(n-2)}-(-1)^n)/(q^2-1)$$

points, in case Π is a tangent hyperplane to $\mathcal{H}(n,q^2)$; see [21].

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Quasi-Hermitian varieties are generalizations of non-singular Hermitian varieties such that they have the same size and the same intersection numbers with respect to hyperplanes.

Actually, a point set $\mathcal{S}$ of $\mathrm{PG}(n, q^2)$, $n > 2$, having the same intersection numbers with respect to hyperplanes as a non-singular Hermitian variety $\mathcal{H}(n, q^2)$ has also the same number of points of $\mathcal{H}(n, q^2)$; for $n = 2$ the size of $\mathcal{S}$ can be either $q^3 + 1$ that is, the size of a Hermitian curve also called a classical unital or, $q^2 + q + 1$ which is the number of points of a Baer subplane of $\mathrm{PG}(2, q^2)$; see [7].

As far as we know, the only quasi-Hermitian varieties of $\mathrm{PG}(n, q^2)$, which are not isomorphic to Hermitian varieties were constructed in the following series of papers [1, 2, 5, 6, 18, 19].

The definition of quasi-Hermitian variety can be extended to that of a singular quasi-Hermitian variety, that is point sets which have the same number of points and the same intersection numbers with respect to hyperplanes as singular Hermitian varieties. Each cone over a quasi-Hermitian variety is a singular quasi-Hermitian variety thus, a natural question is also whether such a cone is isomorphic to a singular Hermitian variety.

Various characterizations of a non-singular Hermitian variety among the quasi-Hermitian ones in $PG(n, q^2)$, with $n \in \{2, 3\}$ have been given, but very few in higher dimensional cases; see [3, 7, 8, 17]. In [3] singular Hermitian varieties were also characterized among singular quasi-Hermitian ones.

Here we first consider point sets of $\mathrm{PG}(3, q^2)$ such that their intersection numbers with respect to planes takes three values as well as the Hermitian cone with one singular point that is, $q^2 + 1$, $q^3 + 1$ or $q^3 + q^2 + 1$.

Combining geometric and combinatorial arguments with algebraic geometry we prove the following result.

THEOREM 1.1. *Let $\mathcal{S}$ be a surface of $\mathrm{PG}(3, q^2)$, of degree $q + 1$. If every plane meets $\mathcal{S}$ in either $q^2 + 1$, $q^3 + 1$, or $q^3 + q^2 + 1$ points of $\mathrm{PG}(3, q^2)$ then, $\mathcal{S}$ is a cone projecting a Hermitian curve in a plane π from a point V not in π.*

Next, we also provide the following characterization of non-singular Hermitian varieties of $\mathrm{PG}(4, q^2)$.

THEOREM 1.2. *Let $\mathcal{S}$ be a quasi-Hermitian variety of $PG(4, q^2)$. If $\mathcal{S}$ is a hypersurface of degree $q + 1$ then $\mathcal{S}$ is a non-singular Hermitian variety.*

2. Preliminaries

Let $\Sigma = \mathrm{PG}(n, q^2)$ be the Desarguesian projective space of dimension n over $\mathrm{GF}(q^2)$ and denote by $X = (x_1, x_2, \ldots, x_{n+1})$ homogeneous coordinates for its points.

We use σ to write the involutory automorphism of $\mathrm{GF}(q^2)$ which leaves all the elements of the subfield $\mathrm{GF}(q)$ invariant. A Hermitian variety $\mathcal{H}(n,q^2)$ is the set of all points X of Σ which are self conjugate under a Hermitian polarity h. If H is the Hermitian $(n+1)\times(n+1)$–matrix associated with h, then the Hermitian variety $\mathcal{H}(n,q^2)$ has equation

$$XH(X^\sigma)^T = 0.$$

When H is non–singular, the corresponding Hermitian variety is non-singular, whereas if H has rank $r+1$, with $r<n$, the related Hermitian variety is singular and it is a cone $\Pi_{n-r-1}\mathcal{H}(r,q^2)$ with vertex an $n-r-1$-space Π_{n-r-1} and basis a non-singular Hermitian variety $\mathcal{H}(r,q^2)$ of an r-space disjoint from Π_{n-r-1}.

