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NONLINEAR SCHRÖDINGER EQUATION WITH BOUNDED MAGNETIC FIELD

GIUSEPPE DEVILLANOVA AND CYRIL TINTAREV

ABSTRACT. The paper studies existence of solutions for the nonlinear Schrödinger equation

$$(0.1) \quad -(\nabla + \mathbf{i}A(x))^2 u + V(x)u = f(|u|)u$$

with a general bounded external magnetic field. In particular, no lattice periodicity of the magnetic field or presence of external electric field is required. Solutions are obtained by means of a general structural statement about bounded sequences in the magnetic Sobolev space.

1. INTRODUCTION

The present paper studies existence of solutions for the nonlinear Schrödinger equation with bounded external magnetic field B on $\mathbb{R}^N$ without additional assumptions, such as lattice periodicity. External magnetic field enters Schrödinger equation, as well as other equations of quantum mechanics, by addition of a real-valued covector field A , called the magnetic potential to the momentum operator $\frac{\hbar}{\mathbf{i}}d$. Magnetic field, which is a measurable quantity, is a differential 2-form $B = dA$, while the magnetic potential is defined up to an arbitrary additive term $d\varphi$, where $\varphi : \mathbb{R}^N \rightarrow \mathbb{R}$ is an arbitrary scalar function. Magnetic momentum $\frac{\hbar}{\mathbf{i}}d + A$ has the following gauge invariance property:

$$\left(\frac{\hbar}{\mathbf{i}}d + A\right)(e^{\mathbf{i}\frac{\varphi}{\hbar}}u) = e^{\mathbf{i}\frac{\varphi}{\hbar}}\left(\frac{\hbar}{\mathbf{i}}d + A + d\varphi\right)u.$$

In this paper we follow the convention that normalizes the mass of the particle and sets the Planck constant $\hbar$ to be equal to 1 (which is always possible by rescaling the time and space variables).

Up to our knowledge, the earliest existence result for nonlinear magnetic Schrödinger equation is the paper by Esteban and Lions [18] where the magnetic field is assumed to be constant. Their approach was generalized

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to periodic magnetic field by Arioli and Szulkin [2] (see also [22], [10], [4]). In addition to that, there is a number of important existence results for quasiclassical solutions, that is, solutions of the nonlinear magnetic Schrödinger equation that exist if $\hbar$ is sufficiently small. The earliest study of quasiclassical solutions known to us is due to Kurata [17], where existence was obtained under assumptions that involved both electric and magnetic potential, and it was followed by a number of other results concerning quasiclassical solutions by Cingolani, Jeanjean, Secchi, Tanaka and several others, see [8, 6, 9] and references therein. In most of these works existence of solutions is connected to concentration at critical points of the electric potential as the Planck constant tends to zero. Several other papers studied critical nonlinearities in the scalar field term, [5, 7], and the Aharonov-Bohm field, related to critical nonlinear growth, in [11]. The main technical difficulty in proving existence results for the magnetic Schrödinger equation is the lack of compactness of Sobolev embeddings in the whole $\mathbb{R}^N$, and it is overcome by the use of a concentration-compactness argument. For instance, in [2], one controls the loss of compactness in problems with a periodic magnetic field by means of energy-preserving operators

$$(1.1) \quad g_y := u \mapsto e^{i\varphi_y(\cdot)} u(\cdot - y), \quad y \in \mathbb{Z}^N,$$

(where φ_y is a suitable re-phasing function (see (2.10) below)), known as *magnetic shifts* with their inverse

$$g_y^{-1} := v \mapsto e^{-i\varphi_y(\cdot+y)} v(\cdot + y), \quad y \in \mathbb{Z}^N.$$

Another concentration mechanism applies to quasiclassical asymptotics and is not considered here. The present paper aims to extend the approach of [18] and [2] to problems with a generic bounded magnetic field, without any periodicity assumption. This is achieved by a refined concentration-compactness analysis (see Theorem 5.1 and Theorem 5.2) that uses a suitable non-isometric counterpart of magnetic shifts (see Definition 3.3). This analysis allows to prove a variety of existence results which are beyond the scope of the present paper. Our objective here is to describe the concentration mechanism in the non-periodic case and give a sample existence result which, in particular, includes existence of solutions in absence of an electric field.

The paper is organized as follows. Section 2 is dedicated to notation, as well as to conventions for magnetic shifts in the lattice-periodic case. In Section 3 we provide the generalization of magnetic shifts for the non-periodic case, as well as the notions of the magnetic field and of the corresponding energy functional at infinity. In Section 4 we prove a “vanishing lemma”, that asserts, in terms of generalized magnetic shifts, a property of the Sobolev embedding similar to cocompactness. In Section 5 we employ generalized magnetic shifts to express the defect of compactness (i.e. the difference between the weak limit (modulo subsequence) of a bounded sequence $(u_k)_{k \in \mathbb{N}}$

and a particular profile decomposition (i.e. a sum of terms with asymptotically disjoint supports, each of them having the form of suitable magnetic shifts (determined by mutually diverging sequences of points) applied to a *fixed function*). In Section 6 we apply the profile decomposition obtained in Section 5 to show the existence of nontrivial solutions to (0.1) under assumption that the magnetic field is bounded and vanishes at infinity. Finally, in the Appendix, we give some details about the set of magnetic shifts in the lattice periodic case and show that it generates a group modulo re-phrasings.

Main results of this paper are Theorem 5.1 and Theorem 5.2 concerning profile decompositions in magnetic Sobolev space and Theorems 6.5 and 6.6 providing some representative existence results. It should be noted, that while most existence results in literature pose conditions on the magnetic potential which is not a measurable quantity and is not uniquely defined by the magnetic field, this paper provides existence of solution at a specified energy level using simple conditions on the magnetic field itself: a uniform bound and the vanishing at infinity.

2. NOTATION AND BASIC PROPERTIES

In this paper the set $\mathbb{N}$ of natural numbers is fixed as starting with zero. Given a magnetic potential A as a real-valued linear form, we define

$$(2.1) \quad \nabla_A(u) \stackrel{\text{def}}{=} (\nabla + \mathbf{i}A)u,$$

and the scalar product

$$\langle u, v \rangle_A \stackrel{\text{def}}{=} \int_{\mathbb{R}^N} \nabla_A u(x) \overline{\nabla_A v(x)} dx,$$

(the symbol of the scalar product on $\mathbb{R}^N$ will be neglected throughout the paper). Then, we introduce the (Dirichlet-type) energy functional

$$(2.2) \quad E_A(u) \stackrel{\text{def}}{=} \int_{\mathbb{R}^N} |\nabla_A u(x)|^2 dx.$$

Note that the following relation holds true.

$$(2.3) \quad \begin{aligned} |\nabla_A u|^2 &= |\nabla u|^2 + \mathbf{i}A(u\nabla \bar{u} - \bar{u}\nabla u) + |A|^2|u|^2 \\ &\geq |\nabla u|^2 - 4|A||u||\nabla u| + |A|^2|u|^2. \end{aligned}$$

Indeed, by applying Cauchy-Schwarz and triangle inequality, the real number

$$(2.4) \quad \begin{aligned} \mathbf{i}A(u\nabla \bar{u} - \bar{u}\nabla u) &= 2\mathbf{i}^2 A(\Re(u)\nabla \Im(u) - \Im(u)\nabla \Re(u)) \\ &\geq -2|A|(|\Re(u)| |\nabla \Im(u)| + |\Im(u)| |\nabla \Re(u)|) \\ &\geq -4|A||u||\nabla u|. \end{aligned}$$

Moreover, by taking into account the trivial relation $(|\nabla u| - 4|A||u|)^2 \geq 0$, we deduce

$$(2.5) \quad 4|A||u||\nabla u| \leq \frac{1}{2}|\nabla u|^2 + 8|A|^2|u|^2.$$

Then, by combining (2.5) and (2.3), we get

$$(2.6) \quad |\nabla_A u|^2 \geq \frac{1}{2} |\nabla u|^2 - 7|A|^2 |u|^2,$$

or, conversely,

$$(2.7) \quad |\nabla u|^2 \leq 2|\nabla_A u|^2 + 14|A|^2 |u|^2.$$

As far as no ambiguity arises, we will not distinguish, in notation, between the magnetic field and the skew-symmetric $N \times N$ matrix-valued function that represents it in the Euclidean space, and, respectively, between the magnetic potential and its representation as a vector field. In this context we also allow differentiation of scalar functions to be denoted, interchangeably as ∇ and as d , depending if we view the resulting value as a vector field or as a 1-form.

When dealing with a magnetic potential A we shall define $\dot{H}_A^{1,2}(\mathbb{R}^N)$ as the completion of C_0^∞ with respect to the norm $\|u\|_A = (E_A(u))^{\frac{1}{2}}$ (note that $\dot{H}_A^{1,2}(\mathbb{R}^N)$ is a space of measurable functions whenever $N > 2$ or $N = 2$ and $dA \neq 0$, see e.g. [16]), and the space $H_A^{1,2}(\mathbb{R}^N)$ as the intersection $\dot{H}_A^{1,2}(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$ (equipped with the standard intersection norm). Note that $|u|^2$ in this model has the meaning of the probability distribution of a particle in the space, and it doesn't change if u is also affected by the multiplication with $e^{i\varphi}$ (on the other side, as already remarked, the magnetic potential A and $A + \nabla\varphi$ give rise to the same magnetic field). Note also that, when $A \equiv 0$, we obtain the corresponding usual Sobolev spaces $\dot{H}^{1,2}(\mathbb{R}^N)$ and $H^{1,2}(\mathbb{R}^N)$ respectively. Moreover, in such a case, the re-phasing function φ_y in (1.1) can be set to be constant for all y . We shall later normalize the magnetic shifts g_y (see (6.22) and (6.23) below) in such a way that they reduce to Euclidean shifts when the magnetic field is zero.

Setting, as usual, $2^* = \begin{cases} \frac{2N}{N-2}, & N > 2 \\ \infty, & N = 2 \end{cases}$ the critical Sobolev exponent, we

have the following continuous embeddings $H^{1,2}(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$ for every $p \in [2, 2^*)$, and $H^{1,2}(\mathbb{R}^N) \hookrightarrow \dot{H}^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$ when $N > 2$. Furthermore, from the well-known *diamagnetic inequality*

$$(2.8) \quad |\nabla_A u(x)| \geq |\nabla |u(x)||,$$

we deduce that

$$E_A(u) \geq E_0(|u|) \geq C \|u\|_{2^*}^2, \quad \text{if } N > 2,$$

and, as a consequence, the continuous embeddings

$$\dot{H}_A^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N) \quad \text{and} \quad H_A^{1,2}(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N) \quad \forall p \in [2, 2^*),$$

(the latter one extends by analogous argument to the case $N = 2$).

Note that, when the magnetic field B is lattice periodic, i.e. when

$$(2.9) \quad B(\cdot - y) = B(\cdot) \quad \text{for all } y \in \mathbb{Z}^N,$$

we have, in terms of a fixed magnetic potential A ,

$$d(A(\cdot - y) - A(\cdot)) = 0 \quad \text{for all } y \in \mathbb{Z}^N.$$

Therefore,

$$(2.10) \quad \forall y \in \mathbb{Z}^N \exists \varphi_y \text{ s.t. } A(\cdot - y) = A(\cdot) + \nabla \varphi_y(\cdot).$$

From this we can derive the following relation which will be used in the Appendix. Namely for any $y_1, y_2 \in \mathbb{Z}^N$ there exists a constant $\gamma(y_1, y_2) \in \mathbb{R}$ such that

$$(2.11) \quad \varphi_{y_1+y_2} = \varphi_{y_1}(\cdot - y_2) + \varphi_{y_2} + \gamma(y_1, y_2),$$

(indeed, derivatives of the left and of the right hand side coincide by (2.10)). Thus, for every $y \in \mathbb{Z}^N$, we can define, by (1.1), a suitable magnetic shift g_y such that, (still denoting by g_y its extension to vector valued functions) we have the following commutation law

$$\nabla_A(g_y u) = g_y(\nabla_A u),$$

where, actually, for all $x \in \mathbb{R}^N$,

$$g_y(\nabla_A u)(x) \stackrel{\text{def}}{=} e^{i\varphi_y(x)} \nabla_A u(x - y) = e^{i\varphi_y(x)} (\nabla u(x - y) + \mathbf{i}A(x - y)u(x - y)).$$

Indeed, by taking into account ((1.1), (2.1) and) (2.10), we have

$$(2.12) \quad \begin{aligned} \nabla_A(g_y u)(x) &= e^{i\varphi_y(x)} (\nabla u(x - y) + \mathbf{i}(\nabla \varphi_y(x) + A(x))u(x - y)) \\ &= e^{i\varphi_y(x)} (\nabla u(x - y) + \mathbf{i}A(x - y)u(x - y)) \\ &= g_y(\nabla_A u)(x). \end{aligned}$$

As a consequence, we have that the magnetic shifts g_y (given by (1.1) where φ_y satisfies (2.10)) are, for all $y \in \mathbb{Z}^N$, isometries on $\dot{H}_A^{1,2}(\mathbb{R}^N)$, indeed, for all $u, v \in \dot{H}_A^{1,2}(\mathbb{R}^N)$, we have

$$(2.13) \quad \langle g_y u, g_y v \rangle_A = \langle u, v \rangle_A.$$

Indeed, by taking into account (2.12) and since φ_y is a real valued function, we get that

$$(2.14) \quad \begin{aligned} \langle g_y u, g_y v \rangle_A &= \int_{\mathbb{R}^N} \nabla_A(g_y u) \overline{\nabla_A(g_y v)} dx \\ &= \int_{\mathbb{R}^N} g_y(\nabla_A u) \overline{g_y(\nabla_A v)} dx \\ &= \int_{\mathbb{R}^N} \nabla_A u(x - y) \overline{\nabla_A v(x - y)} dx = \langle u, v \rangle_A. \end{aligned}$$

As a consequence of (2.13) we have

$$(2.15) \quad E_A(g_y u) = E_A(e^{i\varphi_y(\cdot)} u(\cdot - y)) = E_A(u) \quad \text{for all } y \in \mathbb{Z}^N.$$

The following lemma can be found elsewhere in literature (e.g. [2, 27]) but we include our version here for the consistency of the paper.

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with piecewise C^1 -boundary and let $A \in C(\Omega, \Lambda_1)$. Then, for any $\lambda > 0$, there exist positive constants C_1 and C_2 such that for all $u \in H_A^{1,2}(\mathbb{R}^N)$:*

$$(2.16) \quad C_1 \int_{\Omega} (|\nabla u|^2 + |u|^2) dx \leq \int_{\Omega} (|\nabla_A u|^2 + \lambda |u|^2) dx \leq C_2 \int_{\Omega} (|\nabla u|^2 + |u|^2) dx.$$

Proof. First note that there exist positive constants Λ , C_1 and C_2 such that

$$(2.17) \quad C_1 \int_{\Omega} (|\nabla u|^2 + |u|^2) dx \leq \int_{\Omega} (|\nabla_A u|^2 + \Lambda |u|^2) dx \leq C_2 \int_{\Omega} (|\nabla u|^2 + |u|^2) dx$$

for all $u \in H_A^{1,2}(\mathbb{R}^N)$. Indeed, the right inequality is elementary and the left one easily follows by (2.7). Let us now recall that by compactness of local Sobolev embeddings, for any $\epsilon > 0$ there exists a subspace $E_{\epsilon} > 0$ of finite codimension such that

$$\int_{\Omega} |u|^2 dx \leq \epsilon \int_{\Omega} (|\nabla u|^2 + |u|^2) dx, \quad u \in E_{\epsilon}.$$

Then (2.16) restricted to E_{ϵ} is immediate from (2.17). Furthermore, since all norms are equivalent on finite-dimensional spaces, (2.16) holds also on a complement of E_{ϵ} , and then, by the triangle inequality, it holds for all $u \in H_A^{1,2}(\mathbb{R}^N)$. $\square$

Corollary 2.2. *For any $A \in C_{\text{loc}}(\mathbb{R}^N, \Lambda_1)$ the Fréchet spaces $H_{A,\text{loc}}^{1,2}(\mathbb{R}^N)$ and $H_{\text{loc}}^{1,2}(\mathbb{R}^N)$ coincide.*

3. ENERGY AT INFINITY

Let $N \geq 2$ and let Λ_1 and Λ_2 denote the linear spaces, respectively, of 1-forms on $\mathbb{R}^N$ and of antisymmetric 2-forms on $\mathbb{R}^N \times \mathbb{R}^N$. Let, for $\alpha \in (0, 1]$, $C^{0,\alpha}$ and $C^{1,\alpha}$ denote, respectively, the space of Hölder-continuous functions and the space of functions with Hölder-continuous first derivative, with Hölder exponent α . Given a magnetic field $B \in C^{0,\alpha}(\mathbb{R}^N, \Lambda_2)$, we will say that $A \in C(\mathbb{R}^N, \Lambda_1)$ is a magnetic potential of B if $B = dA$ in the sense of weak differentiation. Note that, when B is bounded, we can assume that $A(x) = O(|x|)$ near infinity (we are using the Landau O symbol to denote functions of the same order). In particular, when B is a constant magnetic field, A is linear in x . Therefore, generally, we cannot define, for a given function $u \in H_A^{1,2}(\mathbb{R}^N)$, the limit of the magnetic energy functional (2.2) when the function u is shifted at infinity like in the non-magnetic case (e.g. [19]), by taking $\lim_{|y| \rightarrow \infty} E_0(u(\cdot + y))$. This difficulty is easily overcome in the case of a lattice-periodic magnetic field. Indeed, by (2.15), the functional $E_A \circ g_{y_k}$ remains equal to E_A for any sequence $y_k \in \mathbb{Z}^N$.

Below, in Lemma 3.1, we construct a re-phasing function φ_y , $y \in \mathbb{R}^N$, such that the corrected magnetic potential

$$(3.1) \quad A_y = A + \nabla \varphi_y,$$

which still corresponds to the same magnetic field $B = dA$, is bounded on balls $B_r(y)$ by a constant which is dependent on r but is uniform in y . Without periodicity assumptions on B , φ_y will not satisfy (2.10), but the difference between the left and right hand side of (2.10) will be still controlled. This allows us to define magnetic shifts as in (1.1) and to give a definition of the limit value of the energy of a function u subjected to a sequence of magnetic shifts related to a diverging sequence of points in $\mathbb{R}^N$.

In the next lemma we shall make use of the following notation: $\forall x \in \mathbb{R}^N, x = (x_1, \dots, x_N)$ and $\forall i \in \{1, \dots, N\}$ we shall set

$$\underline{X}_i := (x_1, \dots, x_i) \in \mathbb{R}^i \quad \text{and} \quad \vec{X}^i := (x_i, \dots, x_N) \in \mathbb{R}^{N-i+1},$$

so that, for all $i \in \{1, \dots, N-1\}$,

$$\underline{X}_N = \vec{X}^1 = x = (\underline{X}_i, \vec{X}^{i+1}),$$

while

$$\underline{X}_1 = x_1 \quad \text{and} \quad \vec{X}^N = x_N.$$

Lemma 3.1. *Let $A := (A_1, \dots, A_N) \in C_{\text{loc}}(\mathbb{R}^N, \Lambda_1)$. Then, for all $y = (y_1, \dots, y_N) \in \mathbb{R}^N$, there exists a function $\varphi_y \in C(\mathbb{R}^N)$ such that for all $x = (x_1, \dots, x_N)$ the following N partial derivatives of φ_y exist and are expressed by means of the coordinate functions of the magnetic potential as follows:*

$$\begin{aligned} \partial_1 \varphi_y(\vec{X}^1) &= -A_1(\vec{X}^1), \\ \partial_2 \varphi_y(\underline{Y}_1, \vec{X}^2) &= -A_2(\underline{Y}_1, \vec{X}^2), \\ \partial_3 \varphi_y(\underline{Y}_2, \vec{X}^3) &= -A_3(\underline{Y}_2, \vec{X}^3), \\ &\dots \\ \partial_N \varphi_y(\underline{Y}_{N-1}, \vec{X}^N) &= -A_N(\underline{Y}_{N-1}, \vec{X}^N). \end{aligned} \tag{3.2}$$

In particular

$$\nabla \varphi_y(y) = -A(y). \tag{3.3}$$

Proof. To any fixed $x = (x_1, \dots, x_N) \in \mathbb{R}^N$ we shall assign a unique real number, to be denoted as $\psi(x)$, first by arbitrarily fixing the value $\psi(y)$ at $y = (y_1, \dots, y_N)$, then allowing successively to vary, in the descending order, each of the variables y_i , $i = N, N-1, \dots, 1$.

We start with $\psi(y_1, \dots, y_{N-1}, x_N)$:

$$\psi(\underline{Y}_{N-1}, \vec{X}^N) = \psi(\underline{Y}_{N-1}, y_N) + \int_{y_N}^{x_N} A_N(\underline{Y}_{N-1}, t) dt,$$

we can recursively define, for decreasing $m = N-1, \dots, 2$,

$$\psi(\underline{Y}_{m-1}, x_m, \vec{X}^{m+1}) = \psi(\underline{Y}_{m-1}, y_m, \vec{X}^{m+1}) + \int_{y_m}^{x_m} A_m(\underline{Y}_{m-1}, t, \vec{X}^{m+1}) dt.$$

Finally, for $m = 1$, we set

$$\psi(x) = \psi(x_1, \vec{X}^2) = \psi(y_1, \vec{X}^2) + \int_{y_1}^{x_1} A_1(t, \vec{X}^2) dt.$$

Then, the claim follows by setting $\varphi_y \stackrel{\text{def}}{=} -\psi$. Functions φ_y , $y \in \mathbb{R}^N$, satisfy the required properties by construction. Each of them is, of course, defined up to the arbitrary chosen initial value $\varphi_y(y) = -\psi(\underline{Y}_N) = -\psi(y)$. $\square$

Note that, by (3.3), no correction is needed in the point y , i.e. the value of the corrected potential A_y defined by (3.1) is zero at y , i.e.

$$(3.4) \quad A_y(y) = A(y) + \nabla \varphi_y(y) = 0.$$

Moreover, by taking into account (3.1) and (3.2), we deduce that

$$(3.5) \quad (A_y)_n(\underline{Y}_{n-1}, \cdot) = A_n(\underline{Y}_{n-1}, \cdot) + \partial_n \varphi_y(\underline{Y}_{n-1}, \cdot) \equiv 0, \quad \forall n \in \{1, \dots, N\},$$

(where we use the convention $\underline{Y}_0 = \emptyset$ so that the above relation means, for $n = 1$, $(A_y)_1 \equiv 0$).