A *d-singular quasi-Hermitian variety* is a subset of points of $\mathrm{PG}(n,q^2)$ having the same number of points and the same intersection sizes with hyperplanes as a singular Hermitian variety with a singular space of dimension d.

Non-singular Hermitian varieties of $\mathrm{PG}(n,q^2)$ are in particular hypersufaces. We recall that a projective *hypersurface* $\mathcal{S}$ of degree d is a set of points of $\mathrm{PG}(n,q^2)$ whose homogenous coordinates satisfy

$$F(X_0,X_1,\dots,X_n)=0, \tag{1}$$

where F is a form of degree d over $\mathrm{GF}(q^2)$.

However, to understand the geometry of the hypersurface $\mathcal{S}$, the zeros of F over $\mathrm{GF}(q^2)$ and over any extension of $\mathrm{GF}(q^2)$ are required. Thus, $\mathcal{S}$ is viewed as a hypersurface over the algebraic closure of $\mathrm{GF}(q^2)$ and a point of $\mathrm{PG}(n,q^2)$ in $\mathcal{S}$ is called a *$\mathrm{GF}(q^2)$-point* or a *rational point* of $\mathcal{S}$; in general a *$\mathrm{GF}(q^{2i})$-point* of $\mathcal{S}$ is a point $P(a_0,\dots,a_n)$ in $\mathrm{PG}(n,q^{2i})$ such that $F(a_0,\dots,a_n)=0$. The number of $\mathrm{GF}(q^{2i})$-point of $\mathcal{S}$ is denoted by $N_{q^{2i}}(\mathcal{S})$. When $n=2$, a projective hypersurface $\mathcal{S}$ is called a *projective plane curve*, whereas when $n=3$, $\mathcal{S}$ is called a *projective surface*.

The following results will be crucial to our proof.

Lemma 2.1 ([20]). *Let d be an integer with $1\le d\le q+1$ and $\mathcal{C}$ be a curve of degree d in $\mathrm{PG}(2,q)$ defined over $\mathrm{GF}(q)$, which may have $\mathrm{GF}(q)$-linear components. Then the number $N_{q^2}(\mathcal{C})$ of rational points of $\mathcal{C}$ is at most $dq+1$ and $N_q(\mathcal{C})=dq+1$ if and only if $\mathcal{C}$ is a pencil of d lines of $\mathrm{PG}(2,q)$.*

Lemma 2.2 ([13, 14, 15]). *Let d be an integer with $2\le d\le q+2$, and $\mathcal{C}$ be a curve of degree d in $\mathrm{PG}(2,q)$ without $\mathrm{GF}(q)$-line components. Then the number of rational points of $\mathcal{C}$ is at most $(d-1)q+1$ except for a class of plane curves of degree 4 over $\mathrm{GF}(4)$ having 14 points.*

LEMMA 2.3 ([11]). *Suppose $q \neq 2$. Let $\mathcal{C}$ be a plane curve over* $\mathrm{GF}(q^2)$ *of degree $q+1$ without* $\mathrm{GF}(q^2)$*-line components. If $\mathcal{C}$ has q^3+1 points over* $\mathrm{GF}(q^2)$*, then $\mathcal{C}$ is a Hermitian curve.*

LEMMA 2.4 ([16]). *Let $\mathcal{S}$ be a surface in* $\mathrm{PG}(3,q^2)$ *without* $\mathrm{GF}(q^2)$*-plane components. If the degree of $\mathcal{S}$ is $q+1$ and the number of its rational points is $(q^3+1)(q^2+1)$ then $\mathcal{S}$ is a non-singular Hermitian surface.*

Finally, a hyperplane of $\mathrm{PG}(n,q^2)$ intersecting a point set $\mathcal{S}$ of the projective space in i points will be called an *i-hyperplane* whereas, a line meeting $\mathcal{S}$ in s points will be called an *s-secant line* if $s \geq 1$ or an *external line* to $\mathcal{S}$ if $s = 0$.