Remark 3.2. In what follows the corrected magnetic potential A_y is defined by (3.1) where φ_y is the function provided by Lemma 3.1 if $y \neq 0$, while for $y = 0$ we set $\varphi_0 = 0$, so that A_0 , defined by (3.1), equals A . Consequently, the magnetic shift g_0 in the definition below becomes the identity operator.

We now define magnetic shifts (with approximate re-phasing) as follows.

Definition 3.3. Let $B = dA$ be a magnetic field with $A \in C_{\text{loc}}^1(\mathbb{R}^N, \Lambda_1)$. Any map

$$(3.6) \quad g_y := u \mapsto e^{i\varphi_y(\cdot)} u(\cdot - y) \quad \forall u \in C_0^1(\mathbb{R}^N), \quad y \in \mathbb{R}^N,$$

where the $C^1(\mathbb{R}^N)$ function φ_y is provided by Lemma 3.4 when $y \neq 0$ or is the null function when $y = 0$ (see Remark 3.2), is called *a magnetic shift* (relative to the magnetic field B) determined by the vector y .

Arguing as in (2.12), we get, by using (3.1) (instead of (2.10)), that

$$(3.7) \quad \nabla_A(g_y u) = g_y(\nabla_{A_y(\cdot+y)} u) \quad \text{and} \quad (g_y)^{-1}(\nabla_A u) = \nabla_{A_y(\cdot+y)}((g_y)^{-1} u).$$

So, arguing as in (2.14), we get

$$\langle g_y u, g_y v \rangle_A = \langle u, v \rangle_{A_y(\cdot+y)} \quad \text{and} \quad \langle (g_y)^{-1} u, (g_y)^{-1} v \rangle_{A_y(\cdot+y)} = \langle u, v \rangle_A,$$

and, in particular, that

$$(3.8) \quad E_A(g_y u) = E_{A_y(\cdot+y)}(u) \quad \text{and} \quad E_A(u) = E_{A_y(\cdot+y)}((g_y)^{-1} u).$$

In next lemma we prove that, given a bounded continuous magnetic field B , all corrected magnetic potentials A_y (which vanish at y) satisfy a bound of linear type.

Lemma 3.4. *Let $B = dA$ be a bounded magnetic field with $A \in C_{\text{loc}}^1(\mathbb{R}^N, \Lambda_1)$. Let the functions φ_y , $y \in \mathbb{R}^N$ be as in Remark 3.2 and let A_y be the corrected potential defined by (3.1). Then for all $y \in \mathbb{R}^N$,*

$$(3.9) \quad |A_y(x)| \leq \|B\|_\infty |x - y| \quad \forall x \in \mathbb{R}^N,$$

where

$$(3.10) \quad \|B\|_\infty^2 \stackrel{\text{def}}{=} \sum_{n=1}^N \sum_{m < n} \|B_{mn}\|_\infty^2,$$

and

$$(3.11) \quad B_{mn} = \partial_n A_m - \partial_m A_n \quad \forall m, n \in \{1, \dots, N\}.$$

Proof. Consider, for a fixed $y \in \mathbb{R}^N$, the form A_y given by (3.1) and Lemma 3.1. We shall prove (3.10) by showing first that (with the convention that the sum over an empty set of indices is zero) the following estimate holds true

$$(3.12) \quad |(A_y)_n(x)| \leq \sum_{m=1}^{n-1} \|B_{mn}\|_\infty |x_m - y_m|, \quad x \in \mathbb{R}^N.$$

By using (3.5) (with n replaced by $n-1$) and (3.11), we have

$$\begin{aligned} (A_y)_n(\underline{Y}_{n-2}, \vec{X}^{n-1}) &= (A_y)_n(\underline{Y}_n, \vec{X}^{n+1}) + \int_{y_{n-1}}^{x_{n-1}} \partial_{n-1}(A_y)_n(\underline{Y}_{n-2}, t, \vec{X}^n) dt \\ &= \int_{y_{n-1}}^{x_{n-1}} \partial_{n-1}(A_y)_n(\underline{Y}_{n-2}, t, \vec{X}^n) dt \\ &\leq - \int_{y_{n-1}}^{x_{n-1}} \partial_n A_{n-1}(\underline{Y}_{n-2}, t, \vec{X}^n) dt \\ &\quad + \|B_{n-1\ n}\|_\infty |x_{n-1} - y_{n-1}| \\ &= \|B_{n-1\ n}\|_\infty |x_{n-1} - y_{n-1}|. \end{aligned}$$

This argument can be repeated for $-(A_y)_n(\underline{Y}_{n-1}, \vec{X}^n)$ yielding the same upper bound for $|(A_y)_n(\underline{Y}_{n-1}, \vec{X}^n)|$.

Repeating the same argument for $(A_y)_n(\underline{Y}_{n-3}, \vec{X}^{n-2})$ while using integration with respect to x_{n-2} , we get

$$\begin{aligned} |(A_y)_n(\underline{Y}_{n-3}, \vec{X}^{n-2})| &\leq |(A_y)_n(\underline{Y}_{n-1}, \vec{X}^n)| + \|B_{n-2\ n}\|_\infty |x_{n-2} - y_{n-2}| \\ &\leq \|B_{n-2\ n}\|_\infty |x_{n-2} - y_{n-2}| + \|B_{n-1\ n}\|_\infty |x_{n-1} - y_{n-1}|. \end{aligned}$$

Iterating the same estimate $n-1$ times one gets (3.12) and then (3.10) follows from Cauchy inequality. Indeed, setting $a_n \stackrel{\text{def}}{=} |x_n - y_n|$ and $b_{mn} \stackrel{\text{def}}{=} \|B_{mn}\|_\infty$,

we get by (3.10),

$$\begin{aligned} \sum_{n=1}^N |(A_y)_n|^2 &\leq \sum_{n=1}^N \left(\sum_{m < n} b_{mn} a_m \right)^2 \leq \sum_{n=1}^N \left(\sum_{m < n} b_{mn}^2 \sum_{i < n} a_i^2 \right) \\ &\leq \left(\sum_{i=1}^N a_i^2 \right) \sum_{n=1}^N \sum_{m < n} b_{mn}^2 = |x - y|^2 \|B\|_\infty^2. \end{aligned}$$

□

In what follows $\dot{C}^1(\mathbb{R}^N)$ denotes the space of functions with uniformly bounded derivatives and $\dot{C}^{0,1}(\mathbb{R}^N)$ denotes the space of functions satisfying $|f(x) - f(y)| \leq C|x - y|$ for all $x, y \in \mathbb{R}^N$.

Remark 3.5. Let $Y = (y_k)_{k \in \mathbb{N}}$ be a diverging sequence in $\mathbb{R}^N$. If $A \in \dot{C}^1(\mathbb{R}^N, \Lambda_1)$, then applying Arzelà-Ascoli theorem to the sequence $(A_{y_k}(\cdot + y_k))_{k \in \mathbb{N}}$, we get a renamed subsequence such that $(A_{y_k}(\cdot + y_k))_{k \in \mathbb{N}}$ and $(dA_{y_k}(\cdot + y_k))_{k \in \mathbb{N}}$ converges, uniformly on bounded sets, respectively to some Lipschitz function $A_\infty^{(Y)} \in \dot{C}^{0,1}(\mathbb{R}^N, \Lambda_1)$ and a bounded function $B_\infty^{(Y)} = dA_\infty^{(Y)} \in L_\infty^\infty(\mathbb{R}^N, \Lambda_1)$.

Definition 3.6. Let $Y = (y_k)_{k \in \mathbb{N}}$ be a diverging sequence in $\mathbb{R}^N$, and consider the renamed subsequence $(y_k)_{k \in \mathbb{N}}$ and the associated magnetic potential $A_\infty^{(Y)} \in \dot{C}_\text{loc}^{0,1}(\mathbb{R}^N, \Lambda_1)$ described in Remark 3.5. Then, the corresponding functional $E_{A_\infty^{(Y)}}$ (see (2.2)) will be called *the energy at infinity relative to the renamed subsequence* $Y = (y_k)_{k \in \mathbb{N}}$, corresponding to the *magnetic field at infinity* $B_\infty^{(Y)} = dA_\infty^{(Y)}$.

We end this section by defining sequences without concentrations and, to this aim, we shall fix a discretization which shall play the role of $\mathbb{Z}^N$ in the periodic setting.

Definition 3.7. A set $\Xi \subset \mathbb{R}^N$ is called a discretization of $\mathbb{R}^N$ if $0 \in \Xi$,

$$\inf_{x, y \in \Xi, x \neq y} |x - y| > 0,$$

and if, for some $\rho > 0$, $\{B_\rho(x)\}_{x \in \Xi}$ is a covering of $\mathbb{R}^N$ of uniformly finite multiplicity.

Note that if Ξ is a discretization of $\mathbb{R}^N$ then, for any $R > \rho$, the covering $\{B_R(x)\}_{x \in \Xi}$ is still of uniformly finite multiplicity. A trivial example for a discretization of $\mathbb{R}^N$ is $\mathbb{Z}^N$.

Definition 3.8. Let $\Xi \subset \mathbb{R}^N$ be a discretization of $\mathbb{R}^N$ and let

$$(3.13) \quad \mathcal{G}_\Xi \stackrel{\text{def}}{=} \{g_y : u \mapsto e^{i\varphi_y(\cdot)} u(\cdot - y) \mid y \in \Xi \text{ and } \varphi_y \text{ as in Remark (3.2)}\}.$$

We shall say that a sequence $(u_k)_{k \in \mathbb{N}}$ in $H_A^{1,2}(\mathbb{R}^N)$ $\mathcal{G}/Xi$ -converges to zero or is $\mathcal{G}/Xi$ -infinitesimal, writing $u_k \xrightarrow{\mathcal{G}} 0$, if for every sequence $(g_k)_{k \in \mathbb{N}} \subset \mathcal{G}_\Xi$, $g_k^{-1} u_k \rightharpoonup 0$ in $H_\text{loc}^{1,2}(\mathbb{R}^N)$.

Note that, by Corollary 2.2, the weak convergence above is also in $H_{A,\text{loc}}^{1,2}(\mathbb{R}^N)$ for any $A \in C_{\text{loc}}(\mathbb{R}^N, \Lambda_1)$ (including $A \equiv 0$).

Remark 3.9. It is not difficult to show that $\mathcal{G}_\Xi$ -convergence is independent of discretization Ξ (cf. Remark 5.5 below for the non-magnetic case), but will not use this property in any subsequent argument.

4. A VANISHING LEMMA OF COCOMPACTNESS TYPE

In order to describe defect of compactness for bounded sequences in $H_A^{1,2}(\mathbb{R}^N)$ we will first characterize the behavior of sequences without concentrations, i.e. of $\mathcal{G}$ -infinitesimal sequences (see Definition 3.8). We start with a statement about local boundedness.