LEMMA 2.5 ([8]). *If each intersection number with planes and hyperplanes of a point set $\mathcal{H}$ in* $\mathrm{PG}(4,q^2)$ *is also an intersection number with planes and hyperplanes of $\mathcal{H}(4,q^2)$, then $\mathcal{H}$ is a non-singular Hermitian variety $\mathcal{H}(4,q^2)$.*

3. Hermitian cones of $\mathrm{PG}(3,q^2)$

THEOREM 3.1. *Let $\mathcal{S}$ be a surface of* $\mathrm{PG}(3,q^2)$*, of degree $q+1$, q any prime power. If every plane meets $\mathcal{S}$ in either q^2+1, q^3+1, or q^3+q^2+1 points of* $\mathrm{PG}(3,q^2)$ *then $\mathcal{S}$ is a cone projecting a Hermitian curve in a plane π from a point V not in π.*

PROOF. Let π be a q^3+q^2+1-plane. As $\mathcal{S}$ is a surface of degree $q+1$ then $C = \mathcal{S} \cap \pi$ is a plane curve of degree $q+1$. By Lemma 2.2, C must have some $\mathrm{GF}(q^2)$-line component and thus by Lemma 2.1, C turns out to be a pencil of $q+1$ lines of π. Furthermore, each line of π has to meet $\mathcal{S}$ in 1, $q+1$ or q^2+1 rational points and in particular, the surface $\mathcal{S}$ contains lines of $\mathrm{PG}(3,q^2)$.

Now, assume that the plane π is a q^3+1-plane which does not have any $\mathrm{GF}(q^2)$-line of $\mathcal{S}$. In this case $C = \pi \cap \mathcal{S}$ is a plane curve of degree $q+1$ without $\mathrm{GF}(q^2)$-line components and it has q^3+1 $\mathrm{GF}(q^2)$-points; thus, by Lemma 2.3, C is a non-singular Hermitian curve for $q \neq 2$.

We are going to prove that $\mathcal{S}$ meets every line of $\mathrm{PG}(3,q^2)$ that is, $\mathcal{S}$ is a blocking set with respect to lines of the projective space. First, we assume $q \neq 2$ and consider a line r of $\mathrm{PG}(3,q^2)$. If r is on a q^3+q^2+1-plane then r is at least a 1-secant line of $\mathcal{S}$. In the case in which r lies on a q^3+1-plane, say π, either π contains some line of $\mathcal{S}$ or $\pi \cap \mathcal{S}$ is a Hermitian unital of π; in both cases r turns out to be at least 1-secant line of $\mathcal{S}$.

Thus, if r is an external line to $\mathcal{S}$, all planes through r have to be q^2+1-planes and the number $N_{q^2}(\mathcal{S})$ of rational points of $\mathcal{S}$ is $(q^2+1)^2$. Let t be a 1-secant line of $\mathcal{S}$ lying in some q^3+q^2+1-plane and let t_i denote the numbers of i-planes through t. Counting the number of $GF(q^2)$-points of $\mathcal{S}$ by using all planes through t we obtain

$$(q^2+1)^2 = t_{q^2+1}q^2 + t_{q^3+1}q^3 + t_{q^3+q^2+1}(q^3+q^2) + 1$$

that gives

$$1 = (q-1)t_{q^3+1} + qt_{q^3+q^2+1},$$

namely $t_{q^3+1} = 0$ and $t_{q^3+q^2+1} = 1/q$, a contradiction.