Lemma 4.1. *Let $N \geq 3$, $A \in C_{\text{loc}}(\mathbb{R}^N, \Lambda_1)$, let $(u_k)_{k \in \mathbb{N}}$ be a bounded sequence in $H_A^{1,2}(\mathbb{R}^N)$. Then, for any sequence $(y_k)_{k \in \mathbb{N}}$ in $\mathbb{R}^N$, the sequence $(g_{y_k}^{-1}u_k)_{k \in \mathbb{N}}$ is bounded in $H_{\text{loc}}^{1,2}(\mathbb{R}^N)$.*

Proof. The local L^2 -bound follows from the Sobolev inequality and the diamagnetic inequality. Indeed, fixed $R > 0$, there exists a constant $S(R)$, such that

$$\begin{aligned}
 \int_{B_R(0)} |g_{y_k}^{-1}u_k|^2 dx &= \int_{B_R(y_k)} |u_k|^2 dx \\
 &\leq S(R) \left(\int_{B_R(y_k)} |u_k|^{2^*} dx \right)^{\frac{2}{2^*}} \\
 &\leq S(R) \left(\int_{\mathbb{R}^N} |u_k|^{2^*} dx \right)^{\frac{2}{2^*}} \\
 &\leq \tilde{S}(R) \int_{\mathbb{R}^N} |\nabla |u_k||^2 dx \leq \tilde{S}(R) E_A(u_k).
 \end{aligned}
 \tag{4.1}$$

Now, by taking into account (3.8) we get, for any $R > 0$, that

$$E_A(u_k) = E_{A_{y_k}(\cdot + y_k)}(g_{y_k}^{-1}u_k) \geq \int_{B_R(0)} |\nabla_{A_{y_k}(\cdot + y_k)} g_{y_k}^{-1}u_k|^2 dx,$$

and, by applying (2.6) (with A and u replaced by $A_{y_k}(\cdot + y_k)$ and $g_{y_k}^{-1}u_k$ respectively), we deduce that

$$\begin{aligned}
 E_A(u_k) &\geq \frac{1}{2} \int_{B_R(0)} |\nabla g_{y_k}^{-1}u_k|^2 dx - 7 \int_{B_R(0)} |A|^2 |g_{y_k}^{-1}u_k|^2 dx \\
 &\geq \frac{1}{2} \int_{B_R(0)} |\nabla g_{y_k}^{-1}u_k|^2 dx - 7C(R) \int_{B_R(0)} |g_{y_k}^{-1}u_k|^2 dx \\
 &\geq \frac{1}{2} \int_{B_R(0)} |\nabla g_{y_k}^{-1}u_k|^2 dx - 7C(R) \tilde{S}(R) E_A(u_k),
 \end{aligned}$$

where, to get last two inequalities we have used (3.9) and (4.1). So, by combining the above inequalities, we get that for a suitable positive constant $\tilde{C}(R)$

$$\int_{B_R(0)} |\nabla g_{y_k}^{-1} u_k|^2 dx \leq \tilde{C}(R) E_A(u_k).$$

□

Lemma 4.2. *For any bounded sequence $(u_k)_{k \in \mathbb{N}}$ in $H_A^{1,2}(\mathbb{R}^N)$, $A \in C_{\text{loc}}(\mathbb{R}^N, \Lambda_1)$, the following implication holds true*

$$u_k \xrightarrow{\mathcal{G}} 0 \quad \Rightarrow \quad u_k \rightarrow 0 \text{ in } L^p(\mathbb{R}^N) \text{ for any } p \in (2, 2^*).$$

Proof. Let $p \in (2, 2^*)$, Ξ be a discretization of $\mathbb{R}^N$, and fix $y \in \Xi$. By combining the Sobolev inequality on an open ball $B_\rho(y)$, with the diamagnetic inequality (2.8), we get, with some constant $C > 0$ independent of y ,

$$\begin{aligned} \int_{B_\rho(y)} |u_k|^p dx &= \left(\int_{B_\rho(y)} |u_k|^p dx \right)^{2/p} \left(\int_{B_\rho(y)} |u_k|^p dx \right)^{1-2/p} \\ &\leq C \int_{B_\rho(y)} (|\nabla |u_k||^2 + |u_k|^2) dx \left(\int_{B_\rho(y)} |u_k|^p dx \right)^{1-2/p} \\ &\leq C \int_{B_\rho(y)} (|\nabla_A u_k|^2 + |u_k|^2) dx \left(\int_{B_\rho(y)} |u_k|^p dx \right)^{1-2/p} \\ &\leq C \int_{B_\rho(y)} (|\nabla_A u_k|^2 + |u_k|^2) dx \sup_{z \in \Xi} \left(\int_{B_\rho(z)} |u_k|^p dx \right)^{1-2/p}. \end{aligned}$$

Adding over all $y \in \Xi$, and taking into account that the covering $\{B_\rho(y)\}_{y \in \Xi}$ has uniform finite multiplicity, we have, for a suitable choice of $y_k \in \Xi$ (and with a change of variable), that

$$\begin{aligned} \int_{\mathbb{R}^N} |u_k|^p dx &\leq C \|u_k\|_{H_A^{1,2}(\mathbb{R}^N)} \sup_{y \in \Xi} \left(\int_{B_\rho(y)} |u_k|^p dx \right)^{1-2/p} \\ &\leq 2C \|u_k\|_{H_A^{1,2}(\mathbb{R}^N)} \left(\int_{B_\rho(y_k)} |u_k|^p dx \right)^{1-2/p} \\ &= 2C \|u_k\|_{H_A^{1,2}(\mathbb{R}^N)} \left(\int_{B_\rho(0)} |u_k(x + y_k)|^p dx \right)^{1-2/p}. \end{aligned}$$

Since, by assumption, $(g_{y_k}^{-1} u_k)_{k \in \mathbb{N}}$ converges weakly to zero in $H^{1,2}(B_\rho(0))$ which is compactly embedded in $L^p(B_\rho(0))$, we get that $u_k(\cdot + y_k) \rightarrow 0$ in $L^p(B_\rho(0))$ and, by the chain of inequalities above, that $(u_k)_{k \in \mathbb{N}}$ vanishes in $L^p(\mathbb{R}^N)$. □

Note that, despite Corollary 2.2, spaces $H_A^{1,2}(\mathbb{R}^N)$ and $\dot{H}_A^{1,2}(\mathbb{R}^N)$ do not generally coincide, in particular, they are distinct if $B = dA = 0$. In general, unless B is identically zero in $\mathbb{R}^N$ and $N = 2$, there exists a positive function W such that $E_A(u) \geq \int_{\mathbb{R}^N} |u|^2 W(x) dx$ (see e.g. [16]), but W may be not necessarily bounded from below. If dA is lattice-periodic, it is easy to show, via partition of unity, that $E_A(u) \geq C \int_{\mathbb{R}^N} |u|^2 dx$ so that, if $u \in \dot{H}_A^{1,2}(\mathbb{R}^N)$, then $u \in L^2(\mathbb{R}^N)$, i.e. $H_A^{1,2}(\mathbb{R}^N) = \dot{H}_A^{1,2}(\mathbb{R}^N)$. For sequences bounded in $\dot{H}_A^{1,2}(\mathbb{R}^N)$, but not bounded in $L^2(\mathbb{R}^N)$, one can prove an analog of Lemma 4.2.

Lemma 4.3. *Assume that $A \in C_{\text{loc}}(\mathbb{R}^N, \Lambda_1)$ and that for some $r > 0$, $B = dA \not\equiv 0$ on any ball $B_r(x)$, $x \in \mathbb{R}^N$ (so that there is no sequence of balls of radius going to infinity where $dA \equiv 0$). Then, for any $p \in (2, 2^*)$, there is a positive continuous bounded function W_p defined on $\mathbb{R}^N$ such that*

$$u_k \xrightarrow{\mathcal{G}} 0 \text{ in } \dot{H}_{\text{loc}}^{1,2}(\mathbb{R}^N) \quad \Rightarrow \quad u_k \rightarrow 0 \text{ in } L^p(\mathbb{R}^N, W_p dx).$$

Proof. The proof follows the same argument used for Lemma 4.2 with the following elementary modifications. The radius of the balls $B_\rho(y)$, $y \in \Xi$, have to be changed to $R = \max(2\rho, r)$, so that Sobolev inequality on the ball $B_R(y)$, $y \in \Xi$ (with a constant *dependent* on y) will follow from the assumption $dA \not\equiv 0$ on $B_r(y)$. This constant, after summation over Ξ (recall that the covering by $B_R(y)$ remains of finite multiplicity) will produce a piecewise-constant weight in the L^p -norm, which can be replaced by a continuous function W_p . $\square$

5. PROFILE DECOMPOSITION

The purpose of this section is to prove the profile decomposition stated in Theorem 5.1 below with the related energy bounds stated in Theorem 5.2.

When we shall deal with sequences $Y^{(n)} \stackrel{\text{def}}{=} (y_k^{(n)})_{k \in \mathbb{N}}$ in $\mathbb{R}^N$ depending on a parameter $n \in \mathbb{N}$, we will abbreviate the notation for the magnetic potentials at infinity $A_\infty^{(Y^{(n)})}$ as $A_\infty^{(n)}$ (and the corresponding magnetic field as $B_\infty^{(n)}$). Moreover, for any n and for any k , $g_{y_k^{(n)}}$ will represent the magnetic shift defined by (3.6) with $y = y_k^{(n)}$, and we will use the abbreviated notation

$$(5.1) \quad g_k^{(n)} \stackrel{\text{def}}{=} g_{y_k^{(n)}} \quad \text{as well as} \quad A_k^{(n)} \stackrel{\text{def}}{=} A_{y_k^{(n)}}(\cdot + y_k^{(n)}).$$

We warn the reader that we shall always reserve the index $n = 0$ to the trivial sequence $Y^{(0)} \stackrel{\text{def}}{=} (y_k^{(0)})_{k \in \mathbb{N}} = (0)_{k \in \mathbb{N}}$ (which, of course is not diverging) and always set $A_k^{(0)} = A_\infty^{(0)} = A$ (indeed, the magnetic shifts $g_k^{(0)} = g_{y_k^{(0)}}$ coincide with the identity map (see Remark 3.2)). Moreover, when $(u_k)_{k \in \mathbb{N}}$ is bounded, we set (modulo subsequences)

$$(5.2) \quad v^{(0)} \stackrel{\text{def}}{=} w - \lim_{k \rightarrow \infty} u_k.$$

In what follows we use weak convergence in $H_{\text{loc}}^{1,2}(\mathbb{R}^N)$, which is a weak convergence in a Fréchet space. In particular weakly converging sequences will converge in the sense of distributions, strongly in $L^p(\Omega)$ whenever $\Omega \subset \mathbb{R}^N$ is a bounded measurable set and $p \in (2, 2^*)$, as well as almost everywhere.

Theorem 5.1. *Let $\Xi \ni 0$ be a discretization of $\mathbb{R}^N$. Let $A \in \dot{C}^1(\mathbb{R}^N, \Lambda_1)$, $N \geq 3$, and let $(u_k)_{k \in \mathbb{N}}$ be a bounded sequence in $H_A^{1,2}(\mathbb{R}^N)$. Then, there exist $v^{(n)} \in H_{\text{loc}}^{1,2}(\mathbb{R}^N)$ (with $v^{(0)} \in H_A^{1,2}(\mathbb{R}^N)$), $Y^{(n)} := (y_k^{(n)})_{k \in \mathbb{N}} \subset \Xi$, $n \in \mathbb{N}$, such that, on a renamed subsequence,*

$$(5.3) \quad (g_k^{(n)})^{-1} u_k \rightharpoonup v^{(n)} \quad \text{in } H_{\text{loc}}^{1,2}(\mathbb{R}^N),$$

$$(5.4) \quad |y_k^{(n)} - y_k^{(m)}| \rightarrow \infty \text{ for } n \neq m,$$

$$(5.5) \quad r_k \stackrel{\text{def}}{=} u_k - \sum_{n=0}^{\infty} g_k^{(n)} v^{(n)} \rightarrow 0 \text{ in } L^p(\mathbb{R}^N), \forall p \in (2, 2^*),$$

and the series $\sum_{n=0}^{\infty} |v^{(n)}(\cdot - y_k^{(n)})|$ and the series in (5.5) converge unconditionally (with respect to n) and uniformly in k , in $H^{1,2}(\mathbb{R}^N)$ and in $H_{\text{loc}}^{1,2}(\mathbb{R}^N)$ respectively. Moreover, for any n ,

$$(5.6) \quad g_{y_k}^{-1} u_k \rightharpoonup 0 \text{ in } H_{\text{loc}}^{1,2}(\mathbb{R}^N) \quad \text{for all } (y_k)_{k \in \mathbb{N}} \text{ in } \mathbb{R}^N \text{ s.t. } |y_k - y_k^{(n)}| \rightarrow +\infty.$$

Note that, since $(y_k^{(0)})_{k \in \mathbb{N}} = (0)_{k \in \mathbb{N}}$, formula (5.4) implies that each sequence $(y_k^{(n)})_{k \in \mathbb{N}} = (0)_{k \in \mathbb{N}}$ is diverging for every $n \neq 0$, while (5.5) gives the profile decomposition

$$(5.7) \quad u_k = v^{(0)} + \sum_{n=1}^{\infty} g_k^{(n)} v^{(n)} + r_k \quad \text{with } r_k \rightarrow 0 \text{ in } L^p(\mathbb{R}^N), \forall p \in (2, 2^*).$$

Theorem 5.2. *Assume conditions of Theorem 5.1. A subsequence $(u_k)_{k \in \mathbb{N}}$ in $H_A^{1,2}(\mathbb{R}^N)$ provided by Theorem 5.1 satisfies*

$$(5.8) \quad \int_{\mathbb{R}^N} |u_k|^p dx \rightarrow \sum_{n=0}^{\infty} \int_{\mathbb{R}^N} |v^{(n)}|^p dx \quad \text{as } k \rightarrow +\infty \forall p \in [2, 2^*),$$

$$(5.9) \quad \sum_{n=0}^{\infty} \int_{\mathbb{R}^N} |v^{(n)}|^2 dx \leq \liminf_{k \rightarrow \infty} \int_{\mathbb{R}^N} |u_k|^2 dx,$$

and furthermore,

$$(5.10) \quad \sum_{n=0}^{\infty} E_{A_{\infty}^{(n)}}(v^{(n)}) \leq \liminf_{k \rightarrow \infty} E_A(u_k),$$

where $A_{\infty}^{(0)} = A$, and for $n \neq 0$, each $A_{\infty}^{(n)}$ is the magnetic potential at infinity relative to the sequence $Y^{(n)} = (y_k^{(n)})_{k \in \mathbb{N}}$ as in Definition 3.6 and $B_{\infty}^{(n)} = dA_{\infty}^{(n)}$ is the related magnetic field.

Before giving the proof of Theorem 5.1 we like to show that relation (5.3) is at all possible.

Lemma 5.3. *Let $A \in \dot{C}^1(\mathbb{R}^N, \Lambda_1)$, $N \geq 3$, let $(u_k)_{k \in \mathbb{N}}$ be a bounded sequence in $\dot{H}_A^{1,2}(\mathbb{R}^N)$, and let $Y = (y_k)_{k \in \mathbb{N}}$ be a sequence in $\mathbb{R}^N$. Then, $(g_{y_k}^{-1}u_k)_{k \in \mathbb{N}}$ has a subsequence weakly convergent in $H_{\text{loc}}^{1,2}(\mathbb{R}^N)$ to some v in $H_{\text{loc}}^{1,2}(\mathbb{R}^N)$, such that, $(A_{y_k}(\cdot + y_k))_{k \in \mathbb{N}}$ converges uniformly on bounded sets to some $A_\infty \in \dot{C}^{0,1}(\mathbb{R}^N, \Lambda_1)$ such that*

$$(5.11) \quad E_{A_\infty}(v) \leq \liminf_{k \rightarrow \infty} E_A(u_k).$$

Proof. To shorten notation we shall set, for any $k \in \mathbb{N}$,

$$A_k \stackrel{\text{def}}{=} A_{y_k}(\cdot + y_k) \quad \text{and} \quad v_k \stackrel{\text{def}}{=} g_{y_k}^{-1}u_k.$$

By Lemma 4.1 the sequence $(v_k)_{k \in \mathbb{N}}$ is bounded in $H_{\text{loc}}^{1,2}(\mathbb{R}^N)$ and thus it has a subsequence weakly convergent to some $v \in H_{\text{loc}}^{1,2}(\mathbb{R}^N)$. Consider a (renamed) subsequence, given by the Arzelà-Ascoli theorem, such that $(A_k)_{k \in \mathbb{N}}$ converges uniformly on bounded sets to $A_\infty \stackrel{\text{def}}{=} A_\infty^{(Y)}$ (see Definition 3.6). Set for any $R > 0$, $\Omega_R \stackrel{\text{def}}{=} B_R(0)$, by applying (3.7), we have

$$\begin{aligned} E_A(u_k) &= E_A(g_{y_k} v_k) \geq \int_{\Omega_R} |\nabla_{A_{y_k}(\cdot + y_k)}(g_{y_k}^{-1}u_k)|^2 dx \\ &= \int_{\Omega_R} |\nabla_{A_k}(v_k)|^2 dx \quad \text{for all } R > 0. \end{aligned}$$

Now, let us remark that

$$\begin{aligned} |\nabla_{A_k} v_k|^2 &= |\nabla v_k + \mathbf{i}A_\infty v_k - \mathbf{i}(A_\infty - A_k)v_k|^2 = |\nabla_{A_\infty} v_k - \mathbf{i}(A_\infty - A_k)v_k|^2 \\ &= |\nabla_{A_\infty} v_k|^2 + \mathbf{i}(A_\infty - A_k) (\overline{v_k} \nabla_{A_\infty} v_k - v_k \overline{\nabla_{A_\infty} v_k}) + |A_\infty - A_k|^2 |v_k|^2, \end{aligned}$$

and that

$$\overline{v_k} \nabla_{A_\infty} v_k - v_k \overline{\nabla_{A_\infty} v_k} = \overline{v_k} \nabla v_k - v_k \nabla \overline{v_k} + 2\mathbf{i}A_\infty |v_k|^2.$$

So, we deduce the following relation

$$\begin{aligned} |\nabla_{A_k} v_k|^2 &= |\nabla_{A_\infty} v_k|^2 - 2A_\infty(A_\infty - A_k)|v_k|^2 + |A_\infty - A_k|^2 |v_k|^2 \\ &\quad + \mathbf{i}(A_\infty - A_k) (\overline{v_k} \nabla v_k - v_k \nabla \overline{v_k}), \end{aligned}$$

and, by arguing as in (2.4), we get (by applying Cauchy-Schwarz Inequality)

$$|\nabla_{A_k} v_k|^2 \geq |\nabla_{A_\infty} v_k|^2 - 2|A_\infty||A_\infty - A_k||v_k|^2 + |A_\infty - A_k|^2 |v_k|^2 - 4|A_\infty - A_k||v_k| |\nabla v_k|.$$

Then, setting $\alpha_k \stackrel{\text{def}}{=} \|A_k\|_{L^\infty(\Omega_R)}$ and $\delta_k \stackrel{\text{def}}{=} \|A_\infty - A_k\|_{L^\infty(\Omega_R)}$ we get, by Hölder Inequality, that

$$\begin{aligned} E_A(u_k) &\geq \int_{\Omega_R} |\nabla_{A_\infty} v_k|^2 dx - 2\alpha_k \delta_k \int_{\Omega_R} |v_k|^2 dx + \delta_k \int_{\Omega_R} |v_k|^2 dx \\ &\quad - 4\delta_k \|v_k\|_{L^2(\Omega_R)} \|\nabla v_k\|_{L^2(\Omega_R)} \\ &= \int_{\Omega_R} |\nabla_{A_\infty} v_k|^2 dx + o(1) \end{aligned}$$

where we have taken into account that $\delta_k \rightarrow 0$ and the existence of the local bound in $H^{1,2}$ for the sequence $(v_k)_{k \in \mathbb{N}}$ according to Lemma 4.1. So, by weak lower semicontinuity of seminorms, (5.11) follows by taking $R \rightarrow \infty$. $\square$

We will use the following statement [26, Corollary 3.3] (which can also be derived from Solimini's result [24, Theorem 2], with the addition of the property on the unconditional convergence of the series as underlined in the Banach space version of the same theorem in [25]).

Proposition 5.4. *Let $(u_k)_{k \in \mathbb{N}}$ be a bounded sequence in $H^{1,2}(\mathbb{R}^N)$. Then, for all $n \in \mathbb{N}$, there exist $w^{(n)} \in H^{1,2}(\mathbb{R}^N)$, $(y_k^{(n)})_{k \in \mathbb{N}}$ in $\mathbb{Z}^N$, such that, on a renamed subsequence,*

$$(5.12) \quad u_k(\cdot + y_k^{(n)}) \rightharpoonup w^{(n)},$$

$$(5.13) \quad |y_k^{(n)} - y_k^{(m)}| \rightarrow \infty \text{ for } n \neq m,$$

$$(5.14) \quad \sum_{n \in \mathbb{N}} \|w^{(n)}\|_{H^{1,2}}^2 \leq \limsup_{k \rightarrow \infty} \|u_k\|_{H^{1,2}}^2,$$

$$(5.15) \quad u_k - \sum_{n \in \mathbb{N}} w^{(n)}(\cdot - y_k^{(n)}) \rightarrow 0 \text{ in } L^p(\mathbb{R}^N), \forall p \in (2, 2^*),$$

and the series in (5.15) converges in $H^{1,2}(\mathbb{R}^N)$ unconditionally (with respect to n) and uniformly in k . Moreover,

$$(5.16) \quad u_k(\cdot + y_k) \rightharpoonup 0 \quad \text{for all } (y_k)_{k \in \mathbb{N}} \text{ in } \mathbb{R}^N \text{ s.t. } |y_k - y_k^{(n)}| \rightarrow \infty, \forall n \in \mathbb{N}.$$

Remark 5.5. One can replace $\mathbb{Z}^N$ in Proposition 5.4 with any other discretization Ξ of $\mathbb{R}^N$. Indeed, for any sequence $(y_k^{(n)})_{k \in \mathbb{N}}$ in $\mathbb{Z}^N$ there exists a sequence $(\bar{y}_k^{(n)})_{k \in \mathbb{N}}$ in Ξ , such that $(\bar{y}_k^{(n)} - y_k^{(n)})_{k \in \mathbb{N}}$ is bounded. Then, passing to a renamed subsequence and by using standard diagonalization, we may assume that $(\bar{y}_k^{(n)} - y_k^{(n)})_{k \in \mathbb{N}}$ converges to some point $z_n \in \mathbb{R}^N$, and all assertions of Proposition 5.4 will hold with $\bar{y}_k^{(n)}$ and $\bar{w}^{(n)} \stackrel{\text{def}}{=} w^{(n)}(\cdot + z_n)$ replacing $y_k^{(n)}$ and $w^{(n)}$ respectively.