Now we assume $q = 2$. An algebraic plane curve of degree 3 in $PG(2,4)$, with 9 rational points, without $GF(4)$-line components is a unital or is projectively equivalent to the curve $\mathcal{C}' : X_0^3 + wX_1^2 + w^2X_2^3 = 0$, which meets each line in either 0, 2 or 3 rational points; see [10, §11]. Therefore, if r is an external line to $\mathcal{S}$ then, r could be contained either in 5-planes or in 9-planes. Suppose that there is at least a planar section of $\mathcal{S}$ which consists of 5 rational points on a line. In this case, a 9-plane never can intersect $\mathcal{S}$ in an algebraic plane curve which is projectively equivalent to $\mathcal{C}'$ therefore, only 5-planes can pass through an external line r of $\mathcal{S}$. Arguing as in the case $q \neq 2$, we get a contradiction.

Hence, each planar section of $\mathcal{S}$ with 5 points has to be an absolutely irreducible cubic curve with a cusp or a non-singular cubic with one rational inflexion; see [10, §11]. Thus, a line of $\mathcal{S}$ lies either on a 9-plane or on a 13-plane, whereas a 2-secant line lies either on a 5-plane or on a 9-plane. Let m be a 2-secant line of a 5-plane, that we know to exist and denote by x_m the number of 5-plane through m. Next, take a line ℓ of $\mathcal{S}$ and denote by x_ℓ the number of 9-planes through ℓ. Counting the number of $GF(4)$-points of $\mathcal{S}$ by using all planes through ℓ and all planes through m we get

$$x_\ell(9-5) + (5-x_\ell)(13-5) + 5 = x_m(5-2) + (5-x_m)(9-2) + 2,$$

that gives $x_\ell = x_m + 2$. As $x_m \geq 1$, we obtain $x_\ell \in \{3,4,5\}$. Consequently, the number of rational points $N_4(\mathcal{S}) \in \{33, 29, 25\}$. In order to prove that none of the previous possibilities can occur for $N_4(\mathcal{S})$, we count in double way the number of planes, the number of pairs (P,π), where $P \in PG(3,4)$ and π is a plane through P, and the number of pairs $((P,Q),\pi)$, where $P,Q \in PG(3,4)$ and π is a plane through P and Q. Let x,y,z denote the numbers of 5- 9- 13- planes respectively, we get the following equations

$$\begin{cases} x+y+z = 85 \\ \\ 5x+9y+13z = 21N_4(S) \\ \\ 20x+72y+156z = 5N_4(S)(N_4(S)-1). \end{cases} \tag{2}$$

For $N_4(\mathcal{S}) = 25$ or $N_4(\mathcal{S}) = 29$, (2) provide $z = 0$ or $z = -1$ respectively, in both cases a contradiction. When $N_4(\mathcal{S}) = 33$, (2) give $z = 3$ that is, there are 3 13-planes, each of which meets $\mathcal{S}$ in 3 concurrent lines. On the other hand, exactly 2 13-planes have to pass trough each line of $\mathcal{S}$ and hence we get a contradiction. Thus, $\mathcal{S}$ is a blocking set with respect to lines of $\mathrm{PG}(3,q^2)$ for all prime power q.

We recall that a blocking set with respect to lines of $\mathrm{PG}(2,q^2)$ which consists of q^2+1 points is a line; see [4]. Thus, if π is a q^2+1-plane then $\pi\cap\mathcal{S}$ consists of q^2+1 points on a line.

Furthermore, each line which is not contained in $\mathcal{S}$ meets $\mathcal{S}$ in i points with $1\leq i\leq q+1$ as $\mathcal{S}$ is a surface of degree $q+1$ over $\mathrm{GF}(q^2)$.