PROOF THEOREM 5.1. Let $(u_k)_{k \in \mathbb{N}}$ be a bounded sequence in $H_A^{1,2}(\mathbb{R}^N)$, then, by (2.8), we can apply Proposition 5.4, appended by Remark 5.5, to the sequence $(|u_k|)_{k \in \mathbb{N}}$. Denoting the magnetic shift $g_{y_k^{(n)}}$ determined by $y_k^{(n)}$

as $g_k^{(n)}$ (see Definition 3.3 and (5.1)) and the magnetic potential $A_{y_k^{(n)}}(\cdot + y_k^{(n)})$ as $A_k^{(n)}$, we have, by Lemma 5.3, that, on a renamed subsequence, $\left((g_k^{(n)})^{-1}u_k\right)_{k \in \mathbb{N}}$ converges weakly in $H_{\text{loc}}^{1,2}(\mathbb{R}^N)$ to some $v^{(n)} \in H_{\text{loc}}^{1,2}(\mathbb{R}^N)$, i.e.

$$v^{(n)} \stackrel{\text{def}}{=} w - \lim_{k \rightarrow \infty} (g_k^{(n)})^{-1}u_k,$$

while $A_k^{(n)} \rightarrow A_\infty^{(n)} \in C^{0,1}$ uniformly on bounded sets. It follows from the equality $|g_k^{(n)}u_k| = |u_k(\cdot + y_k^{(n)})|$ and from (5.12) that $|v^{(n)}| = w^{(n)}$ for all n . Furthermore, by taking into account (5.16), we have, for the same reason as above, that $g_{y_k}u_k \rightarrow 0$ whenever $|y_k - y_k^{(n)}| \rightarrow \infty$.

It easily follows from convergence properties of the series in (5.15) that the series in (5.5) converges unconditionally (with respect to n) and uniformly in k in $H_{\text{loc}}^{1,2}(\mathbb{R}^N)$.

Now, we shall prove (5.5) by means of Lemma 4.2, i.e. by proving that the sequence $(r_k)_{k \in \mathbb{N}}$ (therein defined) $\mathcal{G}$ -converges to zero. So, given a sequence $(y_k)_{k \in \mathbb{N}}$ in Ξ , we have to prove that, for the corresponding sequence of magnetic shifts $(g_{y_k})_{k \in \mathbb{N}}$, we have $g_{y_k}^{-1}r_k \rightharpoonup 0$ in $H_{\text{loc}}^{1,2}(\mathbb{R}^N)$. Actually, we have to face two cases.

If (first case), for all n , $|y_k - y_k^{(n)}| \rightarrow +\infty$ as $k \rightarrow +\infty$, we have, by (5.6), that not only $g_{y_k}u_k \rightharpoonup 0$, but also each term in the series of concentrations vanishes as well, i.e. $g_{y_k}g_k^{(n)}v^{(n)} \rightharpoonup 0$, so $g_{y_k}r_k \rightharpoonup 0$.

If otherwise (second case), $(y_k)_{k \in \mathbb{N}}$ in $\mathbb{R}^N$ is such that $(|y_k - y_k^{(n)}|)_{k \in \mathbb{N}}$ is bounded for some $n \in \mathbb{N}$, then passing to a renamed subsequence we may assume that $y_k - y_k^{(n)} \rightarrow \bar{y}^{(n)}$ with some $\bar{y}^{(n)} \in \mathbb{R}^N$. Note that, in such a case, $|g_{y_k}(g_k^{(n)})^{-1}u| \rightarrow |u(\cdot - \bar{y}^{(n)})|$ in $H^{1,2}(\mathbb{R}^N)$ and that the same is true to their adjoint operators as well. Then, denoting by $o^w(1)$ a sequence weakly convergent to zero, we have

$$\begin{aligned} \left| g_{y_k} \left(u_k - (g_k^{(n)})^{-1}v^{(n)} \right) \right| &= \left| g_{y_k} (g_k^{(n)})^{-1} (g_k^{(n)}u_k - v^{(n)}) \right| \\ &= \left| \left(g_k^{(n)}u_k - v^{(n)} \right) (\cdot - \bar{y}^{(n)}) \right| + o^w(1) \rightharpoonup 0, \end{aligned}$$

from which we conclude that $g_{y_k}r_k \rightharpoonup 0$. Then, by Lemma 4.2, we deduce that $r_k \rightarrow 0$ in $L^p(\mathbb{R}^N)$, for all $p \in (2, 2^*)$ getting (5.5). $\square$

PROOF OF THEOREM 5.2. The first assertion is an iterated version of Brezis-Lieb Lemma for the sequence $(|u_k|)_{k \in \mathbb{N}}$ and its profile decomposition in $H^{1,2}(\mathbb{R}^N)$ (see e.g. [12] : note that $|u_k|(\cdot - y_k^{(n)}) = |g_k^{(n)}u_k| \rightharpoonup |v^{(n)}|$ in $H^{1,2}(\mathbb{R}^N)$ since for sequences bounded in $H^{1,2}(\mathbb{R}^N)$ weak and *a.e.* convergence coincide).

Let us prove the third assertion. Following the proof of Lemma 5.3 to $v = v^{(n)}$, $y_k = y_k^{(n)}$, we have

$$(5.17) \quad \int_{B_R(y_k^{(n)})} |\nabla_A u_k|^2 dx \geq \int_{B_R(0)} |\nabla_{A_\infty^{(n)}} v^{(n)}|^2 dx + o_{k \rightarrow \infty}(1).$$

On the other hand, due to (5.13), we can assume that, for any $R > 0$, for k large enough, $B_R(y_k^{(n)}) \cap B_R(y_k^{(m)}) = \emptyset$ for all $m \neq n$, so for any $M \in \mathbb{N}$, $M \neq 0$, $E_A(u_k) \geq \sum_{n=0}^M \int_{B_R(y_k^{(n)})} |\nabla_A u_k|^2 dx$. So, by (5.17), we get

$$\liminf_{k \rightarrow \infty} E_A(u_k) \geq \sum_{n=0}^M \int_{B_R(0)} |\nabla_{A_\infty^{(n)}} v^{(n)}|^2 dx.$$

Then, (5.10) follows since M and R are arbitrary. $\square$

6. APPLICATIONS

Given a bounded electric potential V , described as a positive measurable function on $\mathbb{R}^N$, we introduce the following functional defined by setting for all $u \in H_A^{1,2}(\mathbb{R}^N)$

$$(6.1) \quad J_{A,V}(u) = \int_{\mathbb{R}^N} (|\nabla_A u(x)|^2 + V(x)|u(x)|^2) dx.$$

We shall emphasize the case in which the electric potential is constant, i.e. when $V \equiv \lambda \in \mathbb{R}$, by using the notation $J_{A,\lambda}$ instead of $J_{A,V}$.

6.1. A ground state problem. Set $\lambda_0 \stackrel{\text{def}}{=} \inf_{\|u\|_2=1} E_A(u)$, we shall introduce, for $\lambda > -\lambda_0$, the following minimization problems:

$$(6.2) \quad C(p, N, A, \lambda) \stackrel{\text{def}}{=} \inf_{\substack{u \in H_A^{1,2}(\mathbb{R}^N) \\ \|u\|_p=1}} J_{A,\lambda}(u), \quad p \in (2, 2^*).$$

We will show that, for any $p \in (2, 2^*)$, the problem (6.2) attains no minimum if the magnetic field $B = dA$ vanishes at infinity.

It is worth to remark that several known existence results (see [2], [17], [22]) prove existence of minimum when the electric potential increases at infinity and offsets the possible decrease of energy (due to the magnetic field) at infinity, penalizing (in minimizing sequences) any possible translation towards infinity. Of course, this setting excludes the constant potential $V \equiv \lambda$. Below, we show in an elementary statement, that if the magnetic field at infinity is zero and the electric potential $V \equiv \lambda$ is constant (so there is no penalty mechanism) the ground state does not exist. Note that the statement is restricted only to positive values of λ , leaving it as an elementary exercise for the reader to prove that, when $V \equiv \lambda \leq 0$, the infimum in (6.2) is zero.

Proposition 6.1. *Let $N \geq 3$ and let $B = dA$ be a bounded magnetic field with $A \in C^{1,\alpha}$ for some $\alpha \in (0, 1)$. Assume moreover that $\text{supp} B = \mathbb{R}^N$ and that $\lim_{|x| \rightarrow \infty} B(x) = 0$. Then, for all $\lambda > 0$, Problem (6.2) has no solution, in particular the infimum is positive but it is not attained.*

Proof. Nonnegativity of the infimum easily follows by the assumption $\lambda > 0 > -\lambda_0$, while positivity follows from diamagnetic inequality (2.8) and from the Sobolev embedding applied to $|u|$.

Assuming, by contradiction, that the infimum in (6.2) is attained at some function u , we get $C(N, p, A, \lambda) > C(N, p, 0, \lambda)$. Indeed, by taking into account that the diamagnetic inequality (2.8) is strict on $\text{supp} B \cap \text{supp} u$, and by using (3.9), we have

$$(6.3) \quad C(N, p, A, \lambda) = J_{A,\lambda}(u) > J_{0,\lambda}(|u|) = J_{0,\lambda}(u) \geq C(N, p, 0, \lambda).$$

On the other hand, since the constant $C(N, p, 0, \lambda)$ (see (6.2) with $A \equiv 0$) is attained by the well-known minimizer w_0 , (which decays exponentially at infinity), and since, by (3.9), we have that the magnetic potential $A_\infty^{(Y)}$ (see Definition 3.6) associated to any diverging sequence $Y \stackrel{\text{def}}{=} (y_k)_{k \in \mathbb{N}} \subset \mathbb{R}^N$ is zero, we have

$$C(N, p, A, \lambda) \leq J_{A,\lambda}(w_0(\cdot - y_k)) \rightarrow J_{0,\lambda}(w_0) = C(N, p, 0, \lambda),$$

getting in contradiction to (6.3). $\square$

6.2. A minimax problem. Let $N \geq 3$, $\lambda > 0$ and $p \in (2, 2^*)$. We consider on $H_A^{1,2}(\mathbb{R}^N)$ the following Hamiltonian functional, (see (6.1)):

$$(6.4) \quad \begin{aligned} I_{A,\lambda}(u) &= \frac{1}{2} J_{A,\lambda}(u) - \frac{1}{p} \int_{\mathbb{R}^N} |u|^p dx \\ &= \frac{1}{2} (E_A(u) + \lambda \|u\|_2^2) - \frac{1}{p} \|u\|_p^p, \end{aligned}$$

Since in this section we shall deal with the case in which the magnetic potential A vanishes at infinity, we shall set

$$(6.5) \quad \begin{aligned} I_\infty(u) &\stackrel{\text{def}}{=} I_{0,\lambda}(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u(x)|^2 + \lambda |u(x)|^2) dx - \frac{1}{p} \int_{\mathbb{R}^N} |u|^p dx \\ &= \frac{1}{2} (E_0(u) + \lambda \|u\|_2^2) - \frac{1}{p} \|u\|_p^p. \end{aligned}$$

This functional is defined on defined on $H^{1,2}(\mathbb{R}^N)$.

Note that critical points of the functional $I_{A,\lambda}$ and I_∞ are weak solutions to the problems

$$(P) \quad \begin{cases} -\nabla_A^2 u + \lambda u = |u|^{p-2} u \\ u \in H_A^{1,2}(\mathbb{R}^N), \end{cases}$$

and

$$(P_\infty) \quad \begin{cases} -\Delta u + \lambda u = |u|^{p-2}u \\ u \in H^{1,2}(\mathbb{R}^N), \end{cases}$$

respectively. In particular, since every weak nontrivial solution to (P_∞) belongs to the Nehari manifold

$$\mathcal{N}_\infty \stackrel{\text{def}}{=} \{u \in H^{1,2}(\mathbb{R}^N) \setminus \{0\} \mid J_{0,\lambda}(u) = \|\nabla u\|_2^2 + \lambda \|u\|_2^2 = \|u\|_p^p\},$$

we call ground state level of the functional I_∞ the quantity

$$c_\infty \stackrel{\text{def}}{=} \inf_{\mathcal{N}_\infty} I_\infty.$$

It is well-known that c_∞ is achieved by a unique (up to translations and change of sign) function w_∞ (to which one refers as to the ground state solution to (P_∞)) which is smooth, positive and radial decreasing with exponential decay at infinity. In particular, we have

$$(6.6) \quad J_{0,\lambda}(w_\infty) = \|\nabla w_\infty\|_2^2 + \lambda \|w_\infty\|_2^2 = \|w_\infty\|_p^p,$$

and, as a consequence,

$$c_\infty = \frac{p-2}{2p} \|w_\infty\|_p^p,$$

or, conversely,

$$(6.7) \quad \|w_\infty\|_p^p = \frac{2p}{p-2} c_\infty.$$

It is possible to see c_∞ as a min – max level. Indeed, it is standard to prove that for any $u \in H^{1,2}(\mathbb{R}^N) \setminus \{0\}$ there exists a unique scaling factor $\bar{t} > 0$ such that $\hat{u} \stackrel{\text{def}}{=} \bar{t}u \in \mathcal{N}_\infty$ and

$$I_\infty(\hat{u}) = \max_{t \in [0, \infty)} I_\infty(tu).$$

So,

$$c_\infty = I_\infty(w_\infty) = \inf_{u \in H^{1,2}(\mathbb{R}^N) \setminus \{0\}} I_\infty(\hat{u}) = \inf_{u \in H^{1,2}(\mathbb{R}^N) \setminus \{0\}} \max_{t \in [0, \infty)} I_\infty(tu).$$

Now, by taking the set $\Phi \subset C([0, \infty), H^{1,2}(\mathbb{R}^N))$ consists of all paths satisfying $\sigma(0) = 0$ and $\lim_{t \rightarrow \infty} |\sigma(t)| = +\infty$ (and, as a consequence, $\lim_{t \rightarrow \infty} I_\infty(\sigma(t)) = -\infty$) we have the following further characterization of c_∞

$$c_\infty \stackrel{\text{def}}{=} \min_{\sigma \in \Phi} \max_{t \in [0, \infty)} I_\infty(\sigma(t)).$$

Finally, it is worth to remark that the ground state w_∞ equals, up to a scalar multiple, the (radial) minimizer of $C(p, N, 0, \lambda)$ defined in (6.2), so $w_0 = \frac{w_\infty}{\|w_\infty\|_p}$, i.e.

$$J_{0,\lambda} \left(\frac{w_\infty}{\|w_\infty\|_p} \right) = C(p, N, 0, \lambda).$$

Let $A \in \dot{C}^1(\mathbb{R}^N, \Lambda_1) \setminus \{0\}$ and assume the following conditions:

(A) $B_\infty^{(Y)} = 0$ a.e. for any diverging sequence $Y \stackrel{\text{def}}{=} (y_k)_{k \in \mathbb{N}} \subset \mathbb{R}^N$, (equivalently, $\lim_{|y_k| \rightarrow \infty} A_{y_k}(\cdot + y_k) = 0$);

$$(B) \|B\|_\infty \leq \left(\frac{(2^{\frac{p-2}{p}} - 1) \|w_\infty\|_p^p}{\int_{\mathbb{R}^N} |x|^2 w_\infty^2 dx} \right)^{\frac{1}{2}}.$$

Let us define a map $\eta := (\eta_1, \dots, \eta_{N+1}) : H_A^{1,2}(\mathbb{R}^N) \rightarrow \mathbb{R}^{N+1}$ by setting:

$$(6.8) \quad \eta_i(u) = \begin{cases} \int_{\mathbb{R}^N} \frac{x_i}{1+|x|} |u|^p dx, & i = 1, \dots, N, \\ \int_{\mathbb{R}^N} |u|^p dx, & i = N+1. \end{cases}$$

Note that, since w_∞ is radially symmetric,

$$(6.9) \quad \eta_0 \stackrel{\text{def}}{=} (0_N, \frac{2p}{p-2} c_\infty) = (0_N, \|\omega_\infty\|_p^p) \in \mathbb{R}^{N+1}.$$

Let us fix $T \geq 2$ sufficiently large so that $I_\infty(Tw_\infty) < 0$, and assume that $R > 0$ is large enough (in what follows the value of R will be subjected to a finite number of restrictions). Let B_R denote the closed ball in $\mathbb{R}^N$ centered at the origin with radius $R > 0$, and let

$$(6.10) \quad \Gamma = \Gamma_{R,T} \stackrel{\text{def}}{=} \{\gamma \in C(B_R \times [0, T], \mathbb{R}^{N+1}) \mid \gamma = \gamma_0, \text{ on } \partial B_R \times [0, T]\},$$

where

$$(6.11) \quad \gamma_0(y, t) = t e^{i\varphi_y} w_\infty(\cdot - y) = t g_y w_\infty, \quad \forall (y, t) \in B_R \times [0, T],$$

and φ_y is given by Lemma 3.4 (see (3.6)). Taking η_0 as in (6.9), we have that, by construction,

$$(6.12) \quad \deg(\eta \circ \gamma, B_R \times [0, T], \eta_0) = \deg(\eta \circ \gamma_0, B_R \times [0, T], \eta_0) \neq 0 \quad \forall \gamma \in \Gamma.$$

(Note that the degree of $\eta \circ \gamma_0$ is well-defined. Indeed, by the choice of T , we have $I_\infty(Tw_\infty) < 0$ and, since w_∞ is radially symmetric, we have $\eta \circ \gamma_0(y, t) \neq 0$ whenever $y \neq 0$.) Then, provided R and T are sufficiently large, the following number is well defined

$$(6.13) \quad c = c_{R,T} \stackrel{\text{def}}{=} \inf_{\gamma \in \Gamma} \max_{y \in B_R, t \in [0, T]} I_{A,\lambda}(\gamma(y, t)).$$

Lemma 6.2. *If $A \in C^1(\mathbb{R}^N, \Lambda_1) \setminus \{0\}$, then*

$$(6.14) \quad c_{R,T} > c_\infty,$$

provided R and T are sufficiently large real numbers.

Proof. By (6.12), and by taking into account that the last component of η_0 is $\|w_\infty\|_p^p = \frac{2pc_\infty}{p-2}$, we have

$$\begin{aligned} c &\geq \inf_{\eta(u)=\eta_0} \left(\frac{1}{2} J_{A,\lambda}(u) - \frac{1}{p} \|u\|_p^p \right) \\ &= \inf_{\eta(u)=\eta_0} \frac{1}{2} J_{A,\lambda}(u) - \frac{2}{p-2} c_\infty \\ &\geq \inf_{\|u\|_p = \|w_\infty\|_p} \frac{1}{2} J_{A,\lambda}(u) - \frac{2}{p-2} c_\infty \\ &\geq \inf_{\|u\|_p = \|w_\infty\|_p} \frac{1}{2} J_{0,\lambda}(|u|) - \frac{2}{p-2} c_\infty = c_\infty, \end{aligned}$$

where, in the last equality, we have used (6.6) and (6.7). Then, the thesis follows since the diamagnetic inequality (2.8) is strict on any minimizer $\pm w_\infty(\cdot - y)$, $y \in \mathbb{R}^N$, of c_∞ (since $\text{supp } B \cap \text{supp } w_\infty = \mathbb{R}^N$). $\square$

Lemma 6.3. *If (B) holds true, then*

$$(6.15) \quad c_{R,T} < 2c_\infty,$$

provided R and T are sufficiently large real numbers.

Proof. Since γ_0 (see (6.11)) trivially belongs to the set Γ (see (6.10)), set

$$\sigma \stackrel{\text{def}}{=} \frac{\|B\|_\infty^2 \int |x|^2 w_\infty^2 dx}{\|w_\infty\|_p^p},$$

we get, by (3.9), that

$$\begin{aligned} c &\leq \sup_{y \in B_R, t \in [0, T]} \frac{1}{2} t^2 J_{A,\lambda}(g_y w_\infty) - \frac{1}{p} t^p \|w_\infty\|_p^p \\ &= \sup_{y \in B_R, t \in [0, T]} \frac{1}{2} t^2 J_{A_y(\cdot+y), \lambda}(w_\infty) - \frac{1}{p} t^p \|w_\infty\|_p^p \\ &\leq \sup_{t \in [0, T]} \frac{1}{2} t^2 J_{0,\lambda}(w_\infty) - \frac{1}{p} t^p \|w_\infty\|_p^p + t^2 \|B\|_\infty^2 \int_{\mathbb{R}^N} |x|^2 w_\infty^2 dx \\ &= \sup_{t \in [0, T]} \frac{1}{2} t^2 \|w_\infty\|_p^p - \frac{1}{p} t^p \|w_\infty\|_p^p + t^2 \sigma \|w_\infty\|_p^p \\ &= \|w_\infty\|_p^p \sup_{t \in [0, T]} \left(\frac{1}{2} t^2 (1 + \sigma) - \frac{1}{p} t^p \right) \\ &= \|w_\infty\|_p^p (1 + \sigma)^{\frac{p}{p-2}} \left(\frac{1}{2} - \frac{1}{p} \right) = c_\infty (1 + \sigma)^{\frac{p}{p-2}}, \end{aligned}$$

where, in the last equality, we have used (6.7). Then, the thesis follows since assumption (B) just implies $(1 + \sigma)^{\frac{p}{p-2}} < 2$. $\square$

Remark 6.4. Note that without assumption **(B)** one has $c_{R,T} \leq 2c_\infty + o_{R \rightarrow \infty}(1)$. This can be shown by considering the inequality

$$c_{R,T} \leq \sup_{y \in B_R, t \in [0, T]} I_{A,\lambda}(\gamma(y, t))$$

with $\gamma \in \Gamma$ defined by setting, for all $(y, t) \in B_R \times [0, T]$,

$$\gamma(y, t) = t \cos\left(\frac{\pi|y|}{2R}\right) g_{-\frac{R}{|y|}y} w_\infty + t \sin\left(\frac{\pi|y|}{2R}\right) g_{\frac{R}{|y|}y} w_\infty.$$

Theorem 6.5. Assume **(A)** and **(B)**. Then, provided $R > 0$ and $T \geq 2$ are sufficiently large, the number $c_{R,T}$ defined by (6.13) is a critical level for the functional $I_{A,\lambda}$ for any $\lambda > 0$. In particular Problem (P) admits a solution at the energy level $c_{R,T}$.