Next step is to prove that each line meets $\mathcal{S}$ either in one, or $q+1$ or q^2+1 $\mathrm{GF}(q^2)$-points. By way of contradiction, assume that there is an i-secant line to $\mathcal{S}$, say m, with $2\leq i\leq q$. Then, each plane through m has to be a q^3+1-plane containing some line of $\mathcal{S}$. Counting the number of $\mathrm{GF}(q^2)$-points of $\mathcal{S}$ by using all planes through m we obtain $N_{q^2}(\mathcal{S})=(q^2+1)(q^3+1-i)+i=q^5+q^3+q^2-iq^2+1$. Let us consider a q^2+1-plane, say α and let ℓ be the line $\alpha\cap\mathcal{S}$. By considering that each plane through ℓ has at most q^3+q^2+1 $\mathrm{GF}(q^2)$-points of $\mathcal{S}$ we get that $N_{q^2}(\mathcal{S})\leq q^2q^3+q^2+1=q^5+q^2+1$. Then, $i\geq q$ and hence $i=q$ which gives $N_{q^2}(\mathcal{S})=q^5+q^2+1$. In particular each line of $\mathcal{S}$ is contained in at most one q^2+1-plane. Now, let x_i denote the numbers of i-planes with respect to $\mathcal{S}$. In this case double counting arguments give

$$\begin{cases} \sum_i x_i=(q^4+1)(q^2+1) \\ \sum_i ix_i=(q^5+q^2+1)(q^4+q^2+1) \\ \sum_{i=1} i(i-1)x_i=(q^5+q^2+1)(q^5+q^2)(q^2+1). \end{cases} \tag{3}$$

By solving (3) we obtain in particular that the number x_{q^2+1} of q^2+1-planes is q^3+1.

Denote by $\Sigma=\{\alpha_1,\ldots,\alpha_{q^3+1}\}$ the set of all q^2+1-planes to $\mathcal{S}$ and set $\ell_i=\alpha_i\cap\mathcal{S}$ for all $\alpha_i\in\Sigma$. We observe that any two lines ℓ_i and ℓ_j, with $i\neq j$ intersect in a point and never three of these lines form a triangle. In fact, a triangle PQR of such lines would be contained in a q^3+1-plane π; since every line of π would meet $\mathcal{S}$ in at least two points we would obtain in particular that every line of π through P would be at least a q-secant to $\mathcal{S}$ and hence we would get $|\pi\cap\mathcal{S}|\geq(q^2-1)(q-1)+2q^2+1>q^3+1$, a contradiction.

This means that the q^3+1 lines contained in $\mathcal{S}$ are concurrent at a point V. Since $\mathcal{S}$ has exactly $q^2(q^3+1)+1$ rational points, each other line contained in $\mathcal{S}$ cannot pass through V and has to meet q^2+1 lines among the lines ℓ_i, with $1\leq i\leq q^3+1$. Thus, we find a $\mathrm{GF}(q^2)$-planar component of $\mathcal{S}$ which is excluded. Hence, $\mathcal{S}$ contains exactly q^3+1 lines and for each line ℓ contained in $\mathcal{S}$ exactly one q^2+1-plane through it exists whereas no plane through ℓ are q^3+1-plane. But then, there are no q^3+1-planes containing some line of $\mathcal{S}$, a contradiction.

Thus, each line which is not contained in $\mathcal{S}$ meets $\mathcal{S}$ in either 1, or $q+1$ rational points. For $q \neq 2$, from [12, Th. 23.5.1] $\mathcal{S}$ has to be a cone $\Pi_0\mathcal{S}'$ with $\mathcal{S}'$ of type

I. a unital;

II. a subplane $\mathrm{PG}(2,q)$;

III. a set of type $(0,q)$ plus an external line;

IV. the complement of a set of type $(0,q^2-q)$.

As the possible intersection sizes with planes of $\mathrm{PG}(3,q^2)$ are q^2+1, q^3+1, q^3+q^2+1, possibilities II, III, and IV must be excluded, since their sizes cannot be possible. This implies that $\mathcal{S}=\Pi_0\mathcal{S}'$, where $\mathcal{S}'$ is a unital. On the other hand $\mathcal{S}'$ turns out to be an algebraic curve of degree $q+1$ without linear components and with q^3+1 points over $\mathrm{GF}(q^2)$. Thus, for $q \neq 2$ Lemma 2.3 applies and $\mathcal{S}'$ has to be a Hermitian curve.

For $q=2$, there is just one point set in $\mathrm{PG}(3,4)$ up to equivalence, meeting each line in 1, 3 or 5 points and each plane in 5, 9 or 13 points, that is the Hermitian cone, see [9, Theorem 19.6.8]. Thus also for $q=2$ our theorem follows. $\square$

As an easy consequence of Theorem 3.1 we get the following.

Corollary 3.2. *Let $\mathcal{S}$ be a surface of $\mathrm{PG}(3,q^2)$ of degree d. If every plane meets $\mathcal{S}$ in either q^2+1, q^3+1 or q^3+q^2+1 points over $\mathrm{GF}(q^2)$ then, $d \geq q+1$. If $d=q+1$ then $\mathcal{S}$ is a cone over a Hermitian curve.*

4. A characterization of $\mathcal{H}(4,q^2)$

Theorem 4.1. *Let $\mathcal{S}$ be a quasi-Hermitian variety of $PG(4,q^2)$. If $\mathcal{S}$ is a hypersurface of degree $q+1$ then $\mathcal{S}$ is a non-singular Hermitian variety.*

Proof. We recall that $\mathcal{S}$ has $q^7+q^5+q^2+1$ rational points and its intersection numbers with respect to hyperplanes over $\mathrm{GF}(q^2)$ are q^5+q^2+1 or $q^5+q^3+q^2+1$.

First we prove that $\mathcal{S}$ does not contain any plane of $\mathrm{PG}(4,q^2)$. Suppose on the contrary that there is a plane α which is contained in $\mathcal{S}$. Let us denote by x the number of hyperplanes through α meeting $\mathcal{S}$ in q^5+q^2+1 $\mathrm{GF}(q^2)$-points.

Counting the number $N_{q^2}(\mathcal{S})$ of $\mathrm{GF}(q^2)$-points of $\mathcal{S}$ by using all hyperplanes through α we obtain

$$q^7+q^5+q^2+1 = N_{q^2}(\mathcal{S}) = (q^2+1-x)(q^5+q^3-q^4)+x(q^5-q^4)+q^4+q^2+1$$

that is,

$$xq^3 = -q^6 + q^5 + q^2,$$

a contradiction. This implies that $\Sigma = \mathcal{S} \cap \Pi$ is an algebraic surface of degree $q+1$ without GF(q^2)-plane components. In the case in which $N_{q^2}(\Sigma) = q^5 + q^3 + q^2 + 1$, by Lemma 2.4, Σ is a non-singular Hermitian surface.

Now let Π' be a hyperplane of PG$(4, q^2)$ meeting $\mathcal{S}$ in $q^5 + q^2 + 1$ rational points and set $\Sigma' = \Pi' \cap \mathcal{S}$. We are going to study the planar sections of Σ'. Thus, let us denote by α a plane contained in Π'. If at least one $q^5+q^3+q^2+1$-hyperplane passes through α then, $\alpha \cap \mathcal{S}$ is either a Hermitian curve or a pencil of $q+1$ concurrent lines and hence, $N_{q^2}(\alpha \cap \Sigma') = q^3 + 1$ or $N_{q^2}(\alpha \cap \Sigma') = q^3 + q^2 + 1$.

Suppose that all hyperplanes containing α meet $\mathcal{S}$ in $q^5 + q^2 + 1$ rational points and set $y = N_{q^2}(\alpha \cap \mathcal{S})$. Then we obtain,

$$q^7 + q^5 + q^2 + 1 = N_{q^2}(\mathcal{S}) = (q^2+1)(q^5 + q^2 + 1 - y) + y$$

namely $y = q^2 + 1$ and thus, $N_{q^2}(\alpha \cap \Sigma') = q^2 + 1$.