Before giving the proof of the theorem we shall discuss about profile decompositions of a Palais Smale sequence (P.S. for short) for the functional $I_{A,\lambda}$ at the level $c_{R,T}$ given by (6.13).

Let $(u_k)_{k \in \mathbb{N}}$ be a P.S. sequence for $I_{A,\lambda}$ in $H_A^{1,2}(\mathbb{R}^N)$ at the level $c = c_{R,T}$, i.e. such that $I_{A,\lambda}(u_k) \rightarrow c$ and $I'_{A,\lambda}(u_k) \rightarrow 0$ as $k \rightarrow \infty$. By applying the nowadays standard argument from [1] we deduce that the sequence $(u_k)_{k \in \mathbb{N}}$ is bounded in $H_A^{1,2}(\mathbb{R}^N)$. So we can use a profile decomposition (see (5.7)) of $(u_k)_{k \in \mathbb{N}}$ given by Theorem 5.1. Since each sequence $(y_k^{(n)})_{k \in \mathbb{N}}$ is diverging for $n \neq 0$ we deduce, by assumption **(A)**, that each magnetic potential $A_\infty^{(n)}$ associated to each sequence $(y_k^{(n)})_{k \in \mathbb{N}}$ is 0 if $n \neq 0$ (while $A_\infty^{(0)} = A$). So, (5.10) gives

$$E_A(v^{(0)}) + \sum_{n=1}^{\infty} E_0(v^{(n)}) \leq \liminf_{k \rightarrow \infty} E_A(u_k),$$

where $v^{(0)}$ and $v^{(n)}$ are defined by (5.2) and (5.3) respectively. Then, by taking into account (6.4) and (6.5), we get, by using (5.8), that

$$(6.16) \quad \begin{aligned} I_{A,\lambda}(v^{(0)}) + \sum_{n=1}^{\infty} I_\infty(v^{(n)}) &\leq \liminf_{k \rightarrow \infty} \frac{1}{2} (E_A(u_k) + \lambda \|u_k\|_2^2) - \frac{1}{p} \|u_k\|_p^p \\ &= \lim_{k \rightarrow \infty} I_{A,\lambda}(u_k) = c. \end{aligned}$$

Note that, since $I'_{A,\lambda}$ is a weak-to-weak continuous functional and $(u_k)_{k \in \mathbb{N}}$ is a Palais Smale sequence, we deduce that its weak limit $v^{(0)}$ is a solution to (P). Analogously, due to assumption **(A)**, the other profiles $v^{(n)}$, for $n \neq 0$, are solutions to (P_∞) . Therefore, either $v^{(n)}$ is trivial (actually it is not a profile) or $I_\infty(v^{(n)}) \geq c_\infty$. As a consequence, (6.16) implies that the number m of (nontrivial) profiles is finite and that

$$(6.17) \quad mc_\infty \leq I_{A,\lambda}(v^{(0)}) + mc_\infty = I_{A,\lambda}(v^{(0)}) + \sum_{n=1}^m I_\infty(v^{(n)}) = c,$$

where last equalities hold (instead of inequalities) since the profile decomposition of $(u_k)_{k \in \mathbb{N}}$ is finite and each $v^{(n)}$ is a solution to (P_∞) .

Claim 6.2.1. Assume **(A)** and **(B)**. Then,

$$(6.18) \quad c_0 \stackrel{\text{def}}{=} I_{A,\lambda}(v^{(0)}) \geq c_\infty.$$

Proof. Assume first that $v^{(0)} = 0$. By combining (6.17), (6.15) and (6.14) we have necessarily $1 < m < 2$, a contradiction that shows that $v^{(0)} \neq 0$.

Recalling that $w_0 = \frac{w_\infty}{\|w_\infty\|_p}$ is a minimizer for $J_{0,\lambda}$ over all vectors with unit L^p -norm, we have

$$J_{0,\lambda} \left(\frac{w_\infty}{\|w_\infty\|_p} \right) \leq J_{0,\lambda} \left(\frac{|v^{(0)}|}{\|v^{(0)}\|_p} \right),$$

and, by applying the diamagnetic inequality, we get

$$(6.19) \quad J_{0,\lambda} \left(\frac{w_\infty}{\|w_\infty\|_p} \right) \leq J_{A,\lambda} \left(\frac{v^{(0)}}{\|v^{(0)}\|_p} \right).$$

By taking into account that $v^{(0)}$ and w_∞ are solutions to (P) and (P_∞) respectively, we have $J_{0,\lambda}(v^{(0)}) = \|v^{(0)}\|_p^p$ and that (see (6.6)) $J_{A,\lambda}(w_\infty) = \|w_\infty\|_p^p$. Then, from (6.19), follows that

$$\frac{2p}{p-2}c_\infty = J_{0,\lambda}(w_\infty) \leq J_{A,\lambda}(v^{(0)}) = \frac{2p}{p-2}I_{A,\lambda}(v^{(0)}),$$

which immediately gives (6.18). $\square$

PROOF OF THEOREM 6.5. Let R and T be large enough to apply Lemma 6.2 and Lemma 6.3 and get that $c \in (c_\infty, 2c_\infty)$. Since the diamagnetic inequality is strict on $\text{supp } B \cap \text{supp } v^{(0)}$, we deduce, since $v^{(0)} \neq 0$, that

$$c_0 = I_{A,\lambda}(v^{(0)}) > I_{0,\lambda}(v^{(0)}) \equiv I_\infty(v^{(0)}) \geq c_\infty.$$

So, (6.16) implies that $(m+1)c_\infty \leq mc_\infty + c_0 = c < 2c_\infty$, i.e. that $m = 0$. Then, the thesis follows since (6.17) gives, for $m = 0$, $c_0 = I_{A,\lambda}(v^{(0)}) = c$. $\square$

In conclusion we give a straightforward generalization of Theorem 6.5 for a magnetic Schrödinger equation that also includes a nonconstant electric potential. The proof, which follows step by step the one of Theorem 6.5, is left to the reader.

Theorem 6.6. Let $A \in \dot{C}^1(\mathbb{R}^N, \Lambda_1) \setminus \{0\}$, $V \in L^\infty(\mathbb{R}^N)$, and assume condition **(A)** as well as the following conditions:

$$(B') \quad \|B\|_\infty^2 \int_{\mathbb{R}^N} |x|^2 w_\infty^2 dx + \int_{\mathbb{R}^N} (V(x) - \lambda) w_\infty^2 dx \leq (2^{\frac{p-2}{p}} - 1) \frac{2p}{p-2} c_\infty,$$

$$(V) \quad \lambda \stackrel{\text{def}}{=} \lim_{|y| \rightarrow \infty} V(y) \leq V(x), \text{ for a.e. } x \in \mathbb{R}^N.$$

Then, there exists a solution $u \in H_A^{1,2}(\mathbb{R}^N)$ to the equation

$$(6.20) \quad -\nabla_A^2 u + Vu = |u|^{p-2}u,$$

satisfying $I_{A,V}(u) = c > 0$, where

$$I_{A,V}(u) \stackrel{\text{def}}{=} \frac{1}{2} J_{A,V}(u) - \frac{1}{p} \int_{\mathbb{R}^N} |u|^p dx$$

and c is given by the relation (6.13) (with the functional $I_{A,\lambda}$ replaced by $I_{A,V}$) with $R > 0$ and $T \geq 2$ sufficiently large.

APPENDIX: THE GROUP OF MAGNETIC SHIFTS

Magnetic shifts of a lattice-periodic magnetic field do not generally form a group. Let us look at that in more detail. Note first that, for every $y_1, y_2 \in \mathbb{Z}^N$, by using (2.11), we have

$$(6.21) \quad \begin{aligned} g_{y_2} g_{y_1} u &= e^{i\varphi_{y_2}} e^{i\varphi_{y_1}(\cdot - y_2)} u(\cdot - y_1 - y_2) = \\ e^{-i\gamma(y_1, y_2)} e^{i\varphi_{y_1+y_2}} u(\cdot - y_1 - y_2) &= e^{-i\gamma(y_1, y_2)} g_{y_1+y_2}. \end{aligned}$$

In other words, a product of magnetic shifts is a magnetic shift up to a constant scalar multiple of magnitude 1. This proves that a larger set, namely

$$\mathcal{G}_A \stackrel{\text{def}}{=} \{g_{y,\theta} : u \mapsto e^{i\theta} e^{i\varphi_y} u(\cdot - y), u \in H_A^{1,2}(\mathbb{R}^N)\}_{y \in \mathbb{Z}^N, \theta \in \mathbb{R}},$$

is closed under multiplication law. We see below that $\mathcal{G}_A$ is a group. Without loss of generality, since every function φ_y , $y \in \mathbb{Z}^N$ which satisfies (2.10), is defined up to a constant, we may fix its value at a given point, which we choose as follows:

$$(6.22) \quad \varphi_y(y/2) = 0, \quad y \in \mathbb{Z}^N.$$

Then $\varphi_0(0) = 0$ and since $\nabla \varphi_0 = A(\cdot - 0) - A(\cdot) = 0$, we have

$$(6.23) \quad \varphi_0(x) = 0, \quad x \in \mathbb{R}^N.$$

From (2.11) with $y_2 = 0$ it follows immediately that for every $y \in \mathbb{Z}^N$

$$\gamma(y, 0) = \gamma(0, y) = 0.$$

Moreover, by evaluating (2.11) with $y_1 = y$, $y_2 = -y$ at $x = -y/2$, and by using (6.22), we have

$$(6.24) \quad \gamma(y, -y) = \varphi_y(y/2) + \varphi_{-y}(-y/2) + \gamma(y, -y) = \varphi_0(-y/2) = 0$$

for every $y \in \mathbb{Z}^N$.

Lemma 6.7. *For every $y \in \mathbb{Z}^N$, $\theta \in \mathbb{R}$:*

$$(6.25) \quad g_{y,\theta}^{-1} = g_{-y,-\theta}.$$

Proof. Note first that it is sufficient to prove (6.25) for $\theta = 0$. Fixed $y \in \mathbb{Z}^N$, from (6.23) and by applying (2.11) with $y_1 = y$, $y_2 = -y$, we get by (6.24)

$$\varphi_{-y} = -\varphi_y(\cdot + y).$$

Then, solving the equation $g_{y,0}u = v$, one has

$$u = e^{-i\varphi_y(\cdot+y)}v(\cdot + y) = e^{i\varphi_{-y}}v(\cdot + y) = g_{-y}v.$$

□

Note that by (2.15) the set $\mathcal{G}$, and thus, the set $\mathcal{G}_A$, consists of isometries on $H_A^{1,2}(\mathbb{R}^N)$.

Finally, we can see that $\mathcal{G}_A$ is a multiplicative group because it is closed with respect to multiplication (by (6.21)) which is trivially associative; it has $g_{0,0}$ as neutral element (see (6.23)); and any of its elements $g_{y,\theta}$ has its own inverse $g_{-y,-\theta}$ (see (6.25)).

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