Theorem 3.1 applies and Σ' turns out to be a cone over a Hermitian curve. Then, each intersection number over GF(q^2) with planes and hyperplanes of $\mathcal{S}$ is also an intersection number with planes and hyperplanes of $\mathcal{H}(4, q^2)$. By Lemma 2.5, $\mathcal{S}$ has to be a non-singular Hermitian variety of PG$(4, q^2)$. $\square$

References

[1] A. Aguglia, *Quasi-Hermitian varieties in* PG(r, q^2)*, q even*, Contrib. Discrete Math. **8** (2013), no. 1, 31–37.

[2] A. Aguglia, A. Cossidente, G. Korchmáros, *On quasi-Hermitian varieties*, J. Combin. Des. **20** (2012), no. 10, 433–447.

[3] A. Aguglia, D. Bartoli, L. Storme, Zs. Weiner *A characterization of Hermitian varieties as codewords*, Electron. J. Combin. **25 (1)** (2018).

[4] R. C. Bose, R. C. Burton, *A characterization of flat spaces in a finite geometry and the uniqueness of the Hamming and the MacDonald codes*, J. Combinatorial Theory **1** (1966), 96–104.

[5] F. Buekenhout, *Existence of unitals in finite translation planes of order q^2 with a kernel of order q*, Geom., Dedicata **5** (1976) 189-194.

[6] S. De Winter, J. Schillewaert, *A note on quasi-Hermitian varieties and singular quasi-quadrics*, Bull. Belg. Math. Soc. Simon Stevin **17** (2010), no. 5, 911–918.

[7] S. De Winter, J. Schillewaert, *Characterizations of finite classical polar spaces by intersection numbers with hyperplanes and spaces of codimension* 2. Combinatorica **30** (2010), n. 1, 25–45.

[8] J.Schillewaert, J.A. Thas, *Characterizations of Hermitian varieties by intersection numbers* Designs, Codes and Cryptography **50** (2009), 41–60.

[9] J.W.P. Hirschfeld, *Finite Projective Spaces of Three Dimensions*, Oxford University Press, (1985).

[10] J.W.P.Hirschfeld,*Projective Geometries over Finite Fields*, 2nd edn., Oxford (1998)

[11] J. W. P. Hirschfeld,L. Storme, J.A. Thas, J. F. Voloch, *A characterization of Hermitian curves.* J. Geom. **41** (1991), n. 1-2, 72–78.

[12] J.W.P. Hirschfeld and J.A. Thas, *General Galois Geometries*, Oxford University Press, (1992).

[13] Homma, Masaaki; Kim, Seon Jeong *Around Sziklai's conjecture on the number of points of a plane curve over a finite field.* Finite Fields Appl. **15** (2009), no. 4, 468–474.

[14] Homma, Masaaki; Kim, Seon Jeong *Sziklai's conjecture on the number of points of a plane curve over a finite field III.* Finite Fields Appl. **16** (2010), no. 5, 315–319.

[15] Homma, Masaaki; Kim, Seon Jeong *Sziklai's conjecture on the number of points of a plane curve over a finite field II.* Finite fields: theory and applications, 225–234, Contemp. Math., 518, Amer. Math. Soc., Providence, RI, 2010.

[16] M Homma, S.J. Kim, *The characterization of Hermitian surfaces by the number of points* J. Geometry, **107** (2016) 509–521.

[17] S. Innamorati, M. Zanetti, F. Zuanni, *On two character* $(q^7+q^5+q^2+1)$*-sets in* $\mathrm{PG}(4,q^2)$. J. Geom. **106**, (2015), no. 2, 287–296.

[18] R. Metz, *On a class of unitals*, Geom. Dedicata **8** (1979), 125-126.

[19] F. Pavese *Geometric constructions of two-character sets*, Discrete Math. **338** (2015), no. 3, 202–208.

[20] B. Segre *Le geometrie di Galois* Ann. MAt. Pura Appl.(4) **48**, 1–96 (1959).

[21] B. Segre, *Forme e geometrie Hermitiane, con particolare riguardo al caso finito*, Ann. Mat. Pura Appl. (4) **70** (1965), 1–201.

